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Structure of singularities and its numerical realization in nonlinear elasticity

By

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0. Introduction

0.1 Nonlinear shell theory and numerical analysis

Since the early work of von Kármán and Tsien in 1939 [52], the basic importance of instability phenomena in nonlinear mechanics has been well recognized, and attracting attention of many researchers in the field of both theoretical mechanics and mathematics. Spap-through and bifurcation bucklings of shells under external loadings are instances of such instability phenomena. Similar phenomena of bifurcation and pattern formation are often found not solely in nonlinear elasticity, but also in a variety of fields such as fluid mechanics, chemical reactions and bio-mathematics.

Mathematically, such instabilities are understood in the context of singularities of a nonlinear equation, say in a Banach space. Due to the high nonlinearity of the problem, the methodology of studying such singularity problems falls naturally into one of the following two categories: (*i*) modern operator theoretical methods with the aid of nonlinear functional analysis, and (*ii*) numerical techniques such as the finite difference method and the finite element or the Ritz-Galerkin method. One can also count those semi-numerical techniques as perturbations or asymptotic expansions within the second category. For works among the first category in the field of nonlinear elasticity, one may refer to, e.g., [3], [25], [26] and [43]. (Also, see [9].) Since the work of von Kármán and Tsien [52], a number of papers in the second category have been published on questions concerning the determination of critical loads of snap and bifurcation bucklings, the sensitivity on imperfections, and initial post-buckling behaviors. (For instance, see [2], [7], [12], [38] and [54].) Huang [17] seems the first who made use of the finite difference method in the analysis of bifurcation bucklings of thin shallow spherical shells.

In this direction, after the thesis of Koiter [19], a significant contribution was made by Thompson and Hunt [51], who have developed basic concepts of elastic stability and have given classification theorems of singularities in general *discrete* elastic systems. Based on these results, Hangai and Kawamata developed the *static perturbation* technique, with which Endou, Hangai and Kawamata [12] have persued a complete pre and post-buckling analysis of shallow spherical shells using the finite element method.

Although the methods in two categories complement each other, it is worthy of noting that the role played by numerical or semi-numerical techniques is crucial, especially in problems with nonlinear fundamental paths, i. e., in problems of $<$*class* $N$$>$ as defined in the text.

Emphasis should be made that, despite the basic significance of numerical analysis, there exists, to the authors' knowledge, very few works which *justify* those numerical results concerning elastic stability problems. It is these general backgrounds that have motivated the present paper. The scope of this paper is thus to provide a mathematical foundation for the numerical analysis of nonlinear elasticity systems which include singularities. (This paper also complements and improves the results in Yamaguti and Fujii [55], where complete settings and proofs were not given.)

Our fundamental question is whether the structure of singularities can be realized *numerically*. We clarify how and in what schemes this realization can be established. It is to be noted that this is not merely the question of "convergence" in the limit of $h \to 0$. We are asking whether for a *finite* $h > 0$, the structure of criticality, say a symmetry breaking bifurcation, is realized again *as itself* (namely, as a symmetry breaking bifurcation) in the approximate finite dimensional subspace.

Our standpoint is in the justification of the methods in the second category, while our method of study is in the first category. Of course, we need basic results of the finite element theory and those of the group representation theory in a Hilbert space as well.

For works with similar standpoints, we note the work of Weiss [53] for finite difference approximation of bifurcation problems of ordinary differential equations, Kikuchi [22] for bifurcations of semilinear elliptic equations from the trivial path, and also Kikuchi [23] for snap bucklings of a class of partial differential equations.

0.2 The von Kármán-Donnell-Marguerre shell theory

We proceed the study within the framework of a nonlinear operator equation in a Hilbert space V:

$$(P) \qquad F(\mu, w)=0,$$

where F is a mapping $\boldsymbol{R}\times V\to V$ of $C^p(p\geq 3)$ class. $F'(\mu, w)\equiv\partial F/\partial w(\mu, w)$ is assumed to be Fredholm and self-adjoint, which may characterize the nonlinear elasticity.

The setting (P) involves the shallow arch, and the shallow shell theory of von Kármán, Donnell and Margurre (*See, e.g.*, [3], [32] and [52].) In fact, the von Kármán-Donnell-Marguerre equation is:

$$(von\ K.D.M.)\left|\begin{array}{l}\Delta^2\phi=-\dfrac{1}{2}[w, w]-[w, w_0]\\ \Delta^2 w=[w+w_0, \phi+\phi_0]+\mu p\end{array}\right. \quad \text{in } \Omega\subset\boldsymbol{R}^2$$

where Δ^2 is the biharmonic operator, and

$$\left|[u, v]=\frac{\partial^2 u}{\partial x^2}\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 u}{\partial y^2}\frac{\partial^2 v}{\partial x^2}-2\frac{\partial^2 u}{\partial x\partial y}\frac{\partial^2 v}{\partial x\partial y}.\right.$$

At the boundary $\partial\Omega$, we impose the conditions:

$$\left|w=\frac{\partial w}{\partial n}=\phi=\frac{\partial\phi}{\partial n}=0 \qquad \text{on } \partial\Omega.\right.$$

Here, w represents the radial component of deflection of the shell from its initial deflection w_0 (a known function) ; ϕ is the Airy stress function, and ϕ_0 is the known Airy function of the applied force to the edge ; p is the external load on the shell with the loading parameter μ.

If we let $V=H_0^2(\Omega)$ with inner product $\langle u, v\rangle=\int_\Omega \Delta u\Delta v$,

$$B(u, v)=\Delta^{-2}[u, v] \text{ and } G=\Delta^{-2},$$

$(von\ K.D.M.)$ is reduced to a pair of operator equations:

$$(von\ K.D.M.)'\left|\begin{array}{l}\phi=-\dfrac{1}{2}B(w, w)-B(w, w_0)\\ w=B(w+w_0, \phi+\phi_0)+\mu Gp.\end{array}\right.$$

Eliminating ψ in these equations, we obtain a single operator equation of the form (P). For details, see Appendix A.

Among problems within the setting (P), the *shallow arch* problem may serve as an example for getting basic *idea* about snap and bifurcation bucklings. If we let $\Omega=(0, \pi)$, the shallow arch equation is given by ([24]):

$$(Arch)\quad \frac{d^4w}{dx^4}-\left\{-\lambda+\left\|\frac{dw}{dx}\right\|_0^2+2\left(\frac{dw_0}{dx}, \frac{dw}{dw}\right)_0\right\}\frac{d^2}{dx^2}(w_0+w)=\mu\cdot p \text{ in } \Omega,$$

with appropriate boundary conditions, where $(\ ,\ )_0$ and $\|\cdot\|_0$ denote respectively the inner product and the norm in $L^2(\Omega)$. It is not difficult to reduce the arch equation to an operator equation in $V=H^2(\Omega)\cap H_0^1(\Omega)$ (the simply supported boundary condition) or in $V=H_0^2(\Omega)$ (the rigid boundary condition). We leave the detail to the reader, and instead study its explicit solution for the particular case that $V=H^2(\Omega)\cap H_0^1(\Omega)$ and $w_0\equiv 0$. Let $\phi_j=\sqrt{2/\pi}\sin jx\,(j=1, 2, \ldots)$, and let $w=\sum_{j=1}^{\infty} w_j\phi_j$. Suppose that $p=\phi_1$. We find easily that $(Arch)$ is equlvalent to:

$$\{k^2-\lambda+\sum_{j=1}^{\infty} j^2w_j^2\}k^2w_k=\mu\cdot\delta_{k1}\qquad k=1, 2, \ldots$$

where δ_{ij} is the Kronecker delta. First of all, for all $\lambda\in\boldsymbol{R}$, there is a path of solutions $(\mu, w)\in\boldsymbol{R}\times V$, where $w=w_1\phi_1$, w_1 being the solution of $\mu=(1-\lambda+w_1^2)w_1$. We call this the fundamental path. If λ is a constant such that $m^2<\lambda\leq(m+1)^2$, we conclude that in addition to the fundamental path, there exist $(m-1)$-bifurcated paths $(\mu, w^{(l)})$, $l=2, 3, \ldots m$, where $w^{(l)}=w_1\phi_1+w_l\phi_l$, with the relations $w_1^2+l^2w_l^2=\lambda-l^2$, and $(1-l^2)w_1=\mu$. See, Fig. 0.1 for the case that λ is a constant such that $4<\lambda\leq 9$. The paths of solutions can be represented in the three dimensional load-coordinate space. In the figure, we have two types of critical points i.e., the spap-through points S and S', and the bifurcation points B and B'.

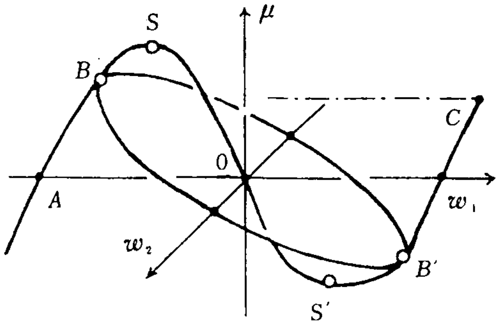


Fig. 0.1

Supposing that the arch is originally in equilibrium with the edge force λ at A, the arch deflects symmetrically by the normal force μ until at B, where it loses stability, jumping to C through the bifurcated path $B \to B' \to C$.

0.3 General outline

The text contains two Chapters and two Appendices. The structure of singularities in nonlinear elasticity is studied in Chapter I, while Chapter II is devoted to the theory of numerical analysis of singularity problems.

In Section 1, Chapter I, we give formal classification theorems of *simple* critical points of the problem (P). We do not assume *a priori* the existence of a fundamental path, which makes it possible to discuss both the snap and bifurcation bucklings in a consistent way. Our classification is parallel to Thompson and Hunt [51], or Hangai and Kawamata [16], in which a *discrete* system is considered by the perturbation method. Local behaviors of solutions in a neighborhood of those critical points are discussed, refinding the famous *exchange of stability* for fold bifurcation points and other stability properties for snap and cusp bifurcation points. These materials themselves are interesting; and more intrincically, we need them in the theory of numerical analysis of singularities.

In Section 2, Chapter I, we go further into the structure of those singularities. We introduce the concept of the *symmetry group* $\mathscr{G}$ and that of *class* <L or N> of the problem. The class of the problem is a *path dependent* notion, which essentially implies that the bifurcation problem is considered either on a *linear* (with respect to the bifurcation parameter) or on a *nonlinear* path. We shall clarify the relation of the type (fold, cusp or etc.) of critical points and the existence or non-existence of a symmetry group. For example, we shall show that a fold bifurcation is, if exists, $\mathscr{G}$-symmetry preserving. An important result in this section is the uniform existence of symmetry breaking bifurcation points with respect to small changes (=perturbations) of the equation under the presence of a symmetry group $\mathscr{G}$ (which we shall call the *structural stability* of the bifurcation points). As an obvious analogue, we have the structural stability of <*class* L> bifurcations under perturbations which do not destroy the <*class* L> property. These are obviously non-generic situations; however, it is this structural stability that guarantees the numerical realization of bifurcation points in the actual world of numerical computations.

The introduction of group theoretical arguments to nonlinear singularities is not indeed new, particularly in pattern formation problems in fluid mechanics. (See, e. g., Ruelle [42] and Sattinger [44-46]. Also, see [36] and [40] for other problems.). However, the emphasis here is on the discussion of structural stability in the sense described in the above. Our main tool is the *standard decomposition* of the Hilbert space V associated with the symmetry group of the problem. This will be again indispensable in the discussions of Chapter II. A remark is that our arguments here exhibit a sharp contrast with the general theory of imperfection sensitivities, e. g., by Thompson and Hunt [51], Hangai and Kawamata [16] and Keener and Keller [20]. See, however, Rooda [41] for discussions of non-generic imperfections.

Chapter II is devoted to the *numerical realization* of those singularities.

We first give an abstract setting (P^h) of a class of approximate schemes defined on a sequence $(h \to 0)$ of a finite element subspaces V^h of V. The setting (P^h) is motivated and is actually satisfied by the von Kármán-Donnell-Marguerre shells (as well as the arch problems). Our primal concern is the compatible class of schemes; however, an extention of (P^h) to, for instance, mixed finite element schemes as proposed by Miyoshi [34], or by Brezzi and Fujii [5] appears to be possible. In the setting (P^h), we assume the approximation properties of V^h in two different norms, namely, the natural energy norm (V-norm) and the L^2-norm. The latter implicitly assumes the situation that Nitsche's trick holds in V^h. (See, e. g., [49].) Our error estimates throughout in Chapter II will be obtained in terms of these two norms.

As a preliminary result on the group property in V^h, we show that if the *mesh pattern* of the finite elements preserves the symmetry group $\mathcal{G}$ of (P), the finite element space V^h is invariant under $\mathcal{G}$, and that the scheme (P^h) constructed in a conforming way is covariant under $\mathcal{G}$.

In section 4, Chapter II, we discuss the numerical realization of ordinary paths. It is proved that ordinary paths always exist in V^h near the original paths *except* in the vicinity of critical points. In Section 5, we prepare theorems on a family of approximate eigenproblems. These are, in a sense, the most crucial part of the numerical buckling theory.

In Section 6, the realization of snap points and neighboring paths in the approximate space V^h is shown. Error estimates of numerical buckling loads, buckling modes and buckling states are also given. Uniform convergence of the neighboring path to that of the original problem is proved as well. In the final section (Sec. 7), we discuss the numerical realization of symmetry breaking bifurcations. The conclusion is that if the scheme (P^h) is covariant under $\mathcal{G}$, uniform numercal realization of symmetry breaking bifurcations is established. (This may be considered as the structural stability of bifurcation points *under numerical perturbations.*) As in the snap buckling case, we obtain error estimates of various quantities.

In Appendix A, we give a complete example of the setting (P), using the von Kármán-Donnell-Marguerre shell defined on a domain $\Omega\subset\boldsymbol{R}^2$. In Appendix B, a compatible scheme for the von Kármán-Donnell-Marguerre equation is given, showing that all the properties required in the setting (P^h) are satisfied. It is noted that Ω is assumed to be either a sufficiently smooth or convex polygonal domain.

Finally, we comment that the problem (P) in Chapter II is of $<$*class* $N>$, the possible bifurcations thus being only the symmetry breaking ones. For a $<$*class* $L>$ problem, for example, for the von Kármán plate buckling problem with respect to the edge force λ, our results on bifurcations in Chapter II can be obtained as obvious corollaries.

Chapter I. Structure of singularities in nonlinear elasticity

1. Snap and bifurcation bucklings. Classification of singularities

The aim of this section is, as a theoretical preparation to the numerical analysis, to give a unified view and terminology to the singularities in nonlinear elasticity theory. We begin with giving classification theorems of singularities which may arise in many contexts in nonlinear elastic systems. We then discuss the behavior of solutions in a neighborhood of those singularities. Notice that this will play a basic role in the theory of numerical analysis of singularities.

We note that these materials are essentially known, for instance, see Crandall and Rabinowitz [11] or Nirenberg [35]. However, our setting is different from [11] in the sense that we do not assume *a priori* the existence of a fundamental path, in order to treat *snap bucklings* as well as *primary or secondary bifurcations* in a unified way. This appears to be more convenient from numerical analysis viewpoints. In fact, the classification theorems and terminologies in this section are in parallel with those found in engineering literatures as, e.g., Thompson and Hunt [51], and Hangai and Kawamata [16].

1.1 Classification of simple critical points

Let V be a Hilbert space with inner product $<\ ,\ >$ and norm $\|\cdot\|_V$. We consider the equation:

$$(P)\quad F(\mu,\ w)=0 \tag{1.1}$$

where F is a continuous mapping $\boldsymbol{R}\times V\to V$.

Envisaging applications to numerical analysis, our object is, for a known solution $\mathcal{O}\equiv(\mu_0, w_0)\in\boldsymbol{R}\times V$, to obtain all the *paths* in $\boldsymbol{R}\times V$ which contain $\mathcal{O}$. By a *path*, we mean a connected component of S, or its subcomponent where S denotes the closure of solutions of (P) in $\boldsymbol{R}\times V$.

Notice that in Eq. (1.1), $F(\mu, 0)=0$ ($^\forall\mu\in\boldsymbol{R}$) is *not* assumed, implying that the problem (P) may not have a trivial path $(\mu, 0)\in\boldsymbol{R}\times V$.

We assume to F the following:

$(A)_1$ F: $\boldsymbol{R}\times V\to V$ of class C^p, $p\geq 3$,

$(A)_2$ F: Fredholm mapping of index 0, namely,
$\dim\ker F'(\mu, w)=\dim\operatorname{coker} F'(\mu, w)=d<+\infty$.

$(A)_3$ $F'(\mu, w)\in\boldsymbol{B}(V)$ *) is self-adjoint.

Here, $F'(\mu, w)$ denotes the Fréchet derivative of F with respect to w at (μ, w):

$$F'(\mu, w)\overset{def}{=}\frac{\partial F}{\partial w}(\mu, w) \tag{1.2}$$

We shall also denote by $\dot{F}(\mu, w)$ the Fréchet derivative of F with respect to μ at (μ, w):

$$\dot{F}(\mu, w)\overset{def}{=}\frac{\partial F}{\partial\mu}(\mu, w). \tag{1.3}$$

Higher order derivatives will be also denoted by, for example, $F''(\mu, w)$, $F'''(\mu, w)$ and so on.

It is noted that every result in this section is applicable to non-self-adjoint cases (with obvious modifications). The assumption $(A)_3$ may characterize the nonlinear elasticity, and since our main object is the application to nonlinear elasticity in numerical analysis aspects, we assume $(A)_3$ in the whole of subsequent discussions.

Definition 1.1 Let $\mathcal{O}\equiv(\mu, w)\in\boldsymbol{R}\times V$ be a solution of $F(\mu, w)=0$. Then, $\mathcal{O}$ is called an *ordinary* (regular) point of (P), if $F'(\mu, w)$ has a bounded inverse, i.e., $F'(\mu, w)^{-1}\in\boldsymbol{B}(V)$, and a *critical* (singular) point if not.

The following lemma is an immediate consequence of the implicit function theorem (see, e.g., Nirenberg [35]).

*) $\boldsymbol{B}(X, Y)$ denotes the set of bounded linear maps $X\to Y$. $\boldsymbol{B}(X)\overset{def}{=}\boldsymbol{B}(X, X)$ and $\boldsymbol{B}_0(X)$ is the set of compact operators in $\boldsymbol{B}(X)$.

Lemma 1.2 *Suppose* $\mathcal{O} \equiv (\mu_0, w_0) \in \boldsymbol{R} \times V$ *is an ordinary point of* (P). *Then, there exist an interval* $I_\delta = \{\mu; |\mu_0| < \delta\}$ *and a unique* C^p *function* $w(\mu)$: $I_\delta \to V$ *such that* $F(\mu, w(\mu)) = 0$ $(\mu \in I_\delta)$.

Suppose now $\mathscr{C} \equiv (\mu_c, w_c) \in \boldsymbol{R} \times V$ is critical. We consider the problem in the particular case that the kernel and the cokernel are one dimensional (which we shall call the *simple* case). Denote by $F'_c, \dot{F}_c, \ldots$ the Fréchet derivatives of F at $\mathscr{C}$.

Let $\mathscr{L}_c \equiv F'_c$. Let $\{\phi_c\} = \ker \mathscr{L}_c$, and denote by Π'_c the functional $\Pi'_c u = \langle u, \phi_c \rangle$, $u \in V$. Let $\mathscr{R}_c = \text{range } \mathscr{L}_c = \{\ker \mathscr{L}_c\}^\perp$ and denote by ω_c the orthogonal projection $V \to \mathscr{R}_c$. We let denote by $\mathscr{L}^\dagger_c$ the bounded map $\mathscr{L}^\dagger_c : V \to V$ such that $\mathscr{L}^\dagger_c \cdot \mathscr{L}_c = \omega_c$.*)

Let:

$$\left.\begin{aligned}
&A_c \equiv \Pi'_c F''_c(\phi_c, \phi_c), \\
&B_c \equiv \Pi'_c F''_c(\phi_c, \phi_c) + \Pi'_c \dot{F}'_c \phi_c \\
&C_c \equiv \Pi'_c F''_c(g_c, g_c) + 2\Pi'_c \dot{F}'_c g_c + \Pi'_c \ddot{F}_c \\
&D_c \equiv \Pi'_c F'''_c(\phi_c, \phi_c, \phi_c) - 3\Pi'_c F''_c(\phi_c, \mathscr{L}^\dagger_c \omega_c F''_c(\phi_c, \phi_c)), \\
&f_c \equiv \dot{F}_c,
\end{aligned}\right\} \quad (1.4)$$

where

$$g_c \equiv -\mathscr{L}^\dagger_c \omega_c f_c.$$

Definition 1.3 A simple, critical point $\mathscr{C} \equiv (\mu_c, w_c) \in \boldsymbol{R} \times V$ is called a *snap* point if $\Pi'_c f_c \neq 0$. Moreover, if $A_c \neq 0$, $\mathscr{C}$ is a *non-degenerate* snap point.

Note 1.4 *A snap point* (a snapping point, a snap-through point) may also be called as *a limit point* (a limiting point) or *a turning point*. See, e.g., [21], [23], [51] and [55].

Definition 1.5 A simple critical point $\mathscr{C} \equiv (\mu_c, w_c) \in \boldsymbol{R} \times V$ is called a *non-degenerate point of bifurcation* if $\Pi'_c f_c = 0$ and $B_c^2 - A_c C_c > 0$. Moreover, if $A_c \neq 0$, $\mathscr{C}$ is called a non-degenerate, *asymmetric* point of bifurcation, and if $A_c = 0$, $D_c \neq 0$, a non-degenerate *symemtric* point of bifurcation.

Note 1.6 The term "symmetric or asymmetric point of bifurcation" often appears in engineering literatures, e.g., [51]. However, as we shall introduce the concept of *group symmetry* to nonlinear singularities, we prefer to call the symmetric and asymmetric points of bifurcations as the

*) $\mathscr{L}^\dagger_c = (\mathscr{L}_c | \mathscr{R}_c)^{-1} \omega_c$

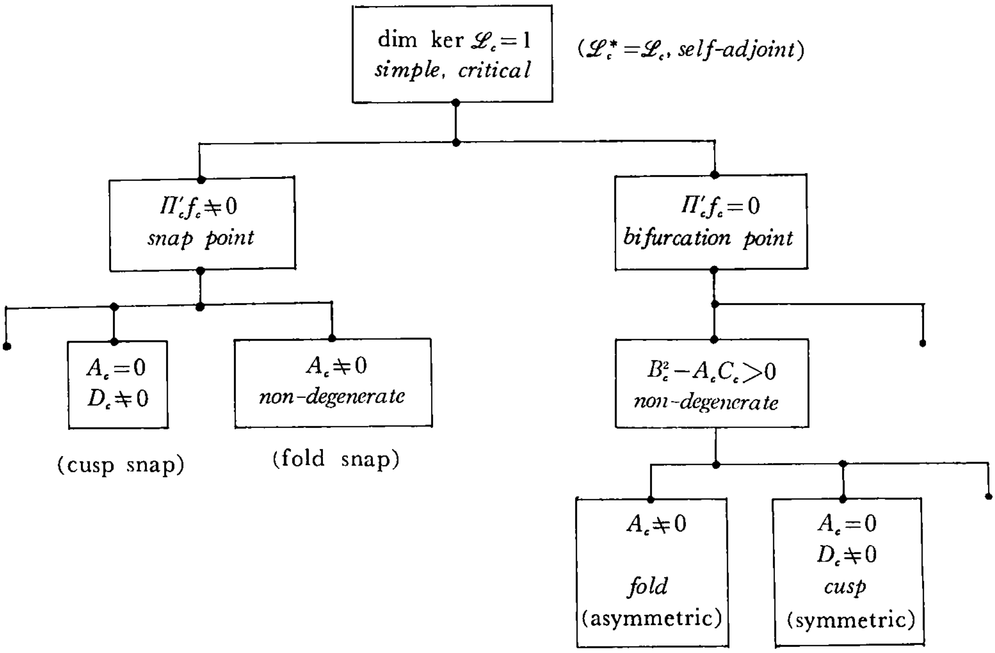


Fig. 1.1 Classification of Simple Critical Points

fold and *cusp* bifurcations, respectively, to avoid possible confusions in terminology. Our terminology corresponds to the first two elementary catastrophes in the theory of universal unfoldings of singularities due to R. Thom [50]

We shall see, however, that the appearance of symmetric or asymmetric points of bifurcation has a crucial relation with the existence or non-existence of symmtry groups.

Remark 1.7 Suppose (P) has a trivial path $(\mu, 0) \in \boldsymbol{R} \times V$. Then, a simple critical point on this path can never be a snap point, since $f_c = \frac{\partial F}{\partial \mu}(\mu, 0) \equiv 0$ for all $\mu \in \boldsymbol{R}$.

1.2 Behavior of solutions in a neighborhood of simple critical points

We now summarize results on local behaviors of solutions of (P) in the vicinity of simple critical points. The knowledge of eigenvalues on the paths will be indispensable in the discussion of numerical solutions about those critical points. Hence, we state the lemmas as well as brief proofs of them.

Firstly,

Proposition 1.8 (*Snap point*)

Suppose $\mathscr{C} \equiv (\mu_c, w_c) \in \boldsymbol{R} \times V$ is a simple, non-degenerate snap point of (P). Then,

(i) *in a neighborhood of $\mathscr{C}$, there is a unique path, say α-path, which meets $\mathscr{C}$. In other words, there exists an interval $I_\delta = \{\alpha ; |\alpha| < \delta\} \subset \boldsymbol{R}(\delta :$ sufficiently small) and two C^p functions $\mu(\alpha) : I_\delta \to \boldsymbol{R}$ and $w(\alpha) : I_\delta \to V$, such that*

$$F(\mu(\alpha), w(\alpha)) = 0,$$

and

$$\mu(0) = \mu_c, \ w(0) = w_c.$$

(ii) *For $\alpha \in I_\delta$, $\mu(\alpha)$ and $w(\alpha)$ satisfy*

$$|\mu(\alpha) - \mu(0)| \leq C\alpha^2, \tag{1.5}$$

and

$$\|w(\alpha) - w_c\|_V \leq C'\alpha.^{*)} \tag{1.6}$$

In fact, they take the form

$$\mu(\alpha) = \mu_c - \frac{A_c}{2\Pi'_c f_c}\alpha^2 + O(\alpha^3) \tag{1.7}$$

and

$$w(\alpha) = w_c + \alpha \cdot \phi_c + \left[-\frac{A_c}{2\Pi'_c f_c} \mathscr{L}_c^{\dagger} \omega_c f_c \right] \alpha^2 + O(\alpha^3) \tag{1.8}$$

(iii) *Furthermore, the linearized eigenproblem on the α-path:*

$$(E)_\alpha \quad F'(\mu(\alpha), w(\alpha)) \cdot \phi(\alpha) = \zeta(\alpha) \cdot \phi(\alpha), \ \alpha \in I_\delta \tag{1.9}$$

has a pair of C^{p-1} functions $\zeta_c(\alpha) : I_\delta \to \boldsymbol{R}$ and $\phi_c(\alpha) : I_\delta \to V$ such that

$$\zeta_c(0) = 0, \ \frac{d\zeta_c}{d\alpha}(0) \neq 0 \quad \text{and} \quad \phi_c(0) = \phi_c. \tag{1.10}$$

---

*) Here and in the sequel, C, C' or C'' denotes a positive generic constant, which may take different values when it appears in different places.

Remark 1.8 The (iii) of the above proposition means that one (and only one) eigenvalue changes its sign when it crosses a non-degenerate snap point of (P). See, Fig. 1.2.

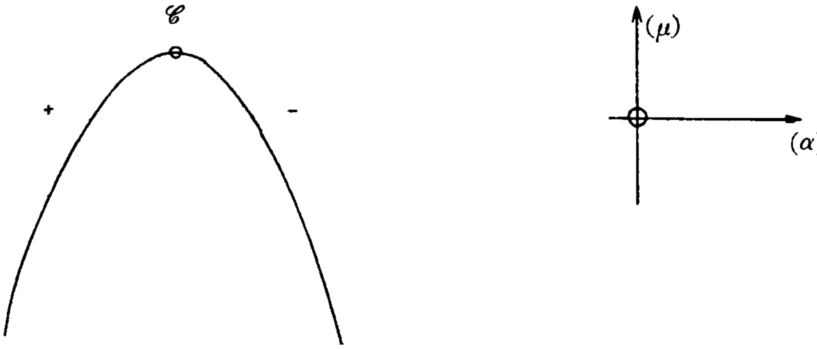


Fig. 1.2. *A non-degenerate snap point* $\mathscr{C}$

Proof. We let in (P) $\mu=\mu_c+\nu$ and $w=w_c+v$. Then,

$$
\begin{aligned}
&G_c(\nu, v) \equiv F(\mu_c+\nu, w_c+v) - F(\mu_c, w_c) \\
(Q) \qquad &= \mathscr{L}_c v + \nu f_c + \frac{1}{2} F_c''(v, v) + \nu \dot{F}_c' \cdot v \qquad (1.11)\\
&+ \frac{1}{2}\nu^2 \ddot{F}_c + \frac{1}{3!} F_c'''(v, v, v) + R_c(\nu, v) = 0,
\end{aligned}
$$

where the remainder term $R_c(\nu, v)$ satisfies $R_c(0, 0)=0$. To solve $G_c(\nu, v)=0$, we apply the Lyapounov-Schmidt decomposition (see, e. g. [35]) $\omega_c G_c(\nu, v)=0$, $\Pi_c' G(\nu, v)=0$ with

$$
v=\alpha\phi_c+\chi, \qquad \chi\in\mathscr{R}_c. \qquad (1.12)
$$

Since $\mathscr{L}_c$ is an isomorphism $\mathscr{R}_c \to \mathscr{R}_c$, the implicit function theorem works for $\omega_c G_c(\nu, \alpha\phi+\chi)=0$. That is, there is a ball $\Omega_{\delta'}=\{(\alpha, \lambda)\ ;\ |\alpha|+|\lambda|<\delta'\}\subset \boldsymbol{R}^2$ and a unique map $\chi=\chi(\alpha, \nu)$: $\Omega_{\delta'}\to\mathscr{R}_c$ such that $\chi(0, 0)=0$ and $\omega_c G_c(\nu, \alpha\phi_c+\chi(\alpha, \nu))=0$. We next substitue $\chi=\chi(\alpha, \nu)$ to $\Pi_c' G_c(\nu, \alpha\phi_c+\chi)=0$, obtaining *the bifurcation equation* for (P) :

$$
\begin{aligned}
\Gamma(\alpha, \nu) &\equiv \nu\Pi_c' f_c + \frac{1}{2}\alpha^2 A_c + \alpha\nu B_c + \frac{1}{2}\nu^2 C_c \\
&+ \frac{1}{3!}\alpha^3 D_c + (h.\ o.\ t.) = 0, \qquad (1.13)
\end{aligned}
$$

where $(h.o.t.)$ denotes higher order terms in α and ν.

Note that Γ is a $C^p (p \geq 3)$ map $\Omega_{\delta'} \to \boldsymbol{R}$.

In view of the relation $\Gamma(0, 0) = 0$ and $\frac{\partial \Gamma}{\partial \nu}(0, 0) = \Pi_c' f_c \neq 0$ by hypothesis, we can again apply the implicit function theorem, which guarantees the existence of an interval $I_\delta = \{\alpha; |\alpha| < \delta\} \subset \boldsymbol{R}$ and a unique C^p map $\nu = \nu(\alpha): I_\delta \to \boldsymbol{R}$ such that $\nu(0) = 0$ and $\Gamma(\alpha, \nu(\alpha)) = 0$. This proves (i), and from which follows (ii).

To prove (iii), we let $(\zeta_c(\alpha), \phi_c(\alpha)) \in \boldsymbol{R} \times V$ $(\alpha \in I_\delta)$ be the continuation of (ζ_c, ϕ_c). Namely, we let

$$(E)_\alpha \qquad \mathscr{L}(\alpha)\phi_c(\alpha) = \zeta_c(\alpha)\phi_c(\alpha), \ \alpha \in I_\delta$$
$$\text{with } \phi_c(0) = \phi_c \text{ and } \zeta_c(0) = \zeta_c, \tag{1.14}$$

where

$$\mathscr{L}(\alpha) = F'(\mu(\alpha), w(\alpha)).$$

Notice that $\mathscr{L}(\alpha)$ is a C^{p-1} map $I_\delta \to \boldsymbol{B}(V)$, since $\mu(\alpha)$, $w(\alpha)$ and F are all maps of C^p class. Accordingly, the implicit function theorem guarantees the unique existence of C^{p-1} map $(\zeta_c(\alpha), \phi_c(\alpha)): I_\delta \to \boldsymbol{R} \times V$. We can also assume (by taking, if necessary, a subinterval $I_{\delta'} \subset I_\delta$) that dim ker $(\mathscr{L}(\alpha) - \zeta_c(\alpha) I) = 1$ for $\alpha \in I_{\delta'}$.

Now, differentiating the both sides of $(E)_\alpha$ by α and taking the inner product with $\phi_c(\alpha)$, we obtain the relation

$$\frac{d\zeta_c}{d\alpha}(\alpha) = \left\langle \frac{d\mathscr{L}}{d\alpha}(\alpha)\phi_c(\alpha), \ \phi_c(\alpha) \right\rangle \tag{1.15}$$

where $\|\phi_c(\alpha)\|_V = 1$ is assumed. It may be a simple work now to show that

$$\frac{d}{d\alpha}\mathscr{L}(\alpha)\,|_{\alpha=0} = \dot{F}'(\mu(\alpha), w(\alpha))\frac{d\mu}{d\alpha}(\alpha) + F''(\mu(\alpha), w(\alpha))\frac{dw}{d\alpha}(\alpha)\,|_{\alpha=0}$$
$$= F_c''(\phi_c, \cdot), \tag{1.16}$$

in view of the relations:

$$\frac{d\mu}{d\alpha}(0) = 0 \text{ and } \frac{dw}{d\alpha}(0) = \phi_c(0) = \phi_c. \tag{1.17}$$

Accordingly, using the hypothesis we have the conclusion:

$$\frac{d\zeta_c}{d\alpha}(0) = \left\langle \frac{d\mathscr{L}}{d\alpha}(0)\phi_c, \ \phi_c \right\rangle = \Pi_c' F_c''(\phi_c, \phi_c)$$
$$= A_c \neq 0.$$

We turn to the bifurcation cases.

Proposition 1.9 (*Fold bifurcation*)

Suppose $\mathscr{C} \equiv (\mu_c, w_c) \in \boldsymbol{R} \times V$ *is a fold bifurcation of* (P). *Then,* (i) *there exist two paths,* μ_+ *and* μ_- *paths, in a neighborhood of* $\mathscr{C}$, *which intersect at* $\mathscr{C}$. *In other words, there is an interval* $I_\delta = \{\nu\ ;\ |\nu| < \delta\} \subset \boldsymbol{R}(^\exists\delta :$ *sufficiently small*), *and two* C^{p-2} *functions* $w_\pm(\nu) : I_\delta \to V$ *such that*

$$F(\mu_c + \nu,\ w_\pm(\nu)) = 0,\ \nu \in I_\delta$$

and

$$w_+(0) = w_-(0) = w_c.$$

(ii) *For* $\nu \in I_\delta$, $w_\pm(\nu)$ *are such that*

$$\|w_\pm(\nu) - w_c\|_V \le C|\nu|. \tag{1.19}$$

In fact, they have the form

$$w_\pm(\nu) = w_c - \nu \mathscr{L}_c^! \omega_c f_c + \alpha_\pm(\nu)\phi_c + O(\nu^2) \tag{1.20}$$

where $\alpha_\pm(\nu)$ *are* C^{p-2} *functions* $I_\delta \to \boldsymbol{R}$ *such that*

$$\alpha_\pm(\nu) = \frac{-B_c \pm \sqrt{B_c^2 - A_c C_c}}{A_c}\nu + O(\nu^2). \tag{1.21}$$

(iii) *Furthermore, each of the linearized operators on the* μ_+ *and* μ_- *paths,* $\mathscr{L}_\pm(\nu) \equiv F'(\mu_c + \nu,\ w_\pm(\nu))$, $\nu \in I_\delta$, *has critical pairs of* C^{p-2} *functions* $(\zeta_c^+(\nu),\ \phi_c^+(\nu)) : I_\delta \to \boldsymbol{R} \times V$ *and* $(\zeta_c^-(\nu),\ \phi_c^-(\nu)) : I_\delta \to \boldsymbol{R} \times V$, *respectively, such that*

$$\zeta_c^+(0) = \zeta_c^-(0) = 0 \text{ and } \phi_c^+(0) = \phi_c^-(0) = \phi_c, \tag{1.22}$$

and

$$\frac{d\zeta_c^+}{d\nu}(0)\frac{d\zeta_c^-}{d\nu}(0) < 0.$$

Remark 1.9′ Assertion (iii) implies that at any point of fold bifurcations, the stability is exchanged from one path to the other path. This is an example of the famous *exchange of stability* of Poincaré. A fold bifurcation may be called a *transcritical* bifurcation by this reason.

Proof. By hypothesis, $\Pi_c' f_c$ vanishes and hence, the bifurcation equation (1.13) becomes

$$\Gamma(\alpha,\ \nu) = \frac{1}{2}\alpha^2 A_c + \alpha\nu B_c + \frac{1}{2}\nu^2 C_c + \frac{1}{3!}\alpha^3 D_c + (h.\ o.\ t.) = 0. \tag{1.23}$$

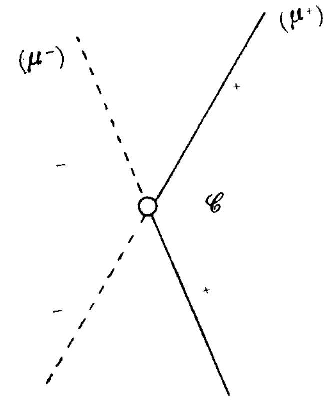


Fig. 1.3 Fold bifurcation and the exchange of stability

Since Γ is of $C^p(p\geq 3)$ and $\Gamma(0, 0)=\Gamma_\alpha(0, 0)=\Gamma_\nu(0, 0)=0$, the Morse lemma can be applied. (See, Nirenberg [35].) Namely, in view of the condition

$$B_c^2-A_cC_c>0 \tag{1.24}$$

there is a C^{p-2} local coordinate change $\tilde{\alpha}=\tilde{\alpha}(\alpha, \lambda)$ and $\tilde{\nu}=\tilde{\nu}(\alpha, \lambda)$ defined in a neighborhood of the origin such that $\tilde{\alpha}(0, 0)=\tilde{\nu}(0, 0)=0$, and that the Jacobian J of the coordinate change at the origin is a unit matrix. See, Fig. 1.4. Γ is written with this new coordinate as

$$\Gamma=\frac{1}{2}\tilde{\alpha}^2A_c+\tilde{\alpha}\tilde{\nu}B_c+\frac{1}{2}\tilde{\nu}^2C_c=0. \tag{1.25}$$

Therefore, by virtue of Eq. (1.24), the set of solutions near the origin consists of two C^{p-2} paths which intersect transversally at the origin. Since $A_c\neq 0$ (by hypothesis) these two paths are explicitly written as

$$\tilde{\alpha}_\pm=\frac{-B_c\pm\sqrt{B_c^2-A_cC_c}}{A_c}\tilde{\nu}, \tag{1.26}$$

or, equivalently

$$\alpha_\pm=\frac{-B_c\pm\sqrt{B_c^2-A_cC_c}}{A_c}\nu+O(\nu^2), \tag{1.27}$$

since $J=I$ at the origin.

To prove (iii), we notice again as in Proposition 1.8, that

$$\frac{d\zeta_c^{\pm}}{d\nu}(\nu) = \left\langle \frac{d\mathscr{L}_{\pm}}{d\nu}(\nu)\phi_c^{\pm}(\nu),\ \phi_c^{\pm}(\nu) \right\rangle. \tag{1.28}$$

It is easily checked that

$$\begin{aligned}\frac{d}{d\nu}\mathscr{L}_{\pm}\big|_{\nu=0} &= \dot{F}'_c + F''_c\left(g_c + \frac{d\alpha^{\pm}}{d\nu}(0)\phi_c, \cdot\right)\\ &= \dot{F}'_c + F''_c\left(g_c + \frac{-B_c \pm \sqrt{B_c^2 - A_c C_c}}{A_c}\phi_c, \cdot\right),\end{aligned} \tag{1.29}$$

and accordingly,

$$\left.\frac{d\zeta_c^{\pm}}{d\nu}\right|_{\nu=0} = \pm\sqrt{B_c^2 - A_c C_c}. \tag{1.30}$$

Thus, we have that

$$\frac{d\zeta_c^{+}}{d\mu}(\mu_c)\,\frac{d\zeta_c^{-}}{d\mu}(\mu_c) = -(B_c^2 - A_c C_c) < 0. \tag{1.31}$$

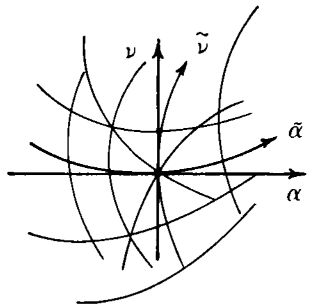


Fig. 1.4

With regards to the cusp bifurcation, we have the following

Proposition 1.10 (*Cusp bifurcation*)

Suppose $\mathscr{C} \equiv (\mu_c, w_c) \in \boldsymbol{R} \times V$ *is a cusp bifurcation point of* (*P*). *Then,* (i) *these exist two paths,* μ- *and* α- *paths, in a neighborhood of* $\mathscr{C}$, *which intersect at* $\mathscr{C}$. *The* μ-*path is parametrized by* $\nu \in I_\delta = \{\nu;\ |\nu| < \delta\} \subset \boldsymbol{R}$, *and is expressed as* $(\mu_c + \nu,\ w^\mu(\nu)) \in \boldsymbol{R} \times V$, $\nu \in I_\delta$, *while the* α-*path is parametrized by* $\alpha \in J_{\delta'} = \{\alpha;\ |\alpha| < \delta'\} \subset \boldsymbol{R}$, *being expressed as* $(\mu_c + \nu(\alpha),\ w^\alpha(\alpha)) \in \boldsymbol{R} \times V$, $\alpha \in J_{\delta'}$. *The functions* $w^\mu(\nu)$, $\nu(\alpha)$, $w^\alpha(\alpha)$ *are all of* C^{p-2} *class, and satisfy the relations* $w^\mu(0) = w^\alpha(0) = w_c$ *and* $\nu(0) = 0$.
(ii) $w^\mu(\nu)$ *is such that for* $\nu \in I_\delta$

$$\|w^\mu(\nu) - w_c\|_V \le C|\nu| \tag{1.32}$$

and has the form

$$w^{\mu}(\nu)=w_c-\nu\Big[\mathscr{L}_c^{!}\omega_c f_c+\frac{C_c}{2B_c}\phi_c\Big]+O(\nu^2). \tag{1.33}$$

On the otherhand, $\nu(\alpha)$ *and* $w^{\alpha}(\alpha)$ *satisfy for* $\alpha\in J_{\delta'}$,

$$|\nu(\alpha)|\le C\alpha^2, \tag{1.34}$$

$$\|w^{\alpha}(\alpha)-w_c\|_V\le C|\alpha|, \tag{1.35}$$

and take the forms:

$$\nu(\alpha)=-\frac{D_c}{6B_c}\alpha^2+O(\alpha^3), \tag{1.36}$$

and

$$w^{\alpha}(\alpha)=w_c+\alpha\phi_c+\alpha^2\Big[\frac{D_c}{6B_c}\mathscr{L}_c^{!}\omega_c f_c-\frac{1}{2}\mathscr{L}_c^{!}\omega_c F_c''(\phi_c,\ \phi_c)\Big]+O(\alpha^3). \tag{1.37}$$

(iii) *Furthermore, let* $\mathscr{L}^{\mu}(\nu)\equiv F'(\mu_c+\nu),\ w^{\mu}(\nu))$ *and* $\mathscr{L}^{\alpha}(\alpha)\equiv F'(\mu_c+\nu(\alpha),\ w^{\alpha}(\alpha))$ *be the linearized operators of* F *on the* μ- *and* α-*paths, respectively. Then,* $\mathscr{L}^{\mu}$ *and* $\mathscr{L}^{\alpha}$ *have, respectively, the critical pairs* $(\zeta_c^{\mu}(\nu),\ \phi_c^{\mu}(\nu))\in \boldsymbol{R}\times V$, $\mu\in I_{\delta}$, *and* $(\zeta_c^{\alpha}(\alpha),\ \phi_c^{\alpha}(\alpha)\in \boldsymbol{R}\times V$, $\alpha\in J_{\delta'}$ *such that*

$$\zeta_c^{\mu}(0)=\zeta_c^{\alpha}(0)=0,$$
$$\phi_c^{\mu}(0)=\phi_c^{\alpha}(0)=\phi_c.$$

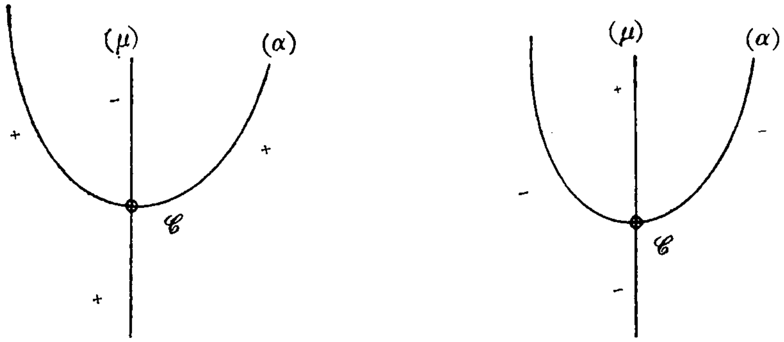
They satisfy the relations:

$$\frac{d\zeta_c^{\mu}}{d\nu}(0)=B_c\neq 0,\quad \frac{d\zeta_c^{\alpha}}{d\alpha}(0)=A_c=0, \tag{1.38$_a$}$$

and moreover, if $p\ge 4$,

$$\frac{d^2\zeta_c^{\alpha}}{d\alpha^2}(0)=\frac{2}{3}D_c=-2\frac{d\zeta_c^{\mu}}{d\nu}(0)\frac{d^2\nu}{d\alpha^2}(0)\neq 0. \tag{1.38$_b$}$$

Remark 1.10′ The relations (1.38) show the stability behavior on the two paths near $\mathscr{C}$. See, Fig. 1.5. If $D_c>0$ ($D_c<0$), $\mathscr{C}$ is called a *stable* (*unstable*) cusp bifurcation point. It is noted that in both cases, the critical eigenvalue $\zeta_c^{\mu}(\lambda)$ on the μ-path changes sign when it crosses $\nu=0$, while on the α-path $\zeta_c^{\alpha}(\alpha)$ does *not* change sign at $\alpha=0$.



"stable cusp bifurcation" *"unstable cusp bifurcation"*

Fig. 1.5

Remark 1.11 As may be seen from the proof, there exist two paths, μ- and α-paths, which intersect at $\mathscr{C}$, *whether or not D_c vanishes*, provided $\mathscr{C}$ is simple and non-degenerate.

Proof. Since $\Pi_c' f_c = 0$, $A_c = 0$, the bifurcation equation is

$$\Gamma(\alpha, \nu) = \alpha\nu B_c + \frac{1}{2}\nu^2 C_c + \frac{1}{3!}\alpha^3 D_c + (h.\ o.\ t.) = 0. \tag{1.39}$$

By virtue of $B_c^2 - A_c C_c = B_c^2 > 0$, $A_c = 0$, one can still apply the Morse lemma. In fact, we have that

$$\Gamma = \tilde{\alpha}\tilde{\nu} B_c + \frac{1}{2}\tilde{\nu}^2 C_c = 0. \tag{1.40}$$

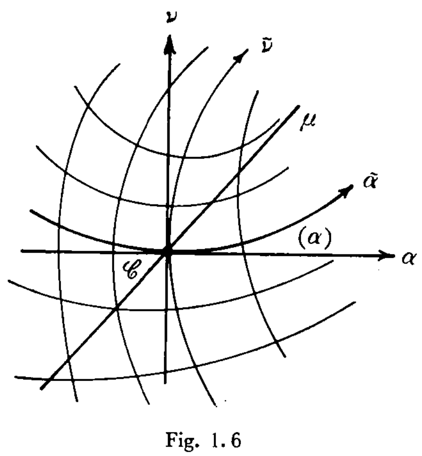


Fig. 1.6

The set of solutions of $\Gamma=0$ now consists of two C^{p-2} curves

$$\tilde{\nu}=0, \text{ and } \tilde{\alpha}=-\frac{C_c}{2_c B}\tilde{\nu}, \tag{1.41}$$

which we call the α- and the μ-path, respectively. $\tilde{\nu}=0$ implies that this α-path can be parametrized by α near the origin. In fact, we have easily that

$$\nu(\alpha)=-\frac{D_c}{6B_c}\alpha^2+O(\alpha^3). \tag{1.42}$$

The μ-path is obviously parametrized by ν near $\mu=\mu_c$, and

$$\alpha(\nu)=-\frac{C_c}{2B_c}\nu+O(\nu^2). \tag{1.43}$$

Now, we turn to (iii). Firstly, we calculate $\frac{d\zeta_c^\mu}{d\nu}(0)$ and $\frac{d\zeta_c^\alpha}{d\alpha}(0)$, which are given by

$$\frac{d\zeta_c^s}{ds}(0)=<\frac{d\mathscr{L}^s}{ds}(s)\phi_c^s(s),\ \phi_c^s(s)>|_{s=0} \tag{1.44}$$

where s denotes either α or ν.

On the α-path, it is easily seen that

$$\frac{d}{d\alpha}\mathscr{L}^\alpha|_{\alpha=0}=F_c''(\phi_c,\cdot) \tag{1.45}$$

as in the proof of Proposition 1.8, and in view of the relations (*cf.* Eqs (1. 36), (1. 37))

$$\frac{d\nu}{d\alpha}(0)=0 \text{ and } \frac{dw^\alpha}{d\alpha}(0)=\phi_c. \tag{1.46}$$

Therefore, we have that

$$\frac{d\zeta_c^\alpha}{d\alpha}(0)=\Pi_c'F_c''(\phi_c,\ \phi_c)=A_c=0. \tag{1.47}$$

On the μ-path, since (*cf.* Eq. (1.33))

$$\frac{dw^\mu}{d\nu}(0)=-\left[\mathscr{L}_c^!\omega_c f_c+\frac{C_c}{2B_c}\phi_c\right], \tag{1.48}$$

we have that

$$\frac{d\zeta_c^\mu}{d\nu}(0)=\Pi_c'\dot{F}_c'\phi_c+\Pi_c'F_c''\left(g_c-\frac{C_c}{2B_c}\phi_c,\ \phi_c\right)=B_c\neq 0 \tag{1.49}$$

in view of the definition (1.4).

Next, we want to calculate $\frac{d^2\zeta_c^\alpha}{d\alpha^2}(0)$. By twice differentiation of $\mathscr{L}^\alpha(\alpha)\phi_c^\alpha(\alpha)=\zeta_c^\alpha(\alpha)\phi_c^\alpha(\alpha)$,

$$\frac{d^2\zeta_c^\alpha}{d\alpha^2}(0)=<\frac{d^2}{d\alpha^2}\mathscr{L}^\alpha(0)\phi_c,\ \phi_c>+2<\frac{d}{d\alpha}\mathscr{L}^\alpha(0)\frac{d}{d\alpha}\phi_c^\alpha(0),\ \phi_c>. \tag{1.50}$$

Thus, it is necessary to calculate $\frac{d^2}{d\alpha^2}\mathscr{L}^\alpha$ and $\frac{d}{d\alpha}\phi_c^\alpha$ at $\alpha=0$.

Firstiy, we can see that

$$\frac{d}{d\alpha}\phi_c^\alpha(0)=-\mathscr{L}_c^\dagger\frac{d\mathscr{L}^\alpha}{d\alpha}(0)\phi_c. \tag{1.51}$$

Indeed, we let $\phi_c^\alpha(\alpha)=\phi_c+\psi_c^\alpha(\alpha)$ with the normalization condition $<\phi_c^\alpha(\alpha),\ \phi_c>=1$, namely $<\psi_c^\alpha(\alpha),\ \phi_c>=0$. Since $\zeta_c^\alpha(0)=\frac{d\zeta_c^\alpha}{d\alpha}(0)=0$, we have that by diferentiation

$$\frac{d\mathscr{L}^\alpha}{d\alpha}(0)\phi_c+\mathscr{L}_c\frac{d}{d\alpha}\phi_c^\alpha(0)=0. \tag{1.52}$$

Noting that

$$\frac{d}{d\alpha}\phi_c^\alpha=\frac{d}{d\alpha}\psi_c^\alpha\in\mathscr{R}_c,$$

and $\frac{d\mathscr{L}^\alpha}{d\alpha}(0)\phi_c=F_c''(\phi_c,\ \phi_c)\in\mathscr{R}_c$, (*cf.* $0=A_c=<F_c''(\phi_c,\ \phi_c),\ \phi_c>$. See, Eqs. (1.4), (1.45).) Eq. (1.51) follows immediately from Eq. (1.52).

Secondly,

$$\begin{aligned}\frac{d^2}{d\alpha^2}\mathscr{L}^\alpha(\alpha)=\Big[\Big(\frac{d\nu}{d\alpha}(\alpha)\Big)^2\ddot{F}'(\alpha)+2\frac{d\nu}{d\alpha}(\alpha)\dot{F}''(\alpha)\Big(\frac{dw^\alpha}{d\alpha}(\alpha),\ \cdot\Big)\\ +F'''(\alpha)\Big(\frac{dw^\alpha}{d\alpha}(\alpha),\ \frac{dw}{d\alpha}(\alpha),\cdot\Big) \qquad (1.53)\\ +\frac{d^2\nu}{d\alpha^2}(\alpha)\dot{F}'(\alpha)+F''(\alpha)\Big(\frac{d^2w^\alpha}{d\alpha^2}(\alpha),\ \cdot\Big)\Big], \qquad (1.53)\end{aligned}$$

where $\ddot{F}'(\alpha)$, etc. are the abreveiations, of $\ddot{F}'(\mu(\alpha),\ w^\alpha(\alpha))$, etc. In view of the relations (1.36), (1.37), we have that

$$\begin{aligned}\frac{d^2}{d\alpha^2}\mathscr{L}^\alpha(0)=F_c'''(\phi_c,\ \phi_c,\ \cdot)-\frac{D_c}{3B_c}\dot{F}_c'-\frac{D_c}{3B_c}F_c''(g_c,\ \cdot)\\ -F_c''(\mathscr{L}_c^\dagger\omega_cF_c''(\phi_c,\ \phi_c),\ \cdot).\end{aligned} \tag{1.54}$$

Therefore, from Eqs. (1.45), (1.50) and (1.54),

$$\frac{d^2\zeta_c^\alpha}{d\alpha^2}(0)=\frac{2}{3}D_c. \tag{1.55}$$

Now, the desired result follows from Eqs. (1.49), (1.36) :

$$\frac{d^2\zeta_c^\alpha}{d\alpha^2}(0)=\frac{2}{3}D_c=-2\frac{d^2\nu}{d\alpha^2}(0)\frac{d\zeta_c^\mu}{d\nu}(0). \tag{1.56}$$

2. Simple bucklings in the presence or non-presence of symmetry groups

So far, we have concentrated on the *formal* classification of simple critical points. In this section, we go more into the *mechanism* of simple bucklings. In other words, we want to know how and when those simple critical points appear *stably*. Two concepts will be introduced for this purpose. The first is the *symmetry group* of the problem (P). We shall make use of some of the results of group representation theory. The second concept to be introduced is whether the problem (P) is of *class L* or *class N*, which respectively implies that the path under consideration is *linear* or *nonlinear* with respect to the parameter $\mu\in\boldsymbol{R}$. (Note that this notion is not purely (P)-dependent, but rather *path*-dependent. For example, even (P) has a linear fundamental path, the secondary bifurcation from the firstly bifurcated path should be considered as a *class N* problem.)

The introduction of group theoretical viewpoints to nonlinear singularity problem is *not* the first here, and in fact, for the pattern formation in Navier-Stokes flow, one can refer to Ruelle [42], Sattinger [45], [46] and other works. Also, for diffusion-reaction problems, there are works of Othmer [36] and so on. See, also Rodrigues [40].

The emphasis here is in the study of "structural stability" of critical points with respect to small changes of the equation, especially that of bifurcation points, though we are only involved in the simple critical cases. Our main tool is *the standard decomposition* of the Hilbert space V associated with the symmetry group $\mathscr{G}$ of the problem. The results here seem interesting by themselves; and the same notion will be indispensable in the theory of numerical deformation or realization of bifurcation points.

2.1 Symmetry group of F

Let $\Omega\subset\boldsymbol{R}^m(1\leq m\leq 3)$ be a bounded domain with a piecewise smooth boundary. Let V be a *complex* separable Hilbert space of functions defined on Ω. Let $<\ ,\ >$ be the inner product of V.

Definition 2.1 $\mathcal{G}$ is the symmetry group of the domain Ω, if

$$\mathcal{G}=\{g\in O(m)\ ;\ g\Omega=\Omega\} \tag{2.1}$$

where $O(m)$ is the classical orthogonal group.

Let T: $\mathcal{G}\to GL(V)$ be a unitary representation of $\mathcal{G}$ on V.*)

Example 2.2 Let u, $v\in H_0^2(\Omega)$ with $\langle u, v\rangle=\int_\Omega \Delta u\overline{\Delta v}$. Let $\mathcal{G}$ be the symmetuy group of $\Omega\subset \boldsymbol{R}^m$. The operators T_g $(g\in\mathcal{G})$:

$$(T_g u)(x)=u(g^{-1}x) \tag{2.2}$$

define an (infinite dimensional) representation of $\mathcal{G}$ on V. $T_g: V\to V(g\in\mathcal{G})$ are *unitary* since

$$\langle T_g u, T_g v\rangle=\langle u, v\rangle,\ u,\ v\in H_0^2(\Omega), \tag{2.3}$$

noting that the Laplacian Δ is invariant under $O(m)$.

We assume *for the present* that $\mathcal{G}\subset O(m)$ is a *finite* group of order $n(\mathcal{G})$. Let $\chi_1, \chi_2, \ldots, \chi_q$ be the complete set of simple characters of non-equivalent irreducible representations $\tau_1, \tau_2, \ldots, \tau_q$. By $n_k(k=1, 2, \ldots, q)$ we denote the dimensions of $\tau_k(k=1, 2, \ldots, q)$. Note that q is equal to the number of conjugacy classes of $\mathcal{G}$. See, e.g., Serre [47] or Miller [33], for details.

We define a $\mathcal{G}$*-invariant* direct sum decomposition of V—*the standard decomposition* of V:

$$V=V_1\oplus V_2\oplus\cdots\oplus V_q. \tag{2.4}$$

The standard decomposition (2.4) is uniquely defined, and indeed, there exists a set of projection operators P_k: $V\to V_k$:

$$P_k=\frac{n_k}{n(\mathcal{G})}\sum_{g\in\mathcal{G}}\overline{\chi_k(g)}T_g,\ k=1, 2, \ldots, q. \tag{2.5}$$

$P_k(k=1, 2, \ldots, q)$ are self-adjoint and commute with $T_g(g\in\mathcal{G})$. It hold that

$$\sum_{k=1}^{q} P_k=\mathrm{I} \text{ and } P_kP_j=\delta_{kj}P_k, \tag{2.6}$$

*) A representation of $\mathcal{G}$ on V is a homomorphism T: $g\to T_g$ of $\mathcal{G}$ into $GL(V)$, where $GL(V)$ denotes the group of all non-singular linear transformations of V onto itself.

where δ_{kj} is the Kronecker delta.

We summarize some of elementary properties of $\mathscr{G}$ and its characters χ which will be used in later discussions.

(i) $\chi_k(e) = n_k$, $k \in \langle 1, 2, \ldots, q \rangle$,
(especially, $\chi_k(e) = 1$ for all k such that $n_k = 1$). (2.7)

(ii) if $k \in \langle 1, 2, \ldots, q \rangle$ such that $n_k = 1$,
$|\chi_k(g)| =$ for all $g \in \mathscr{G}$ (2.8)

and $T_g \phi = \chi_k(g) \phi$ for all $g \in \mathscr{G}$, and fo rall $\phi \in V_k$

(iii) $\chi_k(g) = 1 \, ({}^\forall g \in \mathscr{G})$ if and only if $k = 1$.*) (2.9)

Note that the decomposition (2.4) is reducible, and in fact, each V_k (which is infinite dimensional, in general) can be decomposed into an (infinite number of) direct sum of W_k's which are all homomorphic to τ_k. For the present purpose, we need only the standard decomposition (2.4). The subspaces $V_k (k = 1, 2, \ldots, q)$ may be characterized as: each $u \in V_k$ transforms according to τ_k. Also, with each V_k, one can associate the maximal subgroup $\mathscr{G}_k \subset \mathscr{G}$ under which every element of V_k is invariant, namely, $\mathscr{G}_k = \{g \in \mathscr{G} \,;\, T_g u = u, \, {}^\forall u \in V_k\}$. $\mathscr{G}_k$ is the symmetry group of functions in V_k. We shall call $\mathscr{G}_k$ *the maximal symmetry group of* V_k. Obviously, $\mathscr{G}$ is the maximal symmetry group of V_1, since $T_g P_1 = P_1$ for all $g \in \mathscr{G}$ (*See*, Eq. (2.9)*.) In this sense, we may call V_1 the *$\mathscr{G}$-symmetric space*.

Example 2.3 (a) $C_s \cong C_{1h}$; *the reflection through a plane* $\mathscr{G} = \{e, s\}$, $s^2 = e$.

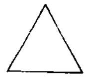
Character table:

	$\{e\}$	$\{s\}$
χ_1	1	1
χ_2	1	-1

Standard decomposition: $V = V^+ \oplus V^-$, *where*

$$(P^\pm u) = \frac{1}{2}(I \pm T_s)u, \quad (T_s u)(x) = u(-x).$$

Example 2.3 (b) $D_3 \cong C_{3v}$; *the group of an equi-lateral triangle in a plane* $D_3 = \{e, g, g^2, s, gs, g^2 s\}$.



There are two generators g, s *with* $g^3 = s^2 = e$, *and* $sgs = g^{-1}$, *where* g *denotes a counter-clockwise rotation through* $120°$, *and* s *a reflection across a median.*

*) Thus, $P_1 = \frac{1}{n(\mathscr{G})} \sum_{g \in \mathscr{G}} T_g$. (2.9)

Character table:

	$\{e\}$	$\{g, g^2\}$	$\{s, gs, g^2s\}$
χ_1	1	1	1
χ_2	1	1	-1
χ_3	2	-1	0

Standard decomposition $V=V_1\oplus V_2\oplus V_3$, *where*

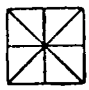
$$P_1=\frac{1}{3}(I+T_g+T_g^2)\frac{1}{2}(I+T_s)$$

$$P_2=\frac{1}{3}(I+T_g+T_g^2)\frac{1}{2}(I-T_s)$$

$$P_3=I-\frac{1}{3}(I+T_g+T_g^2)$$

$$\mathscr{G}_1=D_3,\ \mathscr{G}_2=\{e, g, g^2\},\ \mathscr{G}_3=\{e\}$$

Example 2.3(c) $D_4\cong C_{4V}$; *the group of plane operations which send a square into itself* $D_4=\{s, g, g^2, g^3, s, gs, g^2s, g^3s\}$,



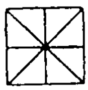
g: *fourfold rotation*,
s: *reflection*,
with $g^4=s^2=e$, $(gs)^2=e$.

Character table:

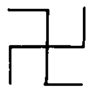
$D_4\cong C_{4V}$	$\mathscr{E}$	C_4^2	$2C_4$	$2C_2$	$2C_2'$
χ_1	1	1	1	1	1
χ_2	1	1	1	-1	-1
χ_3	1	1	-1	1	-1
χ_4	1	1	-1	-1	1
χ_5	2	-2	0	0	0

where $\mathscr{E}=\{e\}$, $C_4^2=\{g^2\}$, $C_4=\{g, g^3\}$, $C_2=\{s, g^2s\}$, $C_2'=\{gs, g^3s\}$.

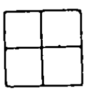
$\mathscr{G}_1=D_4$



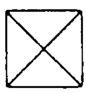
$\mathscr{G}_2=\{e, g, g^2, g^3\}$



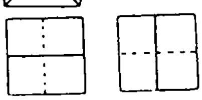
$\mathscr{G}_3=\{e, g^2, s, g^2s\}$



$\mathscr{G}_4=\{e, g^2, gs, g^3s\}$



$\mathscr{G}_5=\{e\}$



Example 2.3(d) D_{2h}; *the group which sends a rectangular solid into itself no matter what the length of the sides may be.*
We omit the details.

We shall now define the notion of *symmetry group* of F, where F is a smooth (at least C^1) mapping of $\boldsymbol{R}\times V$ into V. We shall generally assume that the mapping F is *real* in the sense that $\overline{F(\mu, w)}=F(\mu, \bar{w})$, for all $(\mu, w)\in\boldsymbol{R}\times V$.

Definition 2.4 $\mathscr{G}$ is said to be the symmetry group of F, if F *is covariant under* $\mathscr{G}$ in the sense that

$$F(\mu, T_g w)=T_g F(\mu, w), \tag{2.10}$$

for all $g\in\mathscr{G}$ and $(\mu, w)\in\boldsymbol{R}\times V$.

Example 2.5(a) The Laplacian Δ is covariant under $O(m)$, namely, $T_g\Delta=\Delta T_g$, $^{\forall}g\in O(m)$.

Example 2.5(b) (Sattinger [45]) Let $f(s)$ be a continuous real-valued function. Then, $F(u)(x)=f(u(x))$ is covariant under $O(m)$.

Example 2.5(c) The von Kármán-Donnell Marguerre operator

$$\begin{aligned} w\to&[I-B(\psi_0, \cdot)+B(w_0, B(w_0, \cdot))]w \\ &+\frac{1}{2}[B(w_0, B(w, w)+2B(w, B(w_0, w))] \\ &+\frac{1}{2}B(w, B(w, w)) \end{aligned}$$

is covariant under $\mathscr{G}$, provided ψ_0 and w_0 are $\mathscr{G}$-invariant, where $\mathscr{G}\subset O(2)$ is the symmetry group of the domain Ω. See, Appendix A for details and a proof.

In the sequel, we shall assume that $\mathscr{G}$ is the symmetry group of F. $\mathscr{G}$ may be either *trivial* $\mathscr{G}=\{e\}$ or non-trivial. Note that if $\mathscr{G}$ is trivial (that is, if F has *no* group symmetry), the standard decomposition (2.4) is the trivial one $V=V_1$.

Definition 2.6 Suppose V is decomposed into a direct sum

$$V=V_1\oplus V_2\oplus\cdots\oplus V_q,$$

with P_i: $V\to V_i (i=1, 2, \ldots, q)$ the associate projections. We say that

$F: \boldsymbol{R}\times V\to V$ is *enclosed* in V_1 if

(i) $P_iF'(\mu, w_1)P_j=0$ (2.11)

for $i, j=1, 2, \ldots, q$; $i\neq j$, and for all $(\mu, w_1)\in\boldsymbol{R}\times V_1$, and

(ii) $P_jF(\mu, w_1)=0$ (2.12)

for $j=2, 3, \ldots, q$, and for all $(\mu, w_1)\in\boldsymbol{R}\times V_1$.

That F is enclosed in V_1 implies that the linearized operator of F at $(\mu, w_1)\in\boldsymbol{R}\times V_1$ has a block diagonal form and that the problem $P_jF(\mu, w_1, w_2, \ldots, w_q))=0$ $(j=2, 3, \ldots, q)$ has "a trivial solution" $w_2=w_3=\ldots=w_q\equiv 0$, for all $(\mu, w_1)\in\boldsymbol{R}\times V_1$.
It is almost direct to show the following

Lemma 2.7 *F is enclosed in the $\mathcal{G}$-symmetric space V_1.*

Proof. If $\mathcal{G}=\{e\}$, the proposition is obvious. Hence, we assume $n(\mathcal{G})>1$. Firstly, from Eq. (2.10), we find that

$$\frac{1}{n(\mathcal{G})}\sum_{g\in\mathcal{G}}F(\mu, T_gw)=\frac{1}{n(\mathcal{G})}\sum_{g\in\mathcal{G}}T_gF(\mu, w)$$

for all $(\mu, w)\in\boldsymbol{R}\times V$. In view of the relation $T_gw=w$ for any $w\in V_1$, we have

$$F(\mu, w)=P_1F(\mu, w), \ {}^\forall(\mu, w)\in\boldsymbol{R}\times V_1.$$

Therefore, Eq. (2.12) follows.
Secondly, differentiating Eq. (2.10) with respect to w,

$$F'(\mu, T_gw)T_g=T_gF'(\mu, w), \ {}^\forall g\in\mathcal{G}, \ {}^\forall(\mu, w)\in\boldsymbol{R}\times V.$$

Hence, for any $w\in V_1$, $F'(\mu, w)$ commutes with T_g, i.e.,

$$F'(\mu, w)T_g=T_gF'(\mu, w). \tag{2.13}$$

Multiplying the above relation by $\overline{\chi_i(g)}$ and summing all the $g\in\mathcal{G}$, we have that

$$F'(\mu, w)P_i=P_iF'(\mu, w), \ (i=1, 2, \ldots, q)$$

for all $(\mu, w)\in\boldsymbol{R}\times V_1$, which in turn implies Eq. (2.11)

2.2 Simple bucklings in the presence/non-presence of a symmetry group

Under the existence of a symmetry group $\mathscr{G}$, either trivial or non-trivial, in *any* simple critical points a further structure is built-in there; namely, ($\mathscr{G}$-) *symmetry preserving* and ($\mathscr{G}$-) *symmetry breaking* critical points. We shall see that a symmytry breaking critical point is necessarily a bifurcation point, and which cannot be a fold. (Thus, a fold bifurcation should be, if exists, symmetry preserving.) A symmetry preserving bifurcation *can* exist formally, however, the essential nature of such bifurcations will not become clear until at the next paragraph, where we shall consider them with the viewpoint of structural stability. We remark here that when $\mathscr{G}$ is trivial, only the symmetry preserving case can appear as a critical point. In this paragraph, we shall study such *symmetry structure* of simple critical points.

We begin by recalling that our problem is given by

$$(P) \quad F(\mu, w) = 0 \tag{1.1}$$

where F: $\boldsymbol{R}\times V \to V$ is a $C^p (p\geq 3)$ mapping of Fredholm type. Assume that $\mathscr{G}$ is the symmetry group of F (not necessarily non-trivial). Assume also $\mathscr{G}$ is of finite order. For a compact Lie group e.g., $\mathscr{G}=D_\infty$ case, see Remark 2.15. The standard decomposition of V, Eq. (2.4), is assumed with the corresponding projections $P_i: V\to V_i$ $(i=1, \ldots, q)$, given by Eq. (2.5).*) By Lemma 2.7, F is *enclosed* in V_1—the $\mathscr{G}$-symmetric space. We shall sometimes denote by V^+ the $\mathscr{G}$-symmetric space V_1, and by V^- the $\mathscr{G}$-asymmetric space $V_2 \oplus \ldots \oplus V_q$. Also, P^+ and P^- denote the corresponding projections.

The following lemma may explain why we say that F is enclosed in V^+.

Lemma 2.8 *Suppose* $\mathscr{O}^+ \equiv (\mu_0, w_0^+) \in \boldsymbol{R}\times V^+$ *is an ordinary point of* (P). *Then, the ordinary path which contains* $\mathscr{O}^+$ *lies in* $\boldsymbol{R}\times V^+$. (*cf. Lemma* 1.2, § 1, *Chapter I.*)

Proof. Restricting the problem (P) on V^+, we have an ordinary path which lies in $\boldsymbol{R}\times V^+$ using Lemma 1.2 on the space V^+. Here, the properties (2.11) and (2.12) are essential. The uniqueness of the ordinary path in the whole space V guaranteed by Lemma 1.2 shows the proposition.

This lemma shows that a $\mathscr{G}$-symmetric ordinary path continues to be $\mathscr{G}$-symmetric *until* it arrives at a critical point $\mathscr{C}^+$, *which itself is* $\mathscr{G}$*-symmetric* by the closedness of the subspace V^+.

*) When $\mathscr{G}=\{e\}$ (trivial), $q=1$.

We now suppose $\mathscr{C}^+\equiv(\mu_c, w_c^+)\in \boldsymbol{R}\times V^+$ is a *simple* non-degenerate critical point of (P) on a $\mathscr{G}$-symmetric path. Let $\phi_c\in\ker \mathscr{L}_c$, where $\mathscr{L}_c\equiv F'(\mu_c, w_c^+)$. First, we note that since $F(\mu, w^+)\in V^+$ for all $(\mu, w^+)\in \boldsymbol{R}\times V^+$ by Eq. (2.12), $\dot{F}_c=\frac{\partial}{\partial\mu}F(\mu, w_c^+)|_{\mu=\mu_c}\in V^+$. Next, since $\mathscr{L}_c$ commutes with $T_g({}^\forall g\in\mathscr{G})$ by Eq. (2.13), if $\phi_c\in\ker\mathscr{L}_c$, then $T_g\phi_c\in\ker \mathscr{L}_c$. This fact together with the simpleness assmption of $\mathscr{C}^+$ necessarily implies that ϕ_c belongs to such $V_k({}^\exists k\in\langle 1, 2, \ldots, q\rangle)$ that the corresponding irreducible representation τ_k is one dimensional (i. e., $n_k=1$).

In view of the classification theorems in § 1, Chapter I, we have the following possibilities *formally*:

(i) *Symmetry preserving snap buckling* $(k=1)$:

$$\phi_c\in V^+ \text{ and } \langle\dot{F}_c, \phi_c\rangle\neq 0 \tag{2.14}$$

(ii) *Symmetry preserving bifurcation buckling* $(k=1)$:

$$\phi_c\in V^+ \text{ and } \langle\dot{F}_c, \phi_c\rangle=0. \tag{2.15}$$

(iii) *Symmetry breaking bifurcation buckling* $({}^\exists k\in\langle 2, \ldots, q\rangle)$:

$$\phi_c\in V_k\subset V^- \text{ and hence } \langle\dot{F}_c, \phi_c\rangle=0. \tag{2.16}$$

It may be immediate to see the following

Lemma 2.9

(i) *Suppose $\mathscr{C}^+$ is a symmetry preserving, simple non-degenerate snap point of (P). Then, the unique path emerging from $\mathscr{C}^+$ lies in $\boldsymbol{R}\times V^+$.*

(ii) *Suppose $\mathscr{C}^+$ is a symmetry preserving, simple, non-degenerate bifurcation point of (P). Then, both of two paths emerging from $\mathscr{C}^+$ (see, Lemmas 1.9 and 1.10, § 1, Chapter I) lie in $\boldsymbol{R}\times V^+$.*

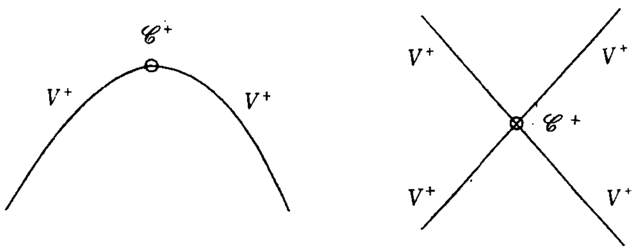


Fig. 2.1 Symmetry preserving critical points

In case of the symmetry breaking bifurcations, we have the following two lemmas, which exhibit an interesting nature of simple, symmetry breaking bifurcations.

Lemma 2. 10 Suppose $(\mathscr{C}^+ ; \phi_c) \equiv (\mu_c, w_c^+ ; \phi_c) \in \boldsymbol{R} \times V_1 \times V_k \subset \boldsymbol{R} \times V^+ \times V^-$ is a simple, non-degenerate symmetry breaking bifurcation point of (P). Then,

(i) there imerges a $\mathscr{G}$-symmetric path $(\mu, w^+(\mu)) \in \boldsymbol{R} \times V^+$ for $\mu - \mu_c \in I_\delta = \{\nu ; |\nu| < \delta\}$ such that $w^+(\mu_c) = w_c^+$.

(ii) The other bifurcating path (*see*, Lemma 1. 10 and Remark 1.10'', § 1, Chapter I) $(\mu(\alpha), w^a(\alpha)) \in \boldsymbol{R} \times V$ for $\alpha \in I_{\delta'} = \{\alpha ; |\alpha| < \delta'\}$ is in the $\mathscr{G}_k$-symmetric space $V_{(k)}^+ \in V$, which is defined by $V_{(k)}^+ = P_{(k)}^+ V$, where

$$P_{(k)}^+ \overset{def}{=} \frac{1}{n(\mathscr{G}_k)} \sum_{g \in \mathscr{G}_k} T_g. \tag{2. 17}$$

Here, $\mathscr{G}_k$ is the maximal symmetry group of V_k in the sense of § 2. 1, V_k being the subspace of V to which ϕ_c belongs. See, Fig. 2. 2.

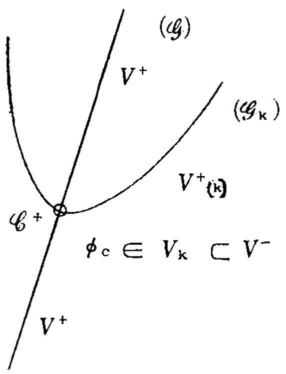


Fig. 2. 2

This lemma shows a situation that *the symmetry group $\mathscr{G}$ on the fundamental $(=\mathscr{G}$-symmetric$)$ path breaks to a subgroup $\mathscr{G}_k$ on the bifurcating $(\mathscr{G}_k$-symmetric$)$ path.*

Proof. Restricting the problem (P) to V^+-space, the assertion (i) is easily checked using a similar reasoning as in Lemma 2. 8. To show (ii) we return to the Lyapounov-Schmidt decomposition of F at (μ_c, w_c^+) :

$$\omega_c G_c(\nu, \alpha\phi_c + \psi) = 0, \tag{2. 18}$$

$$\Pi_c G_c(\nu, \alpha\phi_c + \psi) = 0, \tag{2. 19}$$

where $\psi \in \mathscr{R}_c = \text{range } F_c' = \{\ker F_c'\}^\perp$. Π_c is the projection of V onto ker

F'_c, and $\omega_c = I - \Pi_c$. By Lemma 1. 8, we know the unique existence of $\psi = \psi\ (\alpha, \nu)$ such that Eq. (2. 18) is satisfied. We show that ψ is covariant under $\mathscr{G}$, i. e.,

$$T_g\psi(\alpha, \lambda) = \psi(T_g\alpha, \lambda), \ {}^\forall g \in \mathscr{G}. \tag{2. 20}$$

Here, we understand that if $u = \alpha\phi \in V_k$, $\alpha \in \boldsymbol{C}$,

$$T_g u = T_g \alpha\phi = \alpha\chi_k(g)\phi$$

using the relation (2. 8), so

$$T_g\alpha = \chi_k(g)\alpha. \tag{2. 21}$$

We first note that Π_c and T_g, and hence ω_c and T_g commute. In fact, if we let $u = \alpha\phi_c + \psi({}^\forall u \in V)$, $T_g\Pi_c u = T_g\alpha\phi_c = \alpha\chi_k(g)\phi_c$, but $\Pi_c T_g u = \langle T_g u, \phi_c\rangle\phi_c = \langle u, T_g^*\phi_c\rangle\phi_c = \chi_k(g)\langle u, \phi_c\rangle\phi_c = \chi_k(g)\alpha\phi_c$. Next, the $\mathscr{G}$-covariance of G_c, which follows obviously from Eqs. (1. 11) and (2. 10), and $\omega_c T_g = T_g\omega_c$ yield

$$\begin{aligned} &T_g\omega_c G_c(\nu, \alpha\phi_c + \psi(\alpha, \nu)) \\ &= \omega_c G_c(\nu, \alpha T_g\phi_c + (T_g\psi)(\alpha, \nu)) \\ &= 0. \end{aligned} \tag{2. 23}$$

The uniqueness of the solution of $\psi = \psi(\alpha, \nu)$ in Eq. (2. 18) implies the relation (2. 20).

Now, recalling that $\mathscr{G}_k$ is the maximal symmetry group of V_k (*see*, § 2. 1), we have that

$$T_g\alpha = \alpha \text{ for } {}^\forall g \in \mathscr{G}_k. \tag{2. 24}$$

Accordingly, Eqs. (2. 20) and (2. 24) show that

$$T_g\psi(\alpha, \nu) = \psi(\alpha, \nu), \ {}^\forall g \in \mathscr{G}_k, \tag{2. 25}$$

from which follows

$$P^+_{(k)}\psi(\alpha, \nu) = \psi(\alpha, \nu). \tag{2. 26}$$

Thus, (ii) is proved.

Lemma 2. 11 *A simple, symmetry breaking bifurcation point* $(\mathscr{C}^+; \phi_c^-) \equiv (\mu_c, w_c^+; \phi_c^-) \in \boldsymbol{R} \times V^+ \times V^-$ *can not be a fold bifurcation. Namely, it holds that*

$$A_c = \langle F''(\mu_c, w_c^+)(\phi_c^-, \phi_c^-), \phi_c^-\rangle = 0. \tag{2. 27}$$

Remark 2.12 Accordingly, a fold (=transcritical) bifurcation should be, if exists, symmetry preserving.

A proof of the above lemma may follow from the following observations. Firstly, the bilinear mapping $F''(\mu_c, w_c^+)(\cdot, \cdot)$ is covariant under $\mathscr{G}$:

$$F_c''(T_g u, T_g v) = T_g F_c''(u, v), \ {}^\forall g \in \mathscr{G}, \ {}^\forall u, v \in V, \tag{2.28}$$

here $F_c''(\cdot, \cdot) \equiv F''(\mu_c, w_c^+)(\cdot, \cdot)$. Indeed, from the $\mathscr{G}$-covariance of F, i.e., Eq. (2.10),

$$F''(\mu, T_g w)(T_g \cdot, T_g \cdot) = T_g F''(\mu, w)(\cdot, \cdot).$$

Using the relation $T_g w = w$ for all $w \in V_1 \equiv V^+$, Eq. (2.28) is immediate. Now since T_g is unitary, the form

$$\mathscr{A}_c(\phi) \overset{def}{=} \langle F_c''(\phi, \phi), \phi \rangle, \ \phi \in V_k \tag{2.29}$$

is invariant under $\mathscr{G}$ in the sense that

$$\mathscr{A}_c(T_g \phi) = \mathscr{A}_c(\phi), \ {}^\forall g \in \mathscr{G}. \tag{2.30}$$

On the other hand, Eqs. (2.7) and (2.8) yield

$$\begin{aligned} \mathscr{A}_c(T_g \phi) &= \chi_k(g) \, |\chi_k(g)|^2 \mathscr{A}_c(\phi), \\ &= \chi_k(g) \mathscr{A}_c(\phi), \end{aligned} \tag{2.31}$$

Therefor,

$$(\chi_k(g) - 1) \mathscr{A}_c(\phi) = 0 \quad \text{for all} \quad g \in \mathscr{G}. \tag{2.32}$$

It is however only for $k=1$ that $\chi_k(g) = 1$ for all $g \in \mathscr{G}$ (*see*, Eq. (2.9)). The symmetry breaking assumption $\phi \in V_k \subset V^-$ i.e., $k \in \langle 2, 3, \ldots, q \rangle$ implies $\mathscr{A}_c(\phi) = 0$. This completes the proof.

We can perform similar arguments to know whether and when other coefficients of the bifurcation equation, for instance D_c, vanish. However, this is a reflection of a more general situation that the $\mathscr{G}$-covariance of the problem is inherited by the bifucation equation as was shown by Sattinger [45].

Lemma 2.13. (*D. Sattinger*) *The bifurcation equation $\Gamma(\alpha, \nu)$ is covariant under $\mathscr{G}$:*

$$T_g \Gamma(\alpha, \nu) = \Gamma(T_g \alpha, \nu), \ g \in \mathscr{G}, \tag{2.33}$$

where $T_g\Gamma$ is understood in the sense of Eq. (2.21).

For completeness, we sketch the proof for our *simple* case. From the $\mathscr{G}$-covariance of G_c and of ψ, we find that

$$\begin{aligned}\Gamma(\alpha, \nu) &= \langle G_c(\nu, \alpha\phi_c+\psi(\alpha, \nu)), \phi_c\rangle \\ &= \langle T_g G_c(\nu, \alpha\phi_c+\psi(\alpha, \nu)), T_g\phi_c\rangle \\ &= \langle G_c(\nu, \alpha T_g\phi_c+\psi(T_g\alpha, \nu)), T_g\phi_c\rangle \\ &= \overline{\chi_k(g)}\Gamma(T_g\alpha, \nu)\end{aligned}$$

which is nothing but the relation (2.33).

Remark 2.14 We return to the question: whether and/or when the coefficient D_c vanishes. We have similarly that $(1-\chi_k(g)^2)D_c=0$ for all $g\in\mathscr{G}$. We may have to check whether/when $\chi_k(g)^2=1$ for all $g\in\mathscr{G}$. In every case in Example 2.3, $\chi_k(g)=\pm 1$ for all $g\in\mathscr{G}$ provided $\chi_k(e)=1$ (i.e., $n_k=1$), implying thus D_c does not necessarily vanish (at least, not by group theoretical reasonings). A simple symmetry breaking cusp bifurcation may actually realize.

However, a remark should be given to, for example, a problem with the symmetry group C_3—the cyclic group of order 3 consisting of a rotation through 120° and its powers, which may correspond to, e.g., a shell of revolution with C_3-loadings. The character table of C_3 is given by

C_3	$\mathscr{E}$	C_3	C_3^2
χ_1	1	1	1
χ_2	1	ω	$\bar{\omega}$
χ_3	1	$\bar{\omega}$	ω

, $\omega=\exp\left(\frac{2\pi}{3}i\right)$.

For $k=2$ or 3 (i.e., a simple symmetry breaking case), it is *not* true that $\chi_k(g)^2=1$ for all $g\in C_3$. Note, however, that such simple cases can not happen here, since the mapping F is assumed to be real in our problem.

Remark 2.15 (*A Remark on Shells of Revolution. D_∞—a compact Lie group case*) So far, we have assumed that $\mathscr{G}$ is a finite group. An important case arises in nonlinear elasticity in which $\mathscr{G}$ is not a finite group, but a compact Lie group. Shells of revolution or any other shells with rotational symmetry are such instances. Most of the techniques we have used so far are, up to modifications, applicable to classical Lie groups.*) However, it should be noted that in, e.g., D_∞—the group of rotations and reflections that sends a plane into itself—the irreduclible representations are two dimensional except two representations, including the identity. This may lead to a bifurcation problem with *double* singularities. However, this *group-theoretical double eigenvalues* are *in a sense* only in appearance, as was pointed out by Sattinger in [44]. There bifurcates a one parameter *sheet* of solutions, which is merely a sheet obtained by rotating *a* one parameter path bifurcating from the double critical points $\mathscr{C}$. Thus, in conclusion, we have only to restrict the problem to the subspace $V^{(s)}=\frac{1}{2}(I+T_s)V$ of V, where s is a reflection, reducing the problem to a simple critical case. For more discussions, we refer to [14].

2.3 Stability of critical points under the presence of a symmetry group

At this paragraph, we would like to discuss the stability of critical points, in particular that of bifurcation points, with respect to small changes of the equation (P).

Suppose we have an ε-family ($\varepsilon\in E\subset \boldsymbol{R}$) of perturbed problems:

$$(P)_\varepsilon\ F(\varepsilon;\ \mu,\ w)=0,\ E\times\boldsymbol{R}\times V\to V \tag{2.34}$$

with the condition that

$$F(0;\ \mu,\ w)\equiv F(\mu,\ w),\ {}^\forall(\mu,\ w)\in\boldsymbol{R}\times V. \tag{2.35}$$

$F(\varepsilon;\ \mu,\ w)$ is assumed to be sufficiently smooth in each variable.

We want to discuss in what class of problem $(P)_\varepsilon$ or, under what kind of perturbations, a bifurcation point appears *stably*, or more precisely appears *for every* ε *with* $|\varepsilon|<\varepsilon_0$ $({}^\exists\varepsilon_0>0)$.

We shall introduce *two* classes of $(P)_\varepsilon$, in which bifurcation points appears stably. Firstly,

Theorem 2.16 *Suppose $F(\varepsilon;\ \mu,\ w)$ is covariant under a non-trivial symmetry group $\mathscr{G}$ uniformly in $\varepsilon\in E$. Suppose $F(0;\ \mu,\ w)$ possesses a simple, symmetry breaking bifurcation point $(\mathscr{C}^+;\ \phi_c)\equiv((\mu_c,\ w_c^+);\ \phi_c)\in\boldsymbol{R}\times V_1\times V_k$, for some $k\in\langle 2,\ 3,\ \ldots,\ q\rangle$. Then, there exits a constant $\varepsilon_0>0$ such that an ε-family of simple, symmetry breaking bifurcations $(\mathscr{C}^+(\varepsilon)\ ;\ \phi_c(\varepsilon))\equiv((\mu_c(\varepsilon),$*

*) The standard decomposition (2.4) equally holds with $q=+\infty$. The projection operators P_l are defined with the aid of Haar measure of D_∞. See, Serre [47] for these materials.

$w_c^+(\varepsilon)$) ; $\phi_c(\varepsilon)) \in \boldsymbol{R} \times V_1 \times V_k$ exists in $(P)_\varepsilon$ uniformly in $|\varepsilon| \in [0, \varepsilon_0[$.

Proof. The standard decomposition (2.4) being taken in mind, we have as the symmetric component:

$$P_1 F(\varepsilon; \mu, w_1) = 0, \quad (\mu, w_1) \in \boldsymbol{R} \times V_1. \tag{2.36}$$

When $\varepsilon = 0$, there exists a $\mathscr{G}$-symmetric path $(\mu, w_1(\mu)) \in \boldsymbol{R} \times V_1$ for $\mu \in I_\delta$, such that $P_1 F\ (0; \mu, w_1(\mu)) = 0$ and $P_k\ F(0; \mu, w_1(\mu)) = 0$ $(k = 2, 3, \ldots, q)$. For each $\mu \in I_\delta$ (fixed), there exists a unique function $w_1 = w_1(\varepsilon; \mu) \in V_1$ for $|\varepsilon| < {}^\exists\varepsilon_0$, such that $w_1(0; \mu) = w_1(\mu)$ and that $\|w_1(\varepsilon; \mu) - w_1(\mu)\|_V \leq C|\varepsilon|$ $({}^\forall|\varepsilon| < \varepsilon_0)$, since $P_1\ F'(0; \mu, w_1(\mu))$ is invertible on the space V_1.

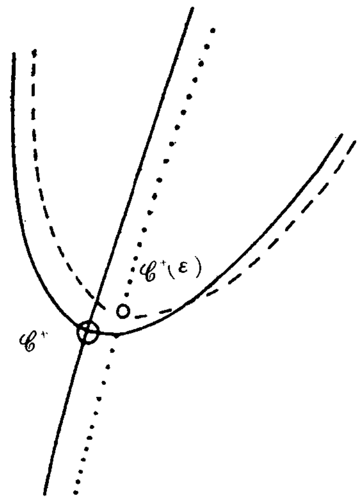


Fig. 2.3 *Stability of a symmetry breaking bifurcation under symmetry preserving perturbations*

The pair $(\mu, w_1(\varepsilon; \mu))$ satisfies Eq. (2.36), and consequently Eq. (2.34) since $(P)_\varepsilon$ is enclosed in V_1. The next stage is to study an (ε, μ)-family of eigenproblems on V_k; for $|\varepsilon| \in [0, \varepsilon_0[$, $\mu \in I_\delta$,

$$\mathscr{L}_k(\varepsilon; \mu)\phi_c(\varepsilon; \mu) = \zeta_c(\varepsilon; \mu)\phi_c(\varepsilon; \mu), \tag{2.37}$$

where

$$\phi_c(\varepsilon; \mu) \in V_k,$$

and

$$\mathscr{L}_k(\varepsilon; \mu) = P_k F'(\varepsilon; \mu, w_1(\varepsilon, \mu)). \tag{2.38}$$

By hypothesis, $\zeta_c(0; \mu)$ vanishes at $\mu = \mu_c$, and

$$\frac{\partial}{\partial\mu}\zeta_c(0\,;\,\mu_c)\neq 0, \tag{2.39}$$

$$\ker\dim\mathscr{L}_k(0\,;\,\mu_c)=1. \tag{2.40}$$

See, Lemma 1.10. Here, $\zeta_c(\varepsilon\,;\,\mu)$ is the continuation of $\zeta_c(0\,;\,\mu)$. We want to seek $\mu=\mu_c(\varepsilon)$ such that

$$\zeta_c(\varepsilon\,;\,\mu)=0 \tag{2.41}$$

holds. By virtue of the relation $\zeta_c(0\,;\,\mu_c)=0$ and (2.39), we have the unique existence of $\mu=\mu_c(\varepsilon)$ such that Eq. (2.41) is satisfied and that $|\mu_c(\varepsilon)-\mu_c|\leq C|\varepsilon|$ for $|\varepsilon|<^{\exists}\varepsilon_1$. ($0<^{\exists}\varepsilon_1\leq\varepsilon_0$: sufficiently small).

Thus, we have again for each $\varepsilon\in[0,\ \varepsilon_1[$ a symmetric breaking bifurcation on a $\mathscr{G}$-symmetric path. Especially, the bifurcation buckling load $\mu_c(\varepsilon)$ is in an ε-neighborhood of that of the unperturbed problem.

Suppose now $(\mathscr{C}^+\,;\,\phi_c)$ is symmetry preserving, where $\mathscr{G}$ may or may not be trivial. There is a class of problems in which symmetry preserving bifurcations may occur stably.

Definition 2.17 *A linear path of* F is a pair $(\mu,\ \mu w_0)\in\boldsymbol{R}\times V$, $\mu\in I\subset\boldsymbol{R}$, such that $F(\mu,\ \mu w_0)=0$ for $\mu\in I$, where I is an open interval $\subset\boldsymbol{R}$ and $w_0\in V$ is a fixed function. In particular, if $w_0\equiv 0$, the pair $(\mu,\ 0)$ is the *trivial* path. *A bifurcation problem* (P) *from a linear (resp. trivial) path is called a problem of class* L *(resp. class* 0*).*

If (P) is neither of class L nor 0, it is called of class N(i.e., nonlinear path).*)

We remark that class L(class 0) problems appear in many engineering and mathematical literatures.

For class L problems, we have an almost trivial analogy of the previous proposition.

Proposition 2.18 *Suppose* (P) *is of class* L, *and that* F *is simple critical at* $(\mathscr{C}\,;\,\phi_c)\equiv(\mu_c,\ \mu_c w_0\,;\,\phi_c)\in\boldsymbol{R}\times V\times V$, $(\mu_c\in I)$. *Then,* $(\mathscr{C}\,;\,\phi_c)$ *is a bifurcation point. Moreover, this bifurcation is, if non-degenerate, stable under any small changes of the equation, provided they do not destroy the class* L *property of* F.

Proof. Since

*) Note that *the class of* (P) is a path-dependent notion. See, remarks at the introduction of § 2.

$$F(\mu,\ \mu w_0)=0,\ {}^\forall\mu\in I, \tag{2.42}$$

we can differentiate (2.42) on the path:

$$\begin{aligned}0&=\frac{d}{d\mu}F(\mu,\ \mu w_0)\\&=\frac{\partial F}{\partial\mu}(\mu,\ \mu w_0)+\frac{\partial F}{\partial w}(\mu,\ \mu w_0)w_0\end{aligned} \tag{2.43}$$

Thus, using the self-adjointness of F', we have at $\mu=\mu_c$,

$$\begin{aligned}<\dot F_c,\ \phi_c>&=-<F'_c w_0,\ \phi_c>\\&=0\end{aligned} \tag{2.44}$$

which shows the first assertion.

Suppose now the perturbed problem

$(P)_\varepsilon$ $F(\varepsilon;\ \mu,\ w)=0$

is still of class L uniformly in $\varepsilon\in E$. Namely, we assume that for each $\varepsilon\in E$, there exists a function $w_0(\varepsilon)\in V$ such that $w_0(0)=w_0$, $\|w_0(\varepsilon)-w_0\|_V\leq C|\varepsilon|$ and that $F(\varepsilon;\ \mu,\ \mu w_0(\varepsilon))=0$ for $\mu\in I$, $\varepsilon\in E$.

We let

$$\mathscr{L}(\varepsilon;\ \mu)\stackrel{def}{=}F'(\varepsilon;\ \mu,\ \mu w_0(\varepsilon)), \tag{2.45}$$

and consider a family of eigenproblems in V:

$$\mathscr{L}(\varepsilon;\ \mu)\phi_c(\varepsilon;\ \mu)=\zeta_c(\varepsilon;\ \mu)\phi_c(\varepsilon;\ \mu),\ \phi_c(\varepsilon;\ \mu)\in V. \tag{2.46}$$

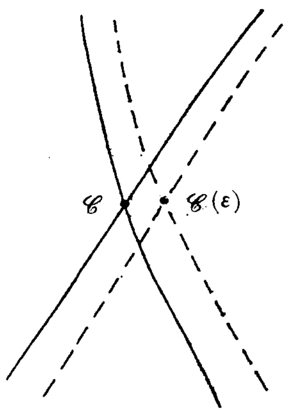


Fig. 2.4 *Stability of (symmetry preserving) bifurcation under perturbations which do not destroy the class L property.*

*) Also, by Lemma 1. 10 for the cusp case.

At $\varepsilon=0$, $\zeta_c(0;\mu)=0$(simple) and $\frac{\partial\zeta_c}{\partial\mu}(0;\mu_c)\neq 0$ by Lemma 1.9.*) Here, $\zeta_c(\varepsilon;\mu)$ is the continuation of $\zeta_c(0;\mu)$. Hence, the implicit function theorem applies to $\zeta_c(\varepsilon;\mu)=0$ at $(\varepsilon,\mu)=(0,\mu_c)$, obtaining a unique $\mu=\mu_c(\varepsilon)$ for each ε, $|\varepsilon|\in[0,\varepsilon_1[$ (ε_1: sufficiently small). Accordingly, we have again a bifurcation for each small ε.

Remark 2.19 As was stated in Remark 2.12, a fold bifurcation is necessarily symmetry preserving, and such *fold* may appear if $(P)_\varepsilon$ preserve the class L property. However, a symmetry preserving bifurcation is *not* necessarily a fold. A cusp or more degenerate bifurcation may appear by virtue of the degeneracy of F itself.

Chapter II. Numerical realization of simple buklings

3. Finite element spaces and preliminary results

We are now at the stage of discussing numerical approximations of (P). The class of schemes which we shall study is primarily that of conforming finite element schemes (P^h). It may be, however, worthnoting that in many applications, for instance, shells of revolution, or shallow cylindrical shells, the approximate space V^h is often taken to be a "hybrid" of finite element and Galerkin spaces, or to be a pure Galerkin space, in order to take into consideration some geometric symmetries of the problem. See, e.g., Yamada [54], Endou, Hangai and Kawamata [12]. In such cases, our setting (P_h) may have to be modified accordingly, but it appears that the essential framework of the theory is still valid. Another remark is that an extention of the setting (P^h) to *mixed* finite element schemes, as was proposed by Miyoshi [34] or by Brezzi and Raviart [6], seems also possible. Such applications will be reported elsewhere. See, also Brezzi and Fujii [5].

In this section, we first restate the problem (P) with precise assumptions. These are essentially motivated, and are actually satisfied by the von Kármán-Donnell-Marguerre equation. One can count also a number of examples within this framework, including *the arch problem* mentioned in the introduction (Chapter 0). Next, we give the abstract form of finite element schemes (P^h) given in a sequence $(h\to 0)$ of approximate finite element subspaces V_h. The hypotheses on the scheme are described in terms of this abstract setting (P^h), which allows one to discuss a class of approximate problems in a general way. A conforming scheme for the von Kármán-Donnell-Marguerre shells is given as an example.

In the fourth subsection, we state the equi-implicit function theorem as applied to a family $(\sigma\in\Sigma)$ of operator equations in a Banach space.

This theorem plays a basic role in many aspects of our discussions.

Lastly, we give a remark on the conservation of *symmetry* covariances in the approximate scheme (P^h). This theorem claims essentially that if the element pattern preserves the symmetry group of the problem (P), the discrete scheme (P^h) inherits the symmetry covariance of (P). Taking the conclusions of §. 7 into consideration, this gives a simple and natural consequence that if the element pattern respects the group symmetry $\mathscr{G}$, symmetry breaking bifurcations are always realized numerically in the approximate space V^h.

3.1 Statement of the problem (P)

In this subsection, we state the precise setting of our problem (P), which will be assumed throughout in the discussion of numerical analysis of (P).

Let U, V and W be separable Hilbert spaces such that

$$W \subsetneq V \subsetneq U, \tag{3.1}$$

where the injections are continuous and dense.*)

The problem is to get the pair $(\mu, w) \in \boldsymbol{R} \times V$ of solutions of $F(\mu, w) = 0$, where F is a $C^p (p \geq 3)$ mapping $\boldsymbol{R} \times V \to V$. We assume the following form to F:

$$(P) \qquad \begin{aligned} F(\mu, w) &\equiv N(w) + \mu f \\ &\equiv (I+L)w + \frac{1}{2!}\mathscr{B}(w, w) + \frac{1}{3!}\mathscr{T}(w, w, w) + \mu G p = 0. \end{aligned} \tag{3.2}$$

N is a smooth and real Fredholm mapping**) $V \to V$, and $N'(w)$ is assumed to be self-adjoint in V. More precisely, we assume the following to L, G, $\mathscr{B}$ and $\mathscr{T}$.

L and G are linear, bounded, compact real mappings and self-adjoint $V \to V$ such that

$$L \in \boldsymbol{B}_0(V) \cap \boldsymbol{B}(U, V) \cap \boldsymbol{B}(V, W) \tag{3.3}$$

$$G \in \boldsymbol{B}_0(V) \cap \boldsymbol{B}(U, W). \tag{3.4}$$

Evidently, the compactness of L and G follows from $L \in \boldsymbol{B}(V, W)$ and $G \in \boldsymbol{B}(U, W)$.

$\mathscr{B}$ is a real symmetric, bilinear mapping $V \times V \to V$ such that

*) It is assumed (whenever necessary) that U, V and W are complexifications of real Hilbert spaces.

**) In fact, N is of C^∞ class.

$\mathscr{B}$: $V\times V\to V$, continuous, separately compact*) and separately self-adjoint, and (3.5)

$$\mathscr{B} : \begin{array}{ll} U\times W\to V & \text{continuous,} \\ V\times W\to W & \text{continuous,} \\ U\times V\to V & \text{continuous.} \end{array} \tag{3.6}$$

$\mathscr{T}$ is a real symmetric, trilinear mapping $V\times V\times V\to V$ such that

$\mathscr{T}$: $V\times V\times V\to V$, continuous, separately compact and separately self-adjoint, and (3.7)

$$\mathscr{T} : \begin{array}{ll} U\times V\times V\to U, & \text{continuous,} \\ U\times V\times W\to V, & \text{continuous,} \\ V\times V\times W\to W, & \text{continuous.} \end{array} \tag{3.8}$$

We assume that $p\in U$ and that solutions (μ, w) of (P) lies in $\boldsymbol{R}\times W$. This assumes implicitly the W-regularity of weak solutions $(\mu, w)\in\boldsymbol{R}\times V$ of (P) for any data $p\in U$.

Example 3.1

An example may be given by the von Kármán-Donnell-Marguerre equation defined on $\Omega\subset\boldsymbol{R}^2$. We may take $U=H^0(\Omega)$, $V=H_0^2(\Omega)$ and $W=H_0^2(\Omega)\cap H^4(\Omega)$ for a smooth domain Ω. When Ω is a convex domain with corners i.e., a convex polygon, it is still possible to apply the above abstract setting with $U=H^0(\Omega)$, $V=H_0^2(\Omega)$ and $W=H_0^2(\Omega)\cap H^{3+\sigma}(\Omega)$, $\sigma\in[0,\sigma_0]$, where $\sigma_0=\sigma_0(\partial\Omega)$ such that $0<\sigma_0\leq 1$.**) See Appendix A for details.

3.2 Finite element spaces and discrete problem (P^h)

Let V^h be a sequence $(h\to 0)$ of finite element subspaces of V, and P^h the orthogonal projection of V onto V^h. In other words, P^h is defined by

$$<(I-P^h)u, v^h>=0 \text{ for all } v^h\in V^h \tag{3.9}$$

where $<\cdot, \cdot>$ denotes the inner product in V.

We assume that there exist two constants $l>0$, $m>0$ such that for any $u\in W$,

$$\|(I-P^h)u\|_V\leq Ch^l\|u\|_W, \tag{3.10}$$

---

*) $\mathscr{B}(u, \cdot)$ and $\mathscr{T}(u, v, \cdot)$ are compact, self-adjoint as linear operators $V\to V$ for $^\forall u, v\in V$.

**) $H^S(\Omega)=W^{2,S}(\Omega)$ and $H_0^2(\Omega)=\left\{u\in H^2(\Omega)\ ;\ u=\frac{\partial u}{\partial n}=0 \text{ on } \partial\Omega\right\}$.

and

$$\|(I-P^h)u\|_U \leq Ch^{l+m}\|u\|_W. \tag{3.11}$$

Let

$$k=\min(l,\ m). \tag{3.12}$$

What the assumptions (3.10) and (3.11) imply is that the linear elliptic problem associated with the inner product $<\ ,\ >$ can be approximated in V^h with $O(h^l)$-convergence in V-norm and $O(h^{l+m})$-convergence in U-norm. The latter implicitly assumes a situation that Nitsche's trick (see, e.g., Strang and Fix [49]) can be applied to the linear elliptic problem associated with $<\ ,\ >$.

Example 3.2 Let U, V and W be as in Example 3.1, with $<u,\ v> \overset{def}{=} \int_\Omega \Delta u \overline{\Delta v}$. Let $V^h(h\to 0)$ be a compatible finite element subspaces of V. Then, Eq. (3.9) reads as

$$\int_\Omega \Delta(P^h u)\overline{\Delta v^h}=\int_\Omega \Delta u \overline{\Delta v^h},\quad {}^\forall v^h \in V^h.$$

$u^h \equiv (P^h u)$ is thus the finite element solution of a linear elliptic problem: $\Delta^2 u=p$ in Ω with u, $\frac{\partial u}{\partial n}=0$ on $\partial\Omega$; namely, u^h is the unique solution of

$$\int_\Omega \Delta u^h \overline{\Delta v^h}=\int_\Omega p\bar{v}^h,\quad {}^\forall v^h \in V^h$$

for $p\in H^0(\Omega)$. Using an appropriate class of subspaces V^h, it is classical to show that $\|u^h-u\|_{H_0^2}\leq Ch^l\|u\|_W$ with $l=1+\sigma$ while the estimate $\|u^h-u\|_{H^0}\leq Ch^{l+m}\|u\|_W$ with $m=1+\sigma$ may be a consequence of Nitsche's trick. See, Appendix B.

The finite element scheme is described in an operator equation in V as:

$$\begin{aligned}(P^h)\quad F^h(\mu,\ w^h) &\equiv N^h(w^h)+\mu f^h\\ &\equiv (I+L^h)w^h+\frac{1}{2!}\mathscr{B}^h(w^h,\ w^h)+\frac{1}{3!}\mathscr{T}^h(w^h,\ w^h,\ w^h)+\mu G^h p\\ &\equiv (I+P^h L)w^h+\frac{1}{2!}P^h\mathscr{B}^{(h)}(w^h,\ w^h)\\ &\quad +\frac{1}{3!}P^h\mathscr{T}^{(h)}(w^h,\ w^h,\ w^h)+\mu^h P^h G p=0 \end{aligned}\tag{3.13}$$

Here, $\mathscr{B}^{(h)}$ is a real symmetric bilinear mapping $V\times V\to V$ such that

$$\mathscr{B}^{(h)}: V\times V\to V, \text{ equi-continuous, separately self-adjoint.} \quad (3.41)^{*)}$$

$\mathscr{B}^{(h)}$ is *close* to $\mathscr{B}$ in the following sense:

$$\text{(i)} \quad \|\mathscr{B}(u, v) - \mathscr{B}^{(h)}(\tilde{u}, \tilde{v})\|_V \leq C\|u\|_W\|v\|_W h^{l+k}, \quad (3.15)$$

$$\text{(ii)} \quad \|\mathscr{B}(u, \cdot) - \mathscr{B}^{(h)}(\tilde{u}, \cdot)\|_{V\to V} \leq C\|u\|_W h^l, \quad (3.16)$$

for any $u, v\in W$ and $\tilde{u}, \tilde{v}\in V$ such that

$$\left.\begin{aligned} &\|u-\tilde{u}\|_V \leq C\|u\|_W h^l, \\ &\|u-\tilde{u}\|_U \leq C\|u\|_W h^{l+k}, \\ &\|v-\tilde{v}\|_V \leq C\|v\|_W h^l, \\ \text{and} \quad &\|v-\tilde{v}\|_U \leq C\|v\|_W h^{l+k}. \end{aligned}\right\} \quad (3.17)$$

$\mathscr{T}^{(h)}$ is a real symmetric trilinear mapping $V\times V\times V\to V$ such that

$$\mathscr{T}^{(h)}: V\times V\times V\to V, \text{ equi-continuous, separately self-adjoint.} \quad (3.18)$$

$\mathscr{T}^{(h)}$ is *close* to $\mathscr{T}$ in the following sense:

$$\text{(i)} \quad \|\mathscr{T}(u, v, w) - \mathscr{T}^{(h)}(\tilde{u}, \tilde{v}, \tilde{w})\|_V \leq C\|u\|_W\|v\|_W\|w\|_W h^{l+k}, \quad (3.19)$$

$$\text{(ii)} \quad \|\mathscr{T}(u, v, \cdot) - \mathscr{T}^{(h)}(\tilde{u}, \tilde{v}, \cdot)\|_{V\to V} \leq C\|u\|_W\|v\|_W h^l, \quad (3.20)$$

for any $u, v, w\in W$ and $\tilde{u}, \tilde{v}, \tilde{w}\in V$ such that Eq. (3.17) and similar relations for w and $\tilde{w}$:

$$\|w-\tilde{w}\|_V \leq C\|w\|_W h^l \quad (3.21)$$

and

$$\|w-\tilde{w}\|_U \leq C\|w\|_W h^{l+k}$$

hold.

Example 3.3 Let U, V, W, V^h and P^h be as in Example 3.2.

The conforming finite element approximate scheme for the (*von K. D. M.*) equation then becomes

$$\begin{aligned} w^h &+ [P^h\mathscr{B}(w_0, P^h\mathscr{B}(w_0, w^h) - P^h\mathscr{B}(\psi_0, w^h)] \\ &+ \frac{1}{2}[P^h\mathscr{B}(w_0, P^h\mathscr{B}(w^h, w^h)) + 2P^h\mathscr{B}(w^h, P^h\mathscr{B}(w^h, w_0))] \\ &+ \frac{1}{2}P^h\mathscr{B}(w^h, P^h\mathscr{B}(w^h, w^h)) = \mu P^h Gp + P^h\mathscr{B}(w_0, \psi_0). \end{aligned} \quad (3.22)$$

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*) For the definition of *equi-continuity*, see Def. 3.4.

Here, $\mathscr{B}(w_0, \psi_0)$ is generally assumed to vanish.

Accordingly, in the setting (P^h) we put:

$$\mathscr{B}^{(h)}(u, v) = \mathscr{T}^{(h)}(u, v, w_0) \tag{3.23}$$

and

$$\mathscr{T}^{(h)}(u, v, w) = \mathscr{B}(u, P^h\mathscr{B}(v, w)) + \mathscr{B}(v, P^h\mathscr{B}(w, u)) + \mathscr{B}(w, P^h\mathscr{B}(u, v)). \tag{3.24}$$

Recalling that

$$\mathscr{B}(u, v) = \mathscr{T}(u, v, w_0), \tag{3.25}$$

and

$$\mathscr{T}(u, v, w) = \mathscr{B}(u, \mathscr{B}(v, w)) + \mathscr{B}(v, \mathscr{B}(w, u)) + \mathscr{B}(w, \mathscr{B}(u, v)), \tag{3.26}$$

the assumptions (3.14)-(3.21) can be checked.

See, Appendix B for details.

3.3 Equi-implicit function theorem

Let X and Y be Banach spaces. Let $G^\sigma (\sigma\in\Sigma)$ be a family of continuous mappings defined on an open subset U of X:

$$G^\sigma: U\subset X\to Y \qquad (\sigma\in\Sigma).$$

Definition 3.4. G^σ is *equi-continuous* in $U\subset X\to Y$ if for all $\varepsilon>0$, there is $\delta>0$ such that

$$\|G^\sigma(x) - G^\sigma(x')\|_Y<\varepsilon, \text{ for all } \sigma\in\Sigma$$

whenever $\|x-x'\|_X<\delta$.

Definition 3.5 A family of equi-continuous mappings G^σ is of class *equi-C^1 if $G^\sigma_x(x_0)$ (Fréchet derivative of G with respect to x)* form a family of equi-continuous linear mappings $x_0\to \boldsymbol{B}(X, Y)$.

Note that a composition of equi-C^p mappings is of equi-C^p.

Now, as a modification of the implicit function theorem (see, e.g., Nirenberg [35]), we have

Theorem 3.6 (*Equi-Implicit Function Theorem*)*)

Let X, Y, Z be Banach spaces. Let G^σ be a family of equi-continuous *mappings on an open subset* $U \subset X \times Y \to Z$, $0 \in U$.

Suppose that

(i) $G^{\sigma}(0, 0) = 0$, for all $\sigma \in \Sigma$,

(ii) *there exist* G_x^{σ} *such that* $G_x^{\sigma}(x, y)$ *is equi-continuous in* U, *and*

(iii) $G_x^{\sigma}(0, 0)$ *is equi-invertible*.**)

Then,

(iv) *there exists a ball* $B_r(0) = \{y \,;\, \|y\| < r\}$, *and a family of equi-continuous mappings* $u^{\sigma}: B_r(0) \to X$, *such that* $u^{\sigma}(0) = 0$, $G^{\sigma}(u^{\sigma}(y), y) = 0$, ${}^{\forall}y \in B_r(0)$. *For each* $\sigma \in \Sigma$, *the mapping* u^{σ} *exists uniquely*.

(v) *If* G^{σ} *is of equi-*C^1, *then*, $u^{\sigma}(y)$ *is of equi-*C^1. *Moreover*, $G_x^{\sigma}(u^{\sigma}(y), y)$ *is equi-invertible, and*

$$u_y^{\sigma}(y) = -[G_x^{\sigma}(u^{\sigma}(y), y)]^{-1} \circ G_y^{\sigma}(u^{\sigma}(y), y) \tag{3.27}$$

As a corollary, we consider a *prototype problem* to which the equi-implicit function theorem is well applied.

Corollary 3.7 *Let* V *be a Banach space and* $V^h \subset V$ *be a family* $(0 < h \leq 1)$ *of subspaces of* V. *Suppose we have a family of operator equations in* V:

$$(\tilde{P}^h) \quad F^h(u^h) \equiv (I + K^h)u^h + N^h(u^h) + f^h = 0, \tag{3.28}$$

where $K^h \in \boldsymbol{B}(V)$ *(uniformly) with range in* V^h, N^h *an equi-*C^1 *mapping* $V \to V^h$ *in a neighorhood of the origin, such that* $N^h(0) = N^{h\prime}(0) = 0$ *and* $f^h \in V^h$. *Assume that* (i) $I + K^h$ *is equi-invertible and* (ii) *for any* $\eta_0 > 0$, *there is a constant* $h(\eta_0) > 0$ *such that* $\|f^h\|_V < \eta_0$ *(for all* $h < h(\eta_0)$*)*.

Then, there is a constant $h_0 > 0$ *such that for all* $0 < h < h_0$, $(\tilde{P}^h)$ *has a unique solution* $u^h \in V^h$ *satisfying*

$$\|u^h\|_V \leq C\|f^h\|_V. \tag{3.29}$$

Proof. Let $g^h \equiv f^h / \|f^h\|_V$ then, $\|g^h\|_V = 1$. We consider the auxiliary problem in $V \times \boldsymbol{R}$:

$$(\tilde{P}^h)_{\eta} \quad G^h(u^h, \eta) \equiv (I + K^h)u^h + N^h(u^h) + \eta g^h = 0 \tag{3.30}$$

The equi-implicit function theorem guarantees the unique existence of a

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*) The proof can be performed in parallel with [35], noting that r of $B_r(0)$ can be taken independent of $\sigma \in \Sigma$.

**) $A^{\sigma} \equiv G_x^{\sigma}(0, 0)$ is equi-invertible if A^{σ} has a bounded inverse $(A^{\sigma})^{-1}: Z \to X$ for each $\sigma \in \Sigma$, and $\|(A^{\sigma})^{-1}\| \leq \rho < +\infty$ for all $\sigma \in \Sigma$.

family of solutions $u^h=u^h(\eta)$, $|\eta|<\eta_0(\eta_0$: independent of $h)$. We choose $h_0>0$ so that $\|f^h\|_V<\eta_0$ for all $h<h_0$. Then, for *each* $h<h_0$, we have a unique $u^h=u^h|_{\eta=\|f^h\|_V}$, which is obviously the desired solution of $(\tilde{P}^h)$.

The estimate (3.29) follows from Eq. (3.27) of the equi-implicit function theorem. In fact, since $G^h_\eta=g^h$ and $u^h(0)=0$,

$$u^h(\eta)=\int_0^\eta u^h_\eta(t)dt$$
$$=-\int_0^\eta [G^h_{u^h}(u^h(t),\ t)]^{-1}g^h\ dt.$$

Thus, $\|u^h(\eta)\|_V\le C|\eta|$ for $|\eta|<\eta_0$ and $0<h<h_0$, from which follows Eq. (3.29).

3.4 A theorem on conservation of symmetry groups in V^h

Let V be a complex, separable Hilbert space with inner product $<\ ,\ >$ and $V^h(h\to 0)$ be a sequence of finite element subspaces of V. Let P^h be the orthogonal projection $V\to V^h$ defined by Eq. (3.9).

Let $\mathscr{G}\subset O(\mathrm{m})$ denotes the symmetry group of F*), T: $\mathscr{G}\to GL(V)$ a unitary representation and $P_i(i=1,\dots,q)$ the projection $V\to V_i$ associated with the standard decompsition of V with respect to $\mathscr{G}$:

$$V=V_1\oplus V_2\oplus\dots\oplus V_q.$$

Recall that $P_i(i=1,\dots,q)$ are defined by

$$P_i=\frac{n_i}{n(\mathscr{G})}\sum_{g\in\mathscr{G}}\overline{\chi_i(g)}T_g \qquad (i=1,\dots,q).$$

See, §2.1 for notations and details.

We would like to clarify in this section under what conditions $\mathscr{G}$ can be the symmetry group of the approximate scheme F^h. This question concerns essentially in what situations the projections $P_i(i=1,\dots,q)$ and P^h commute each other. Intuitively, the answer is that if *the element pattern of* V^h *respects the symmetry group of* Ω, then $P_iP^h=P^hP_i$, $i=1,\dots,q$.

Indeed, we have the following

Lemma 3.8

Assume V^h *is invariant under* $\mathscr{G}$. *Then,*

$$P_iP^h=P^hP_i \qquad i=1,\dots,q. \tag{3.31}$$

*) Here, we assume that $\mathscr{G}$ is finite for brevity. It is immediate to see that the results in this section equally hold for a compact Lie group D_∞.

Proof.

That V^h is invariant under $\mathscr{G}$ implies if $\phi^h \in V^h$ then, $T_g\phi^h \in V^h (\forall g \in \mathscr{G})$. It is enough to show that $T_g P^h = P^h T_g (\forall g \in \mathscr{G})$ since T_g and P_i commute each other. To prove this, we assume the contrary, i. e., that $P^h T_g - T_g P^h \neq 0$. Namely, there exists a non-zero $u \in V$ such that

$$(P^h T_g - T_g P^h) u \equiv \chi^h \neq 0.$$

Notice that since $T_g P^h u \in V^h$ by assumption, $\chi^h \in V^h$.

Now, for any $\phi^h \in V^h$,

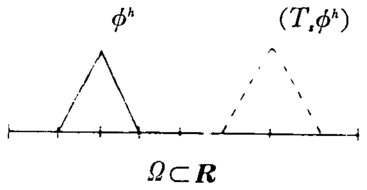
$$\begin{aligned} <\chi^h, \phi^h> &= <(P^h T_g - T_g P^h) u,\ \phi^h> \\ &= <P^h T_g u, \phi^h> - <T_g P^h u,\ \phi^h> \\ &= <T_g u,\ \phi^h> - <P^h u,\ T_g^* \phi^h> \\ &= <T_g u,\ \phi^h> - <u,\ T_g^* \phi^h> \\ &= 0, \end{aligned}$$

here $T_g^* \phi^h \in V^h$ and Eq. (3.9) are used. Thus, if we let $\phi^h = \chi^h$, $||\chi^h||_V = 0$, which is a contradiction.

An implication of the lemma is as follows. It claims that $T_g\phi^h \in V^h$ for all $\phi^h \in V^h$ and for all $g \in \mathscr{G}$. Evidently, enough to claim this only for each basis function of V^h. So, the condition $T_g\phi^h \in V^h$ $(\forall \phi^h \in V^h)$ implies that *the finite elements in V^h, including both the element pattern and basis functions, preserve the symmetry axes of $\mathscr{G}$.*

It is remarked that if the assumption in the above lemma is satisfied, one can generally expect that the $\mathscr{G}$-covariance of F is inherited by F^h. An example is given by Corollary 3. 10.

Example 3. 9 "*Unsymmetric element pattern destroys the group T_s, $T_s^2 = I$*". In fact, $(T_s\phi^h)(x) = \phi^h(-x)$ is *not* included in V^h.



Corollary 3. 10 *Assume V^h is invariant under $\mathscr{G}$, where $\mathscr{G}$ is the symmetry group of the von Kármán-Donnell-Marguerre mapping F. See, Example 3. 1 and 3. 2. $\mathscr{G}$ is the symmetry group of the finite element von K. D. M. scheme F^h defined by Eq.* (3. 22), *provided that w_0 and ϕ_0 are invariant under $\mathscr{G}$.*

4. Numerical realization of ordinary path

The aim of this section is to show that an ordinary path of (P) is always realized in V^h as an ordinary path, except in a neighborhood of critical points. A symmetric ordinary path is certainly a symmetric path in V_h provided the scheme (P^h) respects the same symmetry group $\mathscr{G}$.

Firstly, we show the following

Lemma 4.1 *Let* $(\mu, w)\in \boldsymbol{R}\times W$ *be such that* $F(\mu, w)=0$. *Then, for any* $(\tilde{\mu}, \tilde{w})\in \boldsymbol{R}\times V$ *such that*

$$|\tilde{\mu}-\mu|\leq C''h^{l+k} \tag{4.1}$$

and

$$||\tilde{w}-P^h w||_V\leq C'h^{l+k}, \tag{4.2}$$

it holds that

$$||F^h(\tilde{\mu}, \tilde{w})||_V\leq Ch^{l+k}, \tag{4.3}$$

where C, C' *and* C'' *may depend on* $||w||_W$, $||\tilde{w}||_V$ *and* $|\mu|$, *but not on* h.

Proof. Noting that $P^hF(\mu, w)=0$, we have that

$$\begin{aligned} F^h(\tilde{\mu}, \tilde{w}) &= F^h(\tilde{\mu}, \tilde{w})-P^hF(\mu, w)\\ &=(\tilde{w}-P^h w)+P^hL(\tilde{w}-w)\\ &+\frac{1}{2!}P^h\{\mathscr{B}^{(h)}(\tilde{w}, \tilde{w})-\mathscr{B}(w, w)\}\\ &+\frac{1}{3!}P^h\{\mathscr{T}^{(h)}(\tilde{w}, \tilde{w}, \tilde{w})-\mathscr{T}(w, w, w)\}\\ &+(\tilde{\mu}-\mu)P^hGp. \end{aligned}$$

The assumption (4.2) together with Eqs. (3.10)-(3.11) imply that

$$||\tilde{w}-w||_V=||\tilde{w}-P^h w||_V+||P^h w-w||_V\leq Ch^l,$$

and

$$||\tilde{w}-w||_U\leq Ch^{l+k}.$$

Thus,

$$\begin{aligned} ||F^h(\tilde{\mu}, \tilde{w})||_V &\leq ||\tilde{w}-P^h w||_V+||P^h||_{V\to V}||L||_{U\to V}||\tilde{w}-w||_U\\ &+\frac{1}{2!}||P^h||_{V\to V}||\mathscr{B}^{(h)}(\tilde{w}, \tilde{w})-\mathscr{B}(w, w)||_V\\ &+\frac{1}{3!}||P^h||_{V\to V}||\mathscr{T}^{(h)}(\tilde{w}, \tilde{w}, \tilde{w})-\mathscr{T}(w, w, w)||_V \end{aligned}$$

$$+ |\tilde{\mu}-\mu| \| P^h G p \|_V.$$

By virtue of Eqs. (4. 1), (4. 4), (4. 5), (3. 11), (3. 15) and (3. 19), we have the desired result.

Corollary 4. 2 *Let* $(\mu, w) \in \boldsymbol{R} \times W$ *be such that* $F(\mu, w)=0$. *Then,*

$$\| F^h(\mu, P^h w) \|_V \leq C h^{l+k}, \tag{4.4}$$

where $C=C(|\mu|, \|w\|_W)$.

Definition 4.3 Let

$$\begin{aligned} \mathscr{L}_w &\equiv F'(\mu, w) \\ &\equiv I+\mathscr{K}_w \\ &= I+L+\mathscr{B}(w, \cdot)+\frac{1}{2}\mathscr{T}(w, w, \cdot) \end{aligned} \tag{4.5}$$

and

$$\begin{aligned} \mathscr{L}^h_w &\equiv F^{h\prime}(\mu, w) \\ &\equiv I+\mathscr{K}^h_w \\ &\equiv I+P^h \mathscr{K}^{(h)}_w \\ &= I+P^h \{L+\mathscr{B}^{(h)}(w, \cdot)+\frac{1}{2}\mathscr{T}^{(h)}(w, w, \cdot)\} \end{aligned} \tag{4.6}$$

The following lemma concerns the existence of uniformly bounded inverse of $\mathscr{L}^h_{P^h w}$ when $\mathscr{L}_w$ is invertible.

Lemma 4. 4 *Suppose* $\mathscr{L}_w$ *has a bounded inverse* $\|\mathscr{L}_w^{-1}\|_{V\to V} \leq \rho < +\infty$ *for a given* $w \in W$. *Let* $\tilde{w} \in V$ *be a function close to* w *in the sense that*

$$\| \tilde{w} - P^h w \|_V \leq C' h^{l+k}. \tag{4.7}$$

Then, $\mathscr{L}^h_{\tilde{w}}$ *is equi-invertible in* V. *That is, for* $h < h_0 (\exists h_0 = h_0(\rho)$, *sufficiently small) the approximate operator* $\mathscr{L}^h_{\tilde{w}}$ *is equi-invertible and* $\|(\mathscr{L}^h_{\tilde{w}})^{-1}\|_{V\to V} \leq 2\rho < +\infty$.

Corollary 4. 5 *Suppose* $\mathscr{L}_w$ *has a bounded inverse* $\|(\mathscr{L}_w)^{-1}\|_{V\to V} \leq \rho < +\infty$. *Then,* $\mathscr{L}^h_{P^h w}$ *is equi-invertible and* $\|(\mathscr{L}^h_{P^h w})^{-1}\|_{V\to V} \leq 2\rho < +\infty$ *for* $h < h_0 (\exists h_0 = h_0(\rho)$, *sufficiently small).*

For a proof, we make use of the Kantrovich lemma (Kantrovich and Akilov [18]).

Lemma 4. 6 (*Kantrovich*)

Let X, Y be two Banach spaces. Let $\mathscr{L}: X \to Y$ *be a linear operator. Given* $y \in Y$, *we need to solve*

$$(L) \quad \mathscr{L}x = y, \quad x \in X. \tag{4.8}$$

If for any $y \in Y$, *there is* $x \in X$ *such that*

$$\|\mathscr{L}x - y\| \leq q\|y\| \qquad (q < 1) \tag{4.9}$$

and

$$\|x\| \leq \rho\|y\|, \tag{4.10}$$

then, (L) *has a unique solution* $x \in X$ *with*

$$\|x\| \leq \frac{\rho}{1-q}\|y\|. \tag{4.11}$$

Namely, it holds that

$$\|\mathscr{L}^{-1}\| \leq \frac{\rho}{1-q}. \tag{4.12}$$

Proof of Lemma 4.4 We consider the problem

$$(L^h) \quad \mathscr{L}^h_{\tilde{w}} u = f, \qquad f \in V.$$

For $f \in V$ (given), we let $v = (\mathscr{L}_w)^{-1} f$. By assumption, $\|v\|_V \leq \rho\|f\|_V$. Also,

$$\begin{aligned}\|\mathscr{L}^h_{\tilde{w}} v - f\|_V &= \|\mathscr{L}^h_{\tilde{w}} \mathscr{L}_w^{-1} f - f\|_V \\ &= \|(\mathscr{L}^h_{\tilde{w}} - \mathscr{L}_w)\mathscr{L}_w^{-1} f\|_V \\ &\leq \|\mathscr{L}^h_{\tilde{w}} - \mathscr{L}_w\|_{V\to V}\|\mathscr{L}_w^{-1} f\|_V.\end{aligned}$$

In view of Fqs. (4.5) and (4.6) and Eqs. (3.10), (3.3), (3.6), (3.8), (3.16) and (3.20),

$$\begin{aligned}\|\mathscr{L}^h_{\tilde{w}} - \mathscr{L}_w\|_{V\to V} \leq &\|(I-P^h)L\|_{V\to V} + \|P^h\mathscr{B}^{(h)}(\tilde{w}, \cdot) - \mathscr{B}(w, \cdot)\|_{V\to V} \\ &+ \frac{1}{2}\|P^h\mathscr{T}^{(h)}(\tilde{w}, \tilde{w}, \cdot) - \mathscr{T}(w, w, \cdot)\|_{V\to V} \\ &\leq \|(I-P^h)L\|_{V\to V} + \|(I-P^h)\mathscr{B}(w, \cdot)\|_{V\to V} \\ &+ \frac{1}{2}\|(I-P^h)\mathscr{T}(w, w, \cdot)\|_{V\to V} \\ &+ \|P^h\|_{V\to V}\|\mathscr{B}^{(h)}(\tilde{w}, \cdot) - \mathscr{B}(w, \cdot)\|_{V\to V} \\ &+ \|P^h\|_{V\to V}\|\mathscr{T}^{(h)}(\tilde{w}, \tilde{w}, \cdot) - \mathscr{T}(w, w, \cdot)\|_{V\to V} \\ &\leq C\|w\|_W h^l.\end{aligned} \tag{4.13}$$

Thus, if we take $h_0 = h_0(\rho)$ so small that it holds

$$\|\mathscr{L}_w^h v - f\|_V \le C\|w\|_W h^l \rho \|f\|_V \le q\|f\|_V$$

with $q<1$, the Kantrovich lemma applies to (L^h).

It is noted that in order that $\mathscr{L}_w^h$ is equi-invertible, $h_0=h_0(\rho)$ should be taken smaller as ρ becomes larger, namely, as $\mathscr{L}_w$ comes closer to a critical point In other words, if $h>0$ is kept fixed as is the case in practical computations, we have at this stage no information about the approximate operator $\mathscr{L}^h_{P^h w}$ in a neighborhood (which itself depends on h!) of a critical point where $\mathscr{L}_w$ is not invertible. What we can guarantee from Lemmas 4.1-4.4 and the equi-implicit function theorem is the unique existence of ordinary path of (P^h) which is close to an ordinary path of (P), *except in the vicinity of critical points of* (P).

Proposition 4.6 *Suppose* $\mathcal{O} \equiv (\mu_0, w_0) \in \boldsymbol{R}\times W$ *is an ordinary point of* (P), *such that* $\|(\mathscr{L}_{w_0})^{-1}\|_{V\to V} \le \rho < +\infty$. *Then, there is a constant* $h_0=h_0(\rho)>0$ *and a unique ordinary point of* (P^h) : $\mathcal{O}^h \equiv (\mu_0, w_0^h) \in \boldsymbol{R}\times V^h$, *for all* $h\in]0, h_0[$, *such that*

$$F^h(\mu_0, w_0^h)=0 \tag{4.14}$$

and

$$\|w_0^h - P^h w_0\|_V \le C\|w_0\|_W h^{l+k}.$$

Also,

$$\|w_0^h - w_0\|_V \le C'\|w_0\|_W h^l \tag{4.15}$$

and

$$\|w_0^h - w_0\|_U \le C''\|w_0\|_W h^{l+k}.$$

Moreover, there is a smooth ordinary path $(\mu, w^h(\mu)) \in \boldsymbol{R}\times V^h$ *of* (P^h), *which contains* $\mathcal{O}^h$.

Proof. For the given $(\mu_0, w_0)\in\boldsymbol{R}\times W$, we let

$$w_0^h = P^h w_0 + v_0^h, \quad v_0^h \in V^h. \tag{4.16}$$

Then, $F^h(\mu_0, w_0^h)=0$ is reduced to

$$\begin{aligned} 0 &= F^h(\mu_0, P^h w_0 + v_0^h) \\ &= F^h(\mu_0, P^h w_0) + F^{h\prime}(\mu_0, P^h w_0)v_0^h + R^h(\mu_0, P^h w_0; v_0^h), \end{aligned} \tag{4.17}$$

where $R^h(\mu_0, P^h w_0; v_0^h)$ is the remainder term. Corollary 4.5 guarantees that $\mathscr{L}^h_{P^h w_0} \equiv F^{h\prime}(\mu_0, P^h w_0)$ is equi-invertible for $h \in]0, h_0[$, (P^h) may be rewritten as

$$(Q^h) \qquad v_0^h + (\mathscr{L}^h_{P^h w_0})^{-1} R_0^h(v_0^h) + (\mathscr{L}^h_{P^h w_0})^{-1} F^h(\mu_0, P^h w_0) = 0, \tag{4.18}$$

where $R_0^h(v_0^h)$ denotes $R^h(\mu_0, P^h w_0; v_0^h)$.

We apply to (Q^h) the equi-implicit function theorem (Theorem 3.6) or Corollary 3.7. In fact,

(i) $(\mathscr{L}^h_{P^h w_0})^{-1} R_0^h(v)$ is equi-C^1 mapping $v \to \boldsymbol{B}(V, V)$ in a neighborhood of the origin, and

(ii) $\|(\mathscr{L}^h_{P^h w_0})^{-1} F^h(\mu_0, P^h w_0)\|_V \leq C h^{l+k}$ (4.19)

by Corollaries 4.2 and 4.5.

Since $R_0^h(0) = R_0^{h\prime}(0) = 0$, the conclusion of Corollary 3.7 can be applied to (Q^h). In fact, noting that R_0^h and $R_0^{h\prime}$ are by definition

$$\begin{aligned} R_0^h(v) &= \frac{1}{2!} F^{h\prime\prime}(\mu_0, P^h w_0)(v, v) + \frac{1}{3!} F^{h\prime\prime\prime}(\mu_0, P^h w_0)(v, v, v) \\ &= \frac{1}{2!} P^h \mathscr{B}^{(h)}(v, v) + \frac{1}{2!} P^h \mathscr{T}^{(h)}(P^h w_0, v, v) + \frac{1}{3!} P^h \mathscr{T}^{(h)}(v, v, v), \end{aligned} \tag{4.20}$$

and

$$R_0^{h\prime}(v) = P^h \mathscr{B}^{(h)}(v, \cdot) + P^h \mathscr{T}^{(h)}(P^h w_0, v, \cdot) + \frac{1}{2} P^h \mathscr{T}^{(h)}(v, v, \cdot), \tag{4.21}$$

the conclusion of (i) is immediately seen from the equi-continuity of $\mathscr{B}^{(h)}$ and $\mathscr{T}^{(h)}$ as bi- and tri-linear mappings.

Accordingly, Corollary 3.7 (or the implicit function theorem) yields that there exists a unique $v_0^h \in V^h \subset V$ such that

$$\|v_0^h\|_V \leq C h^{k+l} \text{ for all } h < h_0. \tag{4.22}$$

The second statement of the theorem, namely, the existence of a smooth ordinary path of (P^h) is a direct consequence of the fact that

$$\mathscr{L}^h_{w_0^h} = \mathscr{L}^h_{P^h w_0 + v_0^h} \tag{4.23}$$

is equi-invertible due to (v) of the equi-implicit function theorem.

Corollary 4.7 *Suppose $\mathscr{G} \subset O(m)$ is the symmetry group of F. Suppose also that $\mathscr{G}$ is the symmetry group of F^h, i.e., F^h is covariant under $\mathscr{G}$. Then, the realized ordinary path $(\mu, w^h(\mu))$ of F^h is $\mathscr{G}$-symmetric whenever the original path $(\mu, w^h(\mu))$ of F is $\mathscr{G}$-symmetric.*

See Lemma 2.8 for the $\mathscr{G}$-symmetric path and Lemma 3.8 for the condition that F^h respects the $\mathscr{G}$-symmetry. The proof is obvious since F^h is enclosed in $V^{h+}=V^{h+}$ as F is enclosed in V^+ (*cf.* §7.1, Chapter II).

5. A family of approximate eigenproblems

5.1 Statement of the problem

In this section, we study behaviors of finite element approximate solutions of a one-parameter family of eigenproblems in V:

$$(E) \quad \mathscr{L}(s)\phi(s)=z(s)\phi(s), \qquad s\in S=\{s;\ |s|<s_0\}\subset \boldsymbol{R}. \tag{5.1}$$

Here,

$$\mathscr{L}(s)=I+\mathscr{K}(s), \tag{5.2}$$

and

$(K)_{\mathrm{i}}$ $\mathscr{K}(s)\in \boldsymbol{B}_0(V)$ and is *self-adjoint* for each $s\in S$.

Moreover, we assume that

$$(K)_{\mathrm{ii}} \quad \mathscr{K}(s)\in \boldsymbol{B}(V,\ W)\cap \boldsymbol{B}(U,\ V), \qquad s\in S, \tag{5.3}$$

$$(K)_{\mathrm{iii}} \quad \mathscr{K}(s) \text{ is of } C^1 \text{ class}, \tag{5.4}$$

and

$$\frac{d}{ds}\mathscr{K}(s)\in \boldsymbol{B}(V), \qquad \text{for } s\in S.$$

Notice that every eigenfunction $\phi(s)$ of (E) belongs to W thanks to the regularity of $\mathscr{K}(s)$, Eq. (5.3).

Now, we consider a sequence $(h\to 0)$ of approximate eigenproblems in V:

$$(E^h) \quad \mathscr{L}^h(s)\phi^h(s)=z^h(s)\phi^h(s), \qquad s\in S. \tag{5.5}$$

Here,

$$(K)_{\mathrm{iv}} \quad \mathscr{L}^h(s)=I+P^h\mathscr{K}^{(h)}(s), \qquad s\in S; \tag{5.6}$$

$\mathscr{K}^{(h)}(s)$ is a sequence $(h\to 0)$ of one-parameter family $(s\in S)$ of linear, *uniformly* (in h) bounded operators, i.e.,

$\mathscr{K}^{(h)}(s)\in \boldsymbol{B}(V)$ and is *self-adjoint* for each $s\in S$.

Moreover,

$(K)_{\rm v}$ $\mathscr{K}^{(h)}(s)$ is of C^1 class,

with

$$\frac{d}{ds}\mathscr{K}^{(h)}(s)\in \boldsymbol{B}(V) \tag{5.7}$$

with uniform bounds (with respect to h) for each $s\in S$.

Let us suppose that $\mathscr{K}^{(h)}(s)$ is *close* to $\mathscr{K}(s)$ for each $s\in S$ in the following sense:

$$(K)_{\rm vi}\quad \|\Sigma^{(h)}(s)\|_{V\to V}\leq C_0(s)h^l, \tag{5.8}$$

$$\|\Sigma^{(h)}(s)\|_{W\to V}\leq C_1(s)h^{l+k}, \tag{5.9}$$

$$\|\frac{d}{ds}\Sigma^{(h)}(s)\|_{V\to V}\leq C_2(s)h^l,\qquad \text{for } s\in S, \tag{5.10}$$

where

$$\Sigma^{(h)}(s)\overset{def}{=}\mathscr{K}^{(h)}(s)-\mathscr{K}(s),\qquad s\in S. \tag{5.11}$$

We suppose that at $s=0$ one (and *only* one) of the eigenvalues of (E), say $z_c(s)$, vanishes. We call the pair $(z_c(s),\ \phi_c(s))\in \boldsymbol{R}\times W$ *the critical pair* of (E). More precisely, we assume that

$(K)_{\rm vii}$ there is a critical pair $(z_c(s),\ \phi_c(s))\in \boldsymbol{R}\times W$ at $s=0$, namely, $z_c(0)=0$ and

$$\dim\ker \mathscr{L}(0)\ (=\dim \operatorname{coker} \mathscr{L}(0))=1. \tag{5.12}$$

We may then assume (by taking if necessary a subinterval $S'\subset S$ and which we write again S) that

$$(K)_{\rm viii}\quad \dim\ker(\mathscr{L}(s)-z_c(s)I)=1,\qquad s\in S \tag{5.13}$$

and that

$$\|\{(\mathscr{L}(s)-z_c(s)I)\,|\,\mathscr{R}_c(s)\}^{-1}\|_{V\to V}\leq\rho_c<+\infty \text{ for all } s\in S \tag{5.14}$$

where

$$\mathscr{R}_c(s)=[\ker(\mathscr{L}(s)-z_c(s)I)]^{\perp}=\operatorname{range}(\mathscr{L}(s)-z_c(s)I). \tag{5.15}$$

The hypothesis (5.13) or (5.14) implies that the simple eigenvalue $z_c(0)$ continues to be simple in a small neighborhood S (which is true as may be seen by the implicit function theorem) and that the distances from $z_c(s)$ to the other eigenvalues $z_i(s)$ are strictly positive uniformly in S.

Finally, we assume that all the eigenvalues of $z_i(s)$, except the critical eigenvalue $z_c(s)$, are bounded below uniformly in S. Namely,

$$(K)_{ix} \quad \|(\mathscr{L}(s)\,|\,\mathscr{R}_c(s))^{-1}\|_{V\to V}\leq\rho'_c<+\infty, \qquad \text{for all } s\in S. \quad (5.16)$$

Before proceeding, we note here that both $z_c(s)$ and $\phi_c(s)$ are of C^1 class in $s\in S$ (this fact has been implicitly used in the above setting), and that

$$\left.\begin{array}{ll} |z_c(s)-z_c(0)|\ \leq C|s|, & s\in S, \\ \|\phi_c(s)-\phi_c(0)\|_V\leq C'|s|, & s\in S. \end{array}\right\} \quad (5.17)$$

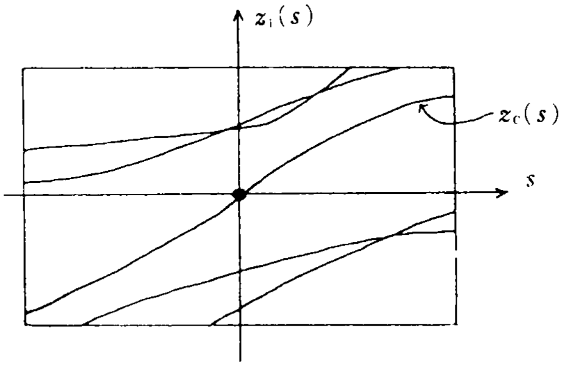
This is a consequence of the implicit function theorem as applied to the pair $(y_c(s),\ \psi_c(s))\in \boldsymbol{R}\times\mathscr{R}_c(0)$, where $z_c(s)=z_c(0)+y_c(s)$, $\phi_c(s)=\phi_c(0)+\psi_c(s)$ with the normalization $<\psi_c(s),\ \phi_c(0)>=0$. (Here, $\phi_c(0)\equiv\phi_c$.)

We consider (E) under the situation which may correspond to the non-degenerate behavior of critical eigenvalues $z_c(s)$. We assume

$$(K)'_x \quad z_c(0)=0 \text{ and } \frac{dz_c}{ds}(0)\neq 0 \qquad (5.18)$$

Taking again a subinterval $S'\subset S$, if necessary, we may assume that

$$(K)_x \quad z_c(0)=0 \text{ and } \left|\frac{dz_c}{ds}(s)\right|\geq d_c>0, \qquad s\in S. \qquad (5.19)$$



Our main theorem states as follows.

Theorem 5.1 *Assume $(K)_i-(K)_x$. Then, there exists a uniqus $s=s(h)$ for each $h\in]0,h_0[$ ($^\exists h_0$: small), such that*

$$\mathscr{L}^h(s(h))\phi_c^h(s(h))=0, \tag{5.20}$$

$$\dim\ker \mathscr{L}^h(s(h))=1, \tag{5.21}$$

$$\frac{d}{ds}z_c^h|_{s=s(h)}\neq 0, \tag{5.22}$$

and

$$|s(h)|\leq Ch^{l+k}. \tag{5.23}$$

Moreover,

$$\|P^h\phi_c(0)-\phi_c^h(s(h))\|_V\leq Ch^{l+k}, \tag{5.24}$$

$$\|\phi_c(0)-\phi_c^h(s(h))\|_V\leq C'h^l, \tag{5.25}$$

and

$$\|\phi_c(0)-\phi_c^h(s(h))\|_U\leq C''h^{l+k}. \tag{5.26}$$

The operator $\mathscr{L}^h(s)|\mathscr{R}_c^h(s)$ *is equi-invertible uniformly in* $s\in S$:

$$\|(\mathscr{L}^h(s)|\mathscr{R}_c^h(s))^{-1}\|_{V\to V}\leq 2\rho_c'<+\infty, \tag{5.27}$$

where

$$\begin{aligned}\mathscr{R}_c^h(s)&=[\ker(\mathscr{L}^h(s)-z_c^h(s)I)]^\perp\\&=\mathrm{range}(\mathscr{L}^h(s)-z_c^h(s)I).\end{aligned} \tag{5.28}$$

Thus, $s(h)$ *is the only critical point of* $\mathscr{L}^h(s)$ *in the interval* $s\in S$.

5.2 Preliminaries. Approximation of eigenproblems for compact operators

We rewrite (E) and (E^h) as:

$$(\tilde{E})\quad \mathscr{K}(s)\phi(s)=\zeta(s)\phi(s),\qquad s\in S, \tag{5.29}$$

and

$$(\tilde{E}^h)\quad P^h\mathscr{K}^{(h)}(s)\phi^h(s)=\zeta^h(s)\phi^h(s),\qquad s\in S. \tag{5.30}$$

Obviously, $\zeta(s)=z(s)-1$ and $\zeta^h(s)=z^h(s)-1$, $s\in S$. Our first task is to show that any simple elgenpair $(\zeta(s),\ \phi(s))\in\boldsymbol{R}\times W$ of $(\tilde{E})$ is *close* to $(\zeta^h(s),\ \phi^h(s))\in\boldsymbol{R}\times V^h$ of $(\tilde{E}^h)$ for each *fixed* $s\in S$ in an appropriate sense (Lemma 5.4).

We begin by considering an intermediate problem (P^hE) which is defined by:

$$(P^hE)\quad P^h\mathscr{K}(s)\psi^h(s)=\xi^h(s)\psi^h(s),\qquad s\in S. \tag{5.31}$$

This is a projection approximation of the eigenproblem (E) for the compact operator $\mathscr{K}(s)$, and we may apply the classical result due Vainikko [28], Chapter 4, § 18 (pp. 269-291).

Lemma 5.2 *Suppose* $(\zeta(s),\ \phi(s))\in\boldsymbol{R}\times W$ *be a simple eigenvalue and eigenvector of* (E) *for a fixed* $s\in S$.

Then, there is a simple eigenvalue and corresponding eigenfuction $(\xi^h(s),\ \psi^h(s))\in\boldsymbol{R}\times V^h$, *such that*

$$|\zeta-\xi^h|\leq Ch^{l+k}\|\phi\|_W, \tag{5.32}$$

and

$$\|P^h\phi-\psi^h\|_V\leq Ch^{l+k}\|\phi\|_W. \tag{5.33}$$

Also,

$$\|\phi-\psi^h\|_V\leq Ch^l\|\phi\|_W, \tag{5.34}$$

and

$$\|\phi-\psi^h\|_U\leq Ch^{l+k}\|\phi\|_W.$$

(*Here, we omitted the parameter s.*)

A proof of the first and third inequalities are due to Vainikko [28], Theorem 18.4 and to Eq. (3.10) :

$$\|\phi-\psi^h\|_V\leq C\|(I-P^h)\phi\|_V\leq Ch^l\|\phi\|_W,$$

and

$$|\zeta-\xi^h|\leq C\|(I-P^h)\phi\|_V^2\leq Ch^{l+k}\|\phi\|_W.$$

To show the second and last estimates, we note first that

$$P^h\mathscr{K}\phi=\zeta P^h\phi \text{ and } P^h\mathscr{K}\psi^h=\xi^hP^h\psi^h,$$

from which

$$(P^h\mathscr{K}-\xi^hI)(\psi^h-P^h\phi)=P^h\{\mathscr{K}(I-P^h)\phi+(\xi^h-\zeta)P^h\phi\}. \tag{3.35}$$

The right-hand side is orthogonal to ψ^h, and hence belongs to

$$\mathscr{R}^h\overset{def}{=}[\ker(P^h\mathscr{K}-\xi^hI)]^\perp. \tag{5.36}$$

By virtue of Lemma 5.3 below*), $(P^h\mathscr{K}-\xi^hI)$ is equi-invertible on $\mathscr{R}^h$, i.e.,

*) By letting $\mathscr{L}_1=I-\frac{1}{\zeta}\mathscr{K}$ and $\mathscr{L}_1^h=I-\frac{1}{\xi^h}P^h\mathscr{K}$ in Lemma 5.3.

$$\|(P^h\mathscr{K}-\xi^h I)^\dagger\|_{V\to V}\le\rho'<+\infty\ (\rho' : \text{independent of } h), \qquad (5.37)$$

where by $A^\dagger$ we denote the bounded map $V\to V$ such that $A^\dagger A=\omega^h$, here ω^h being the projection of V onto $\mathscr{R}^h$. Accordingly,

$$\begin{aligned}\|\phi^h-P^h\phi\|_V&\le\rho'(\|\mathscr{K}(I-P^h)\phi\|_V+|\xi^h-\zeta|\ \|P^h\phi\|_V)\\&\le\rho'(\|\mathscr{K}\|_{U\to V}\|(I-P^h)\phi\|_U+|\xi^h-\zeta|\ \|\phi\|_V)\\&\le C\rho' h^{l+k}\|\phi\|_W,\end{aligned}$$

which is nothing but Eq. (5.33). Eq. (5.34) follows immediately from Eqs. (3.10) and (5.33).

Lemma 5.3 *Let $\tilde{\mathscr{L}}_1=I-\tilde{\mathscr{K}}$ and $\tilde{\mathscr{L}}_1^h=I-\tilde{\mathscr{K}}^h$, where $\tilde{\mathscr{K}}\in \boldsymbol{B}_0(V)$ and $\tilde{\mathscr{K}}^h\in\boldsymbol{B}(V)$ and are assumed to be self-adjoint. We suppose that*

$$\left.\begin{aligned}&(\mathrm{i})\quad \|\tilde{\mathscr{K}}-\tilde{\mathscr{K}}^h\|_{V\to V}\le Ch^l,\\&(\mathrm{ii})\quad \dim\ker\tilde{\mathscr{L}}_1=\dim\ker\tilde{\mathscr{L}}_1^h=1,\\&(\mathrm{iii})\quad \|\phi-\phi^h\|_V\le Ch^l,\end{aligned}\right\}\qquad(5.38)$$

where ϕ and ϕ^h are the normed elements of ker $\tilde{\mathscr{L}}_1$ *and of* ker $\tilde{\mathscr{L}}_1^h$, *respectively. Let $\tilde{\mathscr{R}}=[\ker\tilde{\mathscr{L}}_1]^\perp=\text{range }\tilde{\mathscr{L}}_1$, and $\tilde{\mathscr{R}}^h=[\ker\tilde{\mathscr{L}}_1^h]^\perp=\text{range }\tilde{\mathscr{L}}_1^h$. Let $(\tilde{\mathscr{L}}_1^h)^\dagger$ be the bounded map $V\to V$ such that $(\tilde{\mathscr{L}}_1^h)^\dagger\tilde{\mathscr{L}}_1^h=\omega^h$, where ω^h is the projection onto $\tilde{\mathscr{R}}^h$. Then,*

$$\|(\tilde{\mathscr{L}}_1^h)^\dagger\|_{V\to V}\le\rho'<+\infty\ (\rho' : \textit{independent of } h). \qquad (5.39)$$

Proof. Let Π and Π^h be the projections $V\to\ker\tilde{\mathscr{L}}_1$ and $V\to\ker\tilde{\mathscr{L}}_1^h$, respectiviely, and let $\omega=I-\Pi$ and $\omega^h=I-\Pi^h$. Since $\|\Pi-\Pi^h\|_{V\to V}\le Ch^l$ by Eq. $(5.38)_3$, $\|\omega-\omega^h\|_{V\to V}\le Ch^l$. We take an arbitrary $\chi\in\tilde{\mathscr{R}}^h$, and let $u\in\tilde{\mathscr{R}}^h$ be the unique solution of $\tilde{\mathscr{L}}_1^h u=\chi$. We shall first show that $\|u\|_V\le\rho'\|\chi\|_V$. Noting that $\tilde{\mathscr{L}}_1^\dagger\tilde{\mathscr{L}}_1=\omega$, where $\tilde{\mathscr{L}}_1^\dagger$ is the boudnded map $V\to V$ with $\|\tilde{\mathscr{L}}_1^\dagger\|_{V\to V}\le\rho<+\infty$, the identity $\chi=\tilde{\mathscr{L}}_1 u+(\tilde{\mathscr{K}}^h-\tilde{\mathscr{K}})u$ yields the relation

$$\omega u=\tilde{\mathscr{L}}_1^\dagger\{\chi+(\tilde{\mathscr{K}}-\tilde{\mathscr{K}}^h)u\},$$

from which follows

$$\begin{aligned}\|\omega u\|_V&\le\rho\{\|\chi\|_V+\|\tilde{\mathscr{K}}-\tilde{\mathscr{K}}^h\|_{V\to V}\|u\|_V\}\\&\le\rho\|\chi\|_V+Ch^l\|u\|_V.\end{aligned}$$

Next, since

$$\begin{aligned}u&=\omega u+\Pi u\\&=\omega u+\Pi\omega^h u\\&=\omega u+\Pi(\omega^h-\omega)u,\end{aligned}$$

$$\|u\|_V \leq \|\omega u\|_V + \|\Pi\|_{V\to V}\|\omega^h - \omega\|_{V\to V}\|u\|_V$$
$$\leq \|\omega u\|_V + C'h^l\|u\|_V.$$

Thus, combining the above two inequalities, we have

$$\|u\|_V \leq \frac{1}{1-(C+C')h^l}\rho\|\chi\|_V \leq \rho'\|\chi\|_V, \tag{5.40}$$

for all $h\in]0, h_0[$ (h_0: sufficiently small). Thus, we conclude that $(\tilde{\mathscr{L}}_1^h|\tilde{\mathscr{R}}^h)^{-1}$ is bounded by ρ', and consequently $(\tilde{\mathscr{L}}_1^h)^\dagger = (\tilde{\mathscr{L}}_1^h|\tilde{\mathscr{R}}^h)^{-1}\omega^h$ is bounded by the same constant.

The next lemma concerns the approximation theory for (E^h).

Lemma 5.4 *Under the hypotheses* $(K)_{\rm i}-(K)_{\rm viii}$, *we suppose that* $(\zeta(s), \phi(s))\in \boldsymbol{R}\times W$ *be a family of simple eigenpairs of* $(\tilde{E})$ *for* $s\in S$.*) *Then, there is a family of simple eigenpairs of* $(\tilde{E}^h)$: $(\zeta^h(s), \phi^h(s))\in \boldsymbol{R}\times V^h$ *such that*

$$|\zeta(s)-\zeta^h(s)|\leq C(s)h^{l+k}, \tag{5.41}$$

and

$$\|P^h\phi(s)-\phi^h(s)\|_V \leq C(s)h^{l+k}. \tag{5.42}$$

Also,

$$\|\phi(s)-\phi^h(s)\|_V \leq C(s)h^l, \tag{5.43}$$

and

$$\|\phi(s)-\phi^h(s)\|_U \leq C(s)h^{l+k}. \tag{5.44}$$

Proof. Whenever no confusion arises, we shall drop the parameter s for notational simplicity. We remind first that $(\tilde{E}^h)$ can be regarded as a perturbed elgenproblem in V:

$$(\tilde{E}^h) \quad P^h\mathscr{K}\phi^h + P^h\Sigma^{(h)}\phi^h = \zeta^h\phi^h. \tag{5.45}$$

We let

$$\sigma^{(h)} \equiv \frac{\Sigma^{(h)}}{\|\Sigma^{(h)}\|_{V\to V}}, \quad \|\sigma^{(h)}\|_{V\to V} = 1, \tag{5.46}$$

and consider the following ε-family of elgenproblems $(0\leq\varepsilon<\varepsilon_0)$:

*) $\|(\mathscr{K}(s)-\zeta(s)I)^\dagger\|\leq\rho<+\infty$, $s\in S$.

$$(\tilde{E}^h)_\varepsilon \quad P^h\mathscr{K}\phi^h(\varepsilon)+\varepsilon P^h\sigma^{(h)}\phi^h(\varepsilon)=\zeta^h(\varepsilon)\phi^h(\varepsilon). \tag{5.47}$$

Notice that $(\tilde{E}^h)_\varepsilon$ reduces to $(P^h\tilde{E})$ when $\varepsilon=0$, and to $(\tilde{E}^h)$ when $\varepsilon=||\Sigma^{(h)}||_{V\to V}$. Thus, we may assume

$$\zeta^h(0)=\xi^h,\ \phi^h(0)=\psi^h \tag{5.48}$$

with

$$\dim\ker(\zeta^h(0)I-P^h\mathscr{K})=1,$$

and seek ε-neighborhood solutions. Letting in $(\tilde{E}^h)_\varepsilon$

$$\phi^h(\varepsilon)=\psi^h+\chi^h(\varepsilon) \tag{5.49}$$

and

$$\zeta^h(\varepsilon)=\xi^h+\eta^h(\varepsilon), \tag{5.50}$$

with the normalization condition $<\phi^h(\varepsilon),\ \psi^h>=1$, namely, $\chi^h(\varepsilon)\in\mathscr{R}^h=\ker(P^h\mathscr{K}-\xi^hI)^\perp$ and applying the Lyapounov-Schmidt decomposition, we have that

$$(P^h\mathscr{K}-\xi^hI)\chi^h+\varepsilon\omega^hP^h\sigma^{(h)}(\psi^h+\chi^h)-\eta^h\chi^h=0 \tag{5.51}$$

and

$$\eta^h-\varepsilon<P^h\sigma^{(h)}(\psi^h+\chi^h),\ \psi^h>=0. \tag{5.52}$$

Again, as in the proof of Lemma 5.3, $P^h\mathscr{K}-\xi^hI$ is equi-invertible on $\mathscr{R}^h$ for $h<h_1(h_1$: small), which with the aid of the equi-implicit function theorem yields the unique existence of small solutions $(\eta^h(\varepsilon),\ \chi^h(\varepsilon))\in\boldsymbol{R}\times V^h$ for $|\varepsilon|<\varepsilon_1(\varepsilon_1$: independent of $h)$. We have that for $0<h<h_1$,

$$|\zeta^h(\varepsilon)-\xi^h|\leq C|\varepsilon|, \tag{5.53}$$

$$||\phi^h(\varepsilon)-\psi^h||_V\leq C|\varepsilon|. \tag{5.54}$$

Let $h_0=\min(h_1,\ h_2)$, where $h_2>0$ is such that $||\Sigma^{(h_2)}||_{V\to V}\leq Ch_2^l<\varepsilon_1$. Then, for any $h<h_0$, we obtain the existence of $\phi^h\equiv\phi^h(\varepsilon(h))$, $\zeta^h\equiv\zeta^h(\varepsilon(h))$, where $\varepsilon(h)=||\Sigma^{(h)}||_{V\to V}$ such that

$$P^h\mathscr{K}^{(h)}\phi^h=\zeta^h\phi^h \tag{5.55}$$

and,

$$|\zeta^h-\xi^h|\leq C\varepsilon(h)\leq Ch^l, \tag{5.56}$$

$$||\phi^h-\psi^h||_V\leq C\varepsilon(h)\leq Ch^l. \tag{5.57}$$

It can be shown, however, that it is actually with $0(h^{l+k})$ that $|\zeta^h-\xi^h|$ and $||\phi^h-\psi^h||_V$ tend to zero, thanks to the assumed regularity of $\mathscr{K}$, and $O(h^{l+k})$ convergence of $||\Sigma^{(h)}||_{W\to V}$. In fact, taking the inner product with ψ^h in (E^h), and with ϕ^h in (P^hE), and subtracting one from the other, we have that

$$\zeta^h-\xi^h=<\phi^h,\ \psi^h>^{-1}<(\mathscr{K}^{(h)}-\mathscr{K})\phi^h,\ \psi^h> \tag{5.58}$$
$$=<\phi^h,\psi^h>^{-1}\{<(\mathscr{K}^{(h)}-\mathscr{K})\phi,\psi^h>+<(\mathscr{K}^{(h)}-\mathscr{K})(\phi^h-\phi),\psi^h>\}.$$

Here, the self-adjointness of $\mathscr{K}$ and $\mathscr{K}^{(h)}$ is used; ϕ is the eigenfunction of $(\tilde{E})$. Notice that since $||\phi^h-\psi^h||_V\to 0$ $(h\to 0)$, we may assume $|<\phi^h,\psi^h>|\geq\frac{1}{2}$ (for h: small). Hence,

$$\begin{aligned}|\zeta^h-\xi^h|&\leq C||\mathscr{K}^{(h)}-\mathscr{K}||_{W\to V}||\phi||_W||\psi^h||_V\\&+C||\mathscr{K}^h-\mathscr{K}||_{V\to V}||\phi^h-\phi||_V||\psi^h||_V\\&\leq C(h^{l+k}+h^{2l}),\\&\leq Ch^{l+k}\end{aligned} \tag{5.59}$$

Here, $||\phi^h-\phi||_V\leq||\phi_h-\psi_h||_V+||\psi_h-\phi||_V\leq Ch^l$ is used.

Next, since

$$\begin{aligned}&(P^h\mathscr{K}-\xi^hI)(\psi^h-\phi^h)\\&=P^h(\mathscr{K}^{(h)}-\mathscr{K})\phi^h-(\xi^h-\zeta^h)\phi^h\\&=P^h(\mathscr{K}^{(h)}-\mathscr{K})\phi+P^h(\mathscr{K}^{(h)}-\mathscr{K})(\phi^h-\phi)-(\zeta^h-\xi^h)\phi^h,\end{aligned}$$

it holds that

$$\begin{aligned}||\psi^h-\phi^h||_V&\leq C\{||\mathscr{K}^{(h)}-\mathscr{K}||_{W\to V}||\phi||_W\\&+||\mathscr{K}^{(h)}-\mathscr{K}||_{V\to V}||\phi^h-\phi||_V+|\zeta^h-\xi^h|||\phi^h||_V\}\\&\leq Ch^{l+k}.\end{aligned} \tag{5.60}$$

Thus, summing up the above estimates and Lemma 5.2, we have the final results:

$$|\zeta-\zeta^h|\leq|\zeta-\xi^h|+|\xi^h-\zeta^h|\leq Ch^{l+k}, \tag{5.61}$$

$$\begin{aligned}||P^h\phi-\phi^h||_V&\leq||P^h\phi-\psi^h||_V+||\psi^h-\phi^h||_V\\&\leq Ch^{l+k}||\phi||_W.\end{aligned} \tag{5.62}$$

Eqs. (5.43) and (5.44) are a direct consequence of the estimate (5.62) and the approximation property (3.10), (3.11).

Corollary 5.5 *Under the same assumptions as in Lemma 5.4, it holds that*

$$\|\tilde{\mathscr{L}}(s)^{\dagger}-\tilde{\mathscr{L}}^{h}(s)^{\dagger}\|_{V\to V}\leq C(s)h^{l}, \qquad s\in S, \tag{5.63}$$

where

$$\tilde{\mathscr{L}}(s)\equiv\mathscr{K}(s)-\zeta(s)I,$$

and

$$\tilde{\mathscr{L}}^{h}(s)\equiv P^{h}\mathscr{K}^{(h)}(s)-\zeta^{h}(s)I.$$

Proof. Recall that $\tilde{\mathscr{L}}(s)^{\dagger}$ and $\tilde{\mathscr{L}}^{h}(s)^{\dagger}$ are given by

$$\tilde{\mathscr{L}}(s)^{\dagger}=(\tilde{\mathscr{L}}(s)\,|\,\mathscr{R}(s))^{-1}\omega(s),$$

and

$$\tilde{\mathscr{L}}^{h}(s)^{\dagger}=(\tilde{\mathscr{L}}^{h}(s)\,|\,\mathscr{R}^{h}(s))^{-1}\omega^{h}(s),$$

where $\mathscr{R}(s)=$range $\tilde{\mathscr{L}}(s)$ and $\mathscr{R}^{h}(s)=$range $\tilde{\mathscr{L}}^{h}(s)$; $\omega(s)$ and $\omega^{h}(s)$ denote the projections of V onto $\mathscr{R}(s)$ and $\mathscr{R}^{h}(s)$, respectively. Lemma 5.3 shows that

$$\|\tilde{\mathscr{L}}^{h}(s)^{\dagger}\|_{V\to V}\leq\rho'<+\infty, \qquad s\in S \tag{5.64}$$

for all $h\in]0, h_0[$. Next, Eq. (5.43) shows that

$$\|\Pi(s)-\Pi^{h}(s)\|_{V\to V}\leq C_0(s)h^{l}\leq Ch^{l}, \qquad s\in S, \tag{5.65}$$

and consequently,

$$\|\omega(s)-\omega^{h}(s)\|_{V\to V}\leq C_0'(s)h^{l}\leq C'h^{l}, \qquad s\in S,$$

where $\Pi(s)=I-\omega(s)$ and $\Pi^{h}(s)=I-\omega^{h}(s)$.

However, we see that

$$\begin{aligned}\tilde{\mathscr{L}}(s)^{\dagger}-\tilde{\mathscr{L}}^{h}(s)^{\dagger}&=(\tilde{\mathscr{L}}(s)\,|\,\mathscr{R}(s))^{-1}\omega(s)-(\tilde{\mathscr{L}}^{h}(s)\,|\,\mathscr{R}^{h}(s))^{-1}\omega^{h}(s)\\&=\{(\tilde{\mathscr{L}}(s)\,|\,\mathscr{R}(s))^{-1}-(\tilde{\mathscr{L}}^{h}(s)\,|\,\mathscr{R}^{h}(s))^{-1}\}\,\omega(s)\omega^{h}(s)\\&\quad+(\tilde{\mathscr{L}}(s)\,|\,\mathscr{R}(s))^{-1}\omega(s)\,(\Pi^{h}(s)-\Pi(s))\\&\quad-(\tilde{\mathscr{L}}^{h}(s)\,|\,\mathscr{R}^{h}(s))^{-1}(\Pi(s)-\Pi^{h}(s))\,\omega^{h}(s),\end{aligned} \tag{5.66}$$

where the orthogonality $\omega(s)\Pi(s)=0=\Pi^{h}(s)\omega^{h}(s)$ has been used. Hence, in view of Eqs. (5.64) and (5.65), the second and the third terms are bounded by Ch^{l}, for $s\in S$. Since the first term can be rewritten as

$$(\tilde{\mathscr{L}}^{h}(s)\,|\,\mathscr{R}^{h}(s))^{-1}\{\tilde{\mathscr{L}}^{h}(s)-\tilde{\mathscr{L}}(s)\}\,(\tilde{\mathscr{L}}(s)\,|\,\mathscr{R}(s))^{-1}\omega(s)\omega^{h}(s),$$

this is again bounded by Ch^{l}, $s\in S$, taking notice of Eq. (5.64) and that

$$\begin{aligned}&\|\tilde{\mathscr{L}}^h(s)-\tilde{\mathscr{L}}(s)\|_{V\to V}\\&\quad\le\|\mathscr{K}(s)-P^h\mathscr{K}^{(h)}(s)\|_{V\to V}+|\zeta(s)-\zeta^h(s)|\\&\quad\le\|P^h(\mathscr{K}(s)-\mathscr{K}^{(h)}(s))\|_{V\to V}+\|(I-P^h)\mathscr{K}(s)\|_{V\to V}+|\zeta(s)-\zeta^h(s)|,\\&\quad\le C(s)h^l,\end{aligned}$$

which follows from Eqs. (5.8), (5.41), (5.3) and (3.10)*).

We have thus established that for all $s\in S$

$$|\zeta(s)-\zeta^h(s)|\le C(s)h^{l+k}\le C_0h^{l+k},\qquad s\in S, \tag{5.67}$$

where $C_0=\sup C(s)$, is a constant independent of $s\in S$ and of $h\in]0,h_0[$. Our next task is to show that $\left|\frac{d}{ds}(\zeta(s)-\zeta^h(s))\right|$ also tends to zero as $h\to0$.

Lemma 5.6 *Under the hypotheses* $(K)_{\text{i}}-(K)_{\text{x}}$, *we have that*

$$\left|\frac{d}{ds}(\zeta(s)-\zeta^h(s))\right|\le Ch^l\qquad s\in S. \tag{5.68}$$

Proof. Since

$$\zeta(s)=\langle\dot{\mathscr{K}}(s)\phi(s),\ \phi(s)\rangle, \tag{5.69}$$

and

$$\begin{aligned}\dot{\zeta}^h(s)&=\langle P^h\dot{\mathscr{K}}^{(h)}(s)\phi^h(s),\ \phi^h(s)\rangle\\&=\langle\dot{\mathscr{K}}^{(h)}(s)\phi^h(s),\ \phi^h(s)\rangle,\end{aligned} \tag{5.70}$$

$$\begin{aligned}\dot{\zeta}(s)-\dot{\zeta}^h(s)=&\langle\dot{\mathscr{K}}(s)(\phi(s)-\phi^h(s)),\ \phi^h(s)\rangle\\&+\langle\dot{\mathscr{K}}(s)\phi(s),\ \phi(s)-\phi^h(s)\rangle\\&-\langle\dot{\Sigma}^{(h)}(s)\phi^h(s),\ \phi^h(s)\rangle.\end{aligned} \tag{5.71}$$

In view of Eqs. (5.7), (5.43) and (5.10), $|\zeta(s)-\zeta^h(s)|$ is bounded by a constant $C_1(s)$, for each $s\in S$. Letting $C_1=\sup\ C_1(s)$, we have Eq. (5.68).

5.3 Proof of the main theorem

In this subsection, we finish the proof of the main theorem 5.1.

To sum up the situation, we have a C^1 function $z_c(s):S\to\boldsymbol{R}$ and a sequence of C^1 functions $z_c^h(s):S\to\boldsymbol{R}$, such that from Eq. (5.20)

---

*) $\|(I-P_h)\mathscr{K}(s)w\|_V\le C(s)h^l\|\mathscr{K}(s)w\|_W$
$\le C(s)h^l\|\mathscr{K}(s)\|_{V\to W}\|w\|_V$.

$$z_c(0)=0,\ \frac{dz_c}{ds}(s)\geq d_c>0 \tag{5.77}$$

(for simplicity), and that from Eq. (5.41)

$$|z_c(s)-z_c^h(s)|\leq C_0h^{k+l}, \qquad s\in S \tag{5.78}$$

and from Eq. (5.68)

$$\left|\frac{dz_c}{ds}(s)-\frac{dz_c^h}{ds}(s)\right|\leq C_1h^l, \qquad s\in S. \tag{5.79}$$

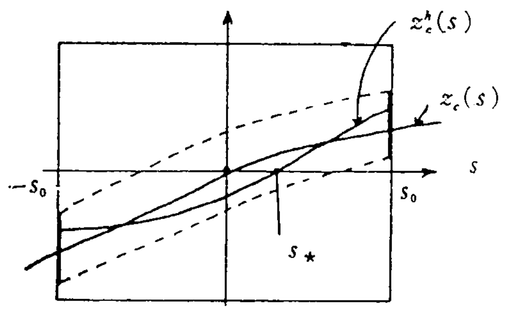
First, we take h_0 so small that $z_c^h(s)$ is strictly monotone in S, i. e.,

$$\frac{dz_c^h}{ds}(s)\geq\frac{d_c}{2}>0 \qquad \text{for } {}^{\forall}h<h_0, \tag{5.80}$$

which is always possible thanks to Eqs. (5.77) and (5.79).

Next, take h_1 so small that at the both ends of S, i. e., at $s=\pm s_0$

$$z_c^h(s_0)>0 \text{ and } z_0^h(-s_0)<0, \tag{5.81}$$



which is possible due to the relations (5.77) and (5.78). In fact, since $z_c(s_0)\geq d_cs_0$,

$$z_c^h(s_0)\geq z_c(s_0)-C_0h^{l+k}\geq d_cs_0-C_0h^{l+k}>0$$

if $h_1<d_cs_0/C_0$.

Then, if $h<\min(h_0, h_1)$, $z_c^h(s)$ has a unique zero $s_*\in S$ due to the monotonicity and to the relation (5.81). It remains to show that $|s_*|\leq Ch^{k+l}$. However, at $s=s_*$, from Eqs. (5.78), (5.80) we have that

$$C_0h^{k+l}\geq|z_c^h(s_*)|\geq d_c|s_*|. \tag{5.82}$$

Hence, we have completed the first part of the proof.

The estimate (5. 24) follows in view of Eqs. (5. 24) and (5. 42) as

$$\begin{aligned}&\|P^h\phi_c(0)-\phi_c^h(s(h))\|_V\\&\leq\|P^h(\phi_c(0)-\phi_c(s(h)))\|_V+\|P^h\phi_c(s(h))-\phi_c^h(s(h))\|_V\\&\leq C|s(h)|+C'h^{l+k}.\end{aligned}\tag{5.83}$$

Thus, Eq. (5. 82) shows the desired result (5. 24).

The last statement of the theorem can be shown as follows. Firstly, we devide the interval S into two subsets S_1 and S_2 in such a way that $|z_c(s)|\rho'<\frac{1}{2}$ for $s\in S_1$, where ρ' is the constant in Eq. (5. 16), and that $S_2=S-S_1$. Noticing the monotonicity of $z_c(s)$, this division is always possible and S_1 is an open interval including $s=0$. Also, the monotonicity of $z_c(s)$ guarantees that in S_2, $(\mathscr{L}(s))^{-1}$ exists and is uniformly bounded. Thus, for sufficiently small h, $\mathscr{L}^h(s)$ is uniformly (in $s\in S_2$) equi-invertible. See, Lemma 4. 3. Accordingly, it is enough to show Eq. (5. 27) for $s\in S_1$.

Suppose $u(s)\in\mathscr{R}(s)$ and $u^h(s)\in\mathscr{R}^h(s)$ be the solutions of

$$\mathscr{L}(s)u(s)=\omega(s)f,\qquad s\in S_1,\tag{5.83}$$

and

$$\mathscr{L}^h(s)u^h(s)=\omega^h(s)f,\qquad s\in S_1,\tag{5.84}$$

for $f\in V$, respectively. It is easily seen that the above equations are equivalent, respectively, to

$$(I+z_h(s)\tilde{\mathscr{L}}(s)^\dagger)u=\tilde{\mathscr{L}}(s)^\dagger f\tag{5.85}$$

and

$$(I+z_c^h(s)\tilde{\mathscr{L}}(s)^\dagger)u^h=\tilde{\mathscr{L}}^h(s)^\dagger f\tag{5.86}$$

where we have put

$$\tilde{\mathscr{L}}(s)=\mathscr{L}(s)-z_c(s)I,$$

and

$$\tilde{\mathscr{L}}^h(s)=\mathscr{L}^h(s)-z_c^h(s)I.$$

If we let

$$g(s)=\tilde{\mathscr{L}}(s)^\dagger f \text{ and } g^h(s)=\tilde{\mathscr{L}}^h(s)f,$$

we have by virtue of Lemma 5. 5 that for all $s\in S$

$$\|g(s)-g^h(s)\|_V\leq C(s)h^l\|f\|_V\leq Ch^l\|f\|_V.\tag{5.87}$$

Lemma 5.5 together with Eq. (5.78) shows that for $s\in S$,

$$\|(I+z_c(s)\tilde{\mathscr{L}}(s)') - (I+z_c^h(s)\tilde{\mathscr{L}}(s)')\|_{V\to V}\le C(s)h^l\le Ch^l. \tag{5.88}$$

Since $(I+z_c(s)\tilde{\mathscr{L}}(s)')$ is uniformly invertible for $s\in S_1$, $(I+z_c^h(s)\tilde{\mathscr{L}}^h(s)')$ is equi-invertible for sufficiently small h, uniformly in $s\in S_1$. Hence the solution u^h of Eq. (5.86) satisfies

$$\|u^h\|_V\le C'\|g^h(s)\|_V\le C''\|f\|_V$$

in view of the relation (5.87). Since f is an arbitrary element of V, the inequality (5.27) follows from the above inequality.

6. Numerical realization of snap points and neighboring paths

Suppose $(\mathscr{C}\ ;\ \phi_c)\equiv(\mu_c,\ w_c\ ;\ \phi_c)\in\boldsymbol{R}\times W\times W$ is a simple, non-degenerate snap point of (P). We shall show in this section that $(\mathscr{C}\ ;\ \phi_c)$ is realized uniquely in $\boldsymbol{R}\times V^h\times V^h$ space as $(\mathscr{C}^h\ ;\ \phi_c^h)$, a simple, non-degenerate snap point of (P^h). The error in the numerical snap-buckling load μ_c^h is $O(h^{l+k})$, and those in the numerical buckling mode ϕ_c^h and buckling state w_c^h will also be obtained with respect to both U and V norms. In the final subsection, we shall also show that the path of (P) in the vicinity of $\mathscr{C}^h$ converges *uniformly* to the path of (P) near $\mathscr{C}$ with the same order as in the numerical buckling mode.

6.1 Critical points of $\mathscr{L}^h_{P^h w(s)}$

Let $(\mathscr{C}\ ;\ \phi_c)\equiv(\mu_c,\ w_c\ ;\ \phi_c)\in\boldsymbol{R}\times W\times W$ be a simple, non-degenerate snap point of (P). By Proposition 1.8 (Chapter I, § 1), we may assume that there is a local parameter $s\in S=\{s\ ;\ |s|<s_0\}$ with which the neighborhood solution of (P) can be expressed as

$$(\mu(s),\ w(s)\ ;\ \phi_c(s))\ :\ s\to\boldsymbol{R}\times W\times W,$$

such that

$$F(\mu(s),\ w(s))=0,\qquad s\in S, \tag{6.1}$$

and

$$\mathscr{L}_{w(s)}(s)\phi_c(s)=\zeta_c(s)\phi_c(s),\qquad s\in S. \tag{6.2}$$

Moreover, they satisfy at $s=0$:

$$\mu(0)=\mu_c,\ \ w(0)=w_c,\ \ \phi_c(0)=\phi_c, \tag{6.3}$$

$$\zeta_c(0)=0,\ \frac{d\zeta_c}{ds}(0)\neq 0, \tag{6.4}$$

and

$$\dim\ker \mathscr{L}_{w(0)}=1.$$

Here, $\mathscr{L}_{w(s)}$ is the Fréchet derivative of F (with respect to w) at $(\mu(s), w(s))$:

$$\mathscr{L}_{w(s)}=I+\mathscr{K}_{w(s)}, \tag{6.5}$$

where

$$\mathscr{K}_{w(s)}=L+\mathscr{B}(w(s),\ \cdot)+\frac{1}{2}\mathscr{T}(w(s),\ w(s),\ \cdot). \tag{6.6}$$

(See, Eq. (4.5) of Definition 4.3.) Since L, $\mathscr{B}(w,\ \cdot)$ and $\mathscr{T}(w,\ w,\ \cdot)\in \boldsymbol{B}(V,\ W)$ for any $w\in W$, we have that $\mathscr{K}_{w(s)}\in \boldsymbol{B}(V,\ W)$. Hence, any eigenfunction ϕ of $\mathscr{L}_{w(s)}$ belongs to W.

Let $\mathscr{L}^h_{P^h w(s)}$ be the Fréchet derivative of F^h with respect to w at $w^h=P^h w(s)$, that is,

$$\mathscr{L}^h_{P^h w(s)}=I+P^h\mathscr{K}^{(h)}_{P^h w(s)} \tag{6.7}$$

where

$$K^{(h)}_{P^h w(s)}=L+\mathscr{B}^{(h)}(P^h w(s),\ \cdot)+\frac{1}{2}\mathscr{T}^{(h)}(P^h w(s),\ P^h w(s),\ \cdot) \tag{6.8}$$

(*see*, Eq. (4.6)).

We consider the eigenproblem for the family of operators $\mathscr{L}^h_{P^h w(s)}$:

$$\mathscr{L}^h_{P^h w(s)}\phi^h(s)=z^h(s)\phi^h(s),\qquad s\in S. \tag{6.9}$$

Lemma 6.1 *There exists a unique $s=s(h)\in S$, and a family of eigenvalues and eigenfunctions $(z^h_c(s),\ \phi^h_c(s))\in \boldsymbol{R}\times V^h(s\in S)$, such that for each $h\in]0,\ h_0[$ ($^{\exists}h_0$: small)*

$$\mathscr{L}^h_{P^h w(s(h))}\phi^h_c(s(h))=0,$$
$$\dim\ker\ \mathscr{L}^h_{P^h w(s(h))}=1,$$
$$\frac{d}{ds}z^h_c|_{s=s(h)}\neq 0, \tag{6.10}$$

and

$$|s(h)|\leq Ch^{l+k}. \tag{6.11}$$

Also, $\psi_c^h(s(h))$ *is close to* ϕ_c *in the sense that*

$$\|P^h\phi_c-\psi_c^h(s(h))\|_V\leq Ch^{l+k}, \tag{6.12}$$

$$\|\phi_c-\psi_c^h(s(h))\|_V\leq C'h^l, \tag{6.13}$$

and

$$\|\phi_c-\psi_c^h(s(h))\|_U\leq C''h^{l+k}. \tag{6.14}$$

Moreover, if we let

$$\mathscr{R}_s^h=\text{range } \mathscr{L}^h_{P^hw(s(h))}=[\ker \mathscr{L}^h_{P^hw(s(h))}]^{\perp}, \tag{6.15}$$

$\mathscr{L}^h_{P^hw(s(h))}$ *is equi-invertible on* $\mathscr{R}_s^h$:

$$\|\mathscr{L}^h_{P^hw(s(h))}{}^{\dagger}\|_{V\to V}\leq\rho'<+\infty, \tag{6.16}$$

where ρ' *is a constant independent of* h.

Proof. We need only to apply Theorem 5.1 with $\mathscr{K}(s)\equiv\mathscr{K}_{w(s)}$, and $\mathscr{K}^{(h)}(s)\equiv\mathscr{K}^{(h)}_{P^hw(s)}$. Firstly, with regards to $\mathscr{K}(s)$, the property that $\mathscr{K}(s)\in\boldsymbol{B}(V, W)\cap\boldsymbol{B}(U, V)$ is immediate from the hypotheses on L, $\mathscr{B}$ and $\mathscr{T}$. Since

$$\dot{\mathscr{K}}(s)=\mathscr{B}(\dot{w}(s),\ \cdot)+\mathscr{T}(w(s),\ \dot{w}(s),\ \cdot) \tag{6.17}$$

and $w(s): s\to V$ is a C^∞ function (since F is of C^∞ class. See, Proposition 1.8), $\dot{K}(s)$ obviously belongs to $\boldsymbol{B}(V)$. Secondly, the uniformly boundedness of $\mathscr{K}^{(h)}(s)$ and $\dot{\mathscr{K}}^{(h)}(s)$ is a consequence of equi-continuity of $\mathscr{B}^{(h)}$ and $\mathscr{T}^{(h)}$. Thirdly, let $\Sigma^{(h)}(s)=\mathscr{K}^{(h)}(s)-\mathscr{K}(s)$. Then,

$$\begin{aligned}\|\Sigma^{(h)}(s)\|_{V\to V}&\leq\|\mathscr{B}^{(h)}(P^hw(s),\ \cdot)-\mathscr{B}(w(s),\ \cdot)\|_{V\to V}\\&\quad+\frac{1}{2}\|\mathscr{T}^{(h)}(P^hw(s),\ P^hw(s),\ \cdot)-\mathscr{T}(w(s),\ w(s),\ \cdot)\|_{V\to V}\\&\leq C\|w(s)\|_W h^l\end{aligned} \tag{6.18}$$

from Eqs. (3.16) and (3.20),

$$\|\Sigma^{(h)}(s)\|_{W\to V}\leq C'\|w(s)\|_W h^{l+k} \tag{6.19}$$

from Eqs. (3 15) and (3.19), and

$$\begin{aligned}\|\dot{\Sigma}^{(h)}(s)\|_{V\to V}&\leq\|\mathscr{B}^{(h)}(P^h\dot{w}(s),\ \cdot)-\mathscr{B}(\dot{w}(s),\ \cdot)\|_{V\to V}\\&\quad+\|\mathscr{T}^{(h)}(P^hw(s),\ P^h\dot{w}(s),\ \cdot)-\mathscr{T}(w(s),\ \dot{w}(s),\ \cdot)\|_{V\to V}\\&\leq C''\|w(s)\|_W\|\dot{w}(s)\|_W\ h^l\end{aligned} \tag{6.20}$$

from Eqs. (3.16) and (3.20). In the last inequality, we have used the regularity of $\dot{w}(s)$, namely, that $\dot{w}(s)\in W$ for $s\in S$. This follows by differentiating (P) with respect to s,

$$\dot{w}(s)=-\{L\dot{w}+\mathscr{B}(w,\ \dot{w})+\frac{1}{2}\mathscr{T}(w,\ w,\ \dot{w})+\dot{\mu}Gp\}, \tag{6.21}$$

and observing that the right-hand side belongs to W for $p\in U$, $w\in W$ and $\dot{w}\in V$ in view of Eqs. (3.3), (3.4), (3.6) and (3.8).

6.2 Unique existence of snap point in V^h

We are now at the stage of studying the behavior of solutions of (P^h) : $F^h(\mu,\ w^h)=0$ in the vicinity of a snap point $(\mathscr{C}\ ;\ \phi_c)$ of (P). The goal of this section is to show that near $\mathscr{C}$, there exists uniquely a snap point $(\mathscr{C}^h;\ \phi_c^h)$ of (P^h).

For this goal, we first denote for simplicity that $\psi_s^h\equiv\psi_c^h(s(h))$, $\mu_s\equiv\mu(s(h))$ and $w_s\equiv w(s(h))$. It is to be noted that for each $h\in]0,\ h_0[$, $s=s(h)$ is a fixed constant.

Let Π_s^h be the orthogonal projection $V\rightarrow\ker\mathscr{L}^h_{P^hw_s}=\{\psi_s^h\}$, and $\omega_s^h=I-\Pi_s^h$ which is the projection $V\rightarrow\mathscr{R}_s^h=\text{range}\ \mathscr{L}^h_{P^hw_s}(=[\ker\mathscr{L}^h_{P^hw_s}]^{\perp})$.

In $F^h(\mu,\ w^h)=0$, we put

$$\mu=\mu_s+\nu \tag{6.22}$$

and

$$w^h=P^hw_s+v^h,\qquad v^h\in V^h.$$

We have then

$$(Q^h)\quad G_s^h(\nu,\ v^h,\ F_s^h)$$

$$\equiv\mathscr{L}_s^hv^h+\frac{1}{2!}\mathscr{Q}_s^h(v^h,\ v^h)+\frac{1}{3!}\mathscr{C}_s^h(v^h,\ v^h,\ v^h)+\nu f^h+F_s^h=0, \tag{6.23}$$

where

$$\begin{aligned}
&\mathscr{L}_s^h\equiv F^{h\prime}(\mu_s,\ P^hw_s)=\mathscr{L}^h_{P^hw_s}\\
&\mathscr{Q}_s^h(u,\ v)\equiv F^{h\prime\prime}(\mu_s,\ P^hw_s)(u,\ v)\\
&\qquad=P^h[\mathscr{B}^{(h)}(u,\ v)+2\mathscr{T}^{(h)}(P^hw_s,\ u,\ v)],\\
&\mathscr{C}_s^h(u,\ v,\ w)\equiv F^{h\prime\prime\prime}(\mu_s,\ P^hw_s)(u,\ v,\ w)\\
&\qquad=2P^h\mathscr{T}^{(h)}(u,\ v,\ w),\\
&F_s^h\equiv F^h(\mu_s,\ P^hw_s),
\end{aligned} \tag{6.24}$$

and

$f^h \equiv P^h G p.$

Notice that $\mathscr{L}^h_s \in \boldsymbol{B}(V)$ has a one-dimensional kernel $\{\psi^h_s\}$, and is equi-invertible on R^h_s; $\mathscr{Q}^h_s$ and $\mathscr{C}^h_s$ are equi-continuous, separately self-adjoint bi- and tri-linear mappings, respectively.

We associate to (Q^h) the linearized elgenproblem:

$$(E^h) \quad \Phi^h_s(v^h, \phi^h) \equiv \{\mathscr{L}^h_s + \mathscr{Q}^h_s(v^h, \cdot) + \frac{1}{2}\mathscr{C}^h_s(v^h, v^h \cdot)\}\phi^h = 0 \tag{6.25}$$

We would like to show in the following that for each $h \in]0, h_0[$, there exists a unique triplet $(\nu^h_c, v^h_c; \phi^h_c) \in \boldsymbol{R} \times V^h \times V^h$ which satisfies (Q^h) and (E^h) simultaneously. In other words, $(\nu^h_c, w^h_c) \in \boldsymbol{R} \times V^h$ is a solution, and at the same time a critical point of the problem (Q^h).

We consider, as before, a family of auxiliary problems $(Q^h)_\varepsilon (0 \leq \varepsilon < \varepsilon_0)$ and corresponding elgenproblems $(E^h)_\varepsilon$:

$$(Q^h)_\varepsilon \quad G^h_s(\nu, v^h; \varepsilon) \equiv \mathscr{L}^h_s v^h + \frac{1}{2!}\mathscr{Q}^h_s(v^h, v^h) + \frac{1}{3!}\mathscr{C}^h_s(v^h, v^h, v^h) + \nu f^h + \varepsilon r^h_s = 0, \tag{6.26}$$

where

$$r^h_s \equiv \frac{F^h_s}{\|F^h_s\|_V}. \tag{6.27}$$

$(E^h)_\varepsilon$ is defined formally by $\Phi^h_s(v^h, \phi^h) = 0$, but v^h should be understood as a solution of $(Q^h)_\varepsilon$. We perform the Lyapounov-Schmidt decomposition both to $(Q^h)_\varepsilon$ and $(E^h)_\varepsilon$, that is, we let

$$v^h = \alpha\psi^h_s + \chi^h, \quad \phi^h = \psi^h_s + \eta^h, \tag{6.28}$$

where $\chi^h, \eta^h \in \mathscr{R}^h_s$ (Notice that the normalization $\langle \phi^h, \psi^h_s \rangle = 1$ is assumed in the above.)

Then, we have a system of four equations:

$$\omega^h_s G^h_s = \mathscr{L}^h_s \chi^h + \frac{1}{2!}\omega^h_s \mathscr{Q}^h_s(\alpha\psi^h_s + \chi^h, \alpha\psi^h_s + \chi^h) + \frac{1}{3!}\omega^h_s \mathscr{C}^h_s(\alpha\psi^h_s + \chi^h, \alpha\psi^h_s + \chi^h, \alpha\psi^h_s + \chi^h) + \varepsilon\omega^h_s r^h_s + \nu\omega^h_s f^h = 0 \tag{6.29}$$

$$\Pi^{h\prime}_s G^h_s = \frac{1}{2!}\Pi^{h\prime}_s \mathscr{Q}^h_s(\alpha\psi^h_s + \chi^h, \alpha\psi^h_s + \chi^h)$$

$$+\frac{1}{3!}\Pi_s^{h\prime}\mathscr{C}_s^h(\alpha\psi_s^h+\chi^h,\ \alpha\psi_s^h+\chi^h,\ \alpha\psi_s^h+\chi^h)$$
$$+\varepsilon\Pi_s^{h\prime}r_s^h+\nu\Pi_s^{h\prime}f^h=0 \tag{6.30}$$

$$\omega_s^h\Phi_s^h=\mathscr{L}_s^h\eta^h+\omega_s^h\mathscr{Q}_s^h(\alpha\psi_s^h+\chi^h,\ \psi_s^h+\eta^h)$$
$$+\frac{1}{2!}\omega_s^h\mathscr{C}_s^h(\alpha\psi_s^h+\chi^h,\ \alpha\psi_s^h+\chi^h,\ \psi_s^h+\eta^h)=0 \tag{6.31}$$

$$\Pi_s^{h\prime}\Phi_s^h=\Pi_s^{h\prime}\mathscr{Q}_s^h(\alpha\psi_s^h+\chi^h,\ \psi_s^h+\eta^h)$$
$$+\frac{1}{2!}\Pi_s^{h\prime}\mathscr{C}_s^h(\alpha\psi_s^h+\chi^h,\ \alpha\psi_s^h+\chi^h,\ \psi_s^h+\eta^h)=0, \tag{6.32}$$

where

$$\Pi_s^{h\prime}u\overset{def}{=}\langle u,\ \psi_s^h\rangle. \tag{6.33}$$

Taking notice of the equi-invertibility of $\mathscr{L}_s^h$ on $\mathscr{R}_s^h$: $\|(\mathscr{L}_s^h)^\dagger\|_{V\to V}\leq\rho'<+\infty$, we can apply the equi-implicit function theorem to solve χ^h and η^h in Eqs. (6.29) and (6.31), respectiveiy. Indeed, we have the following

Lemma 6.2 *There exist a constant* $\gamma>0$ independent of h, a ball $b_\gamma=\{(\alpha,\ \nu,\ \xi);\ |\alpha|+|\nu|+|\varepsilon|<\gamma\}\subset\boldsymbol{R}^3$ *and two functions* $\chi^h=\chi^h(\alpha,\ \nu,\ \varepsilon)\in\mathscr{R}_s^h$, *and* $\eta^h=\eta^h(\alpha,\ \nu,\ \varepsilon)\in\mathscr{R}_s^h$ *which satisfy Eqs.* (6.29) *and* (6.31), *respectively, for* $(\alpha,\ \nu,\ \varepsilon)\in b_\gamma$. *Also,* χ^h *and* η^h *are* (*at least*) *of equi-*C^1 *class.*

It may be easily seen that χ^h and η^h have the form:

$$\chi^h=-\nu(\mathscr{L}_s^h)^\dagger\omega_s^hf^h-\varepsilon(\mathscr{L}_s^h)^\dagger\omega_s^hr_s^h+(h.\ o.\ t.), \tag{6.34}$$

$$\eta^h=-\alpha(\mathscr{L}_s^h)^\dagger\omega_s^h\mathscr{Q}_s^h(\psi_s^h,\ \psi_s^h)$$
$$+\nu(\mathscr{L}_s^h)^\dagger\omega_s^h\mathscr{Q}_s^h(\mathscr{L}_s^{h\dagger}\omega_s^hf^h,\ \psi_s^h)$$
$$+\varepsilon(\mathscr{L}_s^h)^\dagger\omega_s^h\mathscr{Q}_s^h(\mathscr{L}_s^{h\dagger}\omega_s^hr_s^h,\ \psi_s^h)+(h.\ o.\ t), \tag{6.35}$$

where $(h.\ o.\ t.)$ denotes higher order terms in α, ν and ε.

Substituting $\chi^h=\chi^h(\alpha,\ \nu,\ \xi)$ and $\eta^h=\gamma^h(\alpha,\ \nu,\ \xi)$ to Eqs. (6.30) and (6.32), we have the two scalar equations $\Gamma^h: b_\gamma\to\boldsymbol{R}$ and $\Xi^h: b_\gamma\to\boldsymbol{R}$:

$$\Gamma^h(\alpha,\ \nu,\ \varepsilon)\equiv\varepsilon\Pi_s^{h\prime}r_s^h+\nu\Pi_s^{h\prime}f^h$$
$$+\frac{1}{2}A_s^h\alpha^2+B_s^h\alpha\nu+\frac{1}{2}C_s^h\nu^2+\frac{1}{3!}D_s^h\alpha^3+\ldots$$
$$+X_s^h\alpha\varepsilon+Y_s^h\nu\varepsilon+\frac{1}{2}Z_s^h\varepsilon^2+(h.\ o.\ t.)=0, \tag{6.36}$$

and

$$\Xi^h(\alpha,\ \nu,\ \varepsilon)\equiv A_s^h\alpha+B_s^h\nu+X_s^h\varepsilon+(h.\ o.\ t.)=0. \tag{6.37}$$

Here, the coefficients A_s^h, B_s^h, C_s^h, D_s^h, ... are the expressions corresponding to, and converging to A_c, B_c, C_c, D_c, ..., respectively. (*cf.* Chapter *I*, §1, Eq. (1.4.) In fact, they are given by

$$\begin{aligned} &A_s^h \equiv \Pi_s^{h\prime} \mathcal{Q}_s^h(\psi_s^h, \psi_s^h), \\ &B_s^h \equiv \Pi_s^{h\prime}(\psi_s^h, -\mathcal{L}_s^{h\dagger}\omega_s^h f^h), \\ &C_s^h \equiv \Pi_s^{h\prime} \mathcal{Q}_s^h(\mathcal{L}_s^{h\dagger}\omega_s^h f^h, \mathcal{L}_s^{h\dagger}\omega_s^h f^h) \\ &D_s^h \equiv \Pi_s^{h\prime} \mathcal{C}_s^h(\psi_s^h, \psi_s^h, \psi_s^h) - 3\Pi_s^{h\prime} \mathcal{Q}_s^h(\psi_s^h, \mathcal{L}_s^{h\dagger}\omega_s^h \mathcal{Q}_s^h(\psi_s^h, \psi_s^h)), \end{aligned} \tag{6.38}$$

while X_s^h, Y_s^h and Z_s^h are expressions arising from ε-terms:

$$\begin{aligned} &X_s^h \equiv \Pi_s^{h\prime} \mathcal{Q}_s^h(\psi_s^h, \mathcal{L}_s^{h\dagger}\omega_s^h r_s^h), \\ &Y_s^h \equiv \Pi_s^{h\prime} \mathcal{Q}_s^h(-\mathcal{L}_s^{h\dagger}\omega_s^h f_s^h, \mathcal{L}_s^{h\dagger}\omega_s^h r_s^h), \end{aligned} \tag{6.39}$$

and

$$Z_s^h \equiv \Pi_s^{h\prime} Q_s^h(\mathcal{L}_s^h \omega_s^{h\dagger} r_s^h, \mathcal{L}_s^{h\dagger}\omega_s^h r_s^h).$$

We note that $\Xi^h(\alpha, \nu, \varepsilon) = \dfrac{\partial}{\partial \alpha}\Gamma^h(\alpha, \nu, \varepsilon)$.

Lemma 6.3 It *holds that*

$$|A_s^h - A_c| \leq Ch^{l+k} \tag{6.40}$$

$$|B_s^h - B_c| \leq Ch^l \tag{6.41}$$

$$|C_s^h - C_c| \leq Ch^l \tag{6.42}$$

$$|D_s^h - D_c| \leq Ch^l \tag{6.43}$$

and

$$|\Pi_s^{h\prime} f^h - \Pi_c^{\prime} f| \leq Ch^{l+k} \tag{6.44}$$

Proof. We only show the first estimate. The others can be shown similarly, making use of Lemma 5.6. We note first that

$$\mathcal{Q}_c(u, w) = \mathcal{B}(u, v) + 2\mathcal{T}(w_c, u, v),$$

and

$$\mathcal{Q}_s^h(u, v) = P^h[\mathcal{B}^{(h)}(u, v) + 2\mathcal{T}^{(h)}(P^h w_s, u, v)].$$

So,

$$\begin{aligned} A_s^h - A_c &= \langle \mathcal{Q}_s^h(\psi_s^h, \psi_s^h), \psi_s^h \rangle - \langle \mathcal{Q}_c(\phi_c, \phi_c), \phi_c \rangle \\ &= \langle P^h \mathcal{B}^{(h)}(\psi_s^h, \psi_s^h), \psi_s^h \rangle - \langle \mathcal{B}(\phi_c, \phi_c), \phi_c \rangle \end{aligned}$$

$$+2\langle P^h\mathcal{T}^{(h)}(P^h w_s,\ \psi_s^h,\ \psi_s^h),\ \psi_s^h\rangle-2\langle\mathcal{T}(w_c,\ \phi_c,\ \phi_c),\ \phi_c\rangle. \tag{6.45}$$

We estimate the first term which concerns $B^{(h)}$ and B:

$$\begin{aligned}(I)&\equiv\langle P^h\mathcal{B}^{(h)}(\psi_s^h,\ \psi_s^h),\ \psi_s^h\rangle-\langle\mathcal{B}(\phi_c,\ \phi_c),\ \phi_c\rangle\\&=\langle\mathcal{B}^{(h)}(\psi_s^h,\ \psi_s^h)-\mathcal{B}(\phi_c,\ \phi_c),\ \psi_s^h\rangle+\langle\mathcal{B}(\phi_c,\ \phi_c),\ \psi_s^h-\phi_c\rangle\\&=\langle\mathcal{B}^{(h)}(\psi_s^h,\ \psi_s^h)-\mathcal{B}(\phi_c,\ \phi_c),\ \psi_s^h\rangle+\langle\mathcal{B}(\phi_c,\ \psi_s^h-\phi_c),\ \phi_c\rangle,\end{aligned}$$

(where the (separately) self-adjoint property of $\mathcal{B}$ has been used. Thus, by virtue of Eq. (3.15), (3.6) and (6.14),

$$\|(I)\|_V\leq Ch^{l+k}\|\phi_c\|_W^2\|\psi_s^h\|_V+C'\|\phi_c\|_W\|\psi_s^h-\phi_c\|_U\|\phi_c\|_V\leq C''h^{l+k}.$$

The terms corresponding to $\mathcal{T}$ and $\mathcal{T}^{(h)}$ can be similarly estimated.

Lemma 6.3 thus guarantees that $|A_s^h|$ and $|\Pi_s^{h\prime}f^h|$ are bounded *from both above and below*, since $\mathcal{C}$ is a non-degenerate snap point and consequently A_c and $\Pi_c f$ are non-zero quantities. Moreover, it may be easily checked that Γ^h and Ξ^h are (at least) of equi-C^1 class, since $G_s^h(v^h,\ \nu,\ \varepsilon)$ and $\Phi_s^h(v^h)$ are of equi-C^1 and $v^h=\alpha\psi_s^h+\chi^h(\alpha,\ \nu,\ \varepsilon)$ is also equi-C^1 in $b_\gamma\to V^h$.

Hence, by virtue of the equi-implicit function theorem, ν can be solved uniquly as a function of $(\alpha,\ \varepsilon)$ for $(\alpha,\ \varepsilon)\in b_{\gamma'}=\{(\alpha,\ \varepsilon)\ ;\ |\alpha|+|\varepsilon|<\gamma'\}\in\boldsymbol{R}^2$. Namely, there exists an equi-C^1 function $\nu=\nu^h(\alpha,\ \varepsilon):b_{\gamma'}\to\boldsymbol{R}$, such that $\Gamma^h(\alpha,\ \nu^h(\alpha,\ \varepsilon),\ \varepsilon)=0$ for $(\alpha,\ \varepsilon)\in b_{\gamma'}$.

Secondly, substituting $\nu=\nu^h(\alpha,\ \varepsilon)$ in $\Xi^h(\alpha,\ \nu,\ \varepsilon)$, we again have an equi-$C^1$ function $\Xi^h(\alpha,\ \varepsilon)=\Xi^h(\alpha,\ \nu^h(\alpha,\ \varepsilon),\ \varepsilon):(\alpha,\ \varepsilon)\to\boldsymbol{R}$.

Since

$$\frac{\partial}{\partial\alpha}\Xi^h(\alpha,\ \varepsilon)\,|_{(\alpha,\varepsilon)=0}=A_s^h,$$

and $|A_s^h|\geq a_0>0$ (for any $h<h_0$), the equi-implicit function theorem again applies to $\Xi^h(\alpha,\ \varepsilon)=0$. Namely, there exists an equi-C^1 function $\alpha=\alpha^h(\varepsilon):\ \varepsilon\to\boldsymbol{R}$, such that $\Xi^h(\alpha^h(\varepsilon),\ \varepsilon)=0$ for $\varepsilon\in I_\delta=\{\varepsilon\ ;\ |\varepsilon|<\delta\}\subset\boldsymbol{R}$. Tracing back the above arguments, we know, for any $\varepsilon\in I_\delta(\exists\delta>0$, independent of h, for $h\in]0,\ h_0[$), the unique existence of $\alpha=\alpha^h(\varepsilon)$, $\nu=\nu^h(\alpha^h(\varepsilon),\ \varepsilon)\equiv\nu^h(\varepsilon)$, $v^h=\alpha^h(\varepsilon)\psi_s^h+\chi^h(\alpha^h(\varepsilon),\ \nu^h(\varepsilon),\ \varepsilon)\equiv v^h(\varepsilon)$ and $\phi^h=\psi_s^h+\eta^h(\alpha^h(\varepsilon),\ \nu^h(\varepsilon),\ \varepsilon)\equiv\phi^h(\varepsilon)$, which satisfies $G_s^h(v^h,\ \nu,\ \varepsilon)=0$ and $\Phi_s^h(v^h)=0$ simultaneously.

Lemma 6.4 *There is an interval* $I_\delta=\{\varepsilon\ ;\ |\varepsilon|<\delta\}\subset\boldsymbol{R}$, *and equi-*$C^1$ *functions* $\nu=\nu^h(\varepsilon)$ *and* $\alpha=\alpha^h(\varepsilon):\ I_\delta\to\boldsymbol{R}$ *such that* $\Xi^h(\alpha,\ \nu,\ \varepsilon)=\Gamma^h(\alpha,\ \nu,\ \varepsilon)=0$ *for* $\varepsilon\in I_\delta$.

It is noted that $|\nu^h(\varepsilon)|\leq C|\varepsilon|$ and $|\alpha^h(\varepsilon)|\leq C|\varepsilon|$ by (iv) of the equi-implicit function theorem, and they take the form:

$$\begin{aligned}\nu&=\nu^h(\varepsilon)\\&=\frac{\Pi_s^{h'}r_s^h}{\Pi_s^{h'}f^h}\varepsilon+O(\varepsilon^2)\end{aligned}\tag{6.46}$$

and

$$\begin{aligned}\alpha&=\alpha^h(\varepsilon)\\&=-\left(\frac{X_s^h}{A_s^h}+\frac{B_s^h}{A_s^h}\,\frac{\Pi_s^{h'}r_s^h}{\Pi_s^{h'}f^h}\right)\varepsilon+O(\varepsilon^2).\end{aligned}\tag{6.47}$$

As a result, if $h_0>0$ had been chosen so small that

$$||F_s^h||_V\equiv||F^h(\mu_s,\ P^hw_s)||_V\leq Ch^{l+k}<\delta,^{*)}\tag{6.48}$$

for any $h\in]0,\ h_0[$, then the system of equations $G_s^h(v^h(\varepsilon),\ \nu^h(\varepsilon),\ \varepsilon)=0$, $\Phi^h(\nu^h(\varepsilon),\ \phi^h(\varepsilon))=0$ have a unique triplet $(\nu^h(\varepsilon),\ v^h(\varepsilon)\ ;\phi^h(\varepsilon))\in\boldsymbol{R}\times V^h\times V^h$ for $\varepsilon=||F_s^h||_V\equiv\varepsilon(h)$, which is in turn the unique solutions of (Q^h) and (E^h). We let denote

$$\begin{aligned}\nu_c^h&\equiv\nu^h(\varepsilon(h)),\\ v_c^h&\equiv v^h(\varepsilon(h)),\end{aligned}\tag{6.49}$$

and

$$\phi_c^h\equiv\phi^h(\varepsilon(h)).$$

Then,

$$\begin{aligned}|\nu_c^h|&=|\nu^h(\varepsilon(h))|\leq C|\varepsilon(h)|\leq Ch^{l+k},\\ ||v_c^h||_V&=||v^h(\varepsilon(h))||_V\leq C'\,|\varepsilon(h)|\leq C'h^{l+k},\end{aligned}\tag{6.50}$$

and

$$||\phi_c^h-\varphi_s^h||_V=||\phi^h(\varepsilon(h))-\varphi_s^h||_V\leq C''\,|\varepsilon(h)|\leq C''h^{l+k}.$$

We now arrive at the final theorem of this section. Assuming the hypotheses on (P) in § 3. 1 and on (P^h) in §3. 2, we have the following

Theorem 6. 5 *In a neighborhood of* $(\mathscr{C}\ ;\ \phi_c)\equiv(\mu_c,\ w_c\ ;\ \phi_c)\in\boldsymbol{R}\times W\times W$, *there exists a unique, simple, non-degenerate snap point* $(\mathscr{C}^h;\ \phi_c^h)\equiv(\mu_c^h,\ w_c^h\ ;\ \phi_c^h)\in\boldsymbol{R}\times V^h\times V^h$ *of* (P^h) *for any* $h\in]0,\ h_0[$. $(C^h\ ;\ \phi_c^h)$ *is close to* $(\mathscr{C}\ ;\ \phi_c)$ *in the following sense*:

*) By hypothesis $F(\mu(s),\ w(s))=0(s\in S)$, Lemma 4. 1 yields $||F^h(\mu(s),\ P^hw(s))||_V\leq Ch^{l+k}(s\in S)$.

$$|\mu_c-\mu_c^h|\leq Ch^{l+k}, \tag{6.51}$$

$$||P^h w_c-w_c^h||_V\leq Ch^{l+k}, \tag{6.52}$$

and

$$||P^h\phi_c-\phi_c^h||_V\leq Ch^{l+k}. \tag{6.53}$$

Also,

$$||w_c-w_c^h||_V\leq Ch^l, \tag{6.54}$$

and

$$||w_c-w_c^h||_U\leq Ch^{l+k},$$

$$||\phi_c-\phi_c^h||_V\leq Ch^l, \tag{6.55}$$

and

$$||\phi_c-\phi_c^h||_U\leq Ch^{l+k}.$$

Proof. That $(\mathscr{C}^h;\ \phi_c^h)=(\mu_c^h,\ w_c^h;\ \phi_c^h)$ is a *simple critical* point of (P^h) is a consequence of the previous arguments, here by definition (see, Eqs. (6.22) and (6.49)),

$$\mu_c^h=\mu(s(h))+\nu_c^h, \tag{6.56}$$

and

$$w_c^h=P^h w(s(h))+v_c^h. \tag{6.57}$$

Also, it is a snap point, since

$$\begin{aligned}\Pi_c^{h\prime}f^h &\overset{def}{=}\langle f^h,\ \phi_c^h\rangle\\ &=\langle P^h f,\ \phi_c^h\rangle\end{aligned} \tag{6.58}$$

tends to $\Pi_c'f=\langle f,\ \phi_c\rangle$ as $h\to 0$ due to Eq. (6.55) (which we shall show later). This snap point is non-degenerate, since

$$A_c^h\overset{def}{=}\Pi_c^{h\prime}Q_c^h(\phi_c^h,\ \phi_c^h) \tag{6.59}$$

tends to A_c, which is by hypothesis non-zero. (Here, $\mathscr{Q}_c^h(u,v)\overset{def}{=}F''^h(\mu_c^h,\ w_c^h)(u,\ v)$.)

With regards to error estimates, we remind, in a neighborhood of a non-degenerate snap point, that

$$|\mu(s)-\mu_c|=|\mu(s)-\mu(0)|\leq C|s|^2 \tag{6.60}$$

(Eq. (1.5)). Thus, in view of Eqs. (6.50) and (6.56) we have that

$$|\mu_c^h-\mu_c|\leq|\mu(s(h))-\mu_c|+|\nu_c^h|\leq Ch^{l+k}. \tag{6.61}$$

Also, by Eqs. (1.6), (6.50) and (6.57),

$$||w_c^h - P^h w_c||_V \leq ||P^h(w(s(h)) - w_c)||_V + ||v_c^h||_V \leq Ch^{l+k}. \tag{6.62}$$

For the error of critical eigenfunction, we have

$$||\phi_c^h - P^h\phi_c||_V \leq ||\phi_c^h - \psi_s^h||_V + ||\psi_s^h - P^h\phi_c||_V \leq Ch^{l+k} \tag{6.63}$$

from Eqs. (6.12) and (6.50). Equations (6.54) and (6.55) are a result from the approximation properties (3.10) and (3.11) and Eqs. (6.62), (6.63).

6.3 Uniform convergence of paths in a neighborhood of snap point

The arguments of the previous subsection establish the unique existence of a non-degenerate, snap point $(\mathscr{C}^h; \phi_c^h) \equiv (\mu_c^h, w_c^h; \phi_c^h) \in \boldsymbol{R} \times V^h \times V^h$ of (P^h) which lies near a non-degenerate, snap point of (P) : $(\mathscr{C}, \phi_c) \equiv (\mu_c, w_c; \phi_c) \in \boldsymbol{R} \times W \times W$. The general theory as applied to (P^h) guarantees that for each $h \in]0, h_0[$, there is a unique *path* $(\mu^h(\alpha), w^h(\alpha)) \in \boldsymbol{R} \times V^h$ for $\alpha \in I_{\delta^h}$, which passes $\mathscr{C}^h$, namely, $\mu^h(0) = \mu_c^h$ and $w^h(0) = w_c^h$. (*cf.* Proposition 1.8, Chapter *I*.) We call this $(\mu^h(\alpha), w_h(\alpha))$ path as "α^h-*path*". It may then be natural to expect that the α^h-path converges to the corresponding α-path $(\mu(\alpha), w(\alpha)) \in \boldsymbol{R} \times W$ of (P) *uniformly in* $\alpha \in I_{\delta'}$ (δ' : independent of h) *with the same order as in the critical point itself*, and the proof of which is the main subject of this section.

We begin by recalling that the α-path is represented as

$$\mu(\alpha) = \mu_c + \nu(\alpha), \tag{6.64}$$

and

$$\begin{aligned} w(\alpha) &= w_c + v(\alpha) \\ &= w_c + \alpha\phi_c + \chi(\alpha) ; \end{aligned} \tag{6.65}$$

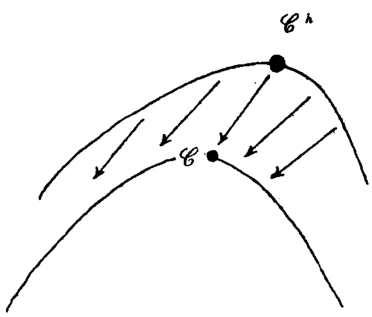


Fig. 6.1

where $\chi(\alpha)$ is orthogonal to ϕ_c. The pair $(\nu(\alpha), v(\alpha))$ satisfies

$$G_c(\nu, v) \equiv F(\mu_c+\nu, w_c+v) - F(\mu_c, w_c) = 0, \tag{6.66}$$

or equivalently, after the Lyapounov-Schmidt decomposition,

$$\mathscr{L}_c\chi + \omega_c R_c(\alpha\phi_c+\chi) + \nu\omega_c f = 0, \tag{6.67}$$

$$\Pi_c' R_c(\alpha\phi_c+\chi) + \nu\Pi_c' f = 0, \tag{6.68}$$

where $\mathscr{L}_c \equiv F'(\mu_c, w_c) = G_c'(0, 0)$ and R_c is the remainder term. (See, the proof of Proposition 1.8.)

We may thus rewrite Eq. (6.66) in the form*):

$$\begin{bmatrix} \Pi_c' f & 0 \\ \omega_c f & \mathscr{L}_c \end{bmatrix} \begin{pmatrix} \nu \\ \chi \end{pmatrix} + \begin{pmatrix} \Pi_c' R_c(\alpha\phi_c+\chi) \\ \omega_c R_c(\alpha\phi_c+\chi) \end{pmatrix} = 0 \tag{6.69}$$

where $(\nu, \chi) \in \boldsymbol{R} \times \mathscr{R}_c$. Notice that the linear operator $\begin{bmatrix} \Pi_c' f & 0 \\ \omega_c f & \mathscr{L}_c \end{bmatrix}$ is invertible on $\boldsymbol{R} \times \mathscr{R}_c$.

For the problem (P^h), we let similarly

$$\mu^h(\alpha) = \mu_c^h + \nu^h(\alpha), \tag{6.70}$$

$$w^h(\alpha) = w_c^h + v^h(\alpha), \tag{6.71}$$

with

$$v^h(\alpha) = \alpha\phi_c^h + \chi^h(\alpha)\ ;\ \chi^h(\alpha) \in \mathscr{R}_c^h, \tag{6.72}$$

where $\mathscr{R}_c^h = \text{range } F^{h\prime}(\mu_c, w_c) = [\ker F^{h\prime}(\mu_c, w_c)]^\perp$.

It is immediate to see that $(\nu^h, \chi^h) \in \boldsymbol{R} \times \mathscr{R}_c^h$ is the solution of*)

$$\begin{bmatrix} \Pi_c^{h\prime} f^h & 0 \\ \omega_c^h f^h & \mathscr{L}_c^h \end{bmatrix} \begin{pmatrix} \nu^h \\ \chi^h \end{pmatrix} + \begin{pmatrix} \Pi_c^{h\prime} R_c^h(\alpha\phi_c^h+\chi^h) \\ \omega_c^h R_c^h(\alpha\phi_c^h+\chi^h) \end{pmatrix} = 0 \tag{6.73}$$

where ω_c^h is the orthogonal projection $V \to \mathscr{R}_c^h$ and $\Pi_c^{h\prime} u \equiv \langle u, \phi_c^h \rangle$.

Firstly, we note the following

Lemma 6.6 *The operator* $\begin{bmatrix} \Pi_c^{h\prime} f^h & 0 \\ \omega_c^h f^h & \mathscr{L}_c^h \end{bmatrix}$ *is equi-invertible on* $\boldsymbol{R} \times \mathscr{R}_c^h$.

Proof. Lemma 5.3 shows that $\mathscr{L}_c^h$ is equi-invertible on $\mathscr{R}_c^h$ for $h \in]0, h_1[$ (h_1: sufficiently small). Since $|\Pi_c^{h\prime} f^h - \Pi_c f| \leq Ch^{l+k}$, which can be shown as in Lemma 7.3, $|\Pi_c^{h\prime} f^h|$ is bounded below as is $\Pi_c f$, uniformly in h,

*) $\mathscr{L}_h(\mathscr{L}_c^h)$ is understood as $\mathscr{L}_c|\mathscr{R}_c(\mathscr{L}_c^h|\mathscr{R}_c^h)$, here.

which shows the assertion.

Secondly, taking notice of the fact that $R_c^h(v)$ is of (at least) equi-C^1 near the origin (see, the proof of Proposition 4.6), $\Pi_c^{h\prime}R_c^h(\alpha\phi_c^h+\chi^h)$ and $\omega_c^h R_c^h(\alpha\phi_c+\chi^h)$ are also of equi-C^1 with respect to α and χ^h near the origin. Thus, the equi-implicit function theorem as applied to Eq. (6.73) guarantees the existence of a constant $\delta'>0$ independent of h, and a solution pair $(\nu^h(\alpha), \chi^h(\alpha))\in \boldsymbol{R}\times\mathscr{R}_c^h$ for $\alpha\in I_{\delta'}=\{\alpha\,;\ |\alpha|<\delta'\}$, which satisfies Eq. (6.73)*). We let $\delta''=\min(\delta, \delta')$ and write it again as δ'.

It is noted that Eq. (6.69) can be regarded as defining an "ordinary path" with respect to the parameter $\alpha\in I_\delta\subset\boldsymbol{R}$. Thus, one might be able to use exactly the same arguments as in the realization of ordinary path (Proposition 4.6). However, since $\mathscr{R}_c^h$ is *not* a subspace of $\mathscr{R}_c$, there is a technical difficulty for performing this program.

Note, however, that for any $\chi\in\mathscr{R}_c\cap W$,

$$\|\Pi_c^h P^h\chi\|_V\le Ch^{l+k}\|\chi\|_W \tag{6.74}$$

since

$$\begin{aligned}\Pi_c^h P^h\chi &= <P^h\chi,\ \phi_c^h>\phi_c^h\\ &= <\chi,\ \phi_c^h-\phi_c>\phi_c^h \quad (\text{since } <\chi,\ \phi_c>=0)\\ &= <\chi,\ \phi_c^h-P^h\phi_c>\phi_c^h+<\chi,\ P^h\phi_c-\phi_c>\phi_c^h\\ &= <\chi,\ \phi_c^h-P^h\phi_c>\phi_c^h+<\chi-P^h\chi,\ P^h\phi_c-\phi_c>\phi_c^h\end{aligned}$$

and Eq. (6.74) follows immediately from Eqs. (6.53) and (3.12) recalling that $k=\min\ (l,\ m)$.

This suggest us to let in Eq. (6.73),

$$\nu^h(\alpha)=\nu(\alpha)+\varepsilon^h(\alpha) \tag{6.75}$$

and

$$\chi^h(\alpha)=\omega_c^h P^h\chi(\alpha)+\eta^h(\alpha)$$

for $\alpha\in I_{\delta'}$. Then, we have

$$\begin{bmatrix}\Pi_c^{h\prime}f^h & 0\\ \omega_c^h f^h & \mathscr{L}_c^h\end{bmatrix}\begin{pmatrix}\varepsilon^h\\ \eta^h\end{pmatrix}+\begin{pmatrix}\Pi_c^{h\prime}\{R_c^h(\alpha\phi_c^h+\omega_c^h P^h\chi(\alpha)+\eta^h)-R_c^h(\alpha\phi_c^h+\omega_c^h P^h\chi(\alpha))\}\\ \omega_c^h\{R_c^h(\alpha\phi_c^h+\omega_c^h P^h\chi(\alpha)+\eta^h)-R_c^h(\alpha\phi_c^h+\omega_c^h P^h\chi(\alpha))\}\end{pmatrix}$$
$$+\begin{pmatrix}\Pi_c^{h\prime}G_c^h(\nu(\alpha),\ \alpha\phi_c^h+\omega_c^h P^h\chi(\alpha))\\ \omega_c^h G_c^h(\nu(\alpha),\ \alpha\phi_c^h+\omega_c^h P^h\chi(\alpha))\end{pmatrix}=0 \tag{6.76}$$

where by definition

*) In Theorem 3.6, $X=\mathscr{R}_c^h$, $Y=\boldsymbol{R}$, $Z=\mathscr{R}_c^h$.

$$G_c^h(\nu, v) = F^h(\mu_c^h + \nu, w_c^h + v) - F^h(\mu_c^h, w_c^h) \\ = F^h(\mu_c^h + \nu, w_c^h + v). \tag{6.77}$$

Lemma 6.7 *It holds that for* $\alpha \in I_{\delta'}$

$$|\Pi_c^{h'} \tilde{G}_c^h(\alpha)| \leq Ch^{l+k} |\alpha|^2, \\ \|\omega_c^h \tilde{G}_c^h(\alpha)\|_V \leq C' h^{l+k} |\alpha|^2, \tag{6.78}$$

where

$$\tilde{G}_c^h(\alpha) \equiv G_c^h(\nu(\alpha), \alpha\phi_c^h + \omega_c^h P^h \chi(\alpha)).$$

Proof. It is enough to show $\|\tilde{G}_c^h(\alpha)\|_V \leq C'' h^{l+k} |\alpha|^2$. However, since

$$F(\mu_c + \nu(\alpha), w_c + \alpha\phi_c + \chi(\alpha)) = 0, \qquad \alpha \in I_{\delta'},$$

the lemma follows by using similar techniques as in Lemma 4.1 and taking note of the fact that $\mathscr{L}_c\phi_c = \mathscr{L}_c^h \phi_c^h = 0$ and $\|\chi(\alpha)\|_V \leq C|\alpha|^2$(Eq. (1.8)), in view of Eqs. (6.51), (6.52), (6.53) and (6.74).

Now, we are at the situation to apply Corollary 3.7 to Eq. (6.76) to have the following

Proposition 6.8 *For each* $h \in]0, h_0[$, *there is a unique path* (α^h), *represented as* $(\mu^h(\alpha), w^h(\alpha)) \in \boldsymbol{R} \times V^h$, $\alpha \in I_{\delta'}$ (δ' : *independent of* h), *which passes the non-degenerate, snap point* $\mathscr{C}^h$ *of* (P^h) *at* $\alpha = 0$. *Moreover, the* (α^h)-*path converges uniformly to the* (α)-*path of* (P), *namely, to* $(\mu(\alpha), w(\alpha)) \in \boldsymbol{R} \times W$, $\alpha \in I_{\delta'}$ *in the following sense*:

$$|\mu^h(\alpha) - \mu(\alpha)| \leq Ch^{l+k}, \tag{6.69}$$

and

$$\|w^h(\alpha) - P^h w(\alpha)\|_V \leq C' h^{l+k}. \tag{6.80}$$

Also,

$$\|w^h(\alpha) - w(\alpha)\|_V \leq C'' h^l, \tag{6.81}$$

and

$$\|w^h(\alpha) - w(\alpha)\|_U \leq C''' h^{l+k}. \tag{6.82}$$

Proof. We first show that for each fixed $\alpha \in I_{\delta''}$ $(\delta'' > 0)$

$$|\varepsilon^h(\alpha)| \leq Ch^{l+k} |\alpha|^2, \tag{6.83}$$

$$\|\eta^h(\alpha)\|_V \le C'h^{l+k}|\alpha|^2. \tag{6.84}$$

The linearized operator of the left-hand side of Eq. (6.76) takes the form

$$\tilde{\mathscr{L}}^h(\alpha) \equiv \begin{bmatrix} \Pi_c^{h\prime} f^h & 0 \\ \omega_c^h f^h & \mathscr{L}_c^h \end{bmatrix} + \begin{bmatrix} 0 & \Pi_c^{h\prime} R_c^{h\prime}(\alpha\phi_c^h + \omega_c^h P^h \chi(\alpha)) \\ 0 & \omega_c^h R_c^{h\prime}(\alpha\phi_c^h + \omega_c^h P^h \chi(\alpha)) \end{bmatrix},$$

where $R_c^{h\prime}(v)$ is the Fréchet derivative of R_c^h at v. It is easily seen that $\tilde{\mathscr{L}}^h(\alpha)$ is equi-invertible as an operator $\boldsymbol{R}\times\mathscr{R}_c^h \to \boldsymbol{R}\times\mathscr{R}_c^h$. In fact, we have by definition that

$$R_c^{h\prime}(\tilde{v}^h(\alpha)) = \mathscr{B}^h(\tilde{v}^h(\alpha), \cdot) + \mathscr{T}^h(w_c^h, \tilde{v}^h(\alpha), \cdot) + \frac{1}{2}\mathscr{T}^h(\tilde{v}^h(\alpha), \tilde{v}^h(\alpha), \cdot),$$

where $\tilde{v}^h(\alpha) = \alpha\phi_c^h + \omega_c^h P^h \chi(\alpha)$. Noting that $\|\chi(\alpha)\|_V \le C|\alpha|^2$(see, Prop. 1.8), we have for $|\alpha| < \delta''$ (sufficiently small) that $\|R_c^{h\prime}(\tilde{v}^h(\alpha))\|_{V\to V} \le C|\alpha|$. Since the first term of $\tilde{\mathscr{L}}^h(\alpha)$ is equi-invertible by Lemma 6.6, $\tilde{\mathscr{L}}^h(\alpha)$ is also equi-invertible for $\alpha \in I_{\delta''} = \{\alpha;\ |\alpha| < \delta''\}$. We can thus apply Corollary 3.7 to Eq. (6.76). (The equi-C^1 property of the mapping can be easily seen as in Prop. 4.6). Hence, follow Eqs. (6.83)-(6.84).

Next, by definition and by Eq. (6.75),

$$|\mu^h(\alpha) - \mu(\alpha)| = |(\mu_c^h + \nu^h(\alpha)) - (\mu_c + \nu(\alpha))| \le |\mu_c^h - \mu_c| + |\varepsilon^h(\alpha)|.$$

Eqs. (6.51) and (6.83) imply Eq. (6.79).

Thirdly, again by definition,

$$\begin{aligned} \|w^h(\alpha) - P^h w(\alpha)\|_V &= \|(w_c^h + \alpha\phi_c^h + \chi^h(\alpha)) - (P^h w_c + \alpha P^h \phi_c + P^h \chi(\alpha))\|_V \\ &\le \|w_c^h - P^h w_c\|_V + |\alpha|\,\|\phi_c^h - P^h \phi_c\|_V \\ &\quad + \|\chi^h(\alpha) - \omega_c^h P^h \chi(\alpha)\|_V + \|\Pi_c^h P^h \chi(\alpha)\|_V. \end{aligned}$$

Thus, in view of Eqs. (6.52), (6.53), (6.74), (6.75) and (6.84), we obtain Eq. (6.80). Eq. (6.80) together with Eqs. (3.10) and (3.11) yields Eq. (6.82).

7. Numerical realization of symmetry breaking bifurcations

Our problem (P) is of *class* N in the sense of Chap. I, §2.3, possible bifurcations may be thus symmetry breaking ones. We assume in this section that the problem (P) has a non-trivial symmetry group $\mathscr{G}$. Lemma 2.11 shows that a symmetry breaking bifurcation $\mathscr{C}$ cannot be a fold (=an asymmetric, =a transcritical) bifurcation, implying that $\mathscr{C}$ is a cusp or, a more degenerate bifurcation.

The aim of this section is to show that if the scheme (P^h) preserves the symmetry group $\mathscr{G}$ (see, § 3.4 for this condition), the symmetry breaking bifurcation $\mathscr{C}$ is *uniformly realized as itself* in the finite element space V^h for $h \in]0, h_0[$. The error in numerical buckling load μ_c^h is shown to be $O(h^{l+k})$, which is as accurate as in the snap buckling case. This shows a sharp contrast with *generic perturbations* for, e.g., a cusp in which 2/3 power law appears in the error. See, e.g. Keener and Keller [20], Thompson and Hunt [51] or Hangai and Kawamata [16].

7.1 Numerical realization of $\mathscr{G}$-symmetric path

We first have to show the numerical realization of symmetric path in a neighborhood of a simple, symmetry breaking bifurcation point $\mathscr{C}^+ \equiv (\mu_c, w^+) \in \boldsymbol{R} \times V^+$. We assume that the scheme

$$(P^h) \quad F(\mu, w^h) = 0, \ (\mu, w^h) \in \boldsymbol{R} \times V^h \tag{7.1}$$

is covariant under $\mathscr{G}$, where $\mathscr{G}$ is the symmetry group of F. See, § 2, Chapter I for notations and definitions. Also, the finite element space V^h is assumed to be invariant under $\mathscr{G}$ in the sense of Lemma 3.8. We assume, *for simplicity* that $\mathscr{G}$ is finite. Then, the standard decomposition of V and V^h associated with $\mathscr{G}$ is given by

$$V = V_1 \oplus V_2 \oplus \dots \oplus V_q, \tag{7.2}$$

and

$$V_1^h = V_1^h \oplus V_2^h \oplus \dots \oplus V_q^h \tag{7.3}$$

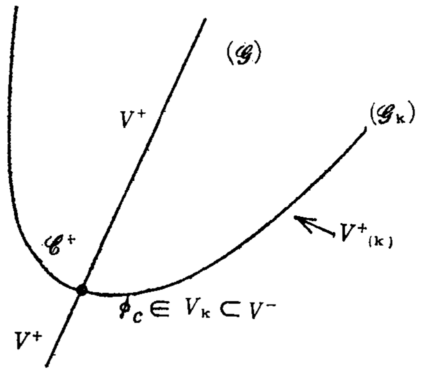
See, § 2 for details. It is noted here that the above assumption on V^h implies that $V_k^h = P_k V^h = P^h V_k$ and $V_k^h \subset V_k (k = 1, 2, \dots, q)$, where $P_k (k = 1, 2, \dots, q)$ is the projection $V \to V_k$ defined by Eq. (2.5).

Suppose $(\mathscr{C}^+ ; \phi_c) \equiv (\mu_c, w_c^+ ; \phi_c) \in \boldsymbol{R} \times V_1 \times V_k \subset \boldsymbol{R} \times V^+ \times V^{-}$ *) be a simple, non-degenerate symmetry breaking bifurcation point of F. To fix our arguments, we assume that $\mathscr{C}^+$ is a *cusp* bifurcation point. Lemma 2.10 shows the existence of a $\mathscr{G}$-symmetric path $(\mu, w^+(\mu)) \in \boldsymbol{R} \times V^+$, $\mu \in I_\delta \equiv \{\mu ; |\mu| < \delta\}$ and a symmetry breaking path $(\alpha, w^\alpha(\alpha)) \in \boldsymbol{R} \times V_{(k)}^+$, $\alpha \in I_{\delta'} \equiv \{\alpha ; |\alpha| < \delta'\}$, which intersect at $\mathscr{C}^+$, i.e., at $\mu = \mu_c$ and $\alpha = 0$. Note that the bifurcated path $(\alpha, w^\alpha(\alpha))$, $\alpha \in I_{\delta'}$, may still have a smaller $(\mathscr{G}_k-)$ symmetry, since $V_{(k)}^+$ is defined by $P_{(k)}^+ V$, $P_{(k)}^+$ being the projection

$$P_{(k)}^+ = \frac{1}{n(\mathscr{G}_k)} \sum_{g \in \mathscr{G}_k} T_g. \tag{7.4}$$

*) $V^+ \equiv V_1$ and $V^- \equiv V_2 \oplus \dots \ominus V_q$.

See, §2 for notations.



Notice that on the $\mathscr{G}$-symmetric path, the linearized operator of F has the block diagonal structure:

$$F'(\mu, w^+(\mu)) \equiv \mathscr{L}^+(\mu) \sim \begin{bmatrix} \mathscr{L}_1^+(\mu) & & & \\ & \mathscr{L}_2^+(\mu) & & \\ & & \ddots & \\ & & & \mathscr{L}_q^+(\mu) \end{bmatrix} \tag{7.5}$$

thanks to Lemma 2.7. By hypothesis (that $\mathscr{C}^+$ is a symmetry breaking bifurcation point), $\mathscr{L}_1^+(\mu)$ is invertible at $\mu=\mu_c$ and consequently in a neighborhood of $\mu=\mu_c$ in view of the C^{p-2}(actually, $p=+\infty$) continuity of $\mathscr{L}_1^+(\mu)$. At the same time, the same hypothesis implies that $\mathscr{L}_k^+(\mu)$ is critical at $\mu=\mu_c$ and that $\{\phi_c\}=\ker \mathscr{L}_k^+(\mu_c)\subset V_k\subset V^-$, where $k\in\langle 2,\ldots,q\rangle$.

We restrict the problems (P) and (P^h) to V_1 and V_1^h, respectively:

$$(P_1P) \qquad P_1F(\mu, w_1)=0, \ (\mu, w_1)\in \boldsymbol{R}\times V_1, \tag{7.6}$$

$$(P_1P^h) \qquad P_1F^h(\mu, w_1^h)=0, \ (\mu, w_1^h)\in \boldsymbol{R}\times V_1^h, \tag{7.7}$$

where P_1 is the projection $V\to V_1$ defined by Eq. (2.9).

Since $P_1P^h=P^hP_1$ by virtue of Lemma 3.8, we can see that (P_1P^h) is equivalent to

$$(P_1F)^h(\mu, w_1^h)=0, \ (\mu, w_1^h)\in \boldsymbol{R}\times V_1^h. \tag{7.8}$$

Here $(P_1F)^h(\mu, w_1^h)$ is understood as

$$(I+P^hP_1L)w^h+\frac{1}{2!}P^hP_1\mathscr{B}^{(k)}(w^h, w^h)$$

$$+\frac{1}{3!}P^h P_1 \mathscr{T}^{(h)}(w^h, w^h, w^h) + \mu P^h P_1 Gp = 0. \tag{7.9}$$

Since F^h is covariant under $\mathscr{G}$,

$$(P_1 F)^h(\mu, w^h) = F^h(\mu, w^h) \tag{7.10}$$

for any $w^h \in V_1^h = P_1 V^h$. Thus, we can apply the results in § 4, and obtain the numerical realization of $\mathscr{G}$-symmetric path in V^h.

As a corollary of Theorem 4. 6, we have

Lemma 7.1 *There exists a unique $\mathscr{G}$-symmetric path $(\mu, w_1^h(\mu)) \in \boldsymbol{R} \times V_1^h$, for $\mu \in I_\delta$, such that*

$$P_1 F^h(\mu, w_1^h(\mu)) = 0.$$

Moreover,

$$\|P^h w_1(\mu) - w_1^h(\mu)\|_V \leq C \|w_1(\mu)\|_W h^{l+k}, \tag{7.11}$$

and

$$\|w_1(\mu) - w_1^h(\mu)\|_V \leq C' \|w_1(\mu)\|_W h^l, \quad \|w_1(\mu) - w_1^h(\mu)\|_U \leq C'' \|w_1(\mu)\|_W h^{l+k}. \tag{7.12}$$

Since F^h is enclosed in V_1^h, it holds that

$$P_j F^h(\mu, w_1^h(\mu)) = 0 \qquad j = 2, 3, \ldots, q, \tag{7.13}$$

which yields the following

Proposition 7.2 *There is a $\mathscr{G}$-symmetric path $(\mu, w_1^h(\mu)) \in \boldsymbol{R} \times V_1^h$, for $\mu \in I_\delta$, of (P^h), where $w_1^h(\mu)$ is given by Lemma 7. 1 and is close to the $\mathscr{G}$-symmetric path of (P) in the sense of Eqs. (7. 11) and (7. 12).*

7.2 Eigenproblem on the $\mathscr{G}$-symmetric path

We investigate the problem (P^h) on the $\mathscr{G}$-symmetric path $(\mu, w_1^h(\mu)) \in \boldsymbol{R} \times V_1^h$, $\mu \in I_\delta$. Since F^h is covariant under $\mathscr{G}$, the linearized operator on this path takes a block diagonal form. We consider the linearized eigenproblem on the subspace $V_k = P_k V$:

$$(P_k E^h) \quad P_k \mathscr{L}^{h,+}(\mu)\phi^h(\mu) = z^h(\mu)\phi^h(\mu), \ \mu \in I_\delta, \tag{7.14}$$

where

$$\mathscr{L}^{h,+}(\mu) = F^{h\prime}(\mu, w_1^h(\mu)). \tag{7.15}$$

Note that in view of the form of $F^{h'}$, Eq. (4.6), any eigenfunction $\phi^h(\mu)$ of $(P_k E^h)$ belongs to V_k^h.

We remind that the corresponding eigenproblem for (P):

$$(P_k E) \quad P_k \mathscr{L}^+(\mu)\phi(\mu) = z(\mu)\phi(\mu), \qquad \mu \in I_\delta \tag{7.16}$$

where $\phi(\mu) \in V_k$, has the property that there is a family $(\mu \in I_\delta)$ of critical pair $(\zeta_c(\mu), \phi_c(\mu)) \in \boldsymbol{R} \times V_k$ such that

$$z_c(\mu_c) = 0, \ \frac{dz_h}{d\mu}(\mu_c) \neq 0,^{*)} \tag{7.17}$$

and

$$\ker(P_k \mathscr{L}^+(\mu_c) P_k) = \{\phi_c(\mu_c)\} = 1 \text{ dimensional.} \tag{7.18}$$

Lemma 7.3 *There is a unique simple, critical point* μ_c^h *of* $(P_k E^h)$, *that is,* $z_c^h(\mu_c^h) = 0$. *Moreover,*

$$|\mu_c - \mu_c^h| \leq C h^{l+k}, \tag{7.19}$$

and

$$\|P^h \phi_c(\mu_c) - \phi_c^h(\mu_c^h)\|_V \leq C' h^{l+k}. \tag{7.20}$$

Also,

$$\|\phi_c(\mu_c) - \phi_c^h(\mu_c^h)\|_V \leq C'' h^l, \tag{7.21}$$

and

$$\|\phi_c(\mu_c) - \phi_c^h(\mu_c^h)\|_U \leq C''' h^{l+k}.$$

Proof. We apply Theorem 5.1 with

$$\begin{aligned}\mathscr{K}(\mu) &= P_k \mathscr{K}_{w_1(\mu)} \\ &= P_k[L + \mathscr{B}(w_1(\mu), \cdot) + \frac{1}{2}\mathscr{T}(w_1(\mu), w_1(\mu), \cdot)],\end{aligned} \tag{7.22}$$

and

$$\mathscr{K}^{(h)}(\mu) = P_k[L + \mathscr{B}^{(h)}(w_1^h(\mu), \cdot) + \frac{1}{2}\mathscr{T}^{(h)}(w_1^h(\mu), w_1^h(\mu), \cdot)]. \tag{7.23}$$

All the assumptions of Theorem 5.1 can be checked in a similar manner as in Lemma 6.1.

The other components $P_j \mathscr{L}^{h,+}(\mu)$ for $j = 1, 2, \ldots, q$; $j \neq k$, are equi-

*) See, Lemma 1.10 (iii). This property is a consequence of the non-degeneracy condition $B_c^2 - A_c C_c > 0$, which in turn implies $B_c \neq 0$ since $A_c = 0$ (symmetry breaking). This holds whether or not D_c vanishes.

invertible, since $P_j\mathscr{L}^+(\mu)$ are invertible for $\mu\in I_\delta$ except $j=k$ by our hypothesis. A proof of this statement is easily obtained by applying Lemma 4.4 to each $P_j\mathscr{L}^{h,+}(\mu)\,(j\neq k)$. Thus, we have shown the unique existence of a simple, symmetry breaking bifurcation point $(\mathscr{C}^{h,+};\phi_c^h)\equiv(\mu_c^h, w_1^h(\mu_c^h);\phi_c^h(\mu_c^h))\in \boldsymbol{R}\times V_1^h\times V_k^h$ on the $\mathscr{G}$-symmetric path $(\mu, w_1^h(\mu))\in\boldsymbol{R}\times V_1^h$. The general theory as applied to (P^h) guarantees that for each $h\in]0, h_0[$, there imerges from $\mathscr{C}^{h,+}$ a symmetry breaking, but $\mathscr{G}_k$-symmetric path $(\mu^h(\alpha), w^h(\alpha))\in\boldsymbol{R}\times V^h$ for $\alpha\in I_{\delta^h}=\{\alpha;\ |\alpha|<\delta^h\}$. We call this $(\mu^h(\alpha), w^h(\alpha))$ path as "α^h-path". We shall show that $\mathscr{C}^{h,+}$ is a *cusp* bifurcation point and that the α^h-path can be extended to $\alpha\in I_{\delta'}$, where $\delta'>0$ is a constant independent of h.

Lemma 7.4 *$\mathscr{C}^{h,+}$ is a cusp bifurcation point of (P^h).*

It is convenient to define here those quantities which appear in the bifurcation equation of (P^h). (*Cf.* Eq. (1.4) and Eq. (6.38).)

$$\begin{aligned}
A_c^h&\equiv\Pi_c^{h\prime}\mathscr{Q}_c^h(\phi_c^h, \phi_c^h),\\
B_c^h&\equiv\Pi_c^{h\prime}\mathscr{Q}_c^h(\phi_c^h, g_c^h),\\
C_c^h&\equiv\Pi_c^{h\prime}\mathscr{Q}_c^h(g_c^h, g_c^h),
\end{aligned}\tag{7.24}$$

and

$$D_c^h\equiv\Pi_c^{h\prime}\mathscr{C}_c^h(\phi_c^h, \phi_c^h, \phi_c^h)-3\Pi_c^{h\prime}\mathscr{Q}_c^h(\phi_c^h, \mathscr{L}_c^{h\dagger}\omega_c^h\mathscr{Q}_c^h(\phi_c^h, \phi_c^h)),$$

where

$$\begin{aligned}
g_c^h&\equiv-\mathscr{L}_c^{h\dagger}\omega_c^h\dot{F}^h(\mu_c^h, w_1^h(\mu_c^h)),\\
&=-\mathscr{L}_c^{h\dagger}\omega_c^h f^h.
\end{aligned}$$

Here, $\mathscr{Q}_c^h$ and $\mathscr{C}_c^h$ are defined as

$$\begin{aligned}
\mathscr{Q}_c^h(u, v)&\equiv F^{h\prime\prime}(\mu_c^h, w_1^h(\mu_c^h))(u, v)\\
&=P^h[\mathscr{B}^{(h)}(u, v)+\mathscr{T}^{(h)}(w_1^h(\mu_c^h), u, v)],
\end{aligned}\tag{7.25}$$

$$\begin{aligned}
\mathscr{C}_c^h(u, v, w)&\equiv F^{h\prime\prime\prime}(\mu_c^h, w_1^h(\mu_c^h))(u, v, w)\\
&=P^h\mathscr{T}^{(h)}(u, v, w),
\end{aligned}$$

$\Pi_c^{h\prime}u\equiv\langle u, \phi_c^h\rangle$; ω_c^h denotes the projection of V onto $\mathscr{R}_c^h=\text{range}\,\mathscr{L}_c^h=\{\ker\mathscr{L}_c^h\}^\perp$, and $\mathscr{L}_c^{h\dagger}$ the bounded inverse $V\to V$ such that $(\mathscr{L}_c^h)^\dagger\mathscr{L}_c^h=\omega_c^h$.

In view of Eqs. (7.11) and (7.20), the quantities A_c^h, B_c^h, C_c^h and D_c^h converge to A_c, B_c, C_c and D_c, respectively, with the same order as in Lemma 6.3.

Lemma 2.11, See. 2.2, shows that a symmetry breaking bifurcation point $\mathscr{C}$ can not be a fold bifurcation point, and consequently, $A_c^h=0$. Since B_c^h converges to B_c, the latter of which is non-zero by hypothesis,

B_c^h is bounded below for sufficiently small h. Moreover, if $D_c \neq 0$ (that is, $\mathscr{C}^+$ ia a cusp), then so is D_c^h. Thus, we have Lemma 7.4.

Lemma 7.5 *There is a constant $\delta' > 0$ independent of $h \in]0, h_0[$, such that the α-path can be extended to $J_{\delta'} = \{\alpha;\ |\alpha| < \delta'\}$.*

Proof. Let $\Gamma^h(\alpha, \nu) = 0$ be the bifurcation equation of (P^h) at $\mathscr{C}^{h,+}$. (*Cf.* Ep. (1.39).) Since $\mathscr{C}^{h,+}$ is a symmetry breaking bifurcation, it follows from Lemma 2.13, Sec. 2.2, that $\Gamma^h(\alpha, \nu) = \alpha\tilde{\Gamma}^h(\alpha, \nu)$. In other words, both the zeroth and second order terms in α vanish in the bifurcation equation Γ^h. Hence,

$$\begin{aligned}\Gamma^h(\alpha, \nu) &= \alpha\tilde{\Gamma}^h(\alpha, \nu)\\ &= \alpha\Big[B_c^h\nu + \frac{1}{3!}D_c^h\alpha^2 + \dots\Big].\end{aligned}$$

Γ^h and consequently $\tilde{\Gamma}^h$, are of equi-C^1, since they are derived from the equi-implicit function theorem as applied to F^h. Since $\tilde{\Gamma}^h(0, 0) = 0$ and $\partial\tilde{\Gamma}^h/\partial\nu(0, 0) = B_c^h \neq 0$ by the previous lemma, the equi-implicit function theorem can be applied to $\tilde{\Gamma}^h(\alpha, \nu) = 0$. Thus, there is a constant $\delta' > 0$ independent of h (sufficiently small), and a function $\nu = \nu^h(\alpha)$ such that $\tilde{\Gamma}^h(\alpha, \nu^h(\alpha)) = 0$ for $\alpha \in J_{\delta'}$.

7.3 Uniform convergence of the bifurcating path

Our final task is to show the uniform convergence of the bifurcating α^h-path to the corresponding α-path of (P). We let denote the α-path of (P) by $(\mu_b(\alpha), w_b(\alpha)) \in \boldsymbol{R} \times W$, where

$$\mu_b(\alpha) = \mu_c + \nu_b(\alpha), \tag{7.26}$$

and

$$w_b(\alpha) = w_1(\mu_c) + \alpha\phi_c + \chi_b(\alpha), \qquad \alpha \in J_{\delta'}.$$

See, Prop. 1.10. Similarly, the α^h-path are written as

$$\begin{aligned}&\mu_b^h(\alpha) = \mu_c^h + \nu_b^h(\alpha),\\ &w_c^h(\alpha) = w_1^h(\mu_c^h) + \alpha\phi_c^h + \chi_b^h(\alpha), \qquad \alpha \in J_{\delta'}.\end{aligned} \tag{7.27}$$

We remind from Prop. 1.10 that,

$$|\nu_b(\alpha)| \leq C|\alpha|^2 \text{ and } \|\chi_b(\alpha)\|_V \leq C'|\alpha|^2, \qquad \alpha \in J_{\delta'}. \tag{7.28}$$

Moreover, it holds that for all $g \in \mathscr{G}$,

$$\nu_b(T_g\alpha)=\nu_b(\alpha) \text{ and } T_g\chi_b(\alpha)=\chi_b(T_g\alpha), \qquad \alpha\in J_{\delta'}. \tag{7.29}$$

The later relation is from Eq. (2.20), and the first is a consequence of the $\mathscr{G}$-covariance of $\Gamma(\alpha, \nu)=\alpha\tilde{\Gamma}(\alpha, \nu)$ (Lemma 2.13). In fact, $\tilde{\Gamma}(T_g\alpha, \nu)=\tilde{\Gamma}(\alpha, \nu)$, for all $g\in\mathscr{G}$, from which follows $\nu_b(\alpha)=\nu_b(T_g\alpha)$ ($^\forall g\in\mathscr{G}$) by the uniqueness of $\nu_b(\alpha)$. We note that the same properties hold for $\nu_b^h(\alpha)$ and $\chi_b^h(\alpha)$, since (P^h) is also a $\mathscr{G}$-covariant problem.

As in Sec. 6.3, we let for $\alpha\in J_{\delta'}$,

$$\nu_b^h(\alpha)=\nu_b(\alpha)+\varepsilon_b^h(\alpha), \tag{7,30$_1$}$$

and

$$\chi_b^h(\alpha)=\omega_c^hP^h\chi_b(\alpha)+\eta_b^h(\alpha), \tag{7.30$_2$}$$

with the conditions

$$\varepsilon_b^h(0)=0 \text{ and } \eta_b^h(0)=0. \tag{7.30$_3$}$$

Substituting Eqs. $(7.30)_{1,2}$ into (P^h), we have as the ω_c^h-component in the Lyapounov-Schmidt decomposition:

$$\begin{aligned}&\mathscr{L}_c^h\eta_c^h+\omega_c^h\mathscr{Q}_c^h(\alpha\phi_c^h+\omega_c^hP^h\chi_b(\alpha), \eta_b^h)+\frac{1}{2}\omega_c^h\mathscr{Q}_c^h(\eta_b^h, \eta_b^h)+\omega_c^h[R_c^h(\alpha\phi_c^h\\&\quad+\omega_c^hP^h\chi_b(\alpha)+\eta_b^h)-R_c^h(\alpha\phi_c^h+\omega_c^hP^h\chi_b(\alpha))]\\&\quad+\varepsilon_b^h\omega_c^hf^h+\omega_c^h\tilde{G}_c^h(\alpha)=0,\end{aligned} \tag{7.31$_1$}$$

where

$$\tilde{G}_c^h(\alpha)\equiv F^h(\mu_c^h+\nu_b(\alpha), w_1^h(\mu_c^h)+\alpha\phi_c^h+\omega_c^hP^h\chi_b(\alpha)), \tag{7.31$_2$}$$

and

$$R_c^h(u)\equiv\mathscr{C}_c^h(u, u, u). \tag{7.31$_3$}$$

Noting the equi-invertibility of $\mathscr{L}_c^h$ on $\mathscr{R}_c^h$ and the estimate $||G_c^h(\alpha)||_V\le C|\alpha|^2h^{k+l}$ (Lemma 6.7), η_b^h can be uniquely solved as a function of α and ε_b^h:

$$\begin{aligned}\eta_b^h&=\eta_b^h(\alpha, \varepsilon_b^h)\\&=-\mathscr{L}_c^{h\dagger}\omega_c^h\tilde{G}_c^h(\alpha)+\varepsilon_b^hg_c^h+\dots,\end{aligned} \tag{7.32$_1$}$$

with

$$||\eta_b^h||_V=O(|\varepsilon_b^h|, ||G_c^h(\alpha)||_V)\le O(|\varepsilon_b^h|, |\alpha|^2) \text{ for } h\in]0, h_0[. \tag{7.32$_2$}$$

Now, we want to solve the kernel component in the Lyapounov-Schmidt decomposition. After the substitution of $\eta_b^h=\eta_b^h(\alpha, \varepsilon_b^h)$, we have

$$\begin{aligned}\Xi^h(\alpha,\ \varepsilon_b^h) \equiv &< \mathcal{Q}_c^h(\alpha\phi_c^h + \omega_c^h P^h \chi_b(\alpha),\ \eta_b^h(\alpha,\ \varepsilon_b^h)),\ \phi_c^h > \\ &+ \frac{1}{2} < \mathcal{Q}_c^h(\eta_b^h(\alpha,\ \varepsilon_b^h),\ \eta_b^b(\alpha,\ \varepsilon_b^h)),\ \phi_c^h > + < R_c^h(\alpha\phi_c^h \\ &+ \omega_c^h P^h \chi_b(\alpha) + \eta_b^h(\alpha,\ \varepsilon_b^h)) - R_c^h(\alpha\phi_c^h + \omega_c^h P^h \chi_b(\alpha)),\ \phi_c^h > \\ &+ < \tilde{G}_c^h(\alpha),\ \phi_c^h > = 0,\end{aligned}$$

in which $<f^h,\ \phi_c^h> = 0$ has been used.

Lemma 7.6 (i) *Ξ^h and $\tilde{G}_c^h$ are covariant under $\mathcal{G}$, i.e.*,

$$T_g \Xi^h(\alpha,\ \varepsilon_b^h) = \Xi^h(T_g\alpha,\ \varepsilon_b^h) \qquad \text{for all } g \in \mathcal{G}, \tag{7.34}$$

and

$$T_g \tilde{G}_c^h(\alpha) = \tilde{G}_c^h(T_g\alpha) \qquad \text{for all } g \in \mathcal{G}. \tag{7.35}$$

(ii) $$|<\tilde{G}_c^h(\alpha),\ \phi_c^h>| \le C|\alpha|^3 h^{k+l}. \tag{7.36}$$

Proof. Eq. (7.35) is a result of the $\mathcal{G}$-covariance of F^h and Eq. (7.29), since w_1^h is $\mathcal{G}$-invariant. (See, Eq. $(7.32)_2$.) Eq. (7.34) follows from this relation and the $\mathcal{G}$-covariance of η_b^h. Next, since $\gamma_c^h(\alpha) \equiv <\tilde{G}_c^h(\alpha),\ \phi_c^h>$ is covariant under $\mathcal{G}$ thanks to Eq. (7.35), the quadratic term in α of $\gamma_c^h(\alpha)$ vanishes. Together with Lemma 6.7, we have Eq. (7.36).

Thus, we may rewrite $\Xi^h(\alpha,\ \varepsilon_b^h) = 0$ as

$$\Xi^h(\alpha,\ \varepsilon_b^h) = \alpha\tilde{\Xi}^h(\alpha,\ \varepsilon_b^h) = 0. \tag{7.37}$$

After a short calculation, we find that $\tilde{\Xi}^h(0,\ 0) = 0$ and $\partial\tilde{\Xi}^h/\partial\varepsilon_b^h(0,0) = B_c^h$. Hence, the equi-implicit function theorem works to solve uniquely $\varepsilon_b^h = \varepsilon_b^h(\alpha)$ as

$$\begin{aligned}\varepsilon_b^h = (B_c^h)^{-1}[&< \mathcal{Q}_c^h(\mathcal{L}_c^{h\dagger}\omega_c^h \tilde{G}_c^h(\alpha),\ \phi_c^h),\ \phi_c^h > - \alpha^{-1} < \tilde{G}_c^h(\alpha),\ \phi_c^h >] \\ &+ (h.\ o.\ t.),\end{aligned} \tag{7.38}$$

with the estimate

$$|\varepsilon_b^h(\alpha)| \le C|\alpha|^2 h^{k+l} \text{ for } \alpha \in J_{\delta''}(^\exists\delta'' > 0), \tag{7.39}$$

due to Eqs. (7.34) and (7.38).

Now, we arrive at

Proposition 7.7 *For each $h \in]0,\ h_0[$, there is a unique bifurcating path* (α^h), $(\mu_b^h(\alpha),\ w_b^h(\alpha)) \in \boldsymbol{R} \times V^h$, $\alpha \in J_{\delta''}(^\exists\delta'' > 0$ independent of $h)$, which crosses $\mathcal{C}^{h,+}$ at $\alpha = 0$. *Moreover, the α-path converges uniformly to the α-path* *of* (P), *namely, to* $(\mu_b(\alpha),\ w_b(\alpha))\in \boldsymbol{R}\times W,\ \alpha\in J_{\delta''}$, *in the following sense:*

$$|\mu_b^h(\alpha)-\mu_b(\alpha)|\leq Ch^{k+l}, \tag{7.40}$$

$$\|w_b^h(\alpha)-P^h w_b(\alpha)\|_V\leq C'h^{k+l}. \tag{7.41}$$

Also,

$$\|w_b^h(\alpha)-w_b(\alpha)\|_V\leq C''h^l, \tag{7.42}$$

and

$$\|w_b^h(\alpha)-w_b(\alpha)\|_U\leq C'''h^{k+l}. \tag{7.43}$$

Proof. In view of Eqs. $(7.32)_2$ and (7.39), we have that

$$\|\eta_b^h(\alpha)\|_V\leq C|\alpha|^2h^{k+l}. \tag{7.44}$$

The proof is then parallel to that of Prop. 6.8.

In a neghborhood of $\mathscr{C}^{h,+}$, we have thus a situation as shown in Fig. 7.2. One may at this stage wonder whether the α^h-path bifurcating from $\mathscr{C}^{h,+}$(plotted as a dotted line) connects to the two ordinary paths (the waved lines), the existence of the latter has been established by Proposition 4.6. This is, however, certainly true when one chooses h sufficiently small, for all $h\in]0,\ h_0[$. In fact, by Lemma 7.5 the α^h-path emerging from $\mathscr{C}^{h,+}$ reaches to a constant value of $\alpha=\delta'$, uniformly for $h\in]0,\ h_0[$. On the other hand, Proposition 4.6 shows that as h_0 is chosen smaller, the outer circle of Fig. 7.2, outside of which is the existence region of the ordinary paths (waved lines), *shrinks*. Thus, choosing h_0 sufficiently

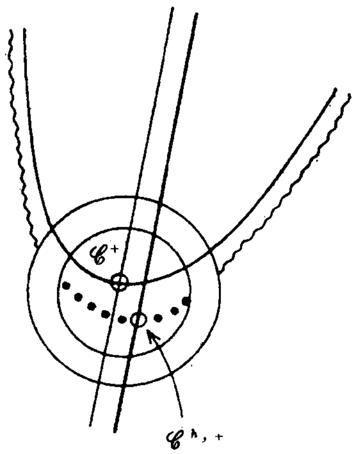


Fig. 7.2

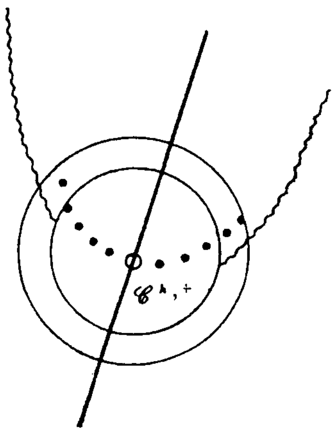


Fig. 7.3

small, the outer circle of Fig. 7.2 comes inside of the other circle, which shows the existence region of the α^h-path. See, Fig. 7.3. We have hence a region where both the α^h-path and the ordinary path coexist. However, both paths are ordinary paths in this region, and hence they are both close to the ordinary path of the original problem (P), the uniqueness of the approximate ordinary path in Proposition 4.6 implies that these two paths should coincide in this region. Accordingly, we have a complete numerical realization of paths in the vicinity of symmetry breaking cusp bifurcation point $\mathscr{C}^+$. See, Fig. 7.4.

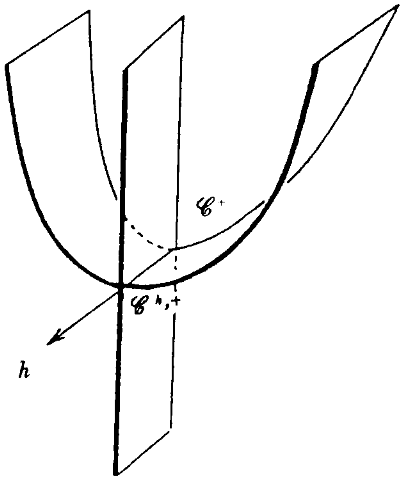


Fig. 7.4

Appendix A Further properties of the von Kármán-Donnell-Marguerre equation

Let $\Omega\subset \boldsymbol{R}^2$ be a bounded domain, and $\partial\Omega$ its boundary. We assume that $\partial\Omega$ is either *smooth* or *convex polygonal*.

We consider the biharmonic problem:

$$(G)\quad \left|\begin{array}{l}\Delta^2\chi=f \text{ in } \Omega,\\ \chi=\dfrac{\partial\chi}{\partial n}=0 \text{ on } \partial\Omega\end{array}\right. \tag{A-1}$$

for $f\in H^{-2}(\Omega)$. The mapping $G: f\rightarrow\chi$ of $H^{-2}(\Omega)$ into $H_0^2(\Omega)$ is continuous. Moreover, G is a linear, continuous operator of $H^{-1+\sigma}(\Omega)$ into $H^{3+\sigma}(\Omega)\cap H_0^2(\Omega)$, $\sigma\in[0, \sigma_0]$; σ_0 being a constant depending only on $\partial\Omega$, such that

$$0<\sigma_0\leq 1. \tag{A-2}$$

For the proof, see P. Grisvard [15]. Note that when $\partial\Omega$ is smooth, $\sigma=1$ and hence, $\chi\in H^4(\Omega)\cap H_0^2(\Omega)$. Also, if Ω is a *rectangle*, $\sigma_0=1$ (Mizutani*)). We let $G_0\equiv G|H^0(\Omega)$.

Definition A-1 Let
$U=H^0(\Omega)(\equiv L^2(\Omega))$, $V=H_0^2(\Omega)$, $W=H^{3+\sigma}(\Omega)\cap H_0^2(\Omega)$, $\sigma\in[0, \sigma_0]$. We equippe with V the inner product

$$<u, v>\overset{def}{=}\int_\Omega \Delta u\Delta v^{**)}$$

Lemma A-2
$G\in \boldsymbol{B}(V', V)\cap \boldsymbol{B}(U, W)$.

We shall later need to consider the problem (G) in a *weaker* sense, i.e., we consider (G) for $f\in W'$. Indeed, we consider the dual problem:

$$(G')_1 \quad \left| \begin{array}{l} \text{For } f\in W', \text{ seek } \chi\in U(=H^0(\Omega)) \text{ such that} \\ (\chi, \psi)_U=\int_\Omega fG_0\psi \text{ for } {}^\forall\psi\in U. \end{array} \right.$$

Here, $\int_\Omega fG_0\psi$ being the duality pair between W' and W, there is a continuous mapping G_0': $W'\to U'=U$ such that $\int_\Omega fG_0\psi=(G_0'f, \psi)_U$. Accordingly, there is a unique solution of $(G')_1$: $\chi=G_0'f$ such that

$$||\chi||_U\leq C||f||_{W'}.$$

Next, we let in $(G')_1$ $\phi=G_0\psi$. Then, $\phi\in W$ and ϕ, $\frac{\partial\phi}{\partial\nu}$ vanish on $\partial\Omega$. We see that $(G')_1$ is equivalent to

$$(G')_2 \quad \int_\Omega \chi\Delta^2\phi=\int_\Omega f\phi \qquad \text{for } {}^\forall\phi\in\mathscr{D}(\Omega)^{***)}$$

Moreover, we have that

---

*) Notice that for any $u\in H_0^1(\Omega)\cap H^2(\Omega)$,

$$(*) \quad ||u||_{H^2(\Omega)}\leq C_1||\Delta u||_{H^0(\Omega)}$$

provided $\partial\Omega$ is smooth or convex polygonal. Hence, $||u||_V=\sqrt{<u, u>}$ can be a norm in V. The inequality $(*)$ follows from the H^2-regularity of the solution of the boundary value problem:

$$-\Delta u=f \quad \text{in } \Omega,$$
$$u=0 \quad \text{on } \partial\Omega$$

where $f\in H^0(\Omega)$. (*cf.* Kondrat'ev [27])

**) *Private communications.*

***) $\mathscr{D}(\Omega)=C_0^\infty(\Omega)$.

$$G_0'|V'=G \text{ on } V',$$

and

$$G_0'|U=G_0 \text{ on } U.$$

In summary,

Proposition A-3 *There is a unique $\chi \in U$, the soluticn of (G'), such that $\chi = G_0' f$ with*

$$||\chi||_U \leq C||f||_{W'}. \tag{A-3}$$

Morever,

$$G_0'|V'=G, \text{ and } G_0'|U=G_0. \tag{A-4}$$

The theory of interpolation spaces (See, Lions-Magenes [30]) together with results in Lemma A-2 yields the following

Lemma A-4 *There exist positive numbers $\delta_0=\delta_0(\partial\Omega)$ and $\varepsilon_0=\varepsilon_0(\partial\Omega)$, such that for any $\delta_0>\delta\geq 0$ and $\varepsilon_0>\varepsilon\geq 0$, G is continuous as a mapping*

$$G: H^{-\delta}(\Omega) \to H^{3+\delta'}(\Omega) \cap H_0^2(\Omega) \qquad ({}^\exists\delta'>0), \tag{A-5}$$

and

$$G: H^{-1-\varepsilon}(\Omega) \to H^{\frac{5}{2}+\varepsilon'}(\Omega) \cap H_0^2(\Omega) \qquad ({}^\exists\varepsilon'>0). \tag{A-6}$$

In fact, choosing $\delta_0=2\sigma/1+\sigma$ and $\varepsilon_0=\sigma/1+\sigma$ in the interpolation space theory we have the desired results.

Definition A-5
For $u, v \in \mathscr{D}(\Omega)$, we define the bracket $[\ ,\ ]$:

$$\begin{aligned}[u, v] &\equiv u_{xx}v_{yy} - 2u_{xy}v_{xy} + u_{yy}v_{xx} \\ &= (u_{yy}v_x - u_{xy}v_y)_x + (u_{xx}v_y - u_{xy}v_x)_y \\ &= (u_{yy}v)_{xx} - 2(u_{xy}v)_{xy} + (u_{xx}v)_{yy}\end{aligned} \tag{A-7}$$

Lemma A-6 *For $u, v, w \in \mathscr{D}(\Omega)$,*

(i) $$\int_\Omega [u, v]w = \int_\Omega [v, u]w = \int_\Omega [w, u]v, \tag{A-8}$$

(ii) $$\left|\int_\Omega [u, v]w\right| \leq C||u||_V||v||_V||w||_V, \tag{A-9}$$

(iii) $$\left|\int_\Omega [u, v]w\right| \leq C||u||_U||v||_V||w||_W. \tag{A-10}$$

Proof. (i) is obvious from the divergence identity and the integration-by parts. To show (ii), we notice that

$$\int_{\Omega}[u, v]w=\int_{\Omega}(u_{xy}v_{y}-u_{yy}v_{x})w_{x}+(u_{xy}v_{x}-u_{xx}v_{y})w_{y}.$$

Since $H_0^2(\Omega)\subsetneq W_0^{1,p}(\Omega)$ for $p<+\infty$ by the Sobolev lemma,

$$\left|\int_{\Omega}[u, v]w\right|\leq C\|u\|_{H_0^2}\|v\|_{W_0^{1,4}}\|w\|_{W_0^{1,4}}$$
$$\leq C'\|u\|_{H_0^2}\|v\|_{H_0^2}\|w\|_{H_0^2}.$$

Also, since $H^{1+\varepsilon}(\Omega)\subsetneq L^{\infty}(\Omega)$ if $n=2$ and $\varepsilon>0$ (Peetre [37]),

$$\left|\int_{\Omega}[v, w]u\right|\leq C\|u\|_{H^0}\|v\|_{H_0^2}|D^2w|_{L^\infty}$$
$$\leq C'\|u\|_{H^0}\|v\|_{H_0^2}\|w\|_{H^{3+\varepsilon}}.$$

Lemma A-6 (ii) implies that the form $\int_{\Omega}[u, v]\phi$ is a bounded, linear functional on V, linear in each u and v in V. By virtue of the Riesz representation theorem, there is a contiuous bilinear mapping $\mathscr{B}: V\times V\rightarrow V$, which is defined by

$$<\mathscr{B}(u, v), \phi>=\int_{\Omega}[u, v]\phi \text{ for } {}^{\forall}\phi\in V. \qquad \text{(A-11)}$$

Lemma A-7

(i) $\mathscr{B}: V\times V\rightarrow V$, *continuous, separately compact and separately self-adjoint.* (A-12)

(ii) $\mathscr{B}: U\times W\rightarrow V$ *continuous,*
$\mathscr{B}: V\times W\rightarrow W$ *continuous,*
$\mathscr{B}: U\times V\rightarrow U$ *continuous.* (A-13)

Proof.

(i) For $u, v\in H_0^2(\Omega)$, $[u, v]\in L^1(\Omega)\subset H^{-1-\varepsilon}({}^{\forall}\varepsilon>0)$ (cf. Peetre [37]). Thanks to Lemma A-4,

$$\mathscr{B}(u, v)=G[u, v]\in H^{3+\varepsilon'}(\Omega) \qquad (\varepsilon'>0).$$

Thus, for $u\in V(=H_0^2(\Omega))$, $\mathscr{B}(u, \cdot): H_0^2(\Omega)\rightarrow H^{3+\varepsilon'}(\Omega)$. The separately self-adjointness of $\mathscr{B}$ follows from Lemma A-6, (i) and (ii). Note that this property means that the form $<\mathscr{B}(u, v), w>$ is symmetric in u, v, $w\in V$.

(ii) For $u\in U$, $v\in V$ and $w\in W$,
$[u, w]\in V'$,

$$[v, w] \in U' = U,$$
$$[u, v] \in W'$$

in view of Lemma A-6 (iii). Thus, Lemma A-1 shows that

$$\mathscr{B}(u, w) = G[u, w] \in V$$

and

$$\mathscr{B}(v, w) = G[v, w] \in W.$$

As for $[u, v] \in W'$, we may have to think of $\mathscr{B}(u, v)$ as distribution-sense, namely, as $G_0'[u, v]$. Then, Proposition A-3 yields that

$$\begin{aligned} \|\mathscr{B}(u, v)\|_U &\leq C\|[u, v]\|_{W'} \\ &\leq C'\|u\|_U\|v\|_V. \end{aligned}$$

Now let us introduce the von Kármán-Donnell-Marguerre equation defined on Ω: (*see*, [3], [32] and [52].)

$$(K. D. M.)_0 \left| \begin{array}{ll} \Delta^2\phi = -\frac{1}{2}[w, w] - [w_0, w] & \\ & \text{in } \Omega \\ \Delta^2 w = [w_0 + w, \phi_0 + \phi] - \mu p & \\ \text{with} & \\ \phi = \frac{\partial\phi}{\partial n} = w = \frac{\partial w}{\partial n} = 0 & \text{on } \partial\Omega. \end{array} \right\} \qquad \text{(A-14)}$$

Here, w_0 represents the known initial deflection; ϕ_0 the known Airy function of the applied force to the edge; p the external normal load on the shell with $\mu \in \boldsymbol{R}$ the loading parameter. We assume that $w_0, \phi_0 \in W$ and $p \in U$. Using the same notations, $(K. D. M.)_0$ is reduced to a pair of nonlinear operator equations in V:

$$(K. D. M.)_1 \left| \begin{array}{l} \phi = -\frac{1}{2}\mathscr{B}(w, w) - \mathscr{B}(w_0, w) \\ w = \mathscr{B}(w_0 + w, \phi_0 + \phi) - \mu G p \end{array} \right. \qquad \text{(A-15)}$$

Eliminating ϕ from $(K. D. M.)_1$, we have the single operator equation in V:

$$(K. D. M.)_2 \left| \begin{array}{l} [I - \mathscr{B}(\phi_0, \cdot) + \mathscr{B}(w_0, \mathscr{B}(w_0, \cdot))]w \\ + \frac{1}{2}[\mathscr{B}(w_0, \mathscr{B}(w, w)) + 2\mathscr{B}(w, \mathscr{B}(w_0, w))] \\ + \frac{1}{2}\mathscr{B}(w, \mathscr{B}(w, w)) = -\mu G p + \mathscr{B}(w_0, \phi_0) \end{array} \right. \qquad \text{(A-16)}$$

We may generally assume that when $\mu = 0$, the shell is in an equilibrium state, i. e., $w\equiv\psi\equiv 0$, which in turn implies $\mathscr{B}(w_0, \psi_0)\equiv 0$. We call a pair $(\mu, w)\in \boldsymbol{R}\times V$ which satisfies $(K. D. M.)_2$ the *weak solution* of the *K. D. M.* equation.

If we put

$$\left.\begin{aligned} &L=-\mathscr{B}(\psi_0, \cdot)+\mathscr{B}(w_0, \mathscr{B}(w_0, \cdot)) \\ &\mathscr{B}(u, v)=\mathscr{T}(u, v, w_0) \\ \text{and}\quad &\mathscr{T}(u, v, w)=\{\mathscr{B}(u, \mathscr{B}(v, w))+\mathscr{B}(v, \mathscr{B}(w, u)) \\ &\qquad +\mathscr{B}(w, \mathscr{B}(u, v))\}, \end{aligned}\right\} \quad \text{(A-17)}$$

$(K. D. M.)$ is exactly the equation (3. 2):

$$(P)\quad F(\mu, w)\equiv(I+L)w+\frac{1}{2!}\mathscr{B}(w, w)+\frac{1}{3!}\mathscr{T}(w, w, w)+\mu Gp=0. \tag{A-18}$$

We collect some results on $\mathscr{B}$ and $\mathscr{T}$, which are merely a Corollary of Lemma A-7.

Lemma A-8 *$\mathscr{B}(\mathscr{T})$ is symmetric, bi-(tri-) linear, continuous map $(V\times) V\times V\to V$, separately compact and separately self-adjoint. Morever, $\mathscr{T}$ is continuous as a mapping*

$$\left.\begin{aligned} \mathscr{T}:\ & U\times V\times V\to U \\ & U\times V\times W\to V, \\ & V\times V\times W\to W. \end{aligned}\right\} \quad \text{(A-19)}$$

$\mathscr{B}$ is continuous as a mapping

$$\left.\begin{aligned} \mathscr{B}:\ & U\times V\to V \\ & U\times W\to V \\ & V\times W\to W \end{aligned}\right\} \quad \text{(A-20)}$$

L is linear, bounded, compact and self-adjoint operator $V\to V$, such that

$$L\in \boldsymbol{B}_0(V)\cap \boldsymbol{B}(U, V)\cap \boldsymbol{B}(V, W). \tag{A-21}$$

Now a result on the regularity of the solutions of $(K. D. M.)$ equation.

Proposition A-9 (*Regularity of K. D. M. solutions*)
Any weak solution of (K. D. M) $(\mu, w)\in \boldsymbol{R}\times V$ is in $\boldsymbol{R}\times W$.

Proof. It may be convenient to work with $(K. D. M.)_0$. Firstly, for any $u, v\in V(=H_0^2(\Omega))$, $[u, v]\in L^1(\Omega)\subset H^{-1-\epsilon}(\Omega)\ ({}^\forall\epsilon>0)$. Hence, $\psi, w\in H^{2.5+\epsilon'}(\Omega)\cap H_0^2(\Omega)\ (\epsilon'>0)$ by Lemma A-4. Next, for any $u\in H^{2.5+\epsilon'}(\Omega)\cap$

$H_0^2(\Omega)$ $(\varepsilon'>0)$, $D^2 u \in H^{\frac{1}{2}+\varepsilon'}(\Omega) \subset L^{\frac{4}{1-2\varepsilon'}}(\Omega) \subset L^4(\Omega)$. Hence, if u, $v \in H^{2.5+\varepsilon'}(\Omega) \cap H_0^2(\Omega)$, $[u, v] \in H^0(\Omega)$ $(=L^2(\Omega))$. Since the right-hand side of Eqs. $(\text{A-14})_{1,2}$ are both in $H^0(\Omega)$, we have that w, $\psi \in H^{3+\varepsilon}(\Omega) \cap H_0^2(\Omega)$ by virtue of Lemma A-2.

Finally, we give a proof of

Proposition A-10 *The von Kármán-Donnell-Marguerre opeator associated with* (A-18) *is covariant under* $\mathcal{O}$, *where* $\mathcal{O} \subset O(2)$ *is the maximal symmetry group of* Ω.

Proof. Let $\mathcal{O} \subset O(2)$ be a classical orthogonal group. We recall that

$$(T_O u)(x) = u(\mathcal{O}^{-1} x), \tag{A-22}$$

or equivalently,

$$(T_O u)(x) = u(y), \tag{A-23}$$

where

$$y_i = O_{ji} x_j. \tag{A-23}$$

Here and in the sequel, the summation convention is understood. By the chain rule, we have that

$$\frac{\partial}{\partial x_i} = \frac{\partial y_m}{\partial x_i} \frac{\partial}{\partial y_m} = O_{im} \frac{\partial}{\partial y_m}. \tag{A-24}$$

We use the notations:

$$u_{i,j} = \frac{\partial^2 u}{\partial x_i \partial x_j},$$
$$u^{l,m} = \frac{\partial^2 u}{\partial y_l \partial y_m}, \tag{A-25}$$

and

$$[u, v] = u_{1,1} v_{2,2} + u_{2,2} v_{1,1} - 2 u_{1,2} v_{1,2} \tag{A-26}$$

Firstly,

Lemma A-11 *It holds that*

$$T_O[u, u] = [T_O u, T_O u], \quad {}^{\forall} u \in V. \tag{A-27}$$

Proof. By the chain rule, it holds that for i, $j=1, 2$,

*) *cf.* Peetre [37]. $H^{1-\varepsilon} \subset L^{\frac{2}{\varepsilon}}$ $(\varepsilon>0)$.

$$(T_0u)_{i,j}(x)=\frac{\partial^2}{\partial x_i \partial x_j}u(y),$$
$$=O_{il}O_{jm}u^{l,m}(y),$$
$$=(\mathcal{O}^T\tilde{U}\mathcal{O})_{ij}, \tag{A-28}$$

where

$$\tilde{U}=\begin{bmatrix} u^{1,1}(y) & u^{1,2}(y) \\ u^{2,1}(y) & u^{2,2}(y) \end{bmatrix},\ (u^{2,1}\equiv u^{1,2}).$$

Noting that

$$[u,\ u]=2\ \det\begin{bmatrix} u_{1,1}(x) & u_{1,2}(x) \\ u_{2,1}(x) & u_{2,2}(x) \end{bmatrix},\ (u_{1,2}\equiv u_{2,1}),$$

it is enough to show that

$$T_o \det\begin{bmatrix} u_{1,1} & u_{1,2} \\ u_{2,1} & u_{2,2} \end{bmatrix}=\det\begin{bmatrix} (T_0u)_{1,1} & (T_0u)_{1,2} \\ (T_0u)_{2,1} & (T_0u)_{2,2} \end{bmatrix}$$

while the right-hand side is equal to det $(\mathcal{O}^T\tilde{U}\mathcal{O})$, which is equal to det $(\tilde{U})$.

Using this lemma, we find easily the following

Lemma A-12 *It holds that*

$$T_0[u,\ v]=[T_0u,\ T_0v],\ {}^{\forall}u,\ v\in V. \tag{A-29}$$

Proof. This follows immediately from the relation

$$[u,\ v]=\frac{1}{2}[u+v,\ u+v]-\frac{1}{2}[u,\ u]-\frac{1}{2}[v,\ v], \tag{A-30}$$

and using the previous lemma.

Now, Proposition A-10 is a direct consequence of Lemma A. 12 and the fact that the Laplacian Δ is covariant under $\mathcal{O}$. It is noted that in order that F is covariant under $\mathcal{O}$ in Eq. (A-18), ψ_0, w_0 and p should be also invariant under $\mathcal{O}\subset O(2)$.

Appendix B Verification of properties of an approximate finite element scheme to the von Kármán-Donnell-Marguerre equation

In this Appendix *B*, we construct a class of conforming finite element schemes for the von Kármán-Donnell-Marguerre equation. We shall show that *the scheme* possesses all the properties which are required in §3. 2.

Notice first that we keep the same symbols as in Appendix A. Remind that

$$U=H^0(\Omega),\ V=H_0^2(\Omega) \text{ and } W=H^{3+\sigma}(\Omega)\cap H_0^2(\Omega),$$

where $\Omega\subset\boldsymbol{R}^2$ is either *a smooth* or *convex polygonal* domain, and $\sigma\in[0, \sigma_0]$ where $\sigma_0=\sigma_0(\partial\Omega)$ is a constant depending on $\partial\Omega$ such that $0<\sigma_0\leq 1$.

We let V^h, $h\in]0, h_0[$ be a family of finite dimensional subspaces of V with the properties:

$$\inf_{v^h\in V^h}||v-v^h||_{H^2(\Omega)}\leq Ch^{r-2}||u||_{H^r(\Omega)} \tag{B-1}$$

for all $u\in H^r(\Omega)\cap H_0^2(\Omega)$, $2\leq r\leq k+1$.

Here, k is a given parameter associated with the family $\{V^h\}$, and in fact k is the degree of the piecewise polynomial approximations. Such examples are known e. g., in [10], [49]. In our problem, we suppose $r\equiv 3+\sigma(0\leq\sigma\leq 1)$ and hence $k\leq 3$. Firstly, we study the projection P^h: $V\rightarrow V^h$, which is defined by: for $u\in V$,

$$<P^hu,\ \phi^h>=<u,\ \phi^h> \qquad {}^\forall\phi^h\in V^h. \tag{B-2}$$

Lemma B-1 *For all $u\in W$, it holds that*

$$||(I-P^h)u||_V\leq Ch^{1+\sigma}||u||_W, \tag{B-3}$$

and

$$||(I-P^h)u||_U\leq C'h^{2(1+\sigma)}||u||_W. \tag{B-4}$$

Proof. The first assertion is classical, and the second is due to Nitsche's trick (cf. Strang and Fix [49]). We sketch the proof for completeness. If we note that

$$||(I-P^h)u||_V\leq \inf_{v^h\in V^h}||u-v^h||_V$$

Eq. (B-3) is immediate from (B-1). Next, we consider an auxiliary problem

$$(X)\left|\begin{array}{ll} \Delta^2\chi=(I-P^h)u & \text{in } \Omega, \\ \chi,\ \dfrac{\partial\chi}{\partial n}=0 & \text{on } \partial\Omega, \end{array}\right. \tag{B-5}$$

we have that by Lemma A-1,

$$||\chi||_W \leq C||(I-P^h)u||_U. \tag{B-6}$$

On the other hand, taking the inner product with $(I-P^h)u$, Eq. (X) becomes after integration-by-parts,

$$\begin{aligned}||(I-P^h)u||_U^2 &= \int_\Omega \Delta\chi\Delta(I-P^h)u \\ &= <\chi,\ (I-P^h)u> \\ &= <\chi-\chi^h,\ (I-P^h)u>\end{aligned} \tag{B-7}$$

for any $\chi^h \in V^h$, in view of (B-2). Thus, choosing $\chi^h = P^h\chi$, we have that

$$\begin{aligned}||(I-P^h)u||_U^2 &\leq ||\chi-\chi^h||_V||(I-P^h)u||_V \\ &\leq Ch^{1+\sigma}||\chi||_W Ch^{1+\sigma}||u||_W \\ &\leq C'h^{2(1+\sigma)}||(I-P^h)u||_U||u||_W\end{aligned} \tag{B-8}$$

Now, let us consider the finite element approximation of the von Kármán-Donnell-Marguerre problem $(K.\ D.\ M.)_0$.

The scheme may be described in the *weak form* as:

Seek ϕ^h, $w^h \in V^h$ *such that*

$$(K.\,D.\,M.)_0^h \left|\begin{array}{l}\int_\Omega \Delta\phi^h\Delta\phi^h = -\frac{1}{2}\int_\Omega [w^h,\ w^h]\phi^h - [\tilde{w}_0,\ w^h]\phi^h \\ \int_\Omega \Delta w^h \Delta v^h = \int_\Omega [\tilde{w}_0 + w^h,\ \tilde{\phi}_0 + \phi^h]v^h - \mu\int_\Omega \tilde{p}v^h \\ \text{for all } \phi^h,\ v^h \in V^h.\end{array}\right. \tag{B-9}$$

Here, $\tilde{w}_0$, $\tilde{\phi}_0$ and $\tilde{p}$ are some functions in V, V and U, respectively, which are close to w_0, ϕ_0 and p in an appropriate sense.*) However, we simply assume in the sequel that $\tilde{w}_0 = w_0$ etc. *only* for the sake of brevity.

$(K.\ D.\ M.)_0^h$ may be written as:

$$\left|\begin{array}{l}<\phi^h,\ \phi^h> = -\frac{1}{2}<\mathscr{B}(w^h,\ w^h),\ \phi^h> - <\mathscr{B}(w_0,\ w^h),\ \phi^h> \\ <w^h,\ v^h> = <\mathscr{B}(w_0+w^h,\ \phi_0+\phi^h),\ v^h> - \mu<Gp,\ v^h> \\ \text{for all } \phi^h,\ v^h \in V^h,\end{array}\right. \tag{B-10}$$

we have a system of operator equations:

$$(K.\,D.\,M.)^h \left|\begin{array}{l}\phi^h = -\frac{1}{2}P^h\mathscr{B}(w^h,\ w^h) - P^h\mathscr{B}(w_0,\ w^h) \\ w^h = P^h\mathscr{B}(w_0+w^h,\ \phi_0+\phi^h) - \mu P^hGp\end{array}\right. \tag{B-11}$$

*) These functions can be in V^h. What should be assumed is that e.g., $||w_0-\tilde{w}_0||_V \leq Ch^{1+\sigma}$, and $||w_0-\tilde{w}_0||_U \leq C'h^{2(1+\sigma)}$, etc.

which corresponds to $(K.\ D.\ M)_1$.

Elimination of ψ from the above equations yields a single operator equation which is equivalent to the scheme $(K.\ D.\ M)_0^h$:

$$(P^h)\quad \left|\begin{aligned} F^h(\mu,\ w^h) &\equiv (I+L^h)w^h+\frac{1}{2!}\mathscr{B}^h(w^h,\ w^h)+\frac{1}{3!}\mathscr{T}^h(w^h,\ w^h,\ w^h) \\ &\quad +\mu G^h p \\ &= (I+P^h L^{(h)})w^h+\frac{1}{2!}P^h\mathscr{B}^{(h)}(w^h,\ w^h) \\ &\quad +\frac{1}{3!}P^h\mathscr{T}^{(h)}(w^h,\ w^h,\ w^h)+\mu P^h G p=0, \end{aligned}\right. \qquad \text{(B-12)}$$

where

$$\left|\begin{aligned} & L^{(h)}=-\mathscr{B}(\psi_0,\ \cdot)+\mathscr{B}(w_0,\ P^h\mathscr{B}(w_0,\ \cdot)) \\ & \mathscr{B}^{(h)}(u,\ v)=\mathscr{T}^{(h)}(u,\ v,\ w_0) \\ & \mathscr{T}^{(h)}(u,\ v,\ w)=\mathscr{B}(u,\ P^h\mathscr{B}(v,\ w))+\mathscr{B}(v,\ P^h\mathscr{B}(w,\ u))+ \\ & \qquad\qquad \mathscr{B}(w,\ P^h\mathscr{B}(u,\ v)). \end{aligned}\right. \qquad \text{(B-13)}$$

and

We would like to show now that the mappings $L^{(h)}$, $\mathscr{B}^{(h)}$ and $\mathscr{T}^{(h)}$ satisfy the assumptions in § 3. 2. We only show the properties of the trilinear mapping $\mathscr{T}^{(h)}$. Similar properties for $L^{(h)}$ and $\mathscr{B}^{(h)}$ are left to the reader to check.

Obviousiy, $\mathscr{T}^{(h)}$ is symmetric trilinear map $V\times V\times V\to V$. Also, $\mathscr{T}^{(h)}$ is separately self-adjoint:

$$<\mathscr{T}^{(h)}(u,\ v,\ w),\ \phi>=<\mathscr{T}^{(h)}(\phi,\ v,\ w),\ u>, \qquad \text{(B-14)}$$

due to the self-adjointness of P^h in V. Equi-continuity of $\mathscr{T}^{(h)}$ may be immediately seen from the continuity of the bilinear map $\mathscr{B}$ and that P^h is uniformly bounded (by unity!) in V. Now, we show that $\mathscr{T}^{(h)}$ is *close* to $\mathscr{T}$ in the following sense:

$$||\mathscr{T}(u,\ v,\ w)-\mathscr{T}^{(h)}(\tilde{u},\ \tilde{v},\ \tilde{w})||_V\le C||u||_W||v||_W||w||_W h^{2(1+\sigma)} \qquad \text{(B-15)}$$

and

$$||\mathscr{T}(u,\ v,\ \cdot)-\mathscr{T}^{(h)}(\tilde{u},\ \tilde{v},\ \cdot)||_{V\to V}\le C||u||_W||v||_W h^{1+\sigma}, \qquad \text{(B-16)}$$

for any u, v, $w\in W$ and $\tilde{u}$, $\tilde{v}$, $\tilde{w}\in V$ such that

$$\begin{aligned} & ||u-\tilde{u}||_V\le C||u||_W h^{1+\sigma} \\ & ||u-\tilde{u}||_U\le C'||u||_W h^{2(1+\sigma)} \end{aligned} \qquad \text{(B-17)}$$

and similar relations for $(v,\ \tilde{v})$, and $(w,\ \tilde{w})$.

Proof of Eq. (B-15). It is enough to show the inequality for $\mathscr{B}$ $(u,\ \mathscr{B}$

$(v, w)) - \mathscr{B}(\tilde{u}, P^h\mathscr{B}(\tilde{v}, \tilde{w}))$. Indeed,

$$\begin{aligned}
&\mathscr{B}(u, \mathscr{B}(v, w)) - \mathscr{B}(\tilde{u}, P^h\mathscr{B}(\tilde{v}, \tilde{w}))\\
&\quad = \mathscr{B}(u, \mathscr{B}(v, w) - P^h\mathscr{B}(\tilde{v}, \tilde{w})) + \mathscr{B}(u-\tilde{u}, \mathscr{B}(v, \tilde{w}))\\
&\qquad - \mathscr{B}(u-\tilde{u}, (I-P^h)\mathscr{B}(v, \tilde{w})) - \mathscr{B}(u-\tilde{u}, P^h\mathscr{B}(v-\tilde{v}, \tilde{w}))\\
&\quad = (I) + (II) + (III) + (IV).
\end{aligned}$$

First, in view of the properties of $\mathscr{B}$ in Lemma A-7,

$$\begin{aligned}
||(II)||_V &\leq C||u-\tilde{u}||_U||\mathscr{B}(v, \tilde{w})||_W\\
&\leq Ch^{2(1+\sigma)}||u||_W||v||_W||\tilde{w}||_V
\end{aligned}$$

$$\begin{aligned}
||(III)||_V &\leq C||u-\tilde{u}||_V||(I-P^h)\mathscr{B}(v, \tilde{w})||_V\\
&\leq Ch^{1+\sigma}||u||_W h^{1+\sigma}||\mathscr{B}(v, \tilde{w})||_W\\
&\leq Ch^{2(1+\sigma)}||u||_W||v||_W||\tilde{w}||_V
\end{aligned}$$

$$\begin{aligned}
||(IV)||_V &\leq C||u-\tilde{u}||_V||P^h\mathscr{B}(v-\tilde{v}, \tilde{w})||_V\\
&\leq C||u-\tilde{u}||_V||P^h||_{V\to V}||v-\tilde{v}||_V||\tilde{w}||_V\\
&\leq Ch^{2(1+\sigma)}||u||_W||v||_W||\tilde{w}||_V.
\end{aligned}$$

And,

$$||(I)||_V \leq C||u||_W||\mathscr{B}(v, w) - P^h\mathscr{B}(\tilde{v}, \tilde{w})||_U.$$

But

$$\begin{aligned}
&||\mathscr{B}(v, w) - P^h\mathscr{B}(\tilde{v}, \tilde{w})||_U\\
&\quad \leq ||\mathscr{B}(v-\tilde{v}, w)||_U + ||(I-P^h)\mathscr{B}(\tilde{v}, w)||_U\\
&\qquad + ||P^h\mathscr{B}(v, w-\tilde{w})||_U + ||P^h\mathscr{B}(\tilde{v}-v, w-\tilde{w})||_U\\
&\quad \leq ||\mathscr{B}(v-\tilde{v}, w)||_U + ||(I-P^h)\mathscr{B}(\tilde{v}, w)||_U\\
&\qquad + ||P^h\mathscr{B}(v, w-\tilde{w})||_V + ||P^h\mathscr{B}(v-\tilde{v}, w-\tilde{w})||_V\\
&\quad \leq C\{||v-\tilde{v}||_U||w||_W + h^{2(1+\sigma)}||\mathscr{B}(\tilde{v}, w)||_W\\
&\qquad + ||v||_W||w-\tilde{w}||_U + ||v-\tilde{v}||_V||w-\tilde{w}||_V\}\\
&\quad \leq C'h^{2(1+\sigma)}\{||v||_W||w||_W + ||\tilde{v}||_V||w||_W\}.
\end{aligned}$$

Thus, combining the above estimates, we have Eq. (B-15). (Note that $||\tilde{u}||_V \leq C||u||_V$ for $0<h<h_0$(and similary for $\tilde{v}$, $\tilde{w}$) due to Eq. (B-17).)

Proof of Eq. (B-16). Let $\tilde{\phi}\in V$ and we estimate

$$(II) \equiv \mathscr{B}(u, \mathscr{B}(v, \tilde{\phi}) - \mathscr{B}(\tilde{u}, P^h\mathscr{B}(\tilde{v}, \tilde{\phi}))$$

and

$$(I) \equiv \mathscr{B}(\tilde{\phi}, \mathscr{B}(u, v)) - \mathscr{B}(\tilde{\phi}, P^h\mathscr{B}(\tilde{u}, \tilde{v})),\ \text{separately.}$$

$$\begin{aligned}
||(I)||_V &= ||\mathscr{B}(\tilde{\phi}, \mathscr{B}(u, v) - P^h\mathscr{B}(\tilde{u}, \tilde{v})||_V\\
&\leq C||\tilde{\phi}||_V||\mathscr{B}(u, v) - P^h\mathscr{B}(\tilde{u}, \tilde{v})||_V
\end{aligned}$$

$$\leq Ch^{1+\sigma}\|u\|_W\|v\|_W\|\tilde{\phi}\|_V.$$

$$\|(II)\|_V=\|\mathscr{B}(u-\tilde{u},\ P^h\mathscr{B}(\tilde{v},\ \tilde{\phi}))+\mathscr{B}(u,\ P^h\mathscr{B}(v-\tilde{v},\ \tilde{\phi}))$$
$$+\mathscr{B}(u,\ (I-P^h)\mathscr{B}(v,\ \tilde{\phi})\|_V$$
$$\leq C\{\|u-\tilde{u}\|_V\|\tilde{v}\|_V\|\tilde{\phi}\|_V+\|u\|_V\|v-\tilde{v}\|_V\|\tilde{\phi}\|_V+\|u\|_V h^{1+\sigma}\|B(v,\tilde{\phi})\|_W\}$$
$$\leq C'h^{1+\sigma}\|u\|_W\|v\|_W\|\tilde{\phi}\|_V.$$

Thus, we have the inequality (B-16).

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