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STRUCTURE OF GENERAL IDEAL SEMIGROUPS OF MONOIDS AND DOMAINS

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ABSTRACT. Let H be a monoid (respectively, an integral domain) and r an ideal system on H. In this paper we investigate the r-ideal semigroup of H. One goal is to specify monoids such that their r-ideal semigroup possesses semigroup-theoretical properties, like almost completeness, π-regularity and completeness. Moreover, if H is an integral domain and $*$ a star operation on H, then we provide conditions on H such that the idempotents of the $*$-ideal semigroup are trivial or such that H is $\pi*$-stable.

0. Introduction. In 1961 Dade, Taussky and Zassenhaus [**9**] investigated the semigroup structure of the ideal class semigroup of one-dimensional Noetherian domains with a focus on non-principal orders in algebraic number fields. In the sequel, this paper seems to have fallen into oblivion. It was reconsidered and generalized by Halter-Koch [**17**], who put the results into the context of the structure theory of semigroups as presented in [**14**]. In recent times, starting with a paper by Zanardo and Zannier [**23**], the structure of the ideal class semigroup of an integral domain attracted the interest of several authors. In particular, the Clifford and Boolean properties of ideal class semigroups was investigated. First, this was done for valuation domains in [**7**]. In the sequel, Bazzoni provided a general theory, focused on Prüfer domains [**2**–**5**]. Among others, she proved that a Prüfer domain has a Clifford semigroup if and only if it has finite character, and she highlighted the connection with stable domains. Following Bazzoni, Kabbaj and Mimouni in [**20**–**22**] investigated the Clifford and Boolean properties of the t-ideal class semigroup. Among others, they characterized Prüfer v-multiplication domains with Clifford t-class semigroup and determined the structure of their constituent groups.

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A more general approach, based on the theory of ideal systems on commutative cancelative monoids, was recently presented by Halter-Koch [**18**]. He observed that the semigroup structure of the ideal class semigroup is essentially determined by the semigroup structure of the ideal semigroup itself. Among others, he proved that the semigroup of fractional r-ideals of an r-Prüfer monoid is a Clifford semigroup if and only if it has the local invertibility property (a generalization of Bazzoni's conjecture for Prüfer domains), and he characterized v-domains with a Clifford v-ideal semigroup.

A valuable overview of Clifford and Boolean properties of (t-)ideal class semigroups of integral domains may be found in the survey article by Bazzoni and Kabbaj [**6**]. A collection of important results concerning the ideal class semigroups of Prüfer domains can be found in [**10**].

In this paper, we reconsider the more general semigroup theoretical properties of ideal semigroups as was done in [**9, 17**], but now we depart from the special cases of one-dimensional or Prüfer domains. Following [**18**], we present our theory as far as possible in the language of ideal systems on commutative monoids [**19**]. We provide conditions on a monoid or domain which entail nice semigroup-theoretic properties (as almost completeness, π-regularity and completeness) of the ideal class semigroup in question.

Our main subject of interest is the v-ideal (class) semigroup of a Mori domain. The main results are in Section 4, and there the case of $* = v$ is of particular interest. The strength of our results in the ring-theoretical case is illustrated by a series of examples and counter-examples in Section 5.

In Section 1 we recall the relevant definitions and facts from the structure theory of commutative semigroups, essentially following [**14**]. In Section 2 we introduce ideal systems and r-ideal semigroups, summarize some of their elementary properties and provide first results concerning stability and completeness. In Section 3 we discuss preparatory ring-theoretical results concerning Mori domains and a general prime avoidance lemma (Lemma 3.2) which seems to be new. As already mentioned, Section 4 contains the main results, in particular, criteria for the idempotents of the ideal semigroup to be trivial and criteria for πr-stability.

1. Commutative semigroups. *Throughout this section, let S be a multiplicative commutative semigroup.* Our main reference for the theory of commutative semigroups is [**14**]. We denote by $\mathsf{E}(S)$ the set of all idempotents of S, endowed with the Rees order $\le$, defined by $e \le f$ if $ef = e$. Note that $ef \le e$, for all $e, f \in \mathsf{E}(S)$. For $a \in S$, let $\mathsf{E}(a) = \{e \in \mathsf{E}(S) \mid ea = a\}$ be the set of all idempotents belonging to a, and set $\mathsf{E}^\infty(a) = \cup_{k\in\mathbf{N}}\mathsf{E}(a^k)$. Then $\mathsf{E}(a) \subseteq \mathsf{E}^\infty(a) \subseteq \mathsf{E}(S) \subseteq S$ are subsemigroups. If $I \subseteq \mathsf{E}(S)$ is a subsemigroup, then every minimal element of I (with respect to the Rees-order) is a least element. In particular, if I is finite, then I has a minimum.

An element $a \in S$ is called

• *regular* if elements $e \in \mathsf{E}(a)$ and $b \in S$ exist such that $ab = e$ (equivalently: some $b \in S$ exist such that $a^2b = a$).

• *π-regular* if a^n is regular for some $n \in \mathbf{N}$.

The semigroup S is called

• *regular* or a *Clifford semigroup* if every $a \in S$ is regular.

• *π-regular* if every $a \in S$ is π-regular.

• *almost complete* if for every $a \in S$, the set $\mathsf{E}(a)$ possesses a minimum (in the Rees-order).

• *complete* if it is π-regular and almost complete.

For $a \in \mathsf{E}(S)$, let $P_a^* = aS \setminus \cup_{b\in\mathsf{E}(S), b<_S a} bS$ be the partial Ponizovski factor of a. Note that the partial Ponizovski factors are essential invariants for the structure of S (see [**14**] and [**17**, Theorems 2.4 and 2.5]).

Lemma 1.1. *Let $a \in S$.*

1. *If $n \in \mathbf{N}$ such that a^n is regular, then a^k is regular for all $k \in \mathbf{N}_{\ge n}$.*

2. *If $k, n \in \mathbf{N}$ exist such that $k \ne n$ and $a^k = a^n$, then a is π-regular.*

3. *If a is π-regular, then $\mathsf{E}^\infty(a)$ has a smallest element in the Rees order.*

Proof. 1. Let $n \in \mathbf{N}$ be such that a^n is regular, and let $k \in \mathbf{N}_{\ge n}$. Some $l \in \mathbf{N}$ exists such that $nl > k$. Since a^n is regular, $e \in \mathsf{E}(a^n)$

and $b \in S$ exist such that $a^n b = e$. Let $b' = a^{nl-k} b^l \in S$. Then $a^k b' = a^{nl} b^l = e^l = e$. Since $e \in \mathsf{E}(a^k)$, it follows that a^k is regular.

2. Let $k, n \in \mathbf{N}$ be such that $k < n$ and $a^k = a^n$. It is straightforward to show that $a^k = a^{k+i(n-k)}$ for all $i \in \mathbf{N}$. Some $j \in \mathbf{N}$ exists such that $j(n-k) > k$; hence, $a^{j(n-k)} = a^k a^{j(n-k)-k} = a^{k+j(n-k)} a^{j(n-k)-k} = a^{2j(n-k)}$. It follows that $a^{j(n-k)}$ is regular, and thus a is π-regular.

3. Let a be π-regular. Then $n \in \mathbf{N}$, $e \in \mathsf{E}(a^n)$ and $b \in S$ exist such that $a^n b = e$. This implies that $e \in \mathsf{E}^\infty(a)$. We assert that $e = \min \mathsf{E}^\infty(a)$. Let $f \in \mathsf{E}^\infty(a)$. There is some $m \in \mathbf{N}$ such that $fa^m = a^m$. If $l \in \mathbf{N}$ is such that $nl > m$, then $ef = e^l f = a^{nl} b^l f = a^{nl-m} a^m f b^l = a^{nl-m} a^m b^l = a^{nl} b^l = e^l = e$. $\square$

2. Ideal systems and r-ideal semigroups. Next we recall the most important facts about ideal systems. For a basic introduction into ideal systems see [**15**]. We briefly gather the required terminology. In the following a monoid is always a multiplicative commutative semigroup that possesses an identity element and a zero element and where every non-zero element is cancelative. A submonoid of a monoid H always contains the zero element and the identity element of H. A monoid K is called quotient monoid of a submonoid H if $K \backslash \{0\}$ is a quotient group of $H \backslash \{0\}$. Quotient monoids of monoids are essentially unique. A submonoid T of a quotient monoid of a monoid H is called overmonoid of H if $H \subseteq T$. For a monoid H, let $H^\bullet = H \backslash \{0\}$.

Throughout this section, let H be a monoid and K a quotient monoid of H. A subset $X \subseteq K$ is called fractional if some $c \in H^\bullet$ exists such that $cX \subseteq H$. Let $\mathbf{F}(H)$ denote the set of all fractional subsets of K. We set $\mathbf{F}^\bullet(H) = \{I \in \mathbf{F}(H) \mid I \backslash \{0\} \neq \varnothing\}$. For arbitrary $X, Y \subseteq K$, let $(X : Y) = \{z \in K \mid zY \subseteq X\}$, $X^{-1} = (H : X)$ and $\mathbf{R}(X) = (X : X)$. Let $\widehat{H}$ denote the complete integral closure of H. Note that for all $X \in \mathbf{F}^\bullet(H)$, $\mathbf{R}(X)$ is a submonoid of $\widehat{H}$. A subset $X \subseteq H$ is called multiplicatively closed if X contains the identity element and $XX \subseteq X$. A subset $\varnothing \neq P \subsetneq H$ is called a prime ideal of H if $HP = P$ and $H \backslash P$ is multiplicatively closed. For a multiplicatively closed subset $T \subseteq H^\bullet$, let $T^{-1}H = \{t^{-1}x \mid t \in T, x \in H\}$, and for a prime ideal P of H, let $H_P = (H \backslash P)^{-1} H$. If T and S are submonoids of K such that $T \subseteq S$, then let $\mathcal{F}_{S/T} = (T : S)$. Especially if T is an overmonoid of H, then $\mathcal{F}_{T/H} = T^{-1}$.

A map $r : \mathbf{F}(H) \to \mathbf{F}(H)$, $I \mapsto I_r$ is called the ideal system on H, if, for all $X, Y \in \mathbf{F}(H)$ and all $c \in K^\bullet$, it follows that:

- $X \cup \{0\} \subseteq X_r$.
- If $X \subseteq Y_r$, then $X_r \subseteq Y_r$.
- $(cX)_r = cX_r$.
- $H_r = H$.

Throughout this section let r be an ideal system on H. Observe that for all $X, Y \in \mathbf{F}^\bullet(H)$, it follows that $(XY)_r = (X_rY)_r = (X_rY_r)_r$ and $(X_r : Y)_r = (X_r : Y) = (X_r : Y_r)$. (For a proof see [**15**, Propositions 2.3 and 11.7]). Furthermore, it is straightforward to show that if $\mathcal{F}_{\widehat{H}/H} \neq \{0\}$, then $\widehat{H}, \mathcal{F}_{\widehat{H}/H} \in \mathcal{F}_r^\bullet(H)$. Next we introduce the most important ideal systems.

Let $s : \mathbf{F}(H) \to \mathbf{F}(H)$ be defined by $s(\varnothing) = \{0\}$ and $s(I) = IH$ for all $I \in \mathbf{F}(H)\backslash\{\varnothing\}$.

Let $v : \mathbf{F}(H) \to \mathbf{F}(H)$ be defined by $I_v = (I^{-1})^{-1}$ for all $I \in \mathbf{F}(H)$.

If H is a domain, then let $d : \mathbf{F}(H) \to \mathbf{F}(H)$ be defined by $I_d = (I)_R$ for all $I \in \mathbf{F}(H)$.

Observe that s, v and d are ideal systems on H. Let $\mathcal{F}_r^\bullet(H) = \{I \in \mathbf{F}^\bullet(H) \mid I_r = I\}$. If $X \in \mathbf{F}(H)$, then let $\mathbf{R}_r(X) = \mathbf{R}(X_r)$. If r' is another ideal system on H, then we set $r \leq r'$ if $\mathcal{F}_{r'}^\bullet(H) \subseteq \mathcal{F}_r^\bullet(H)$ (equivalently: for all $I \in \mathbf{F}^\bullet(H)$, it follows that $I_r \subseteq I_{r'}$). r is called finitary if, for all $X \in \mathbf{F}(H)$, it follows that $X_r = \cup_{F \subseteq X, |F| < \infty} F_r$. Let $\cdot_r : \mathcal{F}_r^\bullet(H) \times \mathcal{F}_r^\bullet(H) \to \mathcal{F}_r^\bullet(H)$ be defined by $I \cdot_r J = (IJ)_r$, for all $I, J \in \mathcal{F}_r^\bullet(H)$. Then $(\mathcal{F}_r^\bullet(H), \cdot_r)$ is a commutative semigroup, called the r-ideal semigroup.

We say that H is r-Noetherian, if H satisfies one of the following equivalent conditions:

- Every ascending sequence of elements of $\{I \in \mathcal{F}_r^\bullet(H) \mid I \subseteq H\}$ becomes stationary.
- Every subset $\varnothing \neq \mathcal{M} \subseteq \{I \in \mathcal{F}_r^\bullet(H) \mid I \subseteq H\}$ has a maximal element.
- If $I \in \mathbf{F}^\bullet(H)$, then some finite $E \subseteq I$ exists such that $I_r = E_r$.

If $r = v$, then these conditions are equivalent to each the following conditions:

• Every descending sequence of elements of $\mathcal{F}_r^\bullet(H)$ with non-zero intersection becomes stationary.

• Every subset $\varnothing \neq \mathcal{M} \subseteq \mathcal{F}_r^\bullet(H)$ with $\cap_{I\in\mathcal{M}} I \neq \{0\}$ has a minimal element.

For $I, J \in \mathcal{F}_r^\bullet(H)$, we set $I \sim_r J$ if some $c \in K^\bullet$ exists such that $I = cJ$. It is easily checked that $\sim_r$ is an equivalence relation on $\mathcal{F}_r^\bullet(H)$. Let $\mathcal{S}_r(H) = \mathcal{F}_r^\bullet(H)/\sim_r$. For $I \in \mathcal{F}_r^\bullet(H)$ we denote by $[I]_{\sim_r}$ the equivalence class of I. Let $\bullet_r : \mathcal{S}_r(H) \times \mathcal{S}_r(H) \to \mathcal{S}_r(H)$ be defined by $[I]_{\sim_r} \bullet_r [J]_{\sim_r} = [(IJ)_r]_{\sim_r}$, for all $I, J \in \mathcal{F}_r^\bullet(H)$. Then $(\mathcal{S}_r(H), \bullet_r)$ is a commutative semigroup, called the r-ideal class semigroup.

From now on we assume that $s \leq r$. Let $r\text{-spec}\,(H) = \{P \in \mathcal{F}_r^\bullet(H) \mid P$ be a prime ideal of $H\}$, and let $r\text{-max}(H)$ be the set of all maximal elements of $\{I \in \mathcal{F}_r^\bullet(H) \mid I \subsetneqq H, IH = I\}$. If r is finitary and $T \subseteq H^\bullet$ is multiplicatively closed, then let $T^{-1}r : \mathbf{F}(T^{-1}H) \to \mathbf{F}(T^{-1}H)$ be defined by $T^{-1}r(T^{-1}X) = T^{-1}X_r$, for all $X \in \mathbf{F}(H)$. Then $T^{-1}r$ is an ideal system on $T^{-1}H$. If P is a prime ideal of H, then let $r_P = (H\backslash P)^{-1}r$. If $H' \in \mathcal{F}_r^\bullet(H)$ is an overmonoid of H, then let $r[H'] : \mathbf{F}(H) \to \mathbf{F}(H)$ be defined by $r[H'](X) = (XH')_r$, for all $X \in \mathbf{F}(H)$. Observe that $r[H']$ is an ideal system on H'. Note that this construction is investigated in [**16**] in the more general context of module systems.

Next we introduce the concept of r- and πr-stability. Note that the r-stability of H is investigated in [**11**] for integral domains and in [**18**] for arbitrary monoids.

An element $I \in \mathcal{F}_r^\bullet(H)$ is called

• *r-invertible* if $(II^{-1})_r = H$.

• *r-regular* if I is regular as an element of $\mathcal{F}_r^\bullet(H)$.

• *πr-regular* if I is π-regular as an element of $\mathcal{F}_r^\bullet(H)$.

• *r-stable* if some $J \in \mathcal{F}_r^\bullet(H)$ exists such that $(IJ)_r = \mathbf{R}(I)$ (equivalently: $(I(\mathbf{R}(I) : I))_r = \mathbf{R}(I)$).

• *πr-stable* if $(I^n)_r$ is r-stable for some $n \in \mathbf{N}$ (equivalently: $(I^n(\mathbf{R}_r(I^n) : I^n))_r = \mathbf{R}_r(I^n)$, for some $n \in \mathbf{N}$).

Note that every r-invertible element of $\mathcal{F}_r^\bullet(H)$ is r-stable and every r-stable element of $\mathcal{F}_r^\bullet(H)$ is r-regular. The monoid H is called

- *r-stable* if every $I \in \mathcal{F}_r^\bullet(H)$ is r-stable.
- *πr-stable* if every $I \in \mathcal{F}_r^\bullet(H)$ is πr-stable.

Lemma 2.1. *Let H be r-Noetherian, T a submonoid of H and $H' \in \mathcal{F}_r^\bullet(H)$ an overmonoid of H.*

1. *H' is $r[H']$-Noetherian.*

2. *$T^{-1}H$ is $T^{-1}r$-Noetherian.*

3. *For all $I, J \in \mathcal{F}_r^\bullet(H)$, $\cap_{P \in r\text{-max}(H)} I_P = I$ and $T^{-1}(I : J) = (T^{-1}I : T^{-1}J)$.*

Proof. 1. Let $I \in \mathbf{F}^\bullet(H') = \mathbf{F}^\bullet(H)$. Then some finite $E \subseteq I$ exists such that $E_r = I_r$. This implies that $E_{r[H']} = (EH')_r = (E_r H')_r = (I_r H')_r = (IH')_r = I_{r[H']}$. Therefore, it follows that H' is $r[H']$-Noetherian.

2. This follows by [**15**, Theorem 4.4].

3. Let $I, J \in \mathcal{F}_r^\bullet(H)$. It follows by [**15**, Theorem 11.3] that $\cap_{P \in r\text{-max}(H)} I_P = I$, and it follows by [**15**, Proposition 11.7] that $T^{-1}(I : J) = (T^{-1}I : T^{-1}J)$. ◻

In the following we investigate the r-ideal semigroup of H with respect to the properties defined in Section 1. Note that $\mathbf{R}(I) \in \mathsf{E}(I) \subseteq \mathsf{E}(\mathcal{F}_r^\bullet(H))$ for all $I \in \mathcal{F}_r^\bullet(H)$. Therefore $\{I \in \mathcal{F}_r^\bullet(H) \mid I = \mathbf{R}(I)\} \subseteq \mathsf{E}(\mathcal{F}_r^\bullet(H))$, and it is natural to ask when equality holds. In this context it is worth remarking that the structure of the idempotents of the t-ideal class semigroup is studied in [**22**].

Lemma 2.2. *Let $I \in \mathcal{F}_r^\bullet(H)$.*

1. *If $J \in \mathcal{F}_r^\bullet(H)$ is such that $(IJ)_r = I$, then $(II^{-1})_r \subseteq (I(J : I))_r \subseteq J \subseteq \mathbf{R}(I)$ and $\mathcal{F}_{\mathbf{R}(I)/H} \subseteq J^{-1} \subseteq \mathbf{R}(I)$.*

2. *If $(I^2 I^{-1})_r = I$, then I is r-regular.*

3. *If $E \in \mathsf{E}(I)$, then $(I(E : I))_r \in \{J \in \mathcal{F}_r^\bullet(H) \mid (II^{-1})_r \subseteq J \subseteq \mathbf{R}(I),\ E \in \mathsf{E}(J) \text{ and } J^2 \subseteq J\}$.*

4. *If $\mathsf{E}^\infty(I)$ possesses a smallest element, then some $n \in \mathbf{N}$ exists such that $\mathbf{R}_r(I^k) = \mathbf{R}_r(I^n)$ for all $k \in \mathbf{N}_{\geq n}$.*

5. *Assume that* $n \in \mathbf{N}$, $E \in \mathsf{E}((I^n)_r)$, *and let* $J \in \mathcal{F}_r^\bullet(H)$ *be such that* $(I^nJ)_r = E$. *Then it follows that* $(I^k(\mathbf{R}_r(I^k) : I^k))_r = E$ *and* $\mathbf{R}((\mathbf{R}_r(I^k) : I^k)) = \mathbf{R}_r(I^k) = \mathbf{R}(E)$ *for all* $k \in \mathbf{N}_{\geq n}$.

Proof. 1. Let $J \in \mathcal{F}_r^\bullet(H)$ be such that $(IJ)_r = I$. Then $IJ \subseteq (IJ)_r = I$; hence, $J \subseteq \mathbf{R}(I)$. Of course, $J^{-1} = (H : J) \subseteq (IH : IJ) \subseteq ((IH)_r : (IJ)_r) = (I : I) = \mathbf{R}(I)$. Clearly, $I^{-1} = (H : I) \subseteq (HJ : IJ) \subseteq ((HJ)_r : (IJ)_r) = (J : I)$ and hence $(II^{-1})_r \subseteq (I(J : I))_r \subseteq J$. Since $J \subseteq \mathbf{R}(I)$, it follows that $\mathcal{F}_{\mathbf{R}(I)/H} = (H : \mathbf{R}(I)) \subseteq (H : J) = J^{-1}$.

2. Let $(I^2I^{-1})_r = I$. Since $I^{-1} \in \mathcal{F}_r^\bullet(H)$, it follows that I is r-regular.

3. Let $E \in \mathsf{E}(I)$. Of course, $(I(E : I))_r \in \mathcal{F}_r^\bullet(H)$ and, by 1, it follows that $(II^{-1})_r \subseteq (I(E : I))_r \subseteq \mathbf{R}(I)$. Since $E \in \mathsf{E}(\mathcal{F}_r^\bullet(H))$ and $((I(E : I))_rE)_r = (IE(E : I))_r = ((IE)_r(E : I))_r = (I(E : I))_r$, it follows that $E \in \mathsf{E}((I(E : I))_r)$. Finally, it follows that $((I(E : I))_r)^2 \subseteq (((I(E : I))_r)^2)_r = (I^2(E : I)^2)_r \subseteq (I(E : I)E)_r = (I(E : I))_r$.

4. Let E be a least element of $\mathsf{E}^\infty(I)$. Some $n \in \mathbf{N}$ exists such that $(EI^n)_r = (I^n)_r$. Let $k \in \mathbf{N}_{\geq n}$. Then some $l \in \mathbf{N}$ exists such that $ln \geq k$. Since $\mathbf{R}_r(I^{ln}) \in \mathsf{E}((I^{ln})_r) \subseteq \mathsf{E}^\infty(I)$, it follows that $E \leq \mathbf{R}_r(I^{ln})$. This implies that $(E\mathbf{R}_r(I^{ln}))_r = E$; hence, $\mathbf{R}_r(I^{ln}) \subseteq \mathbf{R}(E)$. Therefore, it follows that $\mathbf{R}(E) \subseteq \mathbf{R}_r(EI^n) = \mathbf{R}_r(I^n) \subseteq \mathbf{R}_r(I^k) \subseteq \mathbf{R}_r(I^{ln}) \subseteq \mathbf{R}(E)$; hence, $\mathbf{R}_r(I^k) = \mathbf{R}_r(I^n)$.

5. Let $n \in \mathbf{N}$, $k \in \mathbf{N}_{\geq n}$, $E \in \mathsf{E}((I^n)_r)$ and $J \in \mathcal{F}_r^\bullet(H)$ be such that $(I^nJ)_r = E$. Some $l \in \mathbf{N}$ exists such that $ln \geq k$. It follows that $\mathbf{R}_r(I^n) \subseteq \mathbf{R}_r(I^k) \subseteq \mathbf{R}_r(I^{ln}) \subseteq \mathbf{R}_r(I^{ln}J^l) = \mathbf{R}_r(E^l) = \mathbf{R}(E) \subseteq \mathbf{R}_r(EI^n) = \mathbf{R}_r(I^n)$; hence, $\mathbf{R}_r(I^{ln}) = \mathbf{R}_r(I^k) = \mathbf{R}_r(I^n) = \mathbf{R}(E)$. It follows that $E = (I^nJ)_r \subseteq (I^n(E : I^n))_r \subseteq E$, hence $(I^n(E : I^n))_r = E$. Since $(E : I^n) = (E : (I^n)_r) = (E : (EI^n)_r) = (E : EI^n) = (\mathbf{R}(E) : I^n) = (\mathbf{R}_r(I^n) : I^n)$, it follows that $(I^n(\mathbf{R}_r(I^n) : I^n))_r = E$. This implies that $E = (E^l)_r = (I^{ln}(\mathbf{R}_r(I^n) : I^n)^l)_r \subseteq (I^{ln}(\mathbf{R}_r(I^n) : I^{ln}))_r \subseteq (I^k(\mathbf{R}_r(I^n) : I^k))_r \subseteq (I^n(\mathbf{R}_r(I^n) : I^n))_r = E$, hence $(I^k(\mathbf{R}_r(I^k) : I^k))_r = (I^k(\mathbf{R}_r(I^n) : I^k))_r = E$. Finally, this implies that $\mathbf{R}((\mathbf{R}_r(I^k) : I^k)) = (\mathbf{R}_r(I^k) : I^k(\mathbf{R}_r(I^k) : I^k)) = (\mathbf{R}(E) : E) = \mathbf{R}(E)$. ◻

Theorem 2.3. 1. *The following conditions are equivalent*:

a. *If* $I \in \mathsf{E}(\mathcal{F}_r^\bullet(H))$, *then* $I = \mathbf{R}(I)$ (*equivalently*: $\mathsf{E}(\mathcal{F}_r^\bullet(H)) = \{I \in \mathcal{F}_r^\bullet(H) \mid I = \mathbf{R}(I)\}$).

b. *For all* $E, F \in \mathsf{E}(\mathcal{F}_r^\bullet(H))$, $E \le F$ *implies* $F \subseteq E$.

c. *For all* $E, F \in \mathsf{E}(\mathcal{F}_r^\bullet(H))$, $F \subseteq E$ *implies* $E \le F$.

d. *For all* $I \in \mathcal{F}_r^\bullet(H)$, *we have* $\min(\mathsf{E}(I)) = \mathbf{R}(I)$.

If these equivalent conditions are satisfied, then $\mathcal{F}_r^\bullet(H)$ *is almost complete.*

2. *Let* $I, E \in \mathcal{F}_r^\bullet(H)$. *Then the following conditions are equivalent*:

a. E *is a least element of* $\mathsf{E}(I)$ *with respect to the Rees-order.*

b. E *is a least element of* $\{F \in \mathsf{E}(I) \mid F \le \mathbf{R}(I)\}$ *with respect to inclusion.*

c. E *is a minimal element of* $\{F \in \mathsf{E}(I) \mid F \le \mathbf{R}(I)\}$ *with respect to inclusion.*

3. *The following conditions are equivalent*:

a. H *is* πr*-stable.*

b. $\mathcal{F}_r^\bullet(H)$ *is* π*-regular and* $\mathsf{E}(\mathcal{F}_r^\bullet(H)) = \{E \in \mathcal{F}_r^\bullet(H) \mid E = \mathbf{R}(E)\}$.

4. *Suppose that* $\mathsf{E}(\mathcal{F}_r^\bullet(H)) = \{I \in \mathcal{F}_r^\bullet(H) \mid I = \mathbf{R}(I)\}$, *and let* $I \in \mathsf{E}(\mathcal{F}_r^\bullet(H))$. *Then* $I \cdot_r \mathcal{F}_r^\bullet(H) = \{J \in \mathcal{F}_r^\bullet(H) \mid I \subseteq \mathbf{R}(J)\}$ *and* $P_I^* = \{J \in \mathcal{F}_r^\bullet(H) \mid I = \mathbf{R}(J)\}$.

Proof. 1. $a \Rightarrow b$. If $E, F \in \mathsf{E}(\mathcal{F}_r^\bullet(H))$ and $E \le F$, then $(EF)_r = F$. Hence, $F \subseteq (F\mathbf{R}(E))_r = (FE)_r = E$.

$a \Rightarrow c$. If $E, F \in \mathsf{E}(\mathcal{F}_r^\bullet(H))$ and $E \le F$, then $(EF)_r \subseteq (E^2)_r = E$ and $E \subseteq (E\mathbf{R}(F))_r = (EF)_r$. This implies that $(EF)_r = E$, hence $E \le F$.

$b \Rightarrow d$. Let $I \in \mathcal{F}_r^\bullet(H)$ and $F \in \mathsf{E}(I)$. Then we have to show that $(F\mathbf{R}(I))_r = \mathbf{R}(I)$. It follows that $(F\mathbf{R}(I))_r \in \mathsf{E}(I)$ and $(F\mathbf{R}(I))_r \le \mathbf{R}(I)$; hence, $\mathbf{R}(I) \subseteq (F\mathbf{R}(I))_r$. Since $F \in \mathsf{E}(I)$, it follows by Lemma 2.2.1 that $F \subseteq \mathbf{R}(I)$. Hence, $(F\mathbf{R}(I))_r \subseteq (\mathbf{R}(I)^2)_r = \mathbf{R}(I)$. Finally, this implies that $(F\mathbf{R}(I))_r = \mathbf{R}(I)$.

$c \Rightarrow a$. Let $E \in \mathsf{E}(\mathcal{F}_r^\bullet(H))$. Then $E \subseteq \mathbf{R}(E)$ by Lemma 2.2.1; hence, $\mathbf{R}(E) \le E$. Since $(E\mathbf{R}(E))_r = E$, it follows that $E \le \mathbf{R}(E)$, hence $E = \mathbf{R}(E)$.

$d \Rightarrow a$. Let $E \in \mathsf{E}(\mathcal{F}_r^\bullet(H))$. Then $E \in \mathsf{E}(E)$; hence $\mathbf{R}(E) \le E$. Since $(E\mathbf{R}(E))_r = E$, it follows that $E \le \mathbf{R}(E)$; hence, $E = \mathbf{R}(E)$.

2. $a \Rightarrow b$. Of course, $E \in \mathsf{E}(I)$. Since $\mathbf{R}(I) \in \mathsf{E}(I)$, it follows that $E \leq \mathbf{R}(I)$. Let $F \in \mathsf{E}(I)$ be such that $F \leq \mathbf{R}(I)$. It follows that $(EF)_r \in \mathsf{E}(I)$ and $(EF)_r \leq E$, hence $E = (EF)_r$. By Lemma 2.2.1, it follows that $E \subseteq \mathbf{R}(I)$; hence, $E = (EF)_r \subseteq (\mathbf{R}(I)F)_r = F$.

$b \Rightarrow c$. Trivial.

$c \Rightarrow a$. Let $F \in \mathsf{E}(I)$. Then it is sufficient to show that $(EF)_r = E$. It follows that $(EF)_r \leq E \leq \mathbf{R}(I)$ and $(EF)_r \in \mathsf{E}(I)$; hence, $(EF)_r \in \{G \in \mathsf{E}(I) \mid G \leq \mathbf{R}(I)\}$. By Lemma 2.2.1, $F \subseteq \mathbf{R}(I)$; hence, $(EF)_r \subseteq (E\mathbf{R}(I))_r = E$. This implies that $E = (EF)_r$.

3. $a \Rightarrow b$. Let $I \in \mathcal{F}_r^\bullet(H)$. Then there is some $n \in \mathbf{N}$ such that $(I^n(\mathbf{R}_r(I^n) : I^n))_r = \mathbf{R}_r(I^n)$. Since $\mathbf{R}_r(I^n) \in \mathsf{E}((I^n)_r)$, it follows that I is πr-regular. This implies that $\mathcal{F}_r^\bullet(H)$ is π-regular. Let $F \in \mathsf{E}(\mathcal{F}_r^\bullet(H))$. There is some $n \in \mathbf{N}$ such that $(F^n(\mathbf{R}_r(F^n) : F^n))_r = \mathbf{R}_r(F^n)$; hence, $F = (F\mathbf{R}(F))_r = (F(\mathbf{R}(F) : F))_r = (F^n(\mathbf{R}_r(F^n) : F^n))_r = \mathbf{R}_r(F^n) = \mathbf{R}(F)$.

$b \Rightarrow a$. Let $I \in \mathcal{F}_r^\bullet(H)$. Since I is πr-regular, some $n \in \mathbf{N}$, $E \in \mathsf{E}((I^n)_r)$ and $J \in \mathcal{F}_r^\bullet(H)$ exist such that $(I^nJ)_r = E$. Consequently, Lemma 2.2.5 implies that $(I^n(\mathbf{R}_r(I^n) : I^n))_r = E = \mathbf{R}(E) = \mathbf{R}_r(I^n)$.

4. Let us show the first equality. "$\subseteq$:" Let $J \in I \cdot_r \mathcal{F}_r^\bullet(H)$. Then some $A \in \mathcal{F}_r^\bullet(H)$ exists such that $J = (IA)_r$. It is clear that $IJ \subseteq (IJ)_r = (I^2A)_r = ((I^2)_rA)_r = (IA)_r = J$; hence, $I \subseteq \mathbf{R}(J)$. "$\supseteq$:" Let $J \in \mathcal{F}_r^\bullet(H)$ be such that $I \subseteq \mathbf{R}(J)$. It follows that $J \subseteq JI \subseteq J\mathbf{R}(J) = J$; hence, $J = J_r = (IJ)_r = I \cdot_r J \in I \cdot_r \mathcal{F}_r^\bullet(H)$. Now we show the second equality. It follows by 1 that $\cup_{A \in \mathsf{E}(\mathcal{F}_r^\bullet(H)),\ A<I} A \cdot_r \mathcal{F}_r^\bullet(H) = \cup_{A \in \mathsf{E}(\mathcal{F}_r^\bullet(H)),\ I \subsetneq A} \{J \in \mathcal{F}_r^\bullet(H) \mid A \subseteq \mathbf{R}(J)\} = \{J \in \mathcal{F}_r^\bullet(H) \mid \text{some } A \in \mathsf{E}(\mathcal{F}_r^\bullet(H)) \text{ exists such that } I \subsetneq A \subseteq \mathbf{R}(J)\} = \{J \in \mathcal{F}_r^\bullet(H) \mid I \subsetneq \mathbf{R}(J)\}$. This implies that $P_I^* = \{J \in \mathcal{F}_r^\bullet(H) \mid I \subseteq \mathbf{R}(J)\} \setminus \{J \in \mathcal{F}_r^\bullet(H) \mid I \subsetneq \mathbf{R}(J)\} = \{J \in \mathcal{F}_r^\bullet(H) \mid I = \mathbf{R}(J)\}$. ◻

It is not difficult to see that algebraic properties of the r-ideal semigroup $\mathcal{F}_r^\bullet(H)$ induce corresponding properties of the r-ideal class semigroup $\mathcal{S}_r(H)$. Obviously $\mathsf{E}(\mathcal{S}_r(H)) = \{[I]_{\sim_r} \mid I \in \mathsf{E}(\mathcal{F}_r^\bullet(H))\}$ and, if $I, J \in \mathsf{E}(\mathcal{F}_r^\bullet(H))$, then $[I]_{\sim_r} \leq [J]_{\sim_r}$ holds if and only if $I \leq J$. Moreover, if $I \in \mathcal{F}_r^\bullet(H)$, then

- $P^*_{[I]_{\sim_r}} = \{[J]_{\sim_r} \mid J \in P_I^*\}$.
- I is $(\pi)r$-regular if and only if $[I]_{\sim_r}$ is a $(\pi$-$)$regular element of $\mathcal{S}_r(H)$.

Consequently,

- $\mathcal{S}_r(H)$ is regular (π-regular, almost complete, complete) if and only if $\mathcal{F}_r^\bullet(H)$ is regular (π-regular, almost complete, complete).

Moreover let r' be an ideal system on H satisfying $r \le r'$.

- If $\mathcal{F}_r^\bullet(H)$ is regular (π-regular), then $\mathcal{F}_{r'}^\bullet(H)$ is regular (π-regular).
- If H is r-stable[πr-stable], then H is r'-stable ($\pi r'$-stable).

For an abstract semigroup-theoretic formalism behind these statements, see [**18**, Lemma 2.2].

Lemma 2.4. 1. *If $I \in \mathcal{F}_v^\bullet(H)$, then $(II^{-1})_v = \mathcal{F}_{\mathbf{R}(I)/H}$, and if $I \in \mathsf{E}(\mathcal{F}_v^\bullet(H))$, then $\mathcal{F}_{\widehat{H}/H} \subseteq I \subseteq \widehat{H}$ and $\mathcal{F}_{\widehat{H}/H} \subseteq I^{-1} \subseteq \widehat{H}$.*

2. *If $I \in \mathcal{F}_v^\bullet(H)$ and $(I\mathcal{F}_{\mathbf{R}(I)/H})_v = I$, then I is v-regular.*

3. *Let $\widehat{H}$ be factorial and $I \in \mathcal{F}_v^\bullet(H)$. Then some $c \in K^\bullet$ exists such that $((\mathcal{F}_{\widehat{H}/H})^2)_v \subseteq cI \subseteq \widehat{H}$.*

Proof. 1. If $I \in \mathcal{F}_v^\bullet(H)$, then $(II^{-1})_v = (H : (H : II^{-1})) = (H : ((H : I^{-1}) : I)) = (H : (I : I)) = (H : \mathbf{R}(I)) = \mathcal{F}_{\mathbf{R}(I)/H}$. Let $I \in \mathsf{E}(\mathcal{F}_v^\bullet(H))$. Then it follows by Lemma 2.2.1 that $\mathcal{F}_{\widehat{H}/H} \subseteq \mathcal{F}_{\mathbf{R}(I)/H} = (II^{-1})_v \subseteq I \subseteq \mathbf{R}(I) \subseteq \widehat{H}$ and $\mathcal{F}_{\widehat{H}/H} \subseteq \mathcal{F}_{\mathbf{R}(I)/H} \subseteq I^{-1} \subseteq \mathbf{R}(I) \subseteq \widehat{H}$.

2. Let $I \in \mathcal{F}_v^\bullet(H)$ be such that $(I\mathcal{F}_{\mathbf{R}(I)/H})_v = I$. Then it follows by 1 that $(I^2I^{-1})_v = (I(II^{-1})_v)_v = (I\mathcal{F}_{\mathbf{R}(I)/H})_v = I$. Therefore, Lemma 2.2.2 implies that I is v-regular.

3. Since $\widehat{H}$ is factorial, it follows that every $J \in \mathcal{F}_{v_{\widehat{H}}}^\bullet(\widehat{H})$ is principal. Since $(\widehat{H} : I) \in \mathcal{F}_{v_{\widehat{H}}}^\bullet(\widehat{H})$, some $c \in K^\bullet$ exists such that $(\widehat{H} : I) = c\widehat{H}$. By 1 it follows that $\mathcal{F}_{\widehat{H}/H} \subseteq \mathcal{F}_{\mathbf{R}(I)/H} = (II^{-1})_v \subseteq (I(\widehat{H} : I))_v = (Ic\widehat{H})_v$. This implies that $((\mathcal{F}_{\widehat{H}/H})^2)_v \subseteq ((Ic\widehat{H})_v\mathcal{F}_{\widehat{H}/H})_v = c(I\mathcal{F}_{\widehat{H}/H})_v \subseteq cI \subseteq cI\widehat{H} = I(\widehat{H} : I) \subseteq \widehat{H}$. □

Lemma 2.4 shows that every idempotent of the v-ideal semigroup contains the conductor. In Section 5 we present several examples of integral domains, where the conductor itself is an idempotent of the v-ideal semigroup. In the following, we summarize some properties of the

v-ideal semigroup of v-Noetherian monoids and consider the extremal case that the conductor is non-trivial and v-idempotent.

Lemma 2.5. *Let H be v-Noetherian and $\mathcal{F}_{\widehat{H}/H} \in \mathsf{E}(\mathcal{F}_v^\bullet(H))$.*

1. *Let $I \in \mathcal{F}_v^\bullet(H)$ be such that $\mathcal{F}_{\widehat{H}/H} \subseteq I \subseteq \widehat{H}$. If some $k,l \in \mathbf{N}$ exist such that $k \neq l$ and $(I^k)_v \subseteq (I^l)_v$, then I is πv-regular.*

2. *Let $\widehat{H}$ be factorial and $I \in \mathcal{F}_v^\bullet(H)$ such that $\mathbf{R}(I) = \widehat{H}$. Then I is πv-regular.*

Proof. 1. Let $k,l \in \mathbf{N}$ be such that $k \neq l$ and $(I^k)_v \subseteq (I^l)_v$.

Case 1. $k < l$: Since $\mathcal{F}_{\widehat{H}/H} \neq \{0\}$, we have $\widehat{H} = (\mathcal{F}_{\widehat{H}/H})^{-1} \in \mathcal{F}_v^\bullet(H)$. If $0 \neq c \in \mathcal{F}_{\widehat{H}/H}$ and $r \in \mathbf{N}$, then $c(I^{k+r(l-k)})_v \subseteq c(I^{k+(r+1)(l-k)})_v \subseteq c(\widehat{H}^{k+(r+1)(l-k)})_v = c\widehat{H} \subseteq H$. Since H is v-Noetherian, there is some $n \in \mathbf{N}$ such that $c(I^{k+n(l-k)})_v = c(I^{k+(n+1)(l-k)})_v$. This implies that $(I^{k+n(l-k)})_v = (I^{k+(n+1)(l-k)})_v$ and consequently Lemma 1.1.2 implies that I is πv-regular.

Case 2. $l < k$: If $r \in \mathbf{N}$, then $(I^{l+(r+1)(k-l)})_v \subseteq (I^{l+r(k-l)})_v$. This implies that $\{0\} \neq \mathcal{F}_{\widehat{H}/H} = ((\mathcal{F}_{\widehat{H}/H})^{l+(r+1)(k-l)})_v \subseteq (I^{l+(r+1)(k-l)})_v \subseteq (I^{l+r(k-l)})_v$. Since H is v-Noetherian it follows that some $n \in \mathbf{N}$ exists such that $(I^{l+(n+1)(k-l)})_v = (I^{l+n(k-l)})_v$. Therefore, Lemma 1.1.2 implies that I is πv-regular.

2. Of course, $(I\widehat{H})_v = (I\mathbf{R}(I))_v = I_v = I$. It follows by Lemma 2.4.3 that some $c \in K^\bullet$ exists such that $\mathcal{F}_{\widehat{H}/H} \subseteq cI \subseteq \widehat{H}$. Since $(cI)^2 \subseteq cI\widehat{H} \subseteq c(I\widehat{H})_v = cI$, it follows by 1 that cI is πv-regular. This implies that I is πv-regular. □

Theorem 2.6. *Let H be v-Noetherian.*

1. *$\mathcal{F}_v^\bullet(H)$ is almost complete.*

2. *If $I \in \mathcal{F}_v^\bullet(H)$, and if some $n \in \mathbf{N}$ exists such that $\mathbf{R}_v(I^n) = \mathbf{R}_v(I^k)$ for all $k \in \mathbf{N}_{\geq n}$, then $\mathsf{E}^\infty(I)$ has a least element.*

3. *If $\widehat{H}$ is factorial, $\mathcal{F}_{\widehat{H}/H} \in \mathsf{E}(\mathcal{F}_v^\bullet(H))$ and $\{J \in \mathcal{F}_v^\bullet(H) \mid J = \mathbf{R}(J)\} = \{H, \widehat{H}\}$, then $\mathcal{F}_v^\bullet(H)$ is complete.*

Proof. 1. Let $I \in \mathcal{F}_v^\bullet(H)$ and $M = \{F \in \mathsf{E}(I) \mid F \leq \mathbf{R}(I)\}$. Since $\mathbf{R}(I) \in M$, it follows that $M \neq \varnothing$. It follows by Lemma 2.2.1 and Lemma 2.4.1 that $\cap_{B\in M} B \supseteq \mathcal{F}_{\mathbf{R}(I)/H} \neq \{0\}$. Since H is v-Noetherian, there is some $E \in M$ that is minimal with respect to inclusion. Finally, it follows by Theorem 2.3.2 that E is a least element of $\mathsf{E}(I)$ with respect to the Rees-order.

2. Let $I \in \mathcal{F}_v^\bullet(H)$ and $n \in \mathbf{N}$ such that $\mathbf{R}_v(I^k) = \mathbf{R}_v(I^n)$ for all $k \in \mathbf{N}_{\geq n}$. Let $M = \{F \in \mathsf{E}^\infty(I) \mid F \leq \mathbf{R}_v(I^n)\}$. Since $\mathbf{R}_v(I^n) \in \mathsf{E}((I^n)_v) \subseteq \mathsf{E}^\infty(I)$, it follows that $M \neq \varnothing$. Let us show that $\cap_{B\in M} B \supseteq \mathcal{F}_{\mathbf{R}_v(I^n)/H}$. Let $B \in M$. Then there is some $k \in \mathbf{N}_{\geq n}$ such that $B \in \mathsf{E}((I^k)_v)$. It follows by Lemma 2.2.1 and Lemma 2.4.1 that $B \supseteq \mathcal{F}_{\mathbf{R}_v(I^k)/H} = \mathcal{F}_{\mathbf{R}_v(I^n)/H}$. This implies that $\cap_{B\in M} B \neq \{0\}$; hence, (since H is v-Noetherian) some $E \in M$ exists that is minimal with respect to inclusion. Let $F \in \mathsf{E}^\infty(I)$. Then we have to show that $(EF)_v = E$. Some $k \in \mathbf{N}_{\geq n}$ exists such that $F \in \mathsf{E}((I^k)_v)$; hence, Lemma 2.2.1 implies that $(EF)_v \subseteq (E\mathbf{R}_v(I^k))_v = (E\mathbf{R}_v(I^n))_v = E$. It follows that $(EF)_v \in \mathsf{E}^\infty(I)$ and $(EF)_v \leq E \leq \mathbf{R}_v(I^n)$; hence, $(EF)_v \in M$ and $(EF)_v = E$.

3. Let $\widehat{H}$ be factorial, $\mathcal{F}_{\widehat{H}/H} \in \mathsf{E}(\mathcal{F}_v^\bullet(H))$ and $\{J \in \mathcal{F}_v^\bullet(H) \mid J = \mathbf{R}(J)\} = \{H, \widehat{H}\}$. By 1 it follows that $\mathcal{F}_v^\bullet(H)$ is almost complete. Let $I \in \mathcal{F}_v^\bullet(H)$. Then $\mathbf{R}(I) \in \{H, \widehat{H}\}$.

Case 1. $\mathbf{R}(I) = H$: Since $\mathcal{F}_{\mathbf{R}(I)/H} = H \in \mathsf{E}(I)$, it follows by Lemma 2.4.2 that I is v-regular.

Case 2. $\mathbf{R}(I) = \widehat{H}$: It follows by Lemma 2.5.2 that I is πv-regular. □

Note that Theorem 2.6.2 proves the converse of Lemma 2.2.4, if $r = v$ and H is v-Noetherian. We were not able to decide whether the r-ideal semigroup of a r-Noetherian monoid is almost complete. Theorem 2.6.3 gives a hint how to construct examples of monoids which are not πv-stable but whose v-ideal semigroup is π-regular.

3. Preparations for the main results. Let R be an integral domain and K a field of quotients of R. By $\mathcal{F}^\bullet(R)$, we denote the set of all non-zero fractional ideals of R and by $\overline{R}$ we denote the integral closure of R in K. R is called G-domain if $\cap_{P\in\mathrm{spec}(R), P\neq\{0\}} P \neq \{0\}$.

An ideal system r on R is called (extended) star operation on R if $d \leq r$. Note that an ideal system r on R is a star operation if and only if $r|_{\mathcal{F}^\bullet(R)}$ is a star operation on R in the sense of [**13**]. Observe that, for all $X \in \mathcal{F}^\bullet(R)$, $\mathbf{R}(X)$ is an intermediate ring of R and $\widehat{R}$.

Lemma 3.1. *Let R be a Mori domain, $x \in R^\bullet$ and $S \subseteq R^\bullet$ a multiplicatively closed set.*

1. $\mathrm{spec}^1(R) \subseteq v\text{-}\mathrm{spec}(R)$, *and* $\{P \in v\text{-}\mathrm{spec}(R) \mid x \in P\}$ *is finite.*
2. $S^{-1}\widehat{R} = \widehat{S^{-1}R}$.
3. *If R is not a field, then* $R \backslash R^\times = \cup_{M \in v\text{-}\max(R)} M$.
4. *If* $\mathcal{F}_{\widehat{R}/R} \neq \{0\}$, *then $\widehat{R}$ is a Krull domain and* $S^{-1}\mathcal{F}_{\widehat{R}/R} = \mathcal{F}_{\widehat{S^{-1}R}/S^{-1}R}$.
5. *If R is local,* $\dim(R) = 1$ *and $\widehat{R}$ is a Krull domain, then $\widehat{R}$ is a semilocal principal ideal domain.*

Proof. By $\mathrm{spec}^1(R)$ we denote the set of height-one prime ideals of R. Observe that R is a Mori domain if and only if $R^\bullet$ is a v-Noetherian monoid in the sense of [**12**]. Note that, if $\widehat{R^\bullet}$ denotes the complete integral closure of $R^\bullet$ in its quotient group, then $\widehat{R}^\bullet = \widehat{R^\bullet}$.

1. The first assertion follows by [**12**, Proposition 2.2.4.2.], and the second assertion follows by [**12**, Theorem 2.2.5.1].

2. See [**12**, Theorem 2.3.5.2].

3. Let R be not a field. The claim is an immediate consequence of [**12**, Proposition 2.2.4.1].

4. Let $\mathcal{F}_{\widehat{R}/R} \neq \{0\}$. The first assertion follows by [**12**, Theorem 2.3.5.3], and the second assertion follows by 2 and [**12**, Proposition 2.2.8.1].

5. Let R be local, $\dim(R) = 1$ and $\widehat{R}$ a Krull domain, and let M be the maximal ideal of R. It is an immediate consequence of [**12**, Proposition 2.10.5.1.(c)] that $\dim(\widehat{R}) = 1$. Therefore, $\widehat{R}$ is a Dedekind domain. Of course, $M' \cap R = M$ for all $M' \in \max(\widehat{R})$ hence, $\max(\widehat{R}) = \mathcal{P}(M\widehat{R})$. Since $\widehat{R}$ is Noetherian, this implies that $\max(\widehat{R})$ is finite. Therefore, $\widehat{R}$ is a semilocal Dedekind domain; hence, $\widehat{R}$ is a semilocal principal ideal domain. □

Lemma 3.2 (Prime avoidance lemma). *Let S be a commutative ring with identity and $I \subseteq S$ an additively closed subset. For $k \in \mathbf{N}$, let $I^{\langle k \rangle} = \{\prod_{i=1}^{k} z_i \mid (z_i)_{i=1}^{k} \in I^{[1,k]}\}$ and $I^k = \{\sum_{i=1}^{m} z_i \mid m \in \mathbf{N}, (z_i)_{i=1}^{m} \in (I^{\langle k \rangle})^{[1,m]}\}$.*

Let $n \in \mathbf{N}$ and $(P_i)_{i=1}^{n} \in \operatorname{spec}(S)^{[1,n]}$ be such that $I \nsubseteq P_i$ for all $i \in [1,n]$. Then it follows that $I^{(n-1)!} \nsubseteq \cup_{i=1}^{n} P_i$.

Proof. We use induction on n. The assertion is obvious for $n = 1$. $n \to n+1$: Let $n \in \mathbf{N}$ and $(P_i)_{i=1}^{n+1} \in \operatorname{spec}(S)^{[1,n+1]}$ be such that $I \nsubseteq P_i$ for all $i \in [1, n+1]$. If $i \in [1, n+1]$, then $I^{(n-1)!} \nsubseteq \cup_{j=1, j\neq i}^{n+1} P_j$ (by the induction hypothesis), and thus some $(y_i)_{i=1}^{n+1} \in S^{[1,n+1]}$ exists such that $y_i \in I^{(n-1)!} \setminus \cup_{j=1,j\neq i}^{n+1} P_j$ for all $i \in [1, n+1]$. Now we set $y = y_{n+1}^{n} + \prod_{i=1}^{n} y_i \in I^{n!}$ and assume (contrary to our assertion) that $I^{n!} \subseteq \cup_{i=1}^{n+1} P_i$. For every $i \in [1, n+1]$, we have $y_i^n \in I^{n!} \subseteq \cup_{j=1}^{n+1} P_j$, and thus $y_i \in \cup_{j=1}^{n+1} P_j$; hence, $y_i \in P_i$. Since $y \in I^{n!} \subseteq \cup_{i=1}^{n+1} P_i$, some $m \in [1, n+1]$ exists such that $y \in P_m$.

Case 1. $m \in [1,n]$: Since $y_m \in P_m$ it follows that $\prod_{i=1}^{n} y_i \in P_m$. Since $y \in P_m$, we have $y_{n+1}^{n} \in P_m$; hence, $y_{n+1} \in P_m$, a contradiction.

Case 2. $m = n+1$: Since $y_{n+1} \in P_{n+1}$ and $y \in P_{n+1}$, it follows that $\prod_{i=1}^{n} y_i \in P_{n+1}$; hence, some $l \in [1,n]$ exists such that $y_l \in P_{n+1}$, a contradiction. □

4. Main results. In [**9**] it was shown that orders in quadratic number fields are (d-)stable. This result is based on two other results. The first one states that the ideal semigroup of a Noetherian one-dimensional integral domain is πd-stable. The second result gives a common upper bound on the "index of stability" of all elements I (i.e., the smallest $n \in \mathbf{N}$ for which I^n is stable) of the ideal semigroup. The main goal of this paper is to extend the first result to arbitrary $*$-ideal semigroups. As already shown in Section 2 a close connection exists between $\pi*$-stability and the structure of idempotents of the $*$-ideal semigroup. As pointed out in [**9, 17**] the idempotents of the ideal semigroup of a Noetherian integral domain contain the identity. Unfortunately, there is no analogous result for general $*$-ideal semigroups. In Section 5 it will be shown that the idempotents of the v-ideal semigroup of a Mori domain need not contain the identity. It is natural to ask for conditions on R that enforce the $*$-idempotents to be "trivial" (i.e., they contain the identity). The first main result and its corollary deal with this problem.

Theorem 4.1. *Let R be an integral domain, K a field of quotients of R, $*$ a star operation on R and $I, J \in \mathcal{F}_*^\bullet(R)$ such that $(IJ)_* = I$.*

1. *If $\mathcal{F}_{\widehat{R}/R} \neq \{0\}$ and $\widehat{R}$ is a Dedekind domain, then $J\widehat{R} = \widehat{R}$.*

2. *Suppose that one of the following conditions is satisfied:*

a. *$\mathcal{F}_{\widehat{R}/R} \neq \{0\}$ and $\overline{R}$ is a semilocal principal ideal domain.*

b. *R is a $*$-Noetherian G-domain.*

c. *R is $*$-Noetherian and $\dim(R) = 1$.*

Then $J\mathbf{R}(I) = \mathbf{R}(I)$.

3. *If R is $*$-Noetherian and $*$-$\max(R) = \operatorname{spec}^1(R)$, then $(J\mathbf{R}(I))_* = \mathbf{R}(I)$.*

Proof. 1. Let $\mathcal{F}_{\widehat{R}/R} \neq \{0\}$, and let $\widehat{R}$ be a Dedekind domain. Of course, $I\widehat{R}, IJ\widehat{R} \in \mathcal{F}^\bullet(\widehat{R})$. Since $I\widehat{R}$ and $IJ\widehat{R}$ are invertible, this implies that $(I\widehat{R})_{v_{\widehat{R}}} = I\widehat{R}$ and $(IJ\widehat{R})_{v_{\widehat{R}}} = IJ\widehat{R}$. It follows that $IJ\widehat{R} = (\widehat{R} : (\widehat{R} : IJ\widehat{R})) = (\widehat{R} : (\widehat{R} : (IJ\widehat{R})_*)) = (\widehat{R} : (\widehat{R} : (I\widehat{R})_*)) = (\widehat{R} : (\widehat{R} : I\widehat{R})) = I\widehat{R}$. Since $I\widehat{R}$ is invertible, it follows that $J\widehat{R} = \widehat{R}$.

2. a. Let $\mathcal{F}_{\widehat{R}/R} \neq \{0\}$, and let $\overline{R}$ be a semilocal principal ideal domain. Then $\widehat{R} = \overline{R}$. By 1 it follows that $J\mathbf{R}(I)\widehat{R} = J\widehat{R} = \widehat{R}$; hence, $J\mathbf{R}(I) \nsubseteq M$ for all $M \in \max(\widehat{R})$. Since $\widehat{R}$ is semilocal, this implies that $J\mathbf{R}(I) \nsubseteq \cup_{M \in \max(\widehat{R})} M = \widehat{R} \backslash \widehat{R}^\times$. Hence, $J\mathbf{R}(I) \cap \widehat{R}^\times \neq \varnothing$. Of course, $\widehat{R}/\mathbf{R}(I)$ is an integral extension; hence, $\widehat{R}^\times \cap \mathbf{R}(I) = \mathbf{R}(I)^\times$. This implies that $\varnothing \neq J\mathbf{R}(I) \cap \widehat{R}^\times = J\mathbf{R}(I) \cap \mathbf{R}(I) \cap \widehat{R}^\times = J\mathbf{R}(I) \cap \mathbf{R}(I)^\times$, hence $J\mathbf{R}(I) = \mathbf{R}(I)$.

b. Let R be a $*$-Noetherian G-domain. Without restriction we may assume that R is not a field. It follows that $(\mathbf{R}(I) : J\mathbf{R}(I)) = (I : IJ) = (I : (IJ)_*) = \mathbf{R}(I)$; hence, $J\mathbf{R}(I) \subseteq (J\mathbf{R}(I))_{v_{\mathbf{R}(I)}} = \mathbf{R}(I)$. Since $\mathbf{R}(I) \in \mathcal{F}_*^\bullet(R)$ is an overring of R, it follows by Lemma 2.1.1 that $\mathbf{R}(I)$ is $*[\mathbf{R}(I)]$-Noetherian; hence, $\mathbf{R}(I)$ is a Mori domain. If $\mathbf{R}(I) = K$, then $K \in \mathcal{F}_*^\bullet(R)$. Hence, $R = K$, a contradiction. Consequently, Lemma 3.1.3 implies $\mathbf{R}(I)\backslash\mathbf{R}(I)^\times = \cup_{P\in v\text{-max}(\mathbf{R}(I))}P$, and, for all $P \in v\text{-max}(\mathbf{R}(I))$, it follows that $P \neq \{0\}$. Therefore, $P\cap R \neq \{0\}$ for all $P \in v\text{-max}(\mathbf{R}(I))$; hence, $\{0\} \neq \cap_{M\in\text{spec}(R),M\neq\{0\}}M \subseteq R\cap\cap_{P\in v\text{-max}(\mathbf{R}(I))}P$. If $0 \neq x \in \cap_{P\in v\text{-max}(\mathbf{R}(I))}P$, then $\{P \in v\text{-spec}(\mathbf{R}(I)) \mid x\in P\} \supseteq v\text{-max}(\mathbf{R}(I))$, and thus $v\text{-max}(\mathbf{R}(I))$ is finite by Lemma 3.1.1. Assume that $J\mathbf{R}(I) \subsetneqq \mathbf{R}(I)$. Then there is some $Q \in \max(\mathbf{R}(I))$ such that $J\mathbf{R}(I) \subseteq Q$. Since $Q \subseteq \mathbf{R}(I)\backslash\mathbf{R}(I)^\times = \cup_{P\in v\text{-max}(\mathbf{R}(I))}P$, it follows that some $P \in v\text{-max}(\mathbf{R}(I))$ exists such that $Q\subseteq P$; hence, $Q = P$ and $\mathbf{R}(I) = (J\mathbf{R}(I))_{v_{\mathbf{R}(I)}} \subseteq Q_{v_{\mathbf{R}(I)}} = Q$, a contradiction.

c. Let R be $*$-Noetherian and $\dim(R) = 1$. If $P \in \max(R)$, then R_P is a $*_P$-Noetherian G-domain by Lemma 2.1.2. Since $I_P, J_P \in \mathcal{F}^\bullet_{*_P}(R_P)$, and $I_P = ((IJ)_*)_P = (I_PJ_P)_{*_P}$, we obtain $J_P\mathbf{R}(I_P) = \mathbf{R}(I_P)$ by 2 b. Hence, Lemma 2.1.3 implies that $J\mathbf{R}(I) = \cap_{P\in\max(R)}(J\mathbf{R}(I))_P = \cap_{P\in\max(R)}J_P\mathbf{R}(I_P) = \cap_{P\in\max(R)}\mathbf{R}(I_P) = \cap_{P\in\max(R)}\mathbf{R}(I)_P = \mathbf{R}(I)$.

3. Let R be $*$-Noetherian and $*\text{-max}(R) = \text{spec}^1(R)$. If $P \in \text{spec}^1(R)$, then R_P is $*_P$-Noetherian and $I_P = ((IJ)_*)_P = (I_PJ_P)_{*_P}$ by Lemma 2.1.2. Since $I_P, J_P \in \mathcal{F}^\bullet_{*_P}(R_P)$ and $\dim(R_P) = 1$, we obtain $J_P\mathbf{R}(I_P) = \mathbf{R}(I_P)$ by 2 c. Hence, Lemma 2.1.3 implies that $((J\mathbf{R}(I))_*)_P = (J_P\mathbf{R}(I_P))_{*_P} = \mathbf{R}(I_P)_{*_P} = \mathbf{R}(I_P) = \mathbf{R}(I)_P$. Finally, we have $(J\mathbf{R}(I))_* = \cap_{P\in\text{spec}^1(R)}((J\mathbf{R}(I))_*)_P = \cap_{P\in\text{spec}^1(R)}\mathbf{R}(I)_P = \mathbf{R}(I)$ by Lemma 2.1.3. □

Corollary 4.2. *Let R be an integral domain and $*$ a star operation on R.*

1. *If $\widehat{R}$ is a Dedekind domain and $\mathcal{F}_{\widehat{R}/R} \in \mathsf{E}(\mathcal{F}^\bullet_*(R))$, then $R = \widehat{R}$.*

2. *Suppose that one of the following conditions is satisfied*:

a. *$\mathcal{F}_{\widehat{R}/R} \neq \{0\}$ and $\overline{R}$ is a semilocal principal ideal domain.*

b. *R is a $*$-Noetherian G-domain.*

c. *R is $*$-Noetherian and $*\text{-max}(R) = \text{spec}^1(R)$.*

Then $\mathsf{E}(\mathcal{F}^\bullet_*(R)) = \{I \in \mathcal{F}^\bullet_*(R) \mid I = \mathbf{R}(I)\}$.

Proof. 1. Let $\widehat{R}$ be a Dedekind domain and $\mathcal{F}_{\widehat{R}/R} \in \mathsf{E}(\mathcal{F}^\bullet_*(R))$. Then it follows by Theorem 4.1.1 that $\widehat{R} = \mathcal{F}_{\widehat{R}/R}\widehat{R} \subseteq R \subseteq \widehat{R}$; hence, $R = \widehat{R}$.

2. This is an immediate consequence of Theorems 4.1.2 and 4.1.3. □

Theorem 4.3. *Let R be an integral domain, K a field of quotients of R and $*$ a star operation on R such that R is $*$-Noetherian. Suppose that one of the following conditions is fulfilled*:

a. $\widehat{R}$ *is a semilocal principal ideal domain.*

b. $*\text{-max}(R) = \text{spec}^1(R)$ *and* $\widehat{R_P}$ *is a Krull domain for all* $P \in \text{spec}^1(R)$.

Then R is $\pi *$-stable.

Proof. a. Let $\widehat{R}$ be a semilocal principal ideal domain and $I \in \mathcal{F}_*^\bullet(R)$. Then $I\widehat{R} \in \mathcal{F}^\bullet(\widehat{R})$. Since $\widehat{R}$ is a principal ideal domain, some $a \in K^\bullet$ exists such that $I\widehat{R} = a\widehat{R}$; hence, $a^{-1}I\widehat{R} = \widehat{R}$. Hence, it follows that $a^{-1}I \nsubseteq M$ for all $M \in \max(\widehat{R})$. Since $\widehat{R}$ is semilocal, Lemma 3.2 implies that some $u \in \mathbf{N}$ exists satisfying $a^{-u}I^u \nsubseteq \cup_{M\in\max(\widehat{R})} M = \widehat{R}\backslash\widehat{R}^\times$. Consequently, some $b \in \widehat{R}^\times \cap a^{-u}I^u$ exists; hence, $R \subseteq b^{-1}a^{-u}I^u \subseteq \widehat{R}$. Let $J = b^{-1}a^{-u}(I^u)_*$. Then $J \in \mathcal{F}_*^\bullet(R)$. Since R is $*$-Noetherian, some finite subset $E \subseteq b^{-1}a^{-u}I^u$ exists satisfying $J = (E)_*$. Since E is finite and $E \subseteq \widehat{R}$, some finitely generated R-submodule M of K exists such that $R[E] \subseteq M$; hence, $R[E] \in \mathcal{F}^\bullet(R)$. Let $R' = (R[E])_*$. Then $R' \in \mathcal{F}_*^\bullet(R)$ and $J \subseteq R'$. Since $R \subseteq J$, it follows for all $m \in \mathbf{N}$ that $(J^m)_* \subseteq (J^{m+1})_*$ and $(J^m)_* \subseteq ((R')^m)_* = (R[E]^m)_* = R'$. Some $x \in R^\bullet$ exists such that $xR' \subseteq R$; hence, for all $m \in \mathbf{N}$, it follows that $x(J^m)_* \subseteq x(J^{m+1})_*$ and $x(J^m)_* \subseteq R$. Since R is $*$-Noetherian, this implies that some $r \in \mathbf{N}$ exists such that $x(J^r)_* = x(J^k)_*$ for all $k \in \mathbf{N}_{\geq r}$. This implies that $(J^r)_* = (J^{2r})_*$. Consequently, $R \subseteq J \subseteq (J^r)_* \subseteq \mathbf{R}_*(J^r)$; hence, $(J^r)_* = \mathbf{R}_*(J^r)$. This implies that $b^{-r}a^{-ru}(I^{ru})_* = \mathbf{R}_*(I^{ru})$. Let $n = ru \in \mathbf{N}$. Then $\mathbf{R}_*(I^n) = b^{-r}a^{-ru}(I^{ru})_* \subseteq (I^{ru}(\mathbf{R}_*(I^{ru}) : I^{ru}))_* \subseteq \mathbf{R}_*(I^{ru})$; hence, $(I^n(\mathbf{R}_*(I^n) : I^n))_* = \mathbf{R}_*(I^n)$.

b. Let $*\text{-max}(R) = \text{spec}^1(R)$, and let $\widehat{R_P}$ be a Krull domain for all $P \in \text{spec}^1(R)$. Let $I \in \mathcal{F}_*^\bullet(R)$ and $\mathcal{P} = \{P \in \text{spec}^1(R) \mid I_P \neq R_P\}$. There is some $s \in R^\bullet$ such that $sI \subseteq R$ and some $s' \in I\backslash\{0\}$. Let $t = ss' \in R^\bullet$. Then $tR = ss'R \subseteq sIR = sI \subseteq I$. It follows that R is a Mori domain; hence, Lemma 3.1.1 implies that $\{P \in \text{spec}^1(R) \mid st \in P\}$ is finite. Let us show that $\mathcal{P} \subseteq \{Q \in \text{spec}^1(R) \mid st \in Q\}$. Let $Q \in \mathcal{P}$, and assume that $st \in R\backslash Q$. Then $st \in R_Q^\times$; hence $s, t \in R_Q^\times$. This implies that $I_Q = sI_Q = (sI)_Q \subseteq R_Q = tR_Q = (tR)_Q \subseteq I_Q$;

hence, $I_Q = R_Q$, a contradiction. It follows that $\mathcal{P}$ is finite. If $P \in \operatorname{spec}^1(R)$, then R_P is $*_P$-Noetherian by Lemma 2.1.2. Hence, R_P is a Mori domain. Therefore, $\widehat{R_P}$ is a semilocal principal ideal domain by Lemma 3.1.5. Since $I_P \in \mathcal{F}^{\bullet}_{*_P}(R_P)$, it follows by a and Lemma 2.2.5 that a least $n_P \in \mathbf{N}$ exists such that $(I_P^k(\mathbf{R}_{*_P}(I_P^k) : I_P^k))_{*_P} = \mathbf{R}_{*_P}(I_P^k)$ for all $k \in \mathbf{N}_{\geq n_P}$ (note that $n_P = 1$ if $P \notin \mathcal{P}$). If $n = \max(\{n_Q \mid Q \in \operatorname{spec}^1(R)\} \cup \{1\})$, then it follows by Lemma 2.1.3 that $((I^n(\mathbf{R}_*(I^n) : I^n))_*)_Q = (I_Q^n(\mathbf{R}_{*_Q}(I_Q^n) : I_Q^n))_{*_Q} = \mathbf{R}_{*_Q}(I_Q^n) = \mathbf{R}_*(I^n)_Q$ for all $Q \in \operatorname{spec}^1(R)$. Consequently, we have $(I^n(\mathbf{R}_*(I^n) : I^n))_* = \bigcap_{Q \in \operatorname{spec}^1(R)}((I^n(\mathbf{R}_*(I^n) : I^n))_*)_Q = \bigcap_{Q \in \operatorname{spec}^1(R)} \mathbf{R}_*(I^n)_Q = \mathbf{R}_*(I^n)$ by Lemma 2.1.3. □

Corollary 4.4. *Let R be an integral domain and $*$ a star operation on R. Assume that one of the following conditions is fulfilled*:

1. *R is Noetherian and $*$-$\max(R) = \operatorname{spec}^1(R)$.*
2. *R is $*$-Noetherian, $*$-$\max(R) = \operatorname{spec}^1(R)$ and $\mathcal{F}_{\widehat{R}/R} \neq \{0\}$.*

Then R is $\pi*$-stable.

Proof. 1. Let R be Noetherian, $*$-$\max(R) = \operatorname{spec}^1(R)$ and $P \in \operatorname{spec}^1(R)$. Then R_P is Noetherian with $\dim(R_P) = 1$. Therefore, the theorem of Krull-Akizuki implies that $\widehat{R_P}$ is a Dedekind domain. Hence, $\widehat{R_P}$ is a Krull domain. Consequently, Theorem 4.3 implies that R is $\pi*$-stable.

2. Let R be $*$-Noetherian, $*$-$\max(R) = \operatorname{spec}^1(R)$, $\mathcal{F}_{\widehat{R}/R} \neq \{0\}$ and $P \in \operatorname{spec}^1(R)$. Then R_P is a Mori domain by Lemma 2.1.2. Hence, Lemma 3.1.4 implies that $\mathcal{F}_{\widehat{R_P}/R_P} = (\mathcal{F}_{\widehat{R}/R})_P \supseteq \mathcal{F}_{\widehat{R}/R} \neq \{0\}$; consequently, $\widehat{R_P}$ is a Krull domain by Lemma 3.1.4. Therefore, R is $\pi*$-stable by Theorem 4.3. □

Obviously, the conditions of Corollary 4.4 are satisfied for arbitrary Noetherian one-dimensional integral domains. We were not able to decide whether $*$-Noetherian integral domains satisfying $*$-$\max(R) = \operatorname{spec}^1(R)$ are $\pi*$-stable. By Theorem 2.3.3 and Corollary 4.2 it is sufficient to show that the $*$-ideal semigroup of such integral domains is π-regular.

5. Examples. The examples and counterexamples of this section are based on subrings of the ring of formal power series $R[\![X]\!]$ over an integral domain R. For a power series f we denote by $(f_i)_{i\in\mathbf{N}_0}$, the sequence of its coefficients, so that $f=\sum_{i\in\mathbf{N}_0} f_iX^i$. For $d\in R^\bullet$, we define $R_d=\{f\in R[\![X]\!]\mid d|f_1\}$ and, if $b,c\in R$ are such that $b|d$ and $c|d$, then we set $I_{b,c}=\{f\in R[\![X]\!]\mid b|f_0, c|f_1\}$. Observe that R is completely integrally closed if and only if $R[\![X]\!]$ is completely integrally closed (for a proof see [**8**, Theorem 16]). Recall that if $R[\![X]\!]$ is a Mori domain, then R is a Mori domain (since $R[\![X]\!]\cap K=R$ (where K is a field of quotients of R) and this is a consequence of [**1**, Theorem 2.4]).

First we study some ring theoretical properties of R_d and investigate the elements of its v-ideal semigroup. In particular, we investigate the completely integrally closed case where the conductor turns out to be an idempotent of the v-ideal semigroup. In Lemmas 5.3 and 5.4 we study most of the properties introduced in Section 2 with respect to the v-ideal semigroup of R_d. Moreover, we explicitly calculate the set of v-idempotents in some special cases. The most important counterexamples are consolidated in Example 5.5.

Lemma 5.1. *Let R be an integral domain, $d\in R^\bullet$ and K' a field of quotients of $R[\![X]\!]$.*

1. *R_d is an intermediate ring of R and $R[\![X]\!]$, and K' is a quotient field of R_d.*

2. *For all $b,c\in R$ such that $b|d$ and $c|d$ it follows that $I_{b,c}=(b,cX,X^2,X^3)_{R_d}\in\mathcal{F}_v^\bullet(R_d)$, $(I_{b,c})^{-1}=I_{(d/c),(d/b)}$, $\mathbf{R}(I_{b,c})=\{f\in R[\![X]\!]\mid c|bf_1\}$ and $(I_{b,c}^k)^{-1}=\{f\in R[\![X]\!]\mid d|b^kf_1, d|b^{k-1}cf_0\}$ for all $k\in\mathbf{N}$.*

Proof. 1. Of course, $R\subseteq R_d\subseteq R[\![X]\!]$. Therefore, it is sufficient to show that if $f,g\in R_d$, then $f+g, fg\in R_d$. Let $f,g\in R_d$. Then $f+g, fg\in R[\![X]\!]$, $d|(f_1+g_1)=(f+g)_1$ and $d|(f_0g_1+f_1g_0)=(fg)_1$; hence, $f+g,fg\in R_d$. Let $K''\subseteq K'$ be the quotient field of R_d. Since $X=X^3X^{-2}\in K''$, it follows that $R[\![X]\!]\subseteq K''$; hence, $K''=K'$.

2. Let $b,c\in R$ be such that $b|d$ and $c|d$. At first, let us show that $I_{b,c}=(b,cX,X^2,X^3)_{R_d}$.

$\subseteq$: Let $f \in R[\![X]\!]$ be such that $b|f_0$ and $c|f_1$. Then there are $v, w \in R$ such that $f_0 = vb$ and $f_1 = wc$; hence, $f = vb + wcX + (\sum_{i\in\mathbf{N}_{\geq 2}, i\neq 3} f_i X^{i-2})X^2 + f_3X^3 \in (b, cX, X^2, X^3)_{R_d}$.

$\supseteq$: Since $b, cX, X^2, X^3 \in I_{b,c}$, it is sufficient to show that $I_{b,c}$ is an R_d-submodule of $R[\![X]\!]$. Therefore, let $g, h \in I_{b,c}$, and let $p, q \in R_d$. Then $gp + hq \in R[\![X]\!]$ and $b|(g_0p_0 + h_0q_0) = (gp + hq)_0$. It follows that $c|d|p_1$ and $c|d|q_1$. Consequently, $c|(g_1p_0 + g_0p_1 + h_1q_0 + h_0q_1) = (gp + hq)_1$; hence, $gp + hq \in I_{b,c}$. Since $X^2 \in R_d^\bullet$ and $X^2(b, cX, X^2, X^3)_{R_d} = (bX^2, cX^3, X^4, X^5)_{R_d} \subseteq R_d$, it follows that $(b, cX, X^2, X^3)_{R_d} \in \mathcal{F}^\bullet(R_d)$.

Let $k \in \mathbf{N}$. If $g \in K'$ is such that $g(b, cX, X^2, X^3)_{R_d}^k \subseteq R_d$, then $gb^k \in R_d \subseteq R[\![X]\!]$ and $gX^{2k} \in R_d \subseteq R[\![X]\!]$. Hence, $g \in R[\![X]\!]$. This implies that $((b, cX, X^2, X^3)_{R_d}^k)^{-1} = \{f \in R[\![X]\!] \mid f(b, cX, X^2, X^3)_{R_d}^k \subseteq R_d\}$. If $f \in R[\![X]\!]$, then $X^2 f \in R_d$. Hence, $((b, cX, X^2, X^3)_{R_d}^k)^{-1} = \{f \in R[\![X]\!] \mid fb^k, fb^{k-1}cX \in R_d\} = \{f \in R[\![X]\!] \mid d|b^k f_1, d|b^{k-1}cf_0\}$.

Since $(d/b)|d$, $(d/c)|d$ and $b, c \in R$ are arbitrary it follows that $(b, cX, X^2, X^3)_{R_d}^{-1} = \{f \in R[\![X]\!] \mid (d/c)|f_0, (d/b)|f_1\} = I_{(d/c),(d/b)}$ and $(I_{b,c})_v = (I_{(d/c),(d/b)})^{-1} = I_{b,c}$. Finally, it follows that $\mathbf{R}(I_{b,c}) = \{f \in R[\![X]\!] \mid f(b, cX, X^2, X^3)_{R_d} \subseteq I_{b,c}\} = \{f \in R[\![X]\!] \mid fb, fcX, fX^2, fX^3 \in I_{b,c}\} = \{f \in R[\![X]\!] \mid c|bf_1\}$. $\square$

Lemma 5.2. *Let R be a completely integrally closed domain, $d \in R^\bullet$ and K' a field of quotients of $R[\![X]\!]$.*

1. $\overline{R_d} = \widehat{R_d} = R[\![X]\!] = (1, X)_{R_d}$.
2. $\mathcal{F}_{\widehat{R_d}/R_d} = I_{d,d} = (d, dX, X^2, X^3)_{R_d} \in \mathsf{E}(\mathcal{F}_v^\bullet(R_d))$.
3. *Let $Q \in \operatorname{spec}(R_d)$ be such that $\mathcal{F}_{\widehat{R_d}/R_d} \subseteq Q$. Then* $\operatorname{ht}(Q) \geq 2$.
4. $X \in \cap_{P\in\operatorname{spec}^1(R_d)}(R_d)_P$.
5. *Let R_d be a Mori domain and $P \in \operatorname{spec}^1(R_d)$. Then $(R_d)_P$ is a discrete valuation domain.*

Proof. 1. Let us show that $R[\![X]\!] = (1, X)_{R_d}$.

$\subseteq$: Let $f \in R[\![X]\!]$. Then $f = f_1X + (\sum_{i\in\mathbf{N}_0, i\neq 1} f_iX^i)1 \in (1, X)_{R_d}$.

$\supseteq$: Clear. Since $R[\![X]\!]$ is completely integrally closed, this implies that $R[\![X]\!] \subseteq \overline{R_d} \subseteq \widehat{R_d} \subseteq \widehat{R[\![X]\!]} = R[\![X]\!]$, hence $\overline{R_d} = \widehat{R_d} = R[\![X]\!] = (1, X)_{R_d}$.

2. It follows by Lemma 5.1.2 that $I_{d,d} = (d, dX, X^2, X^3)_{R_d} \in \mathcal{F}_v^{\bullet}(R_d)$. Let us show that $\mathcal{F}_{\widehat{R_d}/R_d} = I_{d,d}$.

$\subseteq$: Let $f \in \mathcal{F}_{\widehat{R_d}/R_d}$. Then $f \in R_d$; hence, $d|f_1$. Since $fX \in R_d$, we have $d|(fX)_1 = f_0$.

$\supseteq$: Since $\widehat{R_d} = R[\![X]\!]$, it follows by 1 that $\{d, dX, X^2, X^3\} \subseteq \mathcal{F}_{\widehat{R_d}/R_d}$; hence, $(d, dX, X^2, X^3)_{R_d} \subseteq \mathcal{F}_{\widehat{R_d}/R_d}$. Therefore, Lemma 5.1.2 implies that $(I_{d,d}^2)^{-1} = \{f \in R[\![X]\!] \mid d|d^2 f_0, d|d^2 f_1\} = R[\![X]\!]$; hence, $((\mathcal{F}_{\widehat{R_d}/R_d})^2)_v = (I_{d,d}^2)_v = (R[\![X]\!])^{-1} = \mathcal{F}_{\widehat{R_d}/R_d}$.

3. Let $h : R[\![X]\!] \to R$ be the canonical ringepimorphism and $g = h|_{R_d}$. Then g is a ringepimorphism. Let us show that $\mathrm{Ker}\,(g) = (dX, X^2, X^3)_{R_d}$.

$\subseteq$: Let $f \in \mathrm{Ker}\,(g)$. Then $f_0 = 0$. Of course, $d|f_1$; hence, $f = f_1 X + X^2(\sum_{i \in \mathbf{N}_{\geq 2}, i \neq 3} f_i X^{i-2}) + f_3 X^3 \in (dX, X^2, X^3)_{R_d}$.

$\supseteq$: Trivial. This implies that $R_d/(dX, X^2, X^3)_{R_d} \cong R$; consequently, $(dX, X^2, X^3)_{R_d} \in \mathrm{spec}(R_d)$. Since $\{0\} \subsetneq (dX, X^2, X^3)_{R_d} \subsetneq \mathcal{F}_{\widehat{R_d}/R_d} \subseteq Q$, we have $\mathrm{ht}\,(Q) \geq 2$.

4. Let $P \in \mathrm{spec}^1(R_d)$. By 2 and 3 we have $\{d, dX, X^2, X^3\} \nsubseteq P$.

Case 1. $d \notin P$: Since $d, dX \in R_d$, we have $X = d^{-1} dX \in (R_d)_P$.

Case 2. $dX \notin P$: Since $dX, dX^2 \in R_d$, it follows that $X = (dX)^{-1} dX^2 \in (R_d)_P$.

Case 3. $X^2 \notin P$: Since $X^2, X^3 \in R_d$, we have $X = X^{-2} X^3 \in (R_d)_P$.

Case 4. $X^3 \notin P$: Since $X^3, X^4 \in R_d$, it follows that $X = X^{-3} X^4 \in (R_d)_P$.

5. Since $X \in (R_d)_P$ (by 4), it follows by Lemma 3.1.2 and 1 that $\widehat{(R_d)_P} = (\widehat{R_d})_P = R[\![X]\!]_P \subseteq ((R_d)_P)_P = (R_d)_P$; hence, $(R_d)_P$ is completely integrally closed. Since $(R_d)_P$ is a Mori domain, this implies that $(R_d)_P$ is a Krull domain. Since $(R_d)_P$ is a Krull domain and $\dim((R_d)_P) = 1$, it follows that $(R_d)_P$ is a Dedekind domain and, since $(R_d)_P$ is local, this implies that $(R_d)_P$ is a discrete valuation domain. $\square$

Lemma 5.3. *Let R be a completely integrally closed domain, $d \in R^\bullet$ and K' a field of quotients of $R[\![X]\!]$.*

1. $\{I_{b,c} \mid b,c \in R, b|d, c|d\} \subseteq \{I \in \mathcal{F}_v^\bullet(R_d) \mid \mathcal{F}_{\widehat{R_d}/R_d} \subseteq I$ *and* $I^2 \subseteq I\}$, *and if R is a* GCD-*domain, then* $\{I_{b,c} \mid b,c \in R, b|c|d, \mathrm{GCD}\,(d/b,b) = R^\times\} \subseteq \mathsf{E}(\mathcal{F}_v^\bullet(R_d))$.

2. *Let* $\{I \in \mathcal{F}_v^\bullet(R_d) \mid \mathcal{F}_{\widehat{R_d}/R_d} \subseteq I$ *and* $I^2 \subseteq I\} \subseteq \{I_{b,c} \mid b,c \in R, b|d, c|d\}$. *Then, for all $a \in R$ such that $a^2|d$, it follows that $a \in R^\times$.*

3. *If R/dR is finite, then $\{I \mid I$ is an R_d-submodule of $\widehat{R_d}$ such that $\mathcal{F}_{\widehat{R_d}/R_d} \subseteq I\}$ is finite and $\mathcal{F}_v^\bullet(R_d)$ is almost complete.*

4. *If R/dR is finite, then for all $I \in \mathcal{F}_v^\bullet(R_d)$ such that $\mathcal{F}_{\widehat{R_d}/R_d} \subseteq I \subseteq \widehat{R_d}$ it follows that I is πv-regular.*

5. *Let $I \in \mathcal{F}_v^\bullet(R_d)$ be such that $\mathcal{F}_{\widehat{R_d}/R_d} \subseteq I \subseteq \widehat{R_d}$. Then some $P \subseteq R \times R$ exists such that $I = \{f \in R[\![X]\!] \mid d|(bf_1 + cf_0)$ for all $(b,c) \in P\}$.*

Proof. 1. Let $b,c \in R$ be such that $b|d$ and $c|d$ and $I = I_{b,c}$. Then it follows by Lemma 5.1.2 that $I \in \mathcal{F}_v^\bullet(R_d)$. Since $b|d$ and $c|d$, it follows by Lemma 5.1.2 that $\mathcal{F}_{\widehat{R_d}/R_d} = \{f \in R[\![X]\!] \mid d|f_0, d|f_1\} \subseteq \{f \in R[\![X]\!] \mid b|f_0, c|f_1\} = I$. Since $I^2 = (b^2, bcX, bX^2, bX^3, c^2X^2, cX^3, cX^4, X^4, X^5, X^6)_{R_d}$ and all generators of I^2 are products of elements of I and R_d, it follows that $I^2 \subseteq I$.

Now let R be a GCD-domain, $b|c$ and $\mathrm{GCD}\,((d/b),b) = R^\times$. It follows by Lemma 5.1.2 that $(I^2)^{-1} = \{f \in R[\![X]\!] \mid (d/b)|bf_1, (d/c)|bf_0\}$. Since $b|c$, it follows that $(d/c)|(d/b)$; hence, $\mathrm{GCD}\,((d/c),b) = R^\times$. Since R is a GCD-domain, this implies that $(I^2)^{-1} = \{f \in R[\![X]\!] \mid (d/b)|f_1, (d/c)|f_0\} = ((d/c), (d/b)X, X^2, X^3)_{R_d} = I^{-1}$. Hence, $I = (I^2)_v$, and thus $I \in \mathsf{E}(\mathcal{F}_v^\bullet(R_d))$.

2. Let $a \in R$ be such that $a^2|d$ and $I = (a^2, a^2X, aX + a, X^2, X^3)_{R_d}$. Of course, $I \in \mathcal{F}^\bullet(R_d)$. By Lemma 5.1.2, $I^{-1} = (I_{a^2,a^2} + (aX + a)R_d)^{-1} = (I_{a^2,a^2})^{-1} \cap (aX + a)^{-1}R_d = I_{(d/a^2),(d/a^2)} \cap \{f \in K' \mid (aX + a)f \in R_d\} = \{f \in R[\![X]\!] \mid (d/a^2)|f_0, (d/a^2)|f_1, (d/a)|(f_0 + f_1)\}$.

Let us show that $I^{-1} = ((d/a), (d/a)X, (d/a^2)(1 - X), X^2, X^3)_{R_d}$.

$\subseteq$: Let $f \in R[\![X]\!]$ be such that $(d/a^2)|f_0$, $(d/a^2)|f_1$ and $(d/a)|(f_0 + f_1)$. Then some $r, s \in R$ exist such that $f_0 + f_1 = (d/a)r$ and

$f_0 = (d/a^2)s$; hence, $f = f_0 + f_1X + \sum_{i\in\mathbf{N}_{\geq 2}} f_iX^i = s(d/a^2)(1-X) + r(d/a)X + \sum_{i\in\mathbf{N}_{\geq 2}} f_iX^i \in ((d/a),(d/a)X,(d/a^2)(1-X),X^2,X^3)_{R_d}$.

$\supseteq$: Trivial, since I^{-1} is an R_d-module.

Consequently, it follows by Lemma 5.1.2 that $I_v = (I^{-1})^{-1} = (I_{(d/a),(d/a)} + (d/a^2)(1-X)R_d)^{-1} = (I_{(d/a),(d/a)})^{-1} \cap ((d/a^2)(1-X))^{-1}R_d = I_{a,a} \cap \{f \in K' \mid (d/a^2)(1-X)f \in R_d\} = \{f \in R[\![X]\!] \mid a|f_0, a|f_1, a^2|(f_1-f_0)\}$.

Let us show that $I_v = I$.

$\subseteq$: Let $f \in R[\![X]\!]$ be such that $a|f_0$, $a|f_1$ and $a^2|(f_1-f_0)$. Then some $r,s \in R$ exist such that $f_1 = f_0 + a^2r$ and $f_0 = as$. This implies that $f = f_0 + f_1X + \sum_{i\in\mathbf{N}_{\geq 2}} f_iX^i = s(aX+a) + ra^2X + \sum_{i\in\mathbf{N}_{\geq 2}} f_iX^i \in I$.

$\supseteq$: Trivial. Of course, $I \in \mathcal{F}_v^\bullet(R)$. Since $a^2|d$, it follows by Lemmas 5.1.2 and 5.2.2 that $\mathcal{F}_{\widehat{R_d}/R_d} = I_{d,d} \subseteq I_{a^2,a^2} = (a^2,a^2X,X^2,X^3)_{R_d} \subseteq I$. Of course, $I^2 = (I_{a^2,a^2})^2 + (a^3X+a^3, a^4X+a^4, a^2X^2+2a^2X+a^2, aX^3+aX^2, aX^4+aX^3)_{R_d} \subseteq (a^2,a^2X,X^2,X^3)_{R_d} \subseteq I$. Therefore, there are some $b_0, c_0 \in R$ such that $b_0|d$, $c_0|d$ and $I = I_{b_0,c_0}$. It follows by Lemma 5.1.2 that $aX + a \in I_{b_0,c_0}$; hence, $c_0|a$. Since $c_0X \in I = \{f \in R[\![X]\!] \mid a|f_0, a|f_1, a^2|(f_1-f_0)\}$, it follows that $a^2|c_0$. This implies that $a^2|a$; hence, $a \in R^\times$.

3. Let R/dR be finite and $M = \{I \mid I \text{ is an } R_d\text{-submodule of } \widehat{R_d} \text{ such that } \mathcal{F}_{\widehat{R_d}/R_d} \subseteq I\}$. Let $f : R/dR \times R/dR \to \widehat{R_d}/\mathcal{F}_{\widehat{R_d}/R_d}$ be defined by $f((a+dR, b+dR)) = (a+bX) + \mathcal{F}_{\widehat{R_d}/R_d}$ for all $a,b \in R$. Let $a,a_1,b,b_1 \in R$ be such that $a+dR = a_1+dR$ and $b+dR = b_1+dR$. Then some $c_1,d_1 \in R$ exist such that $a = a_1+c_1d$ and $b = b_1+d_1d$. Therefore, Lemma 5.2.2 implies that $(a+bX) + \mathcal{F}_{\widehat{R_d}/R_d} = (a_1+b_1X) + (c_1d + d_1dX) + \mathcal{F}_{\widehat{R_d}/R_d} = (a_1+b_1X) + \mathcal{F}_{\widehat{R_d}/R_d}$; hence, f is well-defined. Now let $g \in \widehat{R_d} = R[\![X]\!]$. Then, by Lemma 5.2.2, $\sum_{i\in\mathbf{N}_{\geq 2}} g_iX^i \in \mathcal{F}_{\widehat{R_d}/R_d}$; hence, $f((g_0+dR, g_1+dR)) = (g_0+g_1X) + \mathcal{F}_{\widehat{R_d}/R_d} = g + \mathcal{F}_{\widehat{R_d}/R_d}$, and thus f is surjective.

Since $R/dR \times R/dR$ is finite, this implies that $\widehat{R_d}/\mathcal{F}_{\widehat{R_d}/R_d}$ is finite. Since a bijection between the set of all R_d-submodules of $\widehat{R_d}/\mathcal{F}_{\widehat{R_d}/R_d}$ and M exists, it follows that M is finite. It follows by Lemma 2.4.1

that $\mathsf{E}(\mathcal{F}_v^\bullet(R_d)) \subseteq M$; hence, $\mathsf{E}(\mathcal{F}_v^\bullet(R_d))$ is finite. This implies that $\mathcal{F}_v^\bullet(R_d)$ is almost complete.

4. Let R/dR be finite and $I \in \mathcal{F}_v^\bullet(R_d)$ such that $\mathcal{F}_{\widehat{R_d}/R_d} \subseteq I \subseteq \widehat{R_d}$. It follows by Lemma 5.2.2 that $\mathcal{F}_{\widehat{R_d}/R_d} = ((\mathcal{F}_{\widehat{R_d}/R_d})^k)_v \subseteq (I^k)_v \subseteq ((\widehat{R_d})^k)_v = \widehat{R_d}$ for all $k \in \mathbf{N}$. Since $\{I \in \mathcal{F}_v^\bullet(R_d) \mid \mathcal{F}_{\widehat{R_d}/R_d} \subseteq I \subseteq \widehat{R_d}\}$ is finite, there are some $r, s \in \mathbf{N}$ such that $r \neq s$ and $(I^r)_v = (I^s)_v$. Hence, I is πv-regular by Lemma 1.1.2.

5. Let $P = \{(b,c) \in R \times R \mid b + cX \in I^{-1}\}$. Let us show that $I = \{f \in R[\![X]\!] \mid$ for all $(b,c) \in P$ we have $d|(bf_1 + cf_0)\}$.

$\subseteq$: Let $f \in I$ and $(b,c) \in P$. Since $b + cX \in I^{-1}$, it follows that $(b+cX)f \in R_d$, and hence $d|((b+cX)f)_1 = bf_1 + cf_0$.

$\supseteq$: Let $f \in R[\![X]\!]$ be such that $d|(bf_1 + cf_0)$ for all $(b,c) \in P$. It is sufficient to show that $fg \in R_d$ for all $g \in I^{-1}$. Let $g \in I^{-1}$. Then $g \in I^{-1} \subseteq (R_d :_{K'} \mathcal{F}_{\widehat{R_d}/R_d}) = \widehat{R_d} = R[\![X]\!]$. It follows by Lemma 5.2.2 that $\sum_{i \in \mathbf{N}_{\geq 2}} g_i X^i \in (X^2, X^3)_{R_d} \subseteq \mathcal{F}_{\widehat{R_d}/R_d} \subseteq I^{-1}$; hence, $g_0 + g_1 X \in I^{-1}$. Therefore, $(g_0, g_1) \in P$, and thus $d|(g_0 f_1 + g_1 f_0) = (gf)_1$. This implies that $gf \in R_d$. ◻

Lemma 5.4. *Let R be a completely integrally closed domain, $d \in R^\bullet$ and K' a field of quotients of $R[\![X]\!]$.*

1. *The following assertions are equivalent*:

a. $d \in R^\times$.

b. *R_d is completely integrally closed.*

c. *$\mathcal{F}_v^\bullet(R_d)$ is a group.*

d. *$\mathcal{F}_v^\bullet(R_d)$ is a Clifford semigroup.*

e. $\mathsf{E}(\mathcal{F}_v^\bullet(R_d)) = \{F \in \mathcal{F}_v^\bullet(R_d) \mid F = \mathbf{R}(F)\}$.

f. $\mathcal{F}_{\widehat{R_d}/R_d} \in \{F \in \mathcal{F}_v^\bullet(R_d) \mid F = \mathbf{R}(F)\}$.

2. *If d is a product of pairwise non-associated prime elements of R, then $\{I \in \mathcal{F}_v^\bullet(R_d) \mid \mathcal{F}_{\widehat{R_d}/R_d} \subseteq I$ and $I^2 \subseteq I\} = \{I_{b,c} \mid b, c \in R, b|d, c|d\}$.*

3. *If d is a product of prime elements of R, then $\mathsf{E}(\mathcal{F}_v^\bullet(R_d)) = \{I_{b,c} \mid b, c \in R, b|c|d, GCD(b,(d/b)) = R^\times\}$ and $\mathcal{F}_v^\bullet(R_d)$ is almost complete.*

4. *If $R[\![X]\!]$ is factorial and if d is a prime element of R or R/dR is finite, then $\mathcal{F}_v^\bullet(R_d)$ is complete.*

Proof. 1. $a \Rightarrow b$. Let $d \in R^\times$. Then $R_d = \{f \in R[\![X]\!] \mid d|f_1\} = R[\![X]\!] = \widehat{R_d}$ by Lemma 5.2.1; hence, R_d is completely integrally closed.

$b \Rightarrow c$. Clear. $c \Rightarrow d$. Trivial.

$d \Rightarrow a$. Let $\mathcal{F}_v^\bullet(R_d)$ be a Clifford semigroup and $I = \mathcal{F}_{\widehat{R_d}/R_d} + XR_d$. It follows by Lemma 5.2.2 that $I = (d, X, X^2, X^3)_{R_d}$. We show that $(I^2)_v = (I\mathcal{F}_{\widehat{R_d}/R_d})_v = \mathcal{F}_{\widehat{R_d}/R_d}$. It follows by Lemmas 5.1.2 and 5.2.2 that $\mathcal{F}_{\widehat{R_d}/R_d} = ((\mathcal{F}_{\widehat{R_d}/R_d})^2)_v \subseteq (I\mathcal{F}_{\widehat{R_d}/R_d})_v \subseteq (I^2)_v = (\{f \in R[\![X]\!] \mid d|d^2f_1, d|df_0\})^{-1} = (R[\![X]\!])^{-1} = \mathcal{F}_{\widehat{R_d}/R_d}$; hence, $(I^2)_v = (I\mathcal{F}_{\widehat{R_d}/R_d})_v = \mathcal{F}_{\widehat{R_d}/R_d}$. Since I is v-regular, some $E \in \mathsf{E}(I)$ and $J \in \mathcal{F}_v^\bullet(R_d)$ exist such that $(IJ)_v = E$; hence, $(\mathcal{F}_{\widehat{R_d}/R_d}J^2)_v = (I^2J^2)_v = (E^2)_v = E$. This implies that $(\mathcal{F}_{\widehat{R_d}/R_d}E)_v = ((\mathcal{F}_{\widehat{R_d}/R_d})^2J^2)_v = (\mathcal{F}_{\widehat{R_d}/R_d}J^2)_v = E$. It follows by Lemma 2.4.1 that $\mathcal{F}_{\widehat{R_d}/R_d} \subseteq E = (E\mathcal{F}_{\widehat{R_d}/R_d})_v \subseteq (\widehat{R_d}\mathcal{F}_{\widehat{R_d}/R_d})_v = \mathcal{F}_{\widehat{R_d}/R_d}$; hence, $E = \mathcal{F}_{\widehat{R_d}/R_d}$. This implies that $\mathcal{F}_{\widehat{R_d}/R_d} \in \mathsf{E}(I)$; hence, $\mathcal{F}_{\widehat{R_d}/R_d} = (I\mathcal{F}_{\widehat{R_d}/R_d})_v = I$. Finally, it follows by Lemma 5.2.2 that $X \in I = \mathcal{F}_{\widehat{R_d}/R_d} = I_{d,d}$. Therefore, $d|1$ and thus $d \in R^\times$.

$c \Rightarrow e$. Clear. $e \Rightarrow f$. This is an immediate consequence of Lemma 5.2.2.

$f \Rightarrow a$. Let $\mathcal{F}_{\widehat{R_d}/R_d} \in \{F \in \mathcal{F}_v^\bullet(R_d) \mid F = \mathbf{R}(F)\}$. Then, by Lemma 5.2.2, $1 \in \mathcal{F}_{\widehat{R_d}/R_d} = I_{d,d}$. It follows that $d|1$; hence, $d \in R^\times$.

2. Let $n \in \mathbf{N}$, and let $(p_i)_{i=1}^n$ be a finite sequence of pairwise non-associated prime elements of R such that $d = \prod_{i=1}^n p_i$. Now we show the equality.

$\supseteq$: This follows by Lemma 5.3.1.

$\subseteq$: Let $I \in \mathcal{F}_v^\bullet(R_d)$ be such that $\mathcal{F}_{\widehat{R_d}/R_d} \subseteq I$ and $I^2 \subseteq I$. Then $I \subseteq \mathbf{R}(I) \subseteq \widehat{R_d} = R[\![X]\!]$. By Lemma 5.3.5 some $P \subseteq R \times R$ exists such that $I = \{f \in R[\![X]\!] \mid d|(b_0f_1 + c_0f_0)$ for all $(b_0, c_0) \in P\}$. Let $M = \{i \in [1, n] \mid p_i \nmid c_0\}$ for some $(b_0, c_0) \in P\}$. There is some $(b_0, c_0) \in P$ such that $p_i \nmid c_0\}$, $N = \{i \in [1, n] \mid p_i \nmid b_0\}$ for some

$(b_0, c_0) \in P\}$ such that $b = \prod_{i \in M} p_i$ and $c = \prod_{i \in N} p_i$. Next we show that $I = I_{b,c}$.

$\subseteq$: Let $f \in I$. Then $f^2 \in I^2 \subseteq I$. If $(b_1, c_1) \in P$, then $d|(b_1(f^2)_1 + c_1(f^2)_0)$. Therefore, $d|(b_1(2f_0f_1)+c_1f_0^2)$, and thus $d|(b_1f_0f_1+f_0(b_1f_1+c_1f_0))$. Since $d|(b_1f_1 + c_1f_0)$, it follows that $d|b_1f_0f_1$. Now we prove that $b|f_0$. It is sufficient to show that $p_i|f_0$ for all $i \in M$. Let $i \in M$. Then some $(b_0, c_0) \in P$ exists such that $p_i \nmid c_0$.

Case 1. $p_i \nmid b_0$: Since $p_i|d|b_0f_0f_1$, we have $p_i|f_0$ or $p_i|f_1$. If $p_i|f_1$, then since $p_i|d|(b_0f_1 + c_0f_0)$, we have $p_i|f_0$.

Case 2. $p_i|b_0$: Since $p_i|d|(b_0f_1 + c_0f_0)$, we have $p_i|f_0$.

Now we prove that $c|f_1$. It is sufficient to show that, for all $i \in N$, it follows that $p_i|f_1$. Let $i \in N$. Then some $(b_0, c_0) \in P$ exists such that $p_i \nmid b_0$. Since $p_i|d|b_0f_0f_1$, we have $p_i|f_0$ or $p_i|f_1$. If $p_i|f_0$, then $p_i|f_1$, since $p_i|d|(b_0f_1 + c_0f_0)$.

$\supseteq$: Since $X^2, X^3 \in \mathcal{F}_{\widehat{R_d}/R_d} \subseteq I$, we have to show that $b, cX \in I$. Let us show that $b \in I$. It is sufficient to show, that for all $i \in [1, n]$ and all $(b_1, c_1) \in P$, it follows that $p_i|c_1b$. Let $i \in [1, n]$ and $(b_1, c_1) \in P$.

Case 1. For all $(b_0, c_0) \in P$, it follows that $p_i|c_0$. Of course, $p_i|c_1|c_1b$.

Case 2. Some $(b_0, c_0) \in P$ exists such that $p_i \nmid c_0$. Since $i \in M$, we have $p_i|b|c_1b$.

Let us show that $cX \in I$. It is sufficient to show that, for all $i \in [1, n]$ and all $(b_1, c_1) \in P$, it follows that $p_i|b_1c$. Let $i \in [1, n]$ and $(b_1, c_1) \in P$.

Case 1. For all $(b_0, c_0) \in P$, it follows that $p_i|b_0$. Of course, $p_i|b_1|b_1c$.

Case 2. There is some $(b_0, c_0) \in P$ such that $p_i \nmid b_0$. Since $i \in N$, it follows that $p_i|c|b_1c$.

3. Without restriction, let $n \in \mathbf{N}$, $(p_i)_{i=1}^n$ a finite sequence of pairwise non-associated primes of R, and let $(k_i)_{i=1}^n \in \mathbf{N}^{[1,n]}$ be such that $d = \prod_{i=1}^n p_i^{k_i}$.

Claim. $(I_{p_i,1}^{k_i+1})_v = I_{p_i^{k_i}, p_i^{k_i}}$ *for all* $i \in [1, n]$.

Proof of the claim. Let $i \in [1, n]$. Then it follows by Lemma 5.1.2 that $(I_{p_i,1}^{k_i+1})^{-1} = \{f \in R[\![X]\!] \mid d|p_i^{k_i+1}f_1, d|p_i^{k_i}f_0\} = \{f \in R[\![X]\!] \mid$

$(d/p_i^{k_i})|f_1, (d/p_i^{k_i})|f_0\} = I_{(d/p_i^{k_i}),(d/p_i^{k_i})}$; consequently,

$$(I_{p_i,1}^{k_i+1})_v = (I_{(d/p_i^{k_i}),(d/p_i^{k_i})})^{-1} = I_{p_i^{k_i},p_i^{k_i}}$$

by Lemma 5.1.2.

Next we show the equality. $\subseteq$: Let $I \in \mathsf{E}(\mathcal{F}_v^\bullet(R_d))$. Then we have $\mathcal{F}_{\widehat{R_d}/R_d} \subseteq I \subseteq \widehat{R_d}$ by Lemma 2.4.1; hence, some $P \subseteq R \times R$ exists such that $I = \{f \in R[\![X]\!] \mid d|(bf_1 + cf_0)$ for all $(b, c) \in P\}$ by Lemma 5.3.5. Let $M = \{i \in [1, n] \mid p_i|f_0$ for all $f \in I\}$, $l_i = \max\{r \in [0, k_i] \mid p_i^r|f_1$ for all $f \in I\}$ for all $i \in [1, n]$, $b = \prod_{j\in M} p_j^{k_j}$ and $c = \prod_{j=1}^n p_j^{l_j}$. Next we show that $b \mid c$. Let $i \in M$. Then $I \subseteq I_{p_i,1}$ by Lemma 5.1.2. and hence $I = (I^{k_i+1})_v \subseteq (I_{p_i,1}^{k_i+1})_v = I_{p_i^{k_i},p_i^{k_i}}$ by the claim. It follows that $p_i^{k_i}|f_1$ for all $f \in I$. Therefore, $l_i = k_i$; hence, $b \mid c$. Of course, $c \mid d$ and, since $(p_i)_{i=1}^n$ is a sequence of pairwise non-associated prime elements of R, we have GCD $(b, (d/b)) = R^\times$.

It remains to prove that $I = I_{b,c}$. $\subseteq$: Let $f \in I$. If $j \in M$, then $I \subseteq I_{p_j^{k_j},p_j^{k_j}}$. Therefore, $p_j^{k_j}|f_0$; hence, $b|f_0$. Of course, $p_j^{l_j}|f_1$ for all $j \in [1, n]$; hence, $c|f_1$.

$\supseteq$: It follows by Lemma 5.2.2 and Lemma 2.4.1 that $X^2, X^3 \in \mathcal{F}_{\widehat{R_d}/R_d} \subseteq I$. Due to Lemma 5.1.2, it is sufficient to show that $b \in I$ and $cX \in I$. We have to prove that, for all $j \in [1, n]$ and all $(b_1, c_1) \in P$, it follows that $p_j^{k_j}|c_1b$ and $p_j^{k_j}|b_1c$. Let $j \in [1, n]$ and $(b_1, c_1) \in P$.

Case 1. $j \in M$: Since $p_j^{k_j}|b|c$, we have $p_j^{k_j}|c_1b$ and $p_j^{k_j}|cb_1$.

Case 2. $j \notin M$: There is some $f \in I$ such that $p_j \nmid f_0$. Since $f^2 \in I^2 \subseteq I$, we have $d|(b_1(f^2)_1+c_1(f^2)_0)$; hence, $d|(b_1f_0f_1+f_0(b_1f_1+c_1f_0))$. Since $f \in I$, it follows that $p_j^{k_j}|d|b_1f_0f_1$ and, since $p_j \nmid f_0$, we have $p_j^{k_j}|b_1f_1$. Since $p_j^{k_j}|d|(b_1f_1 + c_1f_0)$, it follows that $p_j^{k_j}|c_1|c_1\prod_{i\in M} p_i^{k_i}$. If $l_j = k_j$, then $p_j^{k_j}|b_1c$; hence, we may assume that $l_j < k_j$. Some $g \in I$ exists such that $p_j^{l_j}|g_1$ and $p_j^{l_j+1} \nmid g_1$. It follows that $p_j^{k_j}|d|(b_1g_1 + c_1g_0)$; hence, $p_j^{k_j}|b_1g_1$ and $p_j^{k_j-l_j}|b_1$. This implies that $p_j^{k_j} = p_j^{k_j-l_j}p_j^{l_j}|b_1c$.

$\supseteq$: Let $b, c \in R$ be such that $b|c|d$ and GCD $(b, (d/b)) = R^\times$. We set $I = (b, cX, X^2, X^3)_{R_d}$. It follows by Lemma 5.1.2 that $I \in \mathcal{F}_v^\bullet(R_d)$ and $(I^2)^{-1} = \{f \in R[\![X]\!] \mid (d/b)|bf_1, (d/c)|bf_0\}$.

Let us show that $(I^2)^{-1} = I_{(d/c),(d/b)}$. $\subseteq$: Let $f \in (I^2)^{-1}$. Then $(d/c)|bf_0$ and $(d/b)|bf_1$. Since $(d/c)|(d/b)$, we have $\mathrm{GCD}\,(b,(d/c)) = R^\times$. Of course, d/b and d/c are associate to products of prime elements of R. Therefore, it is straightforward to show that $(d/c)|f_0$ and $(d/b)|f_1$. Consequently, $f \in I_{(d/c),(d/b)}$.

$\supseteq$: Trivial. Therefore, Lemma 5.1.2 implies that $(I^2)^{-1} = I^{-1}$; hence, $(I^2)_v = I$. Now it is straightforward to prove that $\mathsf{E}(\mathcal{F}_v^\bullet(R_d))$ is finite. Hence, it follows that $\mathcal{F}_v^\bullet(R_d)$ is almost complete.

4. Let $R[\![X]\!]$ be factorial. At first, let d be a prime element of R. It follows by Lemma 5.2.2, 2 and 3, that $\{J \in \mathcal{F}_v^\bullet(R_d) \mid \mathcal{F}_{\widehat{R_d}/R_d} \subseteq J$ and $J^2 \subseteq J\} = \{\mathcal{F}_{\widehat{R_d}/R_d}, \mathcal{F}_{\widehat{R_d}/R_d} + XR_d, R_d, \widehat{R_d}\}$, $\mathsf{E}(\mathcal{F}_v^\bullet(R_d)) = \{\mathcal{F}_{\widehat{R_d}/R_d}, R_d, \widehat{R_d}\}$ and $\mathcal{F}_v^\bullet(R_d)$ is almost complete. We have to show that $\mathcal{F}_v^\bullet(R_d)$ is πv-regular. Let $I \in \mathcal{F}_v^\bullet(R_d)$. Then $\mathbf{R}(I) \in \{R_d, \widehat{R_d}\}$.

Case 1. $\mathbf{R}(I) = R_d$: Since $\mathcal{F}_{\mathbf{R}(I)/R_d} = R_d \in \mathsf{E}(I)$, we have I is v-regular by Lemma 2.4.2.

Case 2. $\mathbf{R}(I) = \widehat{R_d}$: Since $\widehat{R_d} \in \mathsf{E}(I)$ and $\widehat{R_d} \notin \mathsf{E}(R_d)$, it follows by Lemmas 2.2.4 and 2.4.1 that $(I(\widehat{R_d} :_{K'} I))_v \in \{J \in \mathcal{F}_v^\bullet(R_d) \mid \mathcal{F}_{\widehat{R_d}/R_d} \subseteq J \subseteq \widehat{R_d}, \widehat{R_d} \in \mathsf{E}(J)$ and $J^2 \subseteq J\} = \{\mathcal{F}_{\widehat{R_d}/R_d}, \mathcal{F}_{\widehat{R_d}/R_d} + XR_d, \widehat{R_d}\}$. Since $\widehat{R_d} = R[\![X]\!]$ is factorial and $(\widehat{R_d} :_{K'} I) \in \mathcal{F}_{v_{\widehat{R_d}}}^\bullet(\widehat{R_d})$, it follows that there is some $c \in K'^\bullet$ such that $(\widehat{R_d} :_{K'} I) = c\widehat{R_d}$. Hence, $(I(\widehat{R_d} :_{K'} I))_v = (cI\widehat{R_d})_v = cI$, and thus $I \in \{c^{-1}\mathcal{F}_{\widehat{R_d}/R_d}, c^{-1}(\mathcal{F}_{\widehat{R_d}/R_d} + XR_d), c^{-1}\widehat{R_d}\}$.

Case 2.1. $I = c^{-1}\mathcal{F}_{\widehat{R_d}/R_d}$: Since $\mathcal{F}_{\mathbf{R}(I)/R_d} \in \mathsf{E}(I)$, we have I is v-regular by Lemma 2.4.2.

Case 2.2. $I = c^{-1}(\mathcal{F}_{\widehat{R_d}/R_d} + XR_d)$: Since $(I^2)_v = c^{-2}\mathcal{F}_{\widehat{R_d}/R_d}$ and $\mathcal{F}_{\mathbf{R}_v(I^2)/R_d} \in \mathsf{E}((I^2)_v)$, it follows by Lemma 2.4.2 that $(I^2)_v$ is v-regular; hence, I is πv-regular.

Case 2.3. $I = c^{-1}\widehat{R_d}$: Since $c\widehat{R_d} \in \mathcal{F}_v^\bullet(R_d)$ and $(Ic\widehat{R_d})_v = \widehat{R_d}$, we have that I is v-regular.

Now let R/dR be finite. It follows by Lemma 5.3.3 that $\mathcal{F}_v^\bullet(R_d)$ is almost complete. Let $J \in \mathcal{F}_v^\bullet(R_d)$. Then, since $\widehat{R_d} = R[\![X]\!]$ is factorial, it follows by Lemmas 2.4.3 and 5.2.2 that some $c \in K'^\bullet$ exists such that $\mathcal{F}_{\widehat{R_d}/R_d} \subseteq cJ \subseteq \widehat{R_d}$. Therefore, Lemma 5.3.4 implies that cJ is πv-regular; hence, J is πv-regular. □

Example 5.5. Each of the following properties is satisfied by some integral domain R:

1. R is Noetherian, $\dim(R) = \dim(\overline{R}) = 2$, $\overline{R}$ is local, Noetherian and factorial, $\mathcal{F}_{\widehat{R}/R} \in \mathsf{E}(\mathcal{F}_v^\bullet(R)) \setminus \{I \in \mathcal{F}_v^\bullet(R) \mid I = \mathbf{R}(I)\}$ and R_P is a discrete valuation domain for all $P \in \mathrm{spec}^1(R)$.

2. R is neither a Mori domain nor completely integrally closed and yet $\mathcal{F}_v^\bullet(R)$ is almost complete.

3. $\mathcal{F}_{\widehat{R}/R} \neq \{0\}$, $\dim(\widehat{R}) = 2$, $\widehat{R}$ is local, Noetherian and factorial, and some $I \in \mathcal{F}_v^\bullet(R)$ exists such that $\mathbf{R}_v(I^n) \subsetneqq \mathbf{R}_v(I^{n+1})$ for all $n \in \mathbf{N}$. In particular, $\mathcal{F}_v^\bullet(R)$ is not π-regular.

Proof. 1. Let S be a discrete valuation domain, L a field of quotients of S, X an indeterminate over L, $d \in S^\bullet \setminus S^\times$, and $R = \{f \in S[\![X]\!] \mid d | f_1\}$. It follows by Lemma 5.2.1 that $\overline{R} = \widehat{R} = S[\![X]\!] = (1, X)_R$; hence, $\overline{R}$ is a local, Noetherian and factorial domain and $\dim(R) = \dim(\overline{R}) = 2$. It follows by the theorem of Eakin-Nagata that R is Noetherian, hence Lemma 5.2.5 implies that R_P is a discrete valuation domain for all $P \in \mathrm{spec}^1(R)$. Since $d \notin S^\times$, it follows by Lemmas 5.2.2 and 5.4.1 that $\mathcal{F}_{\widehat{R}/R} \in \mathsf{E}(\mathcal{F}_v^\bullet(R)) \setminus \{I \in \mathcal{F}_v^\bullet(R) \mid I = \mathbf{R}(I)\}$.

2. Let S be a completely integrally closed domain, that is, not a Mori domain; for example, the ring of algebraic integers. Let Y be an indeterminate over S. It follows that $S[\![Y]\!]$ is a completely integrally closed domain that is not a Mori domain. Of course, Y is a prime element of $S[\![Y]\!]$. Let L be a field of quotients of $S[\![Y]\!]$, X an indeterminate over L and $R = \{f \in (S[\![Y]\!])[\![X]\!] \mid Y|_{S[\![Y]\!]} f_1\}$. Then Lemma 5.4.3 implies that $\mathcal{F}_v^\bullet(R)$ is almost complete. Since $X \notin R$, it follows that R is not completely integrally closed. Assume that R is a Mori domain. Then Lemmas 3.1.4 and 5.2.1 imply that $\widehat{R} = (S[\![Y]\!])[\![X]\!]$ is a Mori domain, hence $S[\![Y]\!]$ is a Mori domain, a contradiction.

3. Let S be an integral domain that is not a field. Let K' be a quotient field of S, Y an indeterminate over K' and $T = S + YK'[\![Y]\!]$. It follows that T and $K'[\![Y]\!]$ have the same field of quotients; we denote it by L. Since $K'[\![Y]\!]$ is a principal ideal domain, it follows

that $\widehat{T} \subseteq \widehat{K'[\![Y]\!]} = K'[\![Y]\!]$. On the other hand, $YK'[\![Y]\!] \subseteq T$. Since $Y \in T^\bullet$, we have $K'[\![Y]\!] \subseteq \widehat{T}$; hence, $\widehat{T} = K'[\![Y]\!]$. It follows that $\widehat{T}$ is a discrete valuation domain. Some $b \in S^\bullet \backslash S^\times$ exists. Of course, $b \in T \backslash T^\times$ and $\cap_{n \in \mathbf{N}} b^n T \supseteq YK'[\![Y]\!] \neq \{0\}$. Let $d \in T^\bullet$ be such that $d \in \cap_{n \in \mathbf{N}} b^n T$.

Let X be an indeterminate over L, $R = \{f \in T[\![X]\!] \mid d|f_1\}$ and $I = (b, dX, X^2, X^3)_R$. It follows that $\widehat{T}[\![X]\!]$ is local, Noetherian, factorial and $\dim(\widehat{T}[\![X]\!]) = 2$. This implies that $\widehat{R} \subseteq \widehat{\widehat{T}[\![X]\!]} = \widehat{T}[\![X]\!]$. Since $X^2Y\widehat{T}[\![X]\!] \subseteq X^2T[\![X]\!] \subseteq R$ and $X^2Y \in R^\bullet$, we have $\widehat{T}[\![X]\!] \subseteq \widehat{R}$. Therefore, $\widehat{R} = \widehat{T}[\![X]\!]$ and $\mathcal{F}_{\widehat{R}/R} \neq \{0\}$. By Lemma 5.1.2, it follows that $(I^n)^{-1} = \{f \in T[\![X]\!] \mid d|b^n f_1, d|b^{n-1} df_0\} = (1, (d/b^n)X, X^2, X^3)_R$ for all $n \in \mathbf{N}$. Hence, it follows by Lemma 5.1.2 that $\mathbf{R}_v(I^n) = ((I^n)_v : I^n) = (R : (I^n)^{-1} I^n) = \mathbf{R}((I^n)^{-1}) = \mathbf{R}((1, (d/b^n)X, X^2, X^3)_R) = \{f \in T[\![X]\!] \mid (d/b^n)|1f_1\} = (1, (d/b^n)X, X^2, X^3)_R$ for all $n \in \mathbf{N}$. Assume that some $m \in \mathbf{N}$ exists such that $\mathbf{R}_v(I^m) = \mathbf{R}_v(I^{m+1})$. Then $(d/b^{m+1})X \in \mathbf{R}_v(I^{m+1}) = \mathbf{R}_v(I^m) = \{f \in T[\![X]\!] \mid (d/b^m)|f_1\}$; hence, $(d/b^m)|(d/b^{m+1})$. This implies that $b \in T^\times$, a contradiction. Hence, Lemmas 1.1.3 and 2.2.4 imply that $\mathcal{F}_v^\bullet(R)$ is not π-regular. □

Note that Example 5.5.1 shows that the idempotents of the v-ideal semigroup of a Mori domain need not be trivial. Moreover, by Example 5.5.2, it follows that the converse of Theorem 2.6.1 does not hold. By Example 5.5.3, we get that the π-regularity of the v-ideal semigroup does not descend from the complete integral closure.

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