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STRUCTURE OF SPACES OF GERMS OF HOLOMORPHIC FUNCTIONS

N. Van Khue and P. Thien Danh

Abstract

Let E be a Frechet (resp. Frechet-Hilbert) space. It is shown that $E \in (\Omega)$ (resp. $E \in (DN)$) if and only if $[\mathcal{H}(O_E)]' \in (\Omega)$ (resp. $[\mathcal{H}(O_E)]' \in (DN)$). Moreover it is also shown that $E \in (DN)$ if and only if $\mathcal{H}_b(E') \in (DN)$. In the nuclear case these results were proved by Meise and Vogt [**2**].

1. Preliminaries

1.1. Let K be a compact set in a Frechet space E. By $\mathcal{H}(K)$ we denote the space of germs of holomorphic functions on K. This space is equipped with the inductive topology

$$\mathcal{H}(K) = \lim_{U \downarrow K} \operatorname{ind} \mathcal{H}^\infty(U).$$

Here for each neighborhood U of K, by $\mathcal{H}^\infty(U)$ we denote the Banach space of bounded holomorphic functions on U with the sup-norm

$$\|f\|_U = \sup\{|f(z) : z \in U\}.$$

1.2. Let E' denote the strong dual space of a Frechet space E. A holomorphic function on E' is said to be of bounded type if it is bounded on every bounded set in E'. By $\mathcal{H}_b(E')$ we denote the metric locally convex space of entire functions of bounded type on E' equipped with the topology the convergence on bounded sets in E'.

For more details concerning holomorphic functions on locally convex spaces we refer to the book of Dineen [**1**].

1.3. Assume the topology of E is defined by an increasing fundamental system of seminorms $\{\|\cdot\|_k\}_{k=1}^{\infty}$. For each subset B of E, define the generalized seminorm $\|\cdot\|_B^* : E' \to [0, +\infty]$, by

$$\|u\|_B^* = \sup\{|u(x)| : x \in B\}.$$

Write $\|\cdot\|_q^*$ for $B = U_q = \{x \in E : \|x\|_q \leq 1\}$.

Using this notation define E to have the property

$$(DN): \quad \exists\, p\, \forall\, q\, \exists\, k, \quad C > 0 : \|\centerdot\|_q^2 \leq C\|\centerdot\|_k\|\centerdot\|_p.$$

$$(\Omega): \forall\, p\, \exists\, q\, \forall\, k\, \exists\, d, \quad C > 0 : \|\cdot\|_q^{*1+d} \leq C\|\cdot\|_k^*\|\cdot\|_p^{*d}.$$

The properties (DN), (Ω) and the other many properties were introduced and investigated by Vogt (see, for example, [**7**], [**8**], etc.). In [**8**] he has proved that $E \in (DN)$ (resp. $E \in (\Omega)$) if and only if E is isomorphic to a subspace (a quotient space) of the space $B\hat{\otimes}_\pi s$ for some Banach space B, where s is the space of rapidly decreasing sequences.

The following three theorems are proved in the present paper.

Theorem 1. *Let E be a Frechet space. Then the following are equivalent*

(i) $E \in (\Omega)$

(ii) $[\mathcal{H}(K)]' \in (\Omega)$ *for some non-empty compact set K in E.*

(iii) $[\mathcal{H}(K)]' \in (\Omega)$ *for all compact sets K in E.*

Theorem 2. *Let E be a Frechet-Hilbert space. Then $E \in (DN)$ if and only if $[\mathcal{H}(O_E)]' \in (DN)$.*

Theorem 3. *Let E be a Frechet space. Then $E \in (DN)$ if and only if $\mathcal{H}_b(E') \in (DN)$.*

The proofs of Theorems 1, 2 and 3 are presented in Sections 2, 3 and 4 respectively.

2. Proof of Theorem 1

2.1. Lemma. *Let E be a Frechet space. Then $E \in (\Omega)$ if and only if E' is isomorphic to a subspace of $B\hat{\otimes}_\pi s'$ for some Banach space B.*

Proof: Suppose E' is isomorphic to a subspace of the space $B\hat{\otimes}_\pi s'$ where B is some Banach space. Then E'' is isomorphic to a quotient space of $(B\hat{\otimes}_\pi s')' \cong B'\hat{\otimes}_\pi s$ and hence $E'' \in (\Omega)$. This implies that $E \in (\Omega)$ since

$$\|u\|_k^* = \sup\{|v(u)| : v \in U_k^{00} \subset E''\} \text{ for } u \in E'.$$

Conversely assume that $E \in (\Omega)$. Consider the canonical resolution

$$0 \longrightarrow E \longrightarrow \prod_{k\geq 1} E_k \xrightarrow{R} \prod_{k\geq 1} E_k \longrightarrow 0$$

as constructed by Palamodov [**4**], where for each $k \geq 1$, E_k stands for the Banach space associated to $\|\,.\,\|_k$.

Since E is isomorphic to a quotient space of $B\hat{\otimes}_\pi s$ with B is some Banach space [**8**], E is quasinormable. Hence we may assume that every bounded set in E_{k+1} can be approximated by a bounded set in E_{k+2} under the canonical map $E_{k+2} \to E_k$. It follows from [**4**] that every bounded set in $\prod_{k\geq 1} E_k$ is the image of a bounded set in $\prod_{k\geq 1} E_k$ under R. By modifying the argument in [**8**] we imply that E is isomorphic to a quotient space of $B\hat{\otimes}_\pi s$ for which E' is isomorphic to a suspace of $(B\hat{\otimes}_\pi s)' \cong B'\hat{\otimes}_\pi s'$. ■

The following lemma is an immediate consequence of the preceding lemma.

2.2. Lemma. *Let E be a Frechet space. Then $E \in (\Omega)$ if and only if $E'' \in (\Omega)$.*

2.3. Lemma. *Let B be a Banach space. Then $[\mathcal{H}(O_{B\hat{\otimes}_\pi s})]' \in (\Omega)$.*

Proof: Let $\{e_j\}$ be the canonical basis of s with the dual basis $\{e_j^*\}$ of s'. Since s is nuclear without loss of generality we may assume that

$$\delta_p = \sum_{j\geq 1} \|e_j^*\|_{p+1}^* \|e_j\|_p < 1/e \text{ for } p \geq 1.$$

For each $p \geq 1$, put

$$|||f|||_{p+1} = \sup\left\{ \left(\frac{1}{p+1}\right)^n \sum_{j_1,\dots,j_n\geq 1} |\widehat{P_n f}(u_1 \otimes e_{j_1}, \dots, u_n \otimes e_{j_n})| \right.$$
$$\left. \times \|e_{j_1}^*\|_{p+1}^* \dots \|e_{j_n}^*\|_{p+1}^* : u_1, \dots, u_n \in W,\, n \geq 0 \right\}$$

for $f \in \mathcal{H}^\infty$ $(\operatorname{conv}(W \otimes U_p))$, where W is the unit ball of B and

$$f(\omega) = \sum_{n\geq 0} P_n f(\omega), \quad \omega \in \operatorname{conv}(W \otimes U_p)$$

is the Taylor expansion of f at $0 \in B\hat{\otimes}_\pi s$.

Since

$$|||f|||_{p+1} = \sup\left\{ \left(\frac{1}{p+1}\right)^n \sum_{j_1,\dots,j_n\geq 1} \left| \widehat{P_n f}\left(u_1 \otimes \frac{e_{j_1}}{\|e_{j_1}\|_p}, \dots, u_n \otimes \frac{e_{j_n}}{\|e_{j_n}\|_p}\right) \right| \right.$$
$$\left. \times \|e_{j_1}^*\|_{p+1}^* \dots \|e_{j_n}\|_p \dots \|e_{j_1}^*\|_{p+1}^* \|e_{j_1}\|_p : u_1, \dots, u_n \in W,\, n \geq 0 \right\}$$
$$\leq C_p \|f\|_{\operatorname{conv}(W\otimes U_p)}$$

for $f \in \mathcal{H}^\infty$ $(\operatorname{conv}(W \otimes U_p))$, where

$$C_p = \sup\left\{ \left(\frac{\delta_p}{p+1}\right)^n \frac{n^n}{n!} : n \geq 0 \right\} < \infty,$$

it follows that $|||\,.\,|||_{p+1}$ is continuous on $\mathcal{H}^\infty$ $(\operatorname{conv}(W \otimes U_p))$.

On the other hand, we have

$$\|f\|_{\operatorname{conv}\left(W\otimes\frac{U_{p+1}}{p+2}\right)} = \sup\left\{\left|f\left(\frac{1}{p+2}\sum_{k\geq 1}\lambda_k u_k\otimes v_k\right)\right| \right.$$

$$\left. : u_k \in W,\, v_k\in U_{p+1},\, \sum_{k\geq 1}|\lambda_k|\leq 1\right\}$$

$$\leq \sup\left\{\sum_{n\geq 0}\left(\frac{p+1}{p+2}\right)^n \sum_{k_1,\dots,k_n\geq 1}|\lambda_{k_1}|\dots|\lambda_{k_n}| \sum_{j_1,\dots,j_n\geq 1}\left(\frac{1}{p+1}\right)^n\right.$$

$$\times |\widehat{P_n f}(u_{k_1}\otimes e_{j_1},\dots,u_{k_n}\otimes e_{j_n})||e^*_{j_1}v_{k_1}|\dots|e^*_{j_n}(v_{k_n})|$$

$$\left. : u_k \in W,\, v_k\in U_{p+1},\, \sum_{k\geq 1}|\lambda_k|\leq 1\right\}$$

$$\leq \sup\left\{\sum_{n\geq 0}\left(\frac{p+1}{p+2}\right)^n \sum_{k_1,\dots,k_n\geq 1}|\lambda_{k_1}|\dots|\lambda_{k_n}| \sum_{j_1,\dots,j_n\geq 1}\left(\frac{1}{p+1}\right)^n\right.$$

$$\times |\widehat{P_n f}(u_{k_1}\otimes e_{j_1},\dots,u_{k_n}\otimes e_{j_n})|\|e^*_{j_1}\|^*_{p+1}\dots\|e^*_{j_n}\|^*_{p+1}$$

$$\left. : u_k \in W,\, \sum_{k\geq 1}|\lambda_k|\leq 1\right\}$$

$$\leq \sup\left\{\sum_{n\geq 0}\left(\frac{p+1}{p+2}\right)^n \sum_{k_1,\dots,k_n\geq 1}|\lambda_{k_1}|\dots|\lambda_{k_n}\||f\||_{p+1} : \sum_{k\geq 1}|\lambda_k|\leq 1\right\}$$

$$\leq \left(\sum_{n\geq 0}\left(\frac{p+1}{p+2}\right)^n\right)\||f\||_{p+1}.$$

Hence

$$\mathcal{H}(O_{B\hat{\otimes}_\pi s}) \cong \lim\operatorname{ind}[\mathcal{H}^\infty(\operatorname{conv}(W\otimes U_p)) : \||\,\boldsymbol{\cdot}\,\||_{p+1}].$$

Given $p\geq 1$, choose $q\geq p$ such that

$$\forall\, k\,\exists\, C,\, d>0 : \|e^*_j\|^{*1+d}_q \leq C\|e^*_j\|^*_k\|e^*_j\|^{*d}_p\,\forall\, j\geq 1.$$

Since $\|\centerdot\|_q^* \le \|\centerdot\|_p^*$, the above inequality holds for every $d' \ge d$. Hence we may assume that $Ckp^d \le q^{1+d}$, then

$$\begin{aligned}
|||f|||_q^{1+d} &= \sup\Bigg\{\left(\frac{1}{q}\right)^n \sum_{j_1,\dots,j_n\ge 1} |\widehat{P_n f}(u_1\otimes e_{j_1},\dots,u_n\otimes e_{j_n})| \\
&\qquad \times \|e_{j_1}^*\|_q^* \dots \|e_{j_n}^*\|_q^* : u_1,\dots,u_n \in W,\, n\ge 0\Bigg\}^{1+d} \\
&\le \sup\Bigg\{\left(\frac{1}{Ckp^d}\right)^{\frac{n}{1+d}} \sum_{j_1,\dots,j_n\ge 1} |\widehat{P_n f}(u_1\otimes e_{j_1},\dots,u_n\otimes e_{j_n})| \\
&\qquad \times C^{\frac{n}{1+d}} \|e_{j_1}^*\|_k^{*1/1+d} \dots \|e_{j_n}^*\|_k^{*1/1+d} \\
&\qquad \times \|e_{j_1}^*\|_p^{*d/1+d} \dots \|e_{j_n}^*\|_p^{*d/1+d} : u_1,\dots,u_n \in W,\, n\ge 0\Bigg\}^{1+d} \\
&\le \sup\Bigg\{(1/k)^n \sum_{j_1,\dots,j_n\ge 1} |\widehat{P_n f}(u_1\otimes e_{j_1},\dots,u_n\otimes e_{j_n})| \\
&\qquad \times \|e_{j_1}^*\|_k^* \dots \|e_{j_n}^*\|_k^* : u_1,\dots,u_n \in W,\, n\ge 0\Bigg\} \\
&\qquad \times \sup\Bigg\{(1/p)^n \sum_{j_1,\dots,j_n\ge 1} |\widehat{P_n f}(u_1\otimes e_{j_1},\dots,u_n\otimes e_{j_n})| \\
&\qquad \times \|e_{j_1}^*\|_p^* \dots \|e_{j_n}^*\|_p^* : u_1,\dots,u_n \in W,\, n\ge 0\Bigg\}^{d} \\
&= |||f|||_k |||f|||_p^d
\end{aligned}$$

for $f \in \mathcal{H}(O_{B\hat{\otimes}_\pi s})$.

Since $B\hat{\otimes}_\pi s$ is quasinormale, according to Mujica [**3**] there exists a Frechet space F such that $F' \cong \mathcal{H}(O_{B\hat{\otimes}_\pi s})$. Combining this fact together with the inequality

$$|||\centerdot|||^{1+d} \le |||\centerdot|||_k |||\centerdot|||_p^d \text{ on } \mathcal{H}(O_{B\hat{\otimes}_\pi s}),$$

by virtue of Lemma 2.2 we obtain $[\mathcal{H}(O_{B\hat{\otimes}_\pi s})]' \cong F''$ has (Ω). ■

Now we are able to prove Theorem 1.

(iii) $\to$ (ii) is trivial.

(ii) $\to$ (i). Fix $x_0 \in K$. Then the form

$$\varphi \to \varphi'(x_0)$$

defines a left inverse of the canonical map $E' \to \mathcal{H}(K)$. Hence $E'' \in (\Omega)$, this implies that $E \in (\Omega)$.

(i) $\to$ (iii). Assume that $E \in (\Omega)$. By Vogt [**5**] there exists a continuous linear map R from $B\hat{\otimes}_\pi s$ onto E for some Banach space B.

Let $\{W_k\}$ be a neighbourhood basis of $0 \in B\hat{\otimes}_\pi s$. Then $\{V_k = R(W_k)\}$ forms a neighbourhood basis of $0 \in E$. Lemma 2.3 gives

$$\forall\, p\, \exists\, q\, \exists\, C,\, d > 0\, \forall\, f \in \mathcal{H}^\infty(W_p) : \|f\|_{W_q}^{1+d} \le C\|f\|_{W_k}\|f\|_{W_p}^{d}.$$

Thus

$$(*) \qquad \|g\|_{V_q}^{1+d} \le \|g\|_{V_k}\|g\|_{V_p}^{d}\ \forall\, g \in \mathcal{H}^\infty(V_p).$$

Next let K be an arbitrary compact set in E. From $(*)$ we deduce that

$$\|g\|_{K+V_q}^{1+d} \le C\|g\|_{K+V_p}\|g\|_{K+V_p}^{d}\ \forall\, f \in \mathcal{H}^\infty(K+V_p).$$

According to Mujica [**3**] there exists a Frechet space F verifying $F' \cong \mathcal{H}(K)$ since E is quasinormale, invoke Lemma 2.2 we conclude that $[\mathcal{H}(K)]' \cong F'' \in (\Omega)$. ■

3. Proof of Theorem 2

3.1. Lemma. *Let E be a Frechet-Hilbert space with $E \in (DN)$. Then E is isomorphic to a subspace of the space $l^2(I)\hat{\otimes}_\pi s$ for some index set.*

Proof: Choose an index set I such that E is isomorphic to a subspace of $[l^2(I)]^N$. Consider the exact sequence of nuclear Frechet spaces

$$0 \to s \to s \to \omega \to 0$$

constructed by Vogt [**8**]. By tensoring this sequence with $l^2(I)$ we get the exact sequence of Frechet-Hilbert spaces [**8**],

$$0 \to l^2(I)\hat{\otimes}_\pi s \to l^2(I)\hat{\otimes}_\pi s \xrightarrow{q} [l^2(I)]^N \to 0$$

Let $\tilde{E} = q^{-1}(E)$. Since

$$0 \to l^2(I)\hat{\otimes}_\pi s \to \hat{E} \xrightarrow{q} E \to 0$$

is a exact sequence of Frechet-Hilbert spaces in which $l^2(I)\hat{\otimes}_\pi s \in (\Omega)$ and $E \in (DN)$ by Vogt [**9**] q has a right inverse. Hence E is isomorphic to subspace of $\tilde{E}$ and hence of $l^2(I)\hat{\otimes}_\pi s$. ■

3.2. Lemma. *Let B be a Banach space. Then $[\mathcal{H}(O_{B\hat{\otimes}_\pi s})]' \in (DN)$.*

Proof: Let W denote the unit ball of B. Write the Taylor expansion of each $f \in \mathcal{H}(O_{B\hat{\otimes}_\pi s})$ at $0 \in B\hat{\otimes}_\pi s$

$$f(\omega) = \sum_{n\geq 0} P_n f(\omega).$$

Formally we have

$$f\left(\sum_{k\geq 1} \lambda_k u_k \otimes v_k\right) = \sum_{n\geq 0} \sum_{k_1,\dots,k_n\geq 1} \lambda_{k_1}\dots\lambda_{k_n}$$
$$\times \sum_{j_1,\dots,j_n\geq 1} \widehat{P_n f}(u_{k_1}\otimes e_{j_1},\dots,u_{k_n}\otimes e_{j_n}) e^*_{j_1}(v_{k_1})\dots e^*_{j_n}(v_{k_n})$$

for $\omega = \sum_{k\geq 1} \lambda_k u_k \otimes v_k \in B\hat{\otimes}_\pi s$.

For each $p \geq 1$ as in Theorem 1 put

$$|||f|||_p = \sup\left\{\frac{1}{p^n} \sum_{j_1,\dots,j_n\geq 1} |\widehat{P_n f}(u_1\otimes e_{j_1},\dots,u_n\otimes e_{j_n})|(j_1\dots j_n)^{-p} \right.$$
$$\left. : n\geq 0,\ u_1,\dots,u_n \in W\right\}$$

and

$$F_p = \{f \in \mathcal{H}(O_{B\hat{\otimes}_\pi s}) : |||f|||_p < +\infty\}.$$

Then

$$\mathcal{H}(O_{B\hat{\otimes}_\pi s}) \cong \lim_p \operatorname{ind} F_p.$$

In order to prove that $[\mathcal{H}(O_{B\hat{\otimes}_\pi s})]' \in (DN)$ we check that

$$\forall\, q\, \exists\, k,\, C > 0 : W_q \subset CsW_1 + \frac{1}{s}W_k \quad \forall\, s > 0 \tag{2}$$

where for each $q \geq 1$ put

$$W_q = \{f \in F_q : |||f|||_q < 1\}.$$

Obviously (4) holds for $0 < s \leq 1$. Let $s > 1$. Choose $k = q^3$.

We have

(3)

$$\sup\left\{\frac{1}{k^n}\sum_{j_1,\dots,j_n \geq 1} |\widehat{P_n f}(u_1 \otimes e_{j_1}, \dots, u_n \otimes e_{j_n})|(j_1 \dots j_n)^{-k} : n \geq \frac{\log s}{\log \frac{k}{q}}, u_1, \dots, u_n \in W\right\}$$

$$+ \sup\left\{\frac{1}{k^n}\sum_{(j_1 \dots j_n \geq s^{\frac{1}{k-q}}} |\widehat{P_n f}(u_1 \otimes e_{j_1}, \dots, u_n \otimes e_{j_n})|(j_1 \dots j_n)^{-k} : n \geq 0, u_1, \dots, u_n \in W\right\}$$

$$\leq \sup\left\{\frac{1}{q^n}\sum_{j_1,\dots,j_n \geq 1} |\widehat{Pn_f}(u_1 \otimes e_{j_1}, \dots, u_n \otimes e_{j_n})|(j_1 \dots j_n)^{-q} : n \geq 0, u_1, \dots, u_n \in W\right\}$$

$$\leq \sup\left\{\left(\frac{q}{k}\right)^n : n \geq \frac{\log s}{\log \frac{k}{q}}\right\} + \sup\{(j_1 \dots j_n)^{q-k} : (j_1 \dots j_n) \geq s^{\frac{1}{k-q}}\}$$

$$\leq \frac{2}{s} \text{ for } f \in W_q$$

and

$$\sup\left\{\sum_{j_1\dots j_n\leq s^{\frac{1}{k-q}}}|\widehat{P_n f}(u_1\otimes e_{j_1},\dots,u_n\otimes e_{j_n})|(j_1\dots j_n)^{-1}\right.$$

$$\left. : n\leq\frac{\log s}{\log\frac{k}{q}}, u_1,\dots,u_n\in W\right\} \tag{4}$$

$$\leq |||f|||_q \sup\left\{q^n(j_1\dots j_n)^q : n\leq\frac{\log s}{\log\frac{k}{q}}, j_1\dots j_n\leq s^{\frac{1}{k-q}}\right\}$$

$$\leq q^{\frac{\log s}{\log\frac{k}{q}}} s^{\frac{1}{q}} \leq s \text{ for } f\in W_q.$$

From (3) and (4) it follows that

$$W_q\subset sW_1+\frac{2}{s}W_k \quad \forall\, s>0.$$

Now by Lemma 2.1 and 2.2 we can complete the proof of Theorem 2. Indeed, let E be a Frechet-Hilbert space with (DN). By Lemma 2.1, E can be considered as a subspace of the space $l^2(I)\hat{\otimes}_\pi s$ for some index I. Since $l^2(I)\hat{\otimes}_\pi s$ has a fundamental system of Hilbert semi-norms, the restriction map $R:\mathcal{H}(O_{l^2(I)\hat{\otimes}_\pi s})\to\mathcal{H}(O_E)$ is surjective. Moreover it is easy to check that every bounded set in $\mathcal{H}(O_E)$ is the image of a bounded set in $\mathcal{H}(O_{l^2(I)\hat{\otimes}_\pi s})$ because of the regularity of $\mathcal{H}(O_E)$ (see [**1**]). Thus $R':[\mathcal{H}(O_E)]'\to[\mathcal{H}(O_{B\hat{\otimes}_\pi s})]'$ is isomorphic onto image. Hence, by Lemma 2.2 it follows that $[\mathcal{H}(O_E)]'\in(DN)$. Conversely, assume that $[\mathcal{H}(O_E)\in(DN)$. Since the form

$$\mathcal{H}(O_E)\ni f\mapsto f'(0)\in E'$$

defines a continuous linear map from $\mathcal{H}(O_E)$ onto E' which is a left inverse of the canonical map $E'\to\mathcal{H}(O_E)$, it follows that E'' is isomorphic to a subspace of $[\mathcal{H}(O_E)]'$. Hence $E''\in(DN)$. On the other hand, since E is reflexive, $E\cong E''\in(DN)$. ■

4. Proof of Theorem 3

By Vogt [**8**] E can be considered as a subspace of $B\hat{\otimes}_\pi s$, where B is some Banach space. Since every bounded set in E' can be lifted from E'

to $(B\hat{\otimes}_\pi s)' \cong B'\hat{\otimes}_\pi s'$ under the restriction map $R : (B\hat{\otimes}_\pi s)' \to E'$, it follows that $\mathcal{H}_b(E')$ is isomorphic to a subspace of $\mathcal{H}_b((B\hat{\otimes}_\pi s))'$. Thus it remains to check that $\mathcal{H}((B\hat{\otimes}_\pi s)') \cong \mathcal{H}_b(B'\hat{\otimes}_\pi s') \in (DN)$.

Given $p \geq 1$. Take $q > p$ such that

$$\delta = \sum_{j\geq 1} \|e_j^*\|_q^* \|e_j\|_p < \frac{1}{e^2 p},$$

where

$$\|u\|_q^* = \sup\{|u(x)| : \|x\|_q \leq 1\}, \quad u \in s'.$$

Let W denote the unit ball of B'. Then for every $f \in \mathcal{H}(B'\hat{\otimes}_\pi s')$ we have

$$\sup\left\{ \sum_{j_1,\dots,j_n\geq 1} p^n |\widehat{P_n f}(u_1 \otimes e_{j_1}^*, \dots, u_n \otimes e_{j_n}^*)| \|e_{j_1}\|_p \dots \|e_{j_n}\|_p \right.$$

$$\left. : u_1, \dots, u_n \in W, n \geq 0 \right\}$$

$$= \sup\left\{ \sum_{j_1,\dots,j_n\geq 1} p^n \left| \widehat{P_n f}\left(u_1 \otimes \frac{e_{j_1}^*}{\|e_{j_1}^*\|_q^*}, \dots, u_n \otimes \frac{e_{j_n}^*}{\|e_{j_n}^*\|_q^*} \right) \right| \right.$$

$$\left. \times \|e_{j_1}^*\|_q^* \|e_{j_1}\|_p \dots \|e_{j_n}^*\|_q^* \|e_{j_n}\|_p : u_1, \dots, u_n \in W, n \geq 0 \right\}$$

$$\leq \sup\left\{ \frac{n^n p^n}{n!} \sum_{j_1,\dots,j_n\geq 1} \|e_{j_1}^*\|_q^* \|e_{j_1}\|_p \dots \|e_{j_n}^*\|_q^* \|e_{j_n}\|_p : n \geq 0 \right\}$$

$$\times \|f\|_{\mathrm{conv}(W\otimes U_q^0)}$$

$$\leq \sup \frac{n^n p^n}{n!} \left(\frac{1}{e^2 p}\right)^n \|f\|_{\mathrm{conv}(W\otimes U_q^0)}$$

$$= C(p) \|f\|_{\mathrm{conv}(W\otimes U_q^0)}$$

where

$$C(p) = \sup \frac{n^n p^n}{n!} \left(\frac{1}{e^2 p}\right)^n < \infty$$

and $f(\omega) = \sum_{n\geq 0} P_n f(\omega)$ is the Taylor expansion of f at $0 \in B'\hat{\otimes}_\pi s'$.

Thus the form (1) defines a continuous semi-norms $|||\,\cdot\,|||_p$ on $\mathcal{H}_b(B'\hat{\otimes}_\pi s')$. On the other hand, since

$$\begin{aligned}
\|f\|_{\mathrm{conv}(W\otimes U_p^0)} &= \sup\left\{\left|f\left(\sum_{k\geq 1}\lambda_k u_k\otimes v_k\right)\right| : u_k\in W, v_k\in U_p^0, \sum_{k\geq 1}|\lambda_k|\leq 1\right\}\\
&\leq \sup\left\{\sum_{n\geq 0}\frac{1}{p^n}\sum_{k_1,\dots,k_n\geq 1}|\lambda_{k_1}|\dots|\lambda_{k_n}|\right.\\
&\quad\times\sum_{j_1,\dots,j_n\geq 1} p^n|\widehat{P_n f}(u_{k_1}\otimes e_{j_1}^*,\dots,u_{k_n}\otimes e_{j_n}^*)|\\
&\quad\left.\times\|e_{j_1}^*(v_{k_1})\|\dots\|e_{j_n}^*(v_{k_n})\| : u_k\in W, v_k\in U_p^0, \sum_{k\geq 1}|\lambda_k|\leq 1\right\}\\
&\leq \sup\left\{\sum_{n\geq 0}\frac{1}{p^n}\sum_{k_1,\dots,k_n\geq 1}|\lambda_{k_1}|\dots|\lambda_{k_n}|\right.\\
&\quad\times\sum_{j_1,\dots,j_n\geq 1} p^n|\widehat{P_n f}(u_{k_1}\otimes e_{j_1}^*,\dots,u_{k_n}\otimes e_{j_n}^*)|\\
&\quad\left.\times\|e_{j_1}^*\|_p\dots\|e_{j_n}^*\|_p : u_k\in W, v_k\in U_p^0, \sum_{k\geq 1}|\lambda_k|\leq 1\right\}\\
&\leq \sup\left\{\sum_{n\geq 0}\frac{1}{p^n}\sum_{k_1,\dots,k_n\geq 1}|\lambda_{k_1}|\dots|\lambda_{k_n}| : \sum_{k\geq 1}|\lambda_k|\leq 1\right\}|||f|||_p\\
&\leq \left(\sum_{n\geq 0}\frac{1}{p^n}\right)|||f|||_p \text{ for } f\in\mathcal{H}_b(B'\hat{\otimes}_\pi s').
\end{aligned}$$

it follows that the topology of $\mathcal{H}(B'\hat{\otimes}_\pi s')$ can be defined by the system of the semi-norms $\{|||\,\cdot\,|||_p\}$. Choose $p\geq 1$ such that (DN) is satisfied. Then

$$\forall\, q\,\exists\, k\,\forall\, j\geq 1 : \|e_j\|_q^2 \leq \|e_j\|_k\|e_j\|_p.$$

Given q. Choose k such that the above condition holds and

$$q^2\leq kp.$$

Then

$$\begin{aligned}
|||f|||_q^2 &= \sup\Big\{ q^n \sum_{j_1,\dots,j_n\geq 1} |\widehat{P_n f}(u_1\otimes e_{j_1}^*,\dots,u_n\otimes e_{j_n}^*)| \|e_{j_1}\|_q \dots \|e_{j_n}\|_q \\
&\qquad : u_1,\dots,u_n \in W, n\geq 0\Big\}^2 \\
&\leq \sup\Big\{ k^{\frac{n}{2}} p^{\frac{n}{2}} \sum_{j_1,\dots,j_n\geq 1} |\widehat{P_n f}(u_1\otimes e_{j_1}^*,\dots,u_n\otimes e_{j_n}^*)| \\
&\qquad \times \|e_{j_1}\|_k^{\frac{1}{2}} \dots \|e_{j_n}\|_k^{\frac{1}{2}} \|e_{j_1}\|_p^{\frac{1}{2}} \dots \|e_{j_n}\|_p^{\frac{1}{2}} : u_1,\dots,u_n \in W, n\geq 0\Big\}^2 \\
&\leq \sup\Big\{ k^n \sum_{j_1,\dots,j_n\geq 1} |\widehat{P_n f}(u_1\otimes e_{j_1}^*,\dots,u_n\otimes e_{j_n}^*)| \|e_{j_1}\|_k \dots \|e_{j_n}\|_k \\
&\qquad : u_1,\dots,u_n \in W, n\geq 0\Big\} \\
&\qquad \times \sup\Big\{ p^n \sum_{j_1,\dots,j_n\geq 1} |\widehat{P_n f}(u_1\otimes e_{j_1}^*,\dots,u_n\otimes e_{j_n}^*)| \|e_{j_1}\|_p \dots \|e_{j_n}\|_p \\
&\qquad : u_1,\dots,u_n \in W, n\geq 0\Big\} \\
&= |||f|||_k |||f|||_p \text{ for } f\in \mathcal{H}_b(B'\hat{\otimes}_\pi s').
\end{aligned}$$

Consequently $\mathcal{H}_b(B'\hat{\otimes}_\pi s') \in (DN)$. ■

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N. Van Khue:
Department of Mathematics
Pedagogical Institute Hanoi 1
Hanoi
(cau Giay, TuLiem, Hanoi)
VIETNAM

P. Thien Danh:
Department of Mathematics
College of Education
280 An Duong Vuong, District 5
Ho Chi Minh City
VIETNAM

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