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Structure of Cooper Pairs in Superconductors

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In the generalized random-phase approximation, introduced by Anderson and applied to the theory of superconductivity, existence of a collective excitation with vanishing excitation energy and momentum was known. It is shown to correspond to the creation of a Cooper pair. The creation operator satisfies a nonlinear operator equation which dose not depend on the magnitude of order parameter. It is, therefore, independent of dynamics, that is, of the strength of the pairing interactions. Nevertheless it has the assumed matrix elements between a given state in an N-particle system and the corresponding state in the $N+2$ particle system. Discussions are confined to the BCS superconductor at absolute zero temperature.

§ 1. Introduction

In the original formulation of the theory of superconductivity, both BCS[1] and Bogoliubov[2] relaxed the constraint of the electron number conservation in order to simplify mathematics. Gor'kov[3] first pointed out the importance of the matrix element of a product of two electron creation operators between ground states of N- and $N+2$ particle systems, in deriving equations for Green's functions. An operator $R^\dagger$ was then introduced, which transforms a given state in an N-particle system into the corresponding state in the $N+2$ particle system. It should satisfy the relations:

$$R^\dagger|0, N\rangle = |0, N+2\rangle\,, \tag{1·1}$$

$$R^\dagger|\boldsymbol{k}, s; N\rangle = |\boldsymbol{k}, s; N+2\rangle\,, \tag{1·2}$$

$$R|0, N+2\rangle = |0, N\rangle\,, \tag{1·3}$$

as mentioned, for instance, in Schrieffer's book.[4] Here, $|0, N\rangle$ is the ground state of the N-particle system, $|\boldsymbol{k}, s; N\rangle$ a low-lying excited state of the N-particle system with momentum $\hbar\boldsymbol{k}$ and spin s. R is the Hermitian conjugate operator to $R^\dagger$. These operators should commute with electron field operators, if small quantities of relative order $(1/N)$ are neglected. The operator $R^\dagger$ evidently adds a Cooper pair to the condensate, while the conjugate R is annihilation operator of the pair. In terms of these operators, the theory could formally restore the number conservation.

The N-particle projection of the wave function written down by BCS is[5]

$$|0, N\rangle = A_N\left[\sum_{\boldsymbol{k}} \frac{E_k - \varepsilon_k}{\varDelta^*}\, a^\dagger_{\boldsymbol{k}\uparrow} a^\dagger_{-\boldsymbol{k}\downarrow}\right]^{N/2}|0\rangle\,, \tag{1·4}$$

where A_N is a normalization constant, $a_{k\downarrow}^\dagger$ and $a_{k\uparrow}^\dagger$ are the creation operators for electrons with momentum $\hbar\boldsymbol{k}$ and with spin down and up, respectively. Δ^* is the complex conjugate quantity to order parameter Δ. ε_k is the kinetic energy of an electron relative to its value at the Fermi surface

$$\varepsilon_k = \frac{\hbar^2 k^2}{2m} - \mu ,$$

m and μ being the electron mass and the chemical potential, respectively. E_k is the energy of an elementary excitation

$$E_k = \sqrt{\varepsilon_k^2 + |\Delta|^2}.$$

$|0\rangle$ is the vacuum. On combining expression (1·4) with a relation satisfied by $R^\dagger$ (1·1), one has hitherto inferred that $R^\dagger$ might have a form[5]

$$R^\dagger = \frac{2}{N} \sum_k \frac{E_k - \varepsilon_k}{\Delta^*} a_{k\uparrow}^\dagger a_{-k\downarrow}^\dagger . \tag{1·5}$$

Operators X, which give an approximate description of elementary excitations, approximately satisfy

$$[H, X] = -\hbar\omega X . \tag{1·6}$$

Here H is the Hamiltonian. If $\hbar\omega$ is positive, it gives the excitation energy and X is the annihilation operator of the excitation. Relations (1·1), (1·2) and (1·3) indicate the fact that the operators $R^\dagger$ and R are the creation and annihilation operators, respectively, of an elementary excitation with a vanishing excitation energy, the energy being measured relatively to μN. The commutator between H and $R^\dagger$ (1·5), however, gives a term like $a^\dagger a$ in the generalized random-phase approximation and does not reproduce such a relation as (1·6). Moreover, the Hermitian conjugate quantity

$$R = \frac{2}{N} \sum_k \frac{E_k - \varepsilon_k}{\Delta} a_{-k\downarrow} a_{k\uparrow}$$

does not satisfy (1·3), when applied to $|0, N+2\rangle$ (1·4). Thus, expression (1·5) should be insufficient for $R^\dagger$.

The generalized random-phase approximation was first introduced by Anderson[6] and applied to the theory of superconductivity. It was shown that most of the elementary excitations have the BCS energy gap spectrum, but there are collective excitations also. The neutral Fermi gas has a low-lying branch of collective modes, while the charged gas has no low-lying collective modes because of the strong plasma effect. In addition, the existence of a single mode with vanishing excitation energy and momentum was pointed out, regardless of whether the system is neutral or charged. The structure of this mode has not been investigated so far.

The purpose of the present paper is to show that the above single mode gives a correct expression for $R^\dagger$. It increases the number of electrons from N to $N+2$. This elementary excitation is clearly the creation of a Cooper pair. Identifying the operator of the elementary excitation with $R^\dagger$, one obtains a non-linear operator equation for $R^\dagger$. $R^\dagger$ and R are found to satisfy relations (1·1), (1·2) and (1·3).

The discussions will be carried out in a self-consistent way. In deriving the equations for the elementary excitation, we assume the existence of $R^\dagger$ and R which satisfy (1·1), (1·2) and (1·3) and commute with electron field operators. We shall then see that $R^\dagger$ and R, thus obtained, actually satisfy the above assumptions.

In § 2, the elementary excitation is determined from the equations derived by Anderson. In § 3, the operator equation for $R^\dagger$ and R are shown to satisfy (1·1), (1·2) and (1·3). The canonical commutation relation between R and $R^\dagger$ is examined. Finally, in § 4, some comments are made on R.

§ 2. Equations of motion

Anderson[6] introduced the notations

$$b_k{}^Q = a_{-k-Q\downarrow}a_{k\uparrow}\,, \qquad \bar{b}_k{}^Q = a_{k+Q\uparrow}^\dagger a_{-k\downarrow}^\dagger\,,$$

$$\rho_k{}^Q = a_{k+Q\uparrow}^\dagger a_{k\uparrow}\,, \qquad \bar{\rho}_k{}^Q = a_{-k\downarrow}^\dagger a_{-k-Q\downarrow}\,,$$

$$n_k = \langle 0, N|a_{k,\sigma}^\dagger a_{k,\sigma}|0, N\rangle = \frac{1}{2}\left(1-\frac{\varepsilon_k}{E_k}\right), \tag{2·1}$$

$$b_k = \langle 0, N|a_{-k\downarrow}a_{k\uparrow}R^\dagger|0, N\rangle = -\frac{\Delta}{2E_k}\,, \tag{2·2}$$

$$\Delta = -\frac{v}{\Omega}\sum_k b_k\,,$$

$$\omega_{kQ} = \varepsilon_{k+Q} - \varepsilon_k\,, \qquad \Omega_{kQ} = \varepsilon_{k+Q} + \varepsilon_k\,,$$

$$n_{kQ} = n_{k+Q} - n_k\,, \qquad z_{kQ} = 1 - n_k - n_{k+Q}\,,$$

where v is the strength of the pairing interactions and is positive. Ω is the volume of the system. In the generalized random-phase approximation, he got

$$[H, \rho_k{}^Q] = \omega_{kQ}\rho_k{}^Q - \frac{1}{\Omega}v(Q)\,n_{kQ}\rho^Q + \Delta^* b_k{}^Q R^\dagger - \Delta\bar{b}_k{}^Q R$$
$$+\frac{v}{\Omega}b_{k+Q}^*\sum_{k'} b_{k'}{}^Q R^\dagger - \frac{v}{\Omega}b_k\sum_{k'}\bar{b}_{k'}^Q R\,,$$

$$[H, \bar{\rho}_k{}^Q] = -\omega_{kQ}\bar{\rho}_k{}^Q + \frac{1}{\Omega}v(Q)\,n_{kQ}\rho^Q + \Delta^* b_k{}^Q R^\dagger - \Delta\bar{b}_k{}^Q R$$
$$+\frac{v}{\Omega}b_k{}^*\sum_{k'} b_{k'}^Q R^\dagger - \frac{v}{\Omega}b_{k+Q}\sum_{k'}\bar{b}_{k'}^Q R\,,$$

$$[H, b_k{}^Q] = -\Omega_{kQ} b_k{}^Q - \frac{v(Q)}{\Omega}(b_k + b_{k+Q})\rho^Q R$$

$$+ \Delta(\rho_k{}^Q + \bar{\rho}_k{}^Q) R + z_{kQ}\frac{v}{\Omega}\sum_{k'} b_{k'}^Q,$$

$$[H, \bar{b}_k{}^Q] = \Omega_{kQ}\bar{b}_k{}^Q + \frac{v(Q)}{\Omega}(b_k{}^* + b_{k+Q}^*)\rho^Q R^\dagger$$

$$- \Delta^*(\rho_k{}^Q + \bar{\rho}_k{}^Q) R^\dagger - z_{kQ}\frac{v}{\Omega}\sum_{k'} \bar{b}_{k'}^Q .$$

The case of neutral Fermi gas was defined by $v(Q) = \text{const}\ (=-v)$ as $Q \to 0$. In the case of charged gas, $v(Q)$ takes a form

$$v(Q) = 4\pi e^2/Q^2.$$

ρ^Q is given by

$$\rho^Q = \sum_k (\rho_k{}^Q + \bar{\rho}_k{}^Q).$$

The operators R and $R^\dagger$ are supplemented on the right-hand side of the equations to satisfy the number conservation. Annihilation operators X of elementary excitations take the form

$$X = \sum_k (\psi_k{}^Q b_k{}^Q + \phi_k{}^Q \bar{\rho}_k{}^Q R + \xi_k{}^Q \rho_k{}^Q R + \chi_k{}^Q \bar{b}_k{}^Q R^2). \tag{2·3}$$

Substitution of this expression into the equation of motion (1·6) gives

$$-\hbar\omega\psi_k{}^Q = -\Omega_{kQ}\psi_k{}^Q + \Delta^*(\phi_k{}^Q + \xi_k{}^Q) + I_1, \tag{2·4}$$

$$-\hbar\omega\phi_k{}^Q = -\omega_{kQ}\phi_k{}^Q + \Delta\psi_k{}^Q - \Delta^*\chi_k{}^Q + I_2, \tag{2·5}$$

$$-\hbar\omega\xi_k{}^Q = \omega_{kQ}\xi_k{}^Q + \Delta\psi_k{}^Q - \Delta^*\chi_k{}^Q + I_2, \tag{2·6}$$

$$-\hbar\omega\chi_k{}^Q = \Omega_{kQ}\chi_k{}^Q - \Delta(\phi_k{}^Q + \xi_k{}^Q) + I_3, \tag{2·7}$$

$$I_1 = \frac{v}{\Omega}\sum_k z_{kQ}\psi_k{}^Q + \frac{v}{\Omega}\sum_k (b_k{}^*\phi_k{}^Q + b_{k+Q}^*\xi_k{}^Q), \tag{2·8}$$

$$I_2 = \frac{v(Q)}{\Omega}\sum_k n_{kQ}(\phi_k{}^Q - \xi_k{}^Q) - \frac{v(Q)}{\Omega}\sum_k \{(b_k + b_{k+Q})\psi_k{}^Q - (b_k{}^* + b_{k+Q}^*)\chi_k{}^Q\}, \tag{2·9}$$

$$I_3 = -\frac{v}{\Omega}\sum_k z_{kQ}\chi_k{}^Q - \frac{v}{\Omega}\sum_k (b_{k+Q}\phi_k{}^Q + b_k\xi_k{}^Q). \tag{2·10}$$

We have assumed that R and H commute, if small quantities of relative order $(1/N)$ are neglected, and $R^\dagger R = 1$, etc. These are deduced from relations (1·1), (1·2) and (1·3).

To find out a new solution, we first assume that the Coulomb interactions in the charged gas are weakly screened, say, as

$$v(Q)=4\pi e^2/(Q^2+\kappa^2)$$

by, for instance, the finite dimensions of the system. The limit of no screening $\kappa\rightarrow 0$ will be taken after one finds the solution. Equation (2·2) tells us that Eqs. (2·5) and (2·6) are satisfied by

$$\phi_k{}^Q=\xi_k{}^Q=0\,, \tag{2·11}$$

$$\Delta\psi_k{}^Q=\Delta^*\chi_k{}^Q\,. \tag{2·12}$$

Equation (2·4) for $\psi_k{}^Q$ gives, in a non-interacting system with $v=0$, an elementary excitation

$$\hbar\omega=\Omega_{kQ}=\varepsilon_{k+Q}+\varepsilon_k\,.$$

This is a two-particle excitation. Therefore, we will take the outgoing wave boundary condition

$$\hbar\omega\rightarrow\hbar\omega-i\delta\,, \qquad (\delta>0)$$

in Eq. (2·4). Equation (2·7) for $\chi_k{}^Q$, however, will be solved with the incoming wave boundary condition

$$h\omega\rightarrow h\omega+i\delta\,,$$

since the corresponding elementary excitation is a two-hole excitation. Substitution of (2·11) into Eqs. (2·4) and (2·7) immediately shows that the possible solution has to have a vanishing excitation energy $\hbar\omega=0$, and $\psi_k{}^Q$ and $\chi_k{}^Q$ satisfy the same equation

$$(\Omega_{kQ}+i\delta)\psi_k{}^Q=\frac{v}{\Omega}\sum_k z_{kQ}\psi_k{}^Q.$$

This equation has a solution only if the total momentum $\boldsymbol{Q}$ vanishes. Then, $\psi_k{}^0$ takes a form

$$\psi_k{}^0=\frac{A}{2\varepsilon_k+i\delta}\,,$$

where

$$A\equiv\frac{v}{\Omega}\sum_k(1-2n_k)\psi_k{}^0=\frac{v}{\Omega}\sum_k\frac{A}{2E_k}\,.$$

The last relation is nothing but the BCS equation for the energy gap. By condition (2·12), one obtains

$$\psi_k{}^0=\frac{I\Delta^*}{2\varepsilon_k+i\delta}\,, \qquad \chi_k{}^0=\frac{I\Delta}{2\varepsilon_k+i\delta}\,, \tag{2·13}$$

I being a constant to be determined later.

Substitution of (2·11) and (2·13) into the expression for X (2·3) gives an operator of the elementary excitation

$$X=\frac{I}{2}\sum_k\frac{1}{\varepsilon_k+i\delta}(\varDelta^*a_{-k\downarrow}a_{k\uparrow}+\varDelta a^\dagger_{k\uparrow}R^2a^\dagger_{-k\downarrow}).$$

The operators R^2 and $a^\dagger$ are assumed to commute, when N is very large. The possible singular interaction $v(Q)$ is included only in the definition of I_2 (2·9). Since I_2 identically vanishes by the requirements (2·11) and (2·12), the operator X has the same form for the charged and neutral Fermi gases.

§ 3. Operator equation for *R*

Since X should be the annihilation operator of a Cooper pair, the expectation value of the number operator $X^\dagger X$ has to give the number of the Cooper pairs:

$$\langle 0, N|X^\dagger X|0, N\rangle\simeq N/2\,.$$

Making a comparison with the relation

$$R^\dagger R\simeq 1\,,$$

we get

$$R=\sqrt{\frac{2}{N}}X=\frac{I}{\sqrt{2N}}\sum_k\frac{1}{\varepsilon_k+i\delta}(\varDelta^*a_{-k\downarrow}a_{k\uparrow}+\varDelta a^\dagger_{k\uparrow}R^2a^\dagger_{-k\downarrow}),\tag{3·1}$$

except a possible phase factor. The constant I will be fixed from relation (1·3). The calculation of $R|0, N+2\rangle$ will be carried out in a self-consistent way: the operator R^2 on the right-hand side of (3·1) is assumed to satisfy a relation like (1·3) and the total R (3·1) is shown to reproduce (1·3), if the constant I is suitably taken.

The normalization constant A_N of the wave function $|0, N\rangle$, (1·4), satisfies

$$NA_N/2\simeq A_{N-2}\,,\tag{3·2}$$

if small quantities of relative order $(1/N)$ are neglected.[7] Equation (3·2) will be derived in the Appendix. The state $R|0, N+2\rangle$ can now be rewritten

$$\begin{aligned}R|0, N+2\rangle&=\frac{I}{\sqrt{2N}}\sum_k\frac{1}{\varepsilon_k+i\delta}\{(E_k-\varepsilon_k)a_{-k\downarrow}a^\dagger_{-k\downarrow}|0, N\rangle+\varDelta a^\dagger_{k\uparrow}a^\dagger_{-k\downarrow}|0, N-2\rangle\}\\&=\frac{I}{\sqrt{2N}}\left\{\sum_k\frac{E_k-\varepsilon_k}{\varepsilon_k+i\delta}|0, N\rangle+\sum_k\frac{\varDelta}{\varepsilon_k+i\delta}\left(1-\frac{(E_k-\varepsilon_k)^2}{|\varDelta|^2}\right)a^\dagger_{k\uparrow}a^\dagger_{-k\downarrow}|0, N-2\rangle\right\}\\&=\frac{I}{\sqrt{2N}}\left\{N(0)\varOmega\int d\varepsilon_k(-i\pi E_k\delta(\varepsilon_k)-1)|0, N\rangle\right.\\&\qquad\left.+2\sum_k\frac{E_k-\varepsilon_k}{\varDelta^*}a^\dagger_{k\uparrow}a^\dagger_{-k\downarrow}|0, N-2\rangle\right\}\\&=(-i\pi|\varDelta|N(0)\varOmega I/\sqrt{2N})|0, N\rangle.\end{aligned}\tag{3·3}$$

Here, $N(0)$ is the density of Bloch states of one spin per unit volume per unit energy at the Fermi surface, and a condition

$$N(0)\Omega \int d\varepsilon_k = N$$

has been used, the integral extending over states within the range where $|\varepsilon_k|$ is less than the average phonon energy. Equation (3·3) conforms to (1·3), if we put

$$I = \frac{\sqrt{2N}i}{\pi|\Delta|N(0)\Omega}, \tag{3·4}$$

which is a small quantity of order $1/\sqrt{N}$. The equation for R (3·1) becomes, with the help of Eq. (3·4),

$$R = \frac{i}{\pi N(0)\Omega}\sum_k \frac{1}{\varepsilon_k + i\delta}(e^{-i\phi}a_{-k\downarrow}a_{k\uparrow} + e^{i\phi}a_{k\uparrow}^\dagger R^2 a_{-k\downarrow}^\dagger), \tag{3·5}$$

where ϕ is the phase of the order parameter

$$\Delta = |\Delta|e^{i\phi}.$$

A similar argument immediately shows that the Hermitian conjugate operator $R^\dagger$ self-consistently satisfies relation (1·1). Commutators $[a_{k\uparrow}, R]$, $[a_{-k\downarrow}, R]$, $[a_{k\uparrow}, R^\dagger]$, etc., give small quantities of order $1/N$. This implies the validity of (1·2), since elementary excitations are usually represented as linear combinations of a's and $a^\dagger$'s or of products of these operators. To obtain these commutators up to terms of order $1/N$, we may write

$$[a_{k\uparrow}, R] = \frac{i}{\pi N(0)\Omega}\left[\frac{e^{i\phi}}{\varepsilon_k + i\delta}a_{-k\downarrow}^\dagger R^2 + \sum_{k'}\frac{2}{\varepsilon_{k'} + i\delta}e^{i\phi}a_{k'\uparrow}^\dagger a_{-k'\downarrow}^\dagger R[a_{k\uparrow}, R]\right]. \tag{3·6}$$

R and $[a_{k\uparrow}, R]$ on the right-hand side may be regarded as if they were c-numbers. Equation (3·6) can be formally solved and gives

$$[a_{k\uparrow}, R] = \frac{ie^{i\phi}}{\pi N(0)\Omega(\varepsilon_k + i\delta)}\cdot J^{-1}a_{-k\downarrow}^\dagger R^2, \tag{3·7}$$

with

$$J = 1 - \frac{2ie^{i\phi}}{\pi N(0)\Omega}\sum_{k'}\frac{a_{k'\uparrow}^\dagger a_{-k'\downarrow}^\dagger R}{\varepsilon_{k'} + i\delta}.$$

Similarly, we find

$$[a_{-k\downarrow}, R] = -\frac{ie^{i\phi}}{\pi N(0)\Omega(\varepsilon_k + i\delta)}\cdot J^{-1}a_{k\uparrow}^\dagger R^2,$$

$$[a_{k\uparrow}^\dagger, R] = -\frac{ie^{-i\phi}}{\pi N(0)\Omega(\varepsilon_k + i\delta)}\cdot J^{-1}a_{-k\downarrow}, \tag{3·8}$$

$$[a^\dagger_{-k\downarrow}, R] = \frac{ie^{-i\phi}}{\pi N(0)\Omega(\varepsilon_k + i\delta)} \cdot J^{-1} a_{k\uparrow} .$$

In the random-phase approximation, the operators X, given in (1·6), have to satisfy the usual commutation relations between creation and annihilation operators as an average. In the present case, the requirement takes the form

$$\langle 0, N | [X, X^\dagger] | 0, N \rangle = 1 ,$$

which, with the help of (3·1), gives

$$\langle 0, N | [R, R^\dagger] | 0, N \rangle = \frac{2}{N} . \tag{3·9}$$

Equation (3·9) will be proved again in a self-consistent way. Supposing R and $R^\dagger$, on the right-hand side of (3·5), have a commutation relation

$$[R, R^\dagger] = \frac{2}{N} , \tag{3·10}$$

at least, among the ground- and low-lying excited states as (3·9), we can show the total R and $R^\dagger$ reproduce (3·9). Equation (3·10) gives

$$[R^2, R^{\dagger 2}] \simeq 4[R, R^\dagger] = \frac{8}{N} , \tag{3·11}$$

if quantities of order $1/N^2$ are neglected. Equation (3·5) and the Hermitian conjugate expression for $R^\dagger$ give the commutation relation

$$\begin{aligned}[R, R^\dagger] = \frac{1}{(\pi N(0)\Omega)^2} \sum_{k,k'} \frac{1}{\varepsilon_k + i\delta} \cdot \frac{1}{\varepsilon_{k'} - i\delta} [& e^{-2i\phi} a_{-k'\downarrow} [a_{-k\downarrow} a_{k\uparrow}, R^{\dagger 2}] a_{k'\uparrow} \\ & - e^{2i\phi} a^\dagger_{k\uparrow} [a^\dagger_{k'\uparrow} a^\dagger_{-k'\downarrow}, R^2] a^\dagger_{-k\downarrow} + a^\dagger_{k\uparrow} [a_{-k'\downarrow}, R^2] a^\dagger_{-k\downarrow} R^{\dagger 2} a_{k'\uparrow} \\ & + a_{-k'\downarrow} \{ a^\dagger_{k\uparrow} R^2 [a^\dagger_{-k\downarrow}, R^{\dagger 2}] + a^\dagger_{k\uparrow} [R^2, R^{\dagger 2}] a^\dagger_{-k\downarrow} + [a^\dagger_{k\uparrow}, R^{\dagger 2}] R^2 a^\dagger_{-k\downarrow} \} a_{k'\uparrow} \\ & + a_{-k'\downarrow} R^{\dagger 2} a^\dagger_{k\uparrow} [a_{k'\uparrow}, R^2] a^\dagger_{-k\downarrow}] .\end{aligned} \tag{3·12}$$

With the help of (3·7) and (3·8), it will be found out that all terms cancel each other except the one with $[R^2, R^{\dagger 2}]$, (3·12) being reduced to

$$[R, R^\dagger] = \frac{8}{(\pi N(0)\Omega)^2 N} \sum_{k,k'} \frac{1}{\varepsilon_k + i\delta} \cdot \frac{1}{\varepsilon_{k'} - i\delta} a_{-k'\downarrow} a^\dagger_{k\uparrow} a^\dagger_{-k\downarrow} a_{k'\uparrow} . \tag{3·13}$$

Use has been made of (3·11). Since the right-hand side of (3·13) is already a quantity of order $1/N$, we may use the wave function of $|0, N\rangle$ in its form of the lowest order approximation, (1·4) or the original BCS form, to calculate the expectation value

$$\langle 0, N | a_{-k'\downarrow} a^\dagger_{k\uparrow} a^\dagger_{-k\downarrow} a_{k'\uparrow} | 0, N \rangle = b_{k'} b_k^* = \frac{|\Delta|^2}{4E_k E_{k'}} .$$

b_k is defined in (2·2). Substitution to (3·13) gives

$$\langle 0, N|[R, R^\dagger]|0, N\rangle = \frac{2|\Delta|^2}{(\pi N(0)\Omega)^2 N}\sum_{k,k'}\frac{1}{(\varepsilon_k+i\delta)(\varepsilon_{k'}-i\delta)E_kE_{k'}} = \frac{2}{N},$$

which is (3·9).

The non-linear equation (3·5) has a formal solution

$$R = \frac{2}{1+\sqrt{1+4U}}\frac{i}{\pi N(0)\Omega}\sum_k \frac{e^{-i\phi}}{\varepsilon_k+i\delta}a_{-k\downarrow}a_{k\uparrow},$$

where

$$U = \frac{1}{(\pi N(0)\Omega)^2}\sum_{k,k'}\frac{1}{(\varepsilon_k+i\delta)(\varepsilon_{k'}+i\delta)}a_{-k\downarrow}a_{k\uparrow}a^\dagger_{k'\uparrow}a^\dagger_{-k'\downarrow}.$$

The annihilation of a Cooper pair turns out to be a superposition of many components. The n-th component annihilates n pairs and creates $n-1$ pairs. The structure of the Cooper pair is, thus, further complicated than that given by the simple formula (1·5).

§ 4. Discussions

The equations of motion for the elementary excitations, derived in the generalized random-phase approximation, have a solution which corresponds to a creation or annihilation of a Cooper pair. One finds a non-linear equation (3·5) for the operator of the pair. It is remarkable that the equation depends only on the phase of the order parameter and not on its absolute magnitude. Therefore, the operators R and $R^\dagger$ do not depend on the strength of the pairing interactions. It may be probable that their structure, investigated in the present work, may be closely related with some symmetry property of the system, since they are practically constants of motion.

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Appendix

Relation $NA_N/2 \simeq A_{N-2}$

Although this relation was implied in a work by Nakamura,[7] we shall briefly summarize here another way of obtaining it, since wave functions are given in a different representation.

Let us put

$$\sum_k\left(\frac{E_k-\varepsilon_k}{|\Delta|}\right)^{2n} = \Omega c_n, \qquad X^\dagger_{n+1} = \sum_k \frac{|\Delta|}{\Delta^*}\left(\frac{E_k-\varepsilon_k}{|\Delta|}\right)^{2n+1}a^\dagger_{k\uparrow}a^\dagger_{-k\downarrow},$$

$$\Psi_M = (X_1^\dagger)^M|0\rangle. \tag{A·1}$$

The ground state of the N-particle system (1·4) is then given by

$$|0, N\rangle = A_N \Psi_{N/2}\,. \tag{A·2}$$

We can also put

$$X_1{}^n \Psi_M = \sum_{\{n_l, m_{l'}\}=0,1,\cdots} \prod_{\substack{l=1,2,\cdots \\ l'=2,3,\cdots}} (\Omega c_l)^{n_l} (X_{l'}^\dagger)^{m_{l'}} \Psi_{M-n-\sum\limits_{l'=2} m_{l'}}$$

$$\times f_n(n_1, n_2, \cdots; m_1, m_2, \cdots)\,. \tag{A·3}$$

The summations are to be taken on condition

$$n = \sum_{l=1} l n_l + \sum_{l'=2} (l'-1) m_{l'}\,,$$

and m_1 is defined by

$$m_1 = M - n - \sum_{l'=2} m_{l'}\,.$$

Multiplication of the operator X_1 on (A·3) leads to a recurrence formula

$$\begin{aligned} &f_{n+1}(n_1, n_2, \cdots; m_1, m_2, \cdots) \\ &\quad = \sum_l (m_l+1) f_n(n_1, \cdots, n_l-1, \cdots; m_1, \cdots, m_l+1, \cdots) \\ &\qquad - \sum_l (m_l+2)(m_l+1) f_n(n_1, n_2, \cdots; m_1, \cdots, m_l+2, \cdots, m_{2l}-1, \cdots) \\ &\qquad - \sum_{l\neq l'} (m_l+1)(m_{l'}+1) \\ &\qquad\qquad \times f_n(n_1, n_2, \cdots; m_1, \cdots, m_l+1, \cdots, m_{l'}+1, \cdots, m_{l+l'}-1, \cdots)\,. \end{aligned} \tag{A·4}$$

Introducing functions g_n by

$$g_n(n_1, n_2, \cdots; m_1, m_2, \cdots) = \sqrt{m_1!\, m_2! \cdots}\, f_n(n_1, n_2, \cdots; m_1, m_2, \cdots)\,, \tag{A·5}$$

we define operators b_i and $b_i{}^\dagger$ for artificial Bose particles by

$$\langle n_1, n_2, \cdots; m_1, \cdots, m_i, \cdots | b_i | g_n\rangle = \sqrt{m_i+1}\, g_n(n_1, n_2, \cdots; m_1, \cdots, m_i+1, \cdots)\,,$$

$$\langle n_1, n_2, \cdots; m_1, \cdots, m_i, \cdots | b_i{}^\dagger | g_n\rangle = \sqrt{m_i}\, g_n(n_1, n_2, \cdots; m_1, \cdots, m_i-1, \cdots)\,,$$

where

$$\langle n_1, n_2, \cdots; m_1, m_2, \cdots | g_n\rangle = g_n(n_1, n_2, \cdots, m_1, m_2, \cdots)\,. \tag{A·6}$$

b_i's and b_j's satisfy

$$b_i b_j{}^\dagger - b_j{}^\dagger b_i = \delta_{ij}\,.$$

One further introduces operators $V_l{}^\dagger$ by

$$\langle n_1, n_2, \cdots, n_l, \cdots; m_1, m_2, \cdots | V_l{}^\dagger | g_n\rangle = g_n(n_1, \cdots, n_l-1, \cdots; m_1, m_2, \cdots)\,,$$

and rewrite (A·4) as

$$g_{n+1} = \Big(\sum_l V_l{}^\dagger b_l - \sum_{l,l'} b_{l+l'}^\dagger b_l b_{l'}\Big) g_n\,.$$

This formula immediately gives

$$\sum_{n=0}^{\infty}\frac{1}{n!}\xi^n g_n=\exp\{\xi(\sum V_l{}^{\dagger}b_l-\sum_{l,l'}b_{l+l'}^{\dagger}b_lb_{l'})\}g_0$$

$$=\exp\{-\xi\sum_{l,l'}b_{l+l'}^{\dagger}b_lb_{l'}\}$$

$$\times\exp\left\{\sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}\xi^{n+1}\sum_{l_1,l_2,\cdots}V_{l_1+l_2+\cdots+l_{n+1}}^{\dagger}b_{l_1}\cdots b_{l_{n+1}}\right\}g_0\,. \tag{A·7}$$

Since we are interested only in g_M, ($M=N/2$), where there are no b-particles, the first exponential factor on the right-hand side of (A·7) can be effectively put at unity. The definition of g_n (A·5), with the help of (A·6), leads to

$$g_0=(b_1{}^{\dagger})^M|0),$$

$|0)$ being the vacuum of b-particles. Therefore, in (A·7), all b_{l_i} should be b_1. It is easily found that

$$\frac{1}{M!}g_M=\sum_{m=0}^{\infty}\frac{(-1)^m}{m!}\sum_{\substack{n_1,n_2,\cdots\\ \Sigma l n_l=M\\ \Sigma n_l=m}}\frac{m!}{n_1!\,n_2!\cdots}\prod_l\left\{\frac{(-1)^l}{l}V_l{}^{\dagger}\right\}^{n_l}(b_1)^M g_0\,. \tag{A·8}$$

Finally, with the help of (A·1), (A·3) and (A·8), one obtains

$$\langle\Psi_M|\Psi_M\rangle=\langle 0|(X_1)^M|\Psi_M\rangle$$

$$=\sum_{\substack{n_l\\ M=\Sigma l n_l}}(\prod_l(\Omega c_l)^{n_l})g_M(n_1,n_2,\cdots;m_1=0,0,\cdots)$$

$$=\sum_{\substack{n_l\\ M=\Sigma l n_l}}(M!)^2\prod_l\frac{(\Omega b_l)^{n_l}}{n_l!}\,,$$

where

$$\Omega b_l=\frac{(-1)^{l-1}}{l}\Omega c_l=\frac{(-1)^{l-1}}{l}\sum_k\left(\frac{E_k-\varepsilon_k}{|\Delta|}\right)^{2l}. \tag{A·9}$$

The normalization factor A_N in (A·2) has a magnitude

$$|A_N|=\left(\left(\frac{N}{2}\right)!\right)^{-1}Q_N{}^{-1/2}, \tag{A·10}$$

with

$$Q_N=\sum_{\substack{n_l\\ N/2=\Sigma l n_l}}\prod_l\frac{(\Omega b_l)^{n_l}}{n_l!}\,.$$

This is the result derived by Nakamura. Since N is even, we put $N=2M$ and

$$\Xi(z)=\sum_{M=0}^{\infty}z^M Q_{2M}=\exp\{\sum_l\Omega b_l z^l\}\,.$$

Then, Q_{2M} takes a form

$$Q_{2M}=\frac{1}{2\pi i}\oint\frac{\Xi(z)}{z^{M+1}}dz=\frac{1}{2\pi i}\oint\frac{1}{z}\exp(\sum_l \Omega b_l z^l - M\log z)\,dz\,. \qquad (\mathrm{A}\cdot 11)$$

The contour is a circle around $z=0$ in a counter-clockwise direction. The integral (A·11) is evaluated by the saddle point method. The saddle point is on the real axis at $z=z_0$, which is determined by

$$\sum_l l\Omega b_l z_0{}^l = M\,. \qquad (\mathrm{A}\cdot 12)$$

Similarly, Q_{2M-2} can be rewritten as

$$Q_{2M-2}=\frac{1}{2\pi i}\oint\exp(\sum_l \Omega b_l z^l - M\log z)\,dz\,,$$

which has the same saddle point z_0. Therefore, if a small quantity of relative order $1/N$ is neglected, we get

$$Q_{2M}/Q_{2M-2}\simeq 1/z_0\,. \qquad (\mathrm{A}\cdot 13)$$

Substitution of (A·9) into (A·12) leads to

$$M=\frac{N}{2}=\sum_k\frac{z_0(E_k-\varepsilon_k)^2}{|\varDelta|^2+z_0(E_k-\varepsilon_k)^2}\,.$$

This equation determines z_0, with the help of (2·1), as

$$z_0\simeq 1\,. \qquad (\mathrm{A}\cdot 14)$$

Equation (A·10), (A·13) and (A·14) gives

$$A_N/A_{N-2}=2/N\,,$$

which is (3·2).

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