TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 234, Number 1, 1977

STRUCTURE OF SYMMETRIC TENSORS OF TYPE (0, 2) AND TENSORS OF TYPE (1, 1) ON THE TANGENT BUNDLE

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ABSTRACT. The concepts of M-tensor and M-connection on the tangent bundle TM of a smooth manifold M are used in a study of symmetric tensors of type (0, 2) and tensors of type (1, 1) on TM. The constructions make use of certain local frames adapted to an M-connection. They involve extending known results on TM using tensors on M to cases in which these tensors are replaced by M-tensors. Particular attention is devoted to (pseudo-) Riemannian metrics on TM, notably those for which the vertical distribution on TM is null or nonnull, and to the construction of almost product and almost complex structures on TM.

1. **Introduction.** Let M be a smooth manifold and TM its tangent bundle. In his 1958 paper [1], S. Sasaki constructed a Riemannian metric on TM from a Riemannian metric on M, heralding the beginning of the differential geometry of the tangent bundle. Since then, other Riemannian metrics on TM have been constructed (see Yano and Ishihara [3, Chapter IV]) but no general method of construction has emerged.

Recently, two of us (Wong and Mok [1]) introduced the concepts of M-tensor and three types of connections on TM, and used them to clarify the relationship between several related known concepts on TM. In this paper, we shall show how the concepts of M-tensor and one of these connections (which we now call M-connection) enable us to have a complete picture of the structures of the symmetric tensors of type (0, 2) and tensors of type (1, 1) on TM. We shall see also that many of the known results on TM arising from tensors on M have a meaning when these tensors are replaced by M-tensors on TM.

In §2, we fix our notations and give some formulas that will be frequently used. In §3, we show that the concepts of M-tensor and M-connection are inherent in the transformation law of the components of a symmetric tensor of type (0, 2) on TM. In §4, we consider certain local frames adapted to an M-connection, and show how any tensor on TM can be expressed in terms of an M-connection and some M-tensors of the same type. In §§5–8, we study

Received by the editors April 2, 1976.

AMS (MOS) subject classifications (1970). Primary 53C05, 53C15, 53C20; Secondary 53C50.

Key words and phrases. Tangent bundle, Sasaki metric, M-tensor, M-connection, adapted frame, (pseudo-) Riemannian metric, almost product structure, almost complex structure.

the structure of a symmetric tensor of type (0, 2) on TM. Two important cases are singled out for detailed discussion: the symmetric tensors of type (0, 2) on TM with respect to which the vertical distribution is respectively null or nonnull. Virtually all the Riemannian metrics on TM that have so far appeared in the literature are of one of these types. It is interesting though not unexpected that the original Sasaki metric occupies a central position among all the possible Riemannian metrics on TM with respect to which the vertical distribution is nonnull. In §§9–11, we carry out a similar study of the structure of a tensor of type (1, 1) on TM, and in particular the structure of a tensor F of type (1, 1) satisfying the condition $F^2 = \lambda E$, where λ is a real number and E the "identity tensor" on TM. Finally in §12, we determine all the compatible Riemannian metrics and almost product (resp. almost complex) structures on TM whose associated M-tensors are everywhere nonsingular or zero.

We remark that the problem for the cotangent bundle T^*M similar to that considered in this paper for the tangent bundle TM can be solved by using the M-tensors on T^*M and a type of connection equivalent to a horizontal distribution on T^*M. Details will be given in a forthcoming paper (Wong and Mok [2]).

2. The tangent bundle TM. Throughout this paper, the indices $a, b, c, \dots$; $h, i, j, \dots$ run over the range $\{1, 2, \dots, n\}$, while the indices $\alpha, \beta, \gamma, \dots$ run over the range $\{1, 2, \dots, n, n+1, \dots, 2n\}$. $\bar{h}$ will denote $n + h$. Summation over repeated indices is always implied.

When matrices are used, we denote their elements by x^i, A_{ij} or F^i_j. In each case, i denotes the row and j denotes the column. A matrix A whose elements are A_{ij} will be denoted by $[A_{ij}]$. The transpose of A is denoted by A^t and the inverse of A, if it exists, is denoted by A^{-1}. The $n \times n$ identity matrix is denoted by I.

Let M be an n-dimensional smooth (i.e., C^∞) manifold which we shall always assume to be connected and paracompact. We denote by T_pM the tangent space to M at the point $p \in M$, and by $TM = \bigcup_{p \in M} T_pM$ the *tangent bundle* of M with base space M, fibers T_pM and projection π: $TM \to M$ which sends the elements of T_pM to p. If U is any subset of M, we denote $\pi^{-1}U$ by TU, so that in particular $T_pM = \pi^{-1}(p) = Tp$. If $\sigma \in TM$ and $\pi\sigma = p$, then $\sigma \in Tp$ and the tangent space to Tp at σ is an n-dimensional subspace V_σ of $T_\sigma(TM)$. The assignment $\sigma \to V_\sigma$ is an integrable distribution of n-planes on TM which we call the *vertical distribution* on TM and denote by V.

Let (U, x) be a chart in M with neighborhood U and coordinate function $x = [x^i]$. If $\sigma \in TM$, then $\sigma \in TU$ for some U so that σ is a tangent vector to M at $p = \pi\sigma \in U$. Suppose that $\sigma = y^i(\partial/\partial x^i)_p$, i.e., y^i are the components of σ in the chart (U, x) in M. Then $(TU, (x, y))$, where $y = [y^i]$, is a chart in TM which we say is induced from the chart (U, x) in M. If $(TU', (x', y'))$ is another induced chart in TM such that $TU \cap TU'$ is nonempty, then the restrictions of the coordinate functions (x, y) and (x', y') to $TU \cap TU'$ are related by

$$x^{i'} = x^{i'}(x^1, \ldots, x^n), \qquad y^{i'} = y^i p_i^{i'}, \quad \text{where } p_i^{i'} = \partial x^{i'}/\partial x^i. \tag{2.1}$$

Here and in what follows, a dash $'$ indicates quantities related to U' or TU', as the case may be. Let us denote by ∂ the operator $y^i\, \partial/\partial x^i$ on functions defined on TU. Then,

$$\partial p_i^{i'} = y^j \frac{\partial^2 x^{i'}}{\partial x^j \partial x^i} \equiv y^j p_{ji}^{i'},$$

and differentiation of (2.1) gives

$$dx' = P\,dx, \qquad dy' = (\partial P)dx + P\,dy, \tag{2.2}$$

where $P = [p_i^{i'}]$, $\partial P = [\partial p_i^{i'}]$. Thus, the Jacobian matrix of the transformation (2.1) is

$$\begin{bmatrix} P & 0 \\ \partial P & P \end{bmatrix}. \tag{2.3}$$

Let ${}^sTM$ be the subset of TM consisting of all the nonzero tangent vectors of M. Then ${}^sTM$ is an open submanifold and also a subbundle of TM which we call the *slit tangent bundle* of M. In ${}^sTM$ we have the induced charts $({}^sTU, (x, y))$ which are the restriction to ${}^sTM$ of the induced charts $(TU, (x, y))$ in TM so that $y \neq 0$ in $({}^sTU, (x, y))$. It will be clear that all the results on TM in this paper also hold for ${}^sTM$.

We now recall the Sasaki metric on TM (see Sasaki [1]) constructed from a Riemannian metric g on M. Let the components of g in (U, x) be g_{ij} and $\{{}^{\,i}_{jk}\}$ the Christoffel symbols of g_{ij}. We denote by g also the matrix $[g_{ij}]$ and by Γ the matrix $[\Gamma^i_j]$, where

$$\Gamma^i_j = \left\{ {i \atop jk} \right\} y^k. \tag{2.4}$$

Then the component matrix in TU of the Sasaki metric is

$$\begin{bmatrix} g + \Gamma^t g \Gamma & \Gamma^t g \\ g\Gamma & g \end{bmatrix}. \tag{2.5}$$

As has been observed by several authors, (2.5) still defines a Riemannian metric on TM if $\{{}^{\,i}_{jk}\}$ in (2.4) are replaced by the components γ^i_{jk} of a linear connection on M.

3. **The concepts of M-tensor and M-connection on TM.** In this section, we shall show how the concepts of M-tensor and M-connection on TM arise naturally from the transformation law for the components of a symmetric tensor of type (0, 2) on TM.

Let G be any symmetric (0, 2)-tensor on TM and

$$\begin{bmatrix} A & B^t \\ B & c \end{bmatrix}$$

the component matrix of G in TU, where A, B, c are $n \times n$ matrix functions of (x, y) of which A and c are symmetric. Then in $TU \cap TU'$, the component matrix of G in TU and that in TU' are related by

$$\begin{bmatrix} A & B^t \\ B & c \end{bmatrix} = \begin{bmatrix} P^t & (\partial P)^t \\ 0 & P^t \end{bmatrix} \begin{bmatrix} A' & B'' \\ B' & c' \end{bmatrix} \begin{bmatrix} P & 0 \\ \partial P & P \end{bmatrix},$$

which is easily seen to be equivalent to the following equations:

$$\text{(3.1)} \qquad \begin{aligned} c &= P^t c' P, \qquad B = P^t (B'P + c'\partial P), \\ A &= \left[P^t A' + (\partial P)^t B'\right] P + \left[P^t B'' + (\partial P)^t c'\right] \partial P. \end{aligned}$$

Suppose now that c is everywhere nonsingular. Guided by the form of the Sasaki metric (2.5), we tentatively put

$$\text{(3.2)} \qquad B = c\Gamma, \qquad A = a + c + \Gamma^t c \Gamma,$$

where $\Gamma = [\Gamma^i_j]$, $a = [a_{ij}]$ are $n \times n$ matrices. Note that since c is nonsingular, first Γ and then a are uniquely determined by (3.2). If we define Γ and a by (3.2), the equations (3.1) imply that

$$\text{(3.3)} \qquad \Gamma' P = P\Gamma - \partial P,$$

$$\text{(3.4)} \qquad a = P^t a' P.$$

In fact, (3.3) is obtained by substituting $(3.2)_1$ in $(3.1)_2$ and then using $(3.1)_1$, and (3.4) is obtained by substituting (3.2) in $(3.1)_3$ and then simplifying the result by $(3.1)_1$ and (3.3). Conversely, if $(3.1)_1$, (3.3) and (3.4) hold, and A, B are given by (3.2), then all of equations (3.1) hold.

Thus, we have proved that equations (3.1) are equivalent to equations $(3.1)_1$, (3.3) and (3.4), the equivalence being achieved by (3.2).

Motivated by equations (3.3) and $(3.1)_1$ and (3.4), the last two of which mean that the elements of the matrices c and a (which are functions of (x, y)) transform like the components of a (0, 2)-tensor on M, we formulate the following definitions.

DEFINITION. An *M-connection on TM* is the geometric object determined by an assignment to each induced chart $(TU, (x, y))$ of an $n \times n$ matrix function $\Gamma = [\Gamma^i_j]$ such that

$$\text{(3.5)} \qquad \Gamma' P = P\Gamma - \partial P \quad \text{on } TU \cap TU'.$$

We call Γ^i_j the *components* of the M-connection and refer to the M-connection simply as the M-connection Γ or Γ^i_j. (Similar terminology will be used for the M-tensors defined below. M-connection has been called *connection of type* (1, 1) in Wong and Mok [1].)

DEFINITION. An *M-tensor of type* (r, s) *on* TM is the geometric object determined by an assignment to each induced chart $(TU, (x, y))$ of a set of n^{r+s} functions $S^{i_1 \cdots i_r}_{j_1 \cdots j_s}(x, y)$ that behave like the components of a tensor of type (r, s) on M, i.e.,

$$S'^{i'_1 \cdots i'_r}_{j'_1 \cdots j'_s} = p^{i'_1}_{i_1} \cdots p^{i'_r}_{i_r} S^{i_1 \cdots i_r}_{j_1 \cdots j_s} p^{j_1}_{j'_1} \cdots p^{j_s}_{j'_s} \quad \text{on } TU \cap TU'. \tag{3.6}$$

For reasons that will be clear in §4 (Theorem 4.2), we shall refer to the symmetric M-tensor c of type (0, 2) appearing in the discussion above as the *M-tensor associated with G*. The following theorem summarises what we have proved so far.

THEOREM 3.1. *The most general symmetric tensor of type* (0, 2) *on TM whose associated M-tensor c is everywhere nonsingular has a component matrix of the form*

$$\begin{bmatrix} a + \Gamma' c \Gamma & \Gamma' c \\ c\Gamma & c \end{bmatrix}, \tag{3.7}$$

where Γ is an M-connection and a is a symmetric M-tensor of type (0, 2) *on TM.*

Similarly we can show that the concepts of M-tensor and M-connection on TM are also inherent in the transformation law for the components of a tensor of type (1, 1) on TM.

We end this section with the following two easy-to-prove theorems.

THEOREM 3.2. (a) *Let $t = 1, \ldots, r$, let λ_t be r real numbers and let Γ_t be r M-connections on TM. Then $\Sigma\lambda_t\Gamma_t$ is an M-connection iff $\Sigma\lambda_t = 1$, and is an M-tensor of type* (1, 1) *iff $\Sigma\lambda_t = 0$. In particular, if Γ is any M-connection, then any other M-connection is of the form $\Gamma + S$, where S is some M-tensor of type* (1, 1).

(b) *If $[\gamma^i_{jk}]$ is a linear connection on M and $\Gamma^i_k = y^j\gamma^i_{jk}$, then $\Gamma = [\Gamma^i_j]$ is an M-connection on TM.*

(c) *Tensors on M can be considered as M-tensors on TM. Addition, scalar multiplication, tensor product and contraction of M-tensors on TM can be defined as for tensors on M. The space of M-tensors on TM is an algebra over the reals.*

(d) *If $S^{i\cdots}_{j\cdots}$ is an M-tensor of type (r, s), then $\partial^t S^{i\cdots}_{j\cdots}/\partial y^{k_1} \ldots y^{k_t}$ is an M-tensor of type $(r, s + t)$.*

(e) *Any M-tensor on TM (but not an M-tensor on ${}^sTM$) determines in a* *natural way a tensor on M. In fact, if S is an M-tensor of type (r, s), then its components $S_{j\cdots}^{i\cdots}(x,y)$ in TU determine the components $S_{j\cdots}^{i\cdots}(x,0)$ in U of a tensor of type (r, s) on M.*

Any M-tensor on TM also determines in a natural way a tensor on TM as stated in the following

THEOREM 3.3. *Let $S_{j_1\cdots j_s}^{i_1\cdots i_r}$ be an M-tensor of type (r, s) on TM. If we put*

$$S = S_{j_1\cdots j_s}^{i_1\cdots i_r}\frac{\partial}{\partial y^{i_1}}\otimes\cdots\otimes\frac{\partial}{\partial y^{i_r}}\otimes dx^{j_1}\otimes\cdots\otimes dx^{j_s}, \tag{3.8}$$

then S is a tensor of type (r, s) on TM. Conversely, if (3.8) *defines a tensor of type (r, s) on TM, then the functions $S_{j_1\cdots j_s}^{i_1\cdots i_r}$ are components in $(TU, (x,y))$ of an M-tensor of type (r, s) on TM.*

4. **Adapted frames.** The M-connection we introduced in §3 has a very simple geometric interpretation which we now explain. Let V be the vertical distribution on TM. A distribution H of n-planes on TM is said to be a *horizontal distribution* if V and H are complementary, i.e., $T_\sigma(TM) = V_\sigma \oplus H_\sigma$ (direct sum) at each $\sigma \in TM$. Using the fact that the vertical distribution V is determined on each TU by the following system of Pfaffian equations:

$$\omega^i \equiv dx^i = 0, \tag{4.1}$$

we can prove (cf. also Yano and Okubo [1])

THEOREM 4.1. *The concept of M-connection on TM is equivalent to the concept of horizontal distribution on TM, the equivalence being achieved by the fact that the restriction of the horizontal distribution to TU is determined by the following system of Pfaffian equations*:

$$\omega^{\bar{i}} \equiv dy^i + \Gamma_j^i dx^j = 0. \tag{4.2}$$

On account of Theorem 4.1, we shall denote by H_Γ the horizontal distribution determined by the M-connection Γ.

It follows that the $2n$ 1-forms

$$[\omega^\alpha] = \begin{bmatrix} \omega^i \\ \omega^{\bar{i}} \end{bmatrix}$$

defined by (4.1) and (4.2) form a coframe on TU. The frame $[D_\beta] = [D_j\ D_{\bar{j}}]$ on TU dual to $[\omega^\alpha]$ consists of the following $2n$ vector fields on TU:

$$D_j = \frac{\partial}{\partial x^j} - \Gamma_j^i\frac{\partial}{\partial y^i}, \qquad D_{\bar{j}} = \frac{\partial}{\partial y^j}, \tag{4.3}$$

of which the n vector fields D_j span the horizontal distribution H_Γ and the n vector fields $D_{\bar{j}}$ span the vertical distribution.

We call $[\omega^\alpha]$ and $[D_\beta]$ respectively the *coframe and frame in TU adapted to* *the M-connection* Γ. If $[\omega^{\alpha'}]$, $[D_{\beta'}]$ are the coframe and frame in TU' adapted to Γ, then in $TU \cap TU'$ we have

$$[\omega^{\alpha'}] = \begin{bmatrix} P & 0 \\ 0 & P \end{bmatrix} [\omega^{\alpha}], \qquad [D_{\beta}] = [D_{\beta'}] \begin{bmatrix} P & 0 \\ 0 & P \end{bmatrix}. \tag{4.4}$$

On the other hand, it follows from (4.1), (4.2) and (4.3) that the adapted frame and the natural frame on TU are related by

$$[\omega^{\alpha}] = N[dx^{\alpha}], \qquad [D_{\beta}] = [\partial/\partial x^{\beta}]N^{-1}, \tag{4.5}$$

where

$$N = \begin{bmatrix} I & 0 \\ \Gamma & I \end{bmatrix}, \qquad N^{-1} = \begin{bmatrix} I & 0 \\ -\Gamma & I \end{bmatrix}.$$

Let S be a tensor, say of type (1, 2), on TM. The frame components of S in TU are the functions $S^{\alpha}_{\beta\gamma}$ appearing in the equation

$$S = S^{\alpha}_{\beta\gamma} D_{\alpha} \otimes \omega^{\beta} \otimes \omega^{\gamma}.$$

It follows from (4.5) that for a tensor G of type (0, 2) on TM, its frame component matrix $\bar{G}$ is related to its component matrix (also denoted by G) by

$$\bar{G} = (N^{-1})^{t} G N^{-1}. \tag{4.6}$$

Similarly, for a tensor F of type (1, 1) on TM, its frame component matrix $\bar{F}$ is related to its component matrix (also denoted by F) by

$$\bar{F} = NFN^{-1}. \tag{4.7}$$

By means of the adapted frames and coframes, we can obtain from a tensor of type (r, s) on TM 2^{r+s} M-tensors of the same type on TM, as explained in the following theorem for a tensor of type (1, 2).

THEOREM 4.2. *Let S be a tensor of type* (1, 2) *on TM. Suppose that it is expressed in frame components, so that on TU,*

$$\begin{aligned} S = {} & S^{i}_{jk} D_{i} \otimes \omega^{j} \otimes \omega^{k} + \cdots + S^{i}_{\bar{j}\bar{k}} D_{i} \otimes \omega^{\bar{j}} \otimes \omega^{\bar{k}} \\ & + \cdots + S^{\bar{i}}_{\bar{j}\bar{k}} D_{\bar{i}} \otimes \omega^{\bar{j}} \otimes \omega^{\bar{k}}. \end{aligned} \tag{4.8}$$

Then the geometric objects $S_1, \ldots, S_4, \ldots, S_8$ on TM whose components are $(S_1)^i_{jk} = S^i_{jk}, \ldots, (S_4)^i_{jk} = S^i_{\bar{j}\bar{k}}, \ldots, (S_8)^i_{jk} = S^{\bar{i}}_{\bar{j}\bar{k}}$ are all M-tensors of type (1, 2).

The M-tensor S_4 is independent of the M-connection Γ whose adapted frame is used in the formulation of (4.8). In fact

$$S^{i}_{\bar{j}\bar{k}} = S(\omega^{i}, D_{\bar{j}}, D_{\bar{k}}) = S(dx^{i}, \partial/\partial y^{j}, \partial/\partial y^{k})$$

so that for any M-connection Γ the set $S^{i}_{\bar{j}\bar{k}}$ of the frame components of S is the same as the corresponding set of ordinary components of S. We call S_4 the *M-tensor associated with* S and denote it by $A(S)$. The associated M-tensors for tensors of other types are defined similarly. It is clear that the associated M-tensor c of a symmetric tensor G of type (0, 2) which appeared in §3 is precisely the M-tensor $A(G)$ associated with G as defined above.

Let $A = [A^i]$ be an M-vector. Then (by Theorem 3.3), $A^i\partial/\partial y^i$ is a vector (field) on TM. This vector is obviously vertical; we call it the *vertical lift* of A and denote it by A^V. In particular, the vertical lift of the vector $\partial/\partial x^i$ on U is the vector $\partial/\partial y^i$ on TU. Thus, the vertical lift gives rise to a natural isomorphism from T_xM to $V_{(x,y)}$ defined by $(\partial/\partial x^i)_x \to (\partial/\partial y^i)_{(x,y)}$.

Now let Γ be an M-connection on TM, and A an M-vector on TM. Then (cf. Wong and Mok [1, §5]) $[{}_{-\Gamma A}^{A}]$ is a vector field on TM which is horizontal with respect to Γ. We call it the *horizontal lift* of A and denote it by A^H. In particular, the horizontal lift of the vector $\partial/\partial x^i$ on U is the vector D_i on TU. Thus, the horizontal lift gives rise to a natural isomorphism from T_xM to $H_\Gamma(x,y)$ defined by $(\partial/\partial x^i)_x \mapsto (D_i)_{(x,y)}$.

5. **Symmetric tensors of type** (0, 2) **on** TM. Let G be a nonzero symmetric tensor of type (0, 2) on TM which at present is not assumed to be of full rank or even of the same rank everywhere on TM, and let Γ be an M-connection. Then as in Theorem 4.2 we can express G on TU as

$$G = a_{ij}\omega^i \otimes \omega^j + h_{ji}\omega^i \otimes \omega^{\bar{j}} + h_{ij}\omega^{\bar{i}} \otimes \omega^j + c_{ij}\omega^{\bar{i}} \otimes \omega^{\bar{j}},$$

where $[\omega^\alpha]$ is the coframe on TU adapted to Γ and $a = [a_{ij}]$, $h = [h_{ij}]$ and $c = [c_{ij}]$ are M-tensors of type (0, 2) on TM of which a and c are symmetric and c is the M-tensor $A(G)$ associated with G and is independent of the choice of Γ.

Relative to the M-connection Γ, the frame component matrix of G is

$$\begin{bmatrix} a & h^t \\ h & c \end{bmatrix}.$$

Using relation (4.5) between the frame components and the ordinary components of G, we can prove

THEOREM 5.1. *The most general symmetric tensor G of type* (0, 2) *on TM has a component matrix of the form*

$$G = \begin{bmatrix} a + \Gamma^t c\Gamma + h^t\Gamma + \Gamma^t h & \Gamma^t c + h^t \\ c\Gamma + h & c \end{bmatrix}, \tag{5.1}$$

where $c = A(G)$ is the M-tensor associated with G, Γ is an M-connection and a (*symmetric*) *and h are M-tensors of type* (0, 2) *on TM. Relative to the M-connection Γ, the frame component matrix of G is*

$$\begin{bmatrix} a & h^t \\ h & c \end{bmatrix}. \tag{5.2}$$

We note that G in (5.1) can be uniquely expressed as the following sum of three symmetric tensors of type (0, 2) on TM:

$$\begin{bmatrix} a - c & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} c + \Gamma^t c \Gamma & \Gamma^t c \\ c\Gamma & c \end{bmatrix} + \begin{bmatrix} h^t\Gamma + \Gamma^t h & h^t \\ h & 0 \end{bmatrix}.$$

We shall call these three tensors the *vertical lift of the M-tensor $a - c$*, the *diagonal lift of the M-tensor c* and the *symmetric lift of the M-tensor h* respectively. The diagonal lift and the symmetric lift (which depend on Γ) will reappear in subsequent discussions.

Let $\tilde{\Gamma}$ be another M-connection, and

$$\begin{bmatrix} \tilde{a} & \tilde{h}^t \\ \tilde{h} & \tilde{c} \end{bmatrix}$$

the frame component of G relative to $\tilde{\Gamma}$. Then we have

$$\tilde{c} = c, \tag{5.3}$$

$$\tilde{h} = h - cT, \qquad \tilde{a} = a + T^t c T - h^t T - T^t h, \tag{5.4}$$

where $T = \tilde{\Gamma} - \Gamma$. These relations suggest that we may express a symmetric tensor of type (0, 2) on TM as a certain equivalence class. In fact, let c be a symmetric M-tensor of type (0, 2) on TM, $\mathfrak{T}$ the set of ordered pairs (h, a) of M-tensors of type (0, 2) of which a is symmetric, and $\sim$ the relation in $\mathfrak{T}$ defined by $(\tilde{h}, \tilde{a}) \sim (h, a)$ if (5.4) is satisfied for some M-tensor T of type (1, 1). Then $\sim$ is an equivalence relation and there is a bijection between the set of symmetric tensors of type (0, 2) on TM and the set of ordered pairs $(c, [h, a]_c)$, where $[h, a]_c$ is the equivalence class containing (h, a).

6. **Symmetric (0, 2)-tensors on *TM*–geometric considerations.** In order to gain a deeper insight into the structure of a symmetric (0, 2)-tensor G on TM, let us study briefly the geometry associated with G. As a symmetric (0, 2)-tensor, G assigns to each point $\sigma \in TM$ a symmetric bilinear form G_σ on $T_\sigma(TM)$ by means of which "orthogonality" can be defined in $T_\sigma(TM)$. Thus, two planes A, B at σ are said to be *orthogonal* (relative to G) if $G_\sigma(X, Y) = 0$ for all the vectors $X \in A$ and $Y \in B$. Let $A^\perp$ be the subspace of $T_\sigma(TM)$ consisting of all the vectors orthogonal to the plane A. Then $\dim(A \cap A^\perp)$ is called the *nullity* of A. In particular, we say that A is *null* if its nullity is equal to $\dim A$, and *nonnull* if its nullity is zero. It is easy to see that A is null iff the restriction of G_σ to A is zero, and A is nonnull iff this restriction is nonsingular.

Let $c = A(G)$ with component matrix $[c_{ij}]$ be the M-tensor associated with

G, and V_σ the n-plane of the vertical distribution V at the point $\sigma \in TM$. Then for any two vectors X, Y of V_σ with component matrices

$$\begin{bmatrix} 0 \\ X^i \end{bmatrix}, \quad \begin{bmatrix} 0 \\ Y^i \end{bmatrix},$$

we have

$$G_\sigma(X, Y) = X^t G_\sigma Y = c_{ij}(\sigma) X^i Y^j.$$

Therefore, c_σ is the restriction of G_σ to V_σ, or to put it simply, c is the restriction of G to V.

Suppose G is expressed in the form given in Theorem 5.1. Applying our discussions above to the n-planes V_σ and $H_\Gamma(\sigma)$ of the vertical distribution V and the horizontal distribution H_Γ, we can prove

THEOREM 6.1. *Let G be a symmetric tensor of type* (0, 2) *and* Γ *an M-connection on TM. Suppose that the frame component matrix of G relative to* Γ *is*

$$\begin{bmatrix} a & h^t \\ h & c \end{bmatrix}.$$

Then, at an arbitrarily fixed point $\sigma \in TM$,

$$\begin{aligned} &\operatorname{rank} c = n - \textit{nullity of } V_\sigma, \\ (6.1) \qquad &\operatorname{rank} a = n - \textit{nullity of } H_\Gamma(\sigma), \\ &\operatorname{rank} h = n - \dim(H_\Gamma(\sigma) \cap V_\sigma^\perp). \end{aligned}$$

PROOF. We shall only prove $(6.1)_3$ as the proof for $(6.1)_1$ and $(6.2)_2$ is similar. We recall that $H_\Gamma(\sigma)$ and V_σ are spanned respectively by the vectors $D_i(\sigma)$ and $D_{\bar{i}}(\sigma)$ of the adapted frame, so that an arbitrary vector in $H_\Gamma(\sigma)$ has frame component matrix of the form $[{X \atop 0}]$. Now this vector lies in $V_\sigma^\perp$ iff

$$[X^t\ 0]\begin{bmatrix} a & h^t \\ h & c \end{bmatrix}\begin{bmatrix} 0 \\ I \end{bmatrix} = 0, \quad \text{i.e., } X^t h^t = 0.$$

It follows from this that $\dim(H_\Gamma(\sigma) \cap V_\sigma^\perp) = n - \operatorname{rank} h$, which is $(6.1)_3$. □

We note that the assignment $\sigma \to V_\sigma^\perp$ is generally not a distribution on TM and that the nullity of V_σ (or, equivalently, the rank of c) generally varies from point to point. In the next two sections, we shall consider in detail the two extreme cases, where the rank of c is everywhere n or everywhere zero.

7. **Symmetric (0, 2)-tensors with nonsingular associated M-tensor.** Let us now consider the case of a symmetric (0, 2)-tensor G whose associated M-tensor $c = A(G)$ is everywhere nonsingular. First, it is easy to prove

THEOREM 7.1. *The following conditions on a symmetric tensor G of type* (0, 2) *on TM are equivalent*:

(i) *The M-tensor $A(G)$ is everywhere of full rank n,*

(ii) *V is nonnull with respect to G,*

(iii) *The assignment $\sigma \to V_\sigma^\perp$ is a horizontal distribution on TM (which we denote by $V^\perp$).*

Suppose that G is a symmetric (0, 2)-tensor satisfying any one of the equivalent conditions in Theorem 7.1. Then there is a unique M-connection Γ such that $H_\Gamma = V^\perp$. Let $[D_\beta]$ be the frame adapted to Γ. Then, $G(D_i, D_{\bar{j}}) = 0$. It follows that the frame component matrix of G is of the form $\left[\begin{smallmatrix} a & 0 \\ 0 & c \end{smallmatrix}\right]$, where $c = A(G)$ and $a = [a_{ij}]$ is the symmetric M-tensor determined by $a_{ij} = G(D_i, D_j)$ and is therefore the restriction of G to $H_\Gamma = V^\perp$. In terms of c, Γ, and a, the component matrix of G is exactly that in (3.7). Hence, we have the following version of Theorem 3.1:

THEOREM 7.2. *The most general symmetric tensor G of type* (0, 2) *on TM whose associated M-tensor $c = A(G)$ is of full rank everywhere has a component matrix of the form*

$$G = \begin{bmatrix} a + \Gamma^t c\Gamma & \Gamma^t c \\ c\Gamma & c \end{bmatrix}, \tag{7.1}$$

where Γ is the (unique) M-connection such that $H_\Gamma = V^\perp$ and a is the symmetric M-tensor of type (0, 2) *which is the restriction of G to $V^\perp$. Relative to this Γ, the frame component matrix of G is*

$$\begin{bmatrix} a & 0 \\ 0 & c \end{bmatrix}. \tag{7.2}$$

Consequently, the rank of G is equal to $n +$ rank a. In particular, G is a (pseudo-) Riemannian metric iff a is of rank n everywhere.

Since c, Γ, a in Theorem 7.2 are uniquely determined when G is given, a consequence of Theorem 7.2 is that there is a bijection between the set of symmetric tensors of type (0, 2) on TM whose associated M-tensor is everywhere nonsingular and the set of ordered triples (c, Γ, a) where c (nonsingular) and a are symmetric M-tensors of type (0, 2) on TM and Γ is an M-connection.

REMARK. The M-tensor a considered above is intrinsically associated with G. It is interesting to see how it can be expressed in terms of the M-tensors $\tilde{c}$, $\tilde{h}$ and $\tilde{a}$ which appear in the frame component matrix of G relative to an arbitrarily chosen M-connection $\tilde{\Gamma}$. To do this, we put $h = 0$ in (5.4) and eliminate T, obtaining $a = \tilde{a} - \tilde{h}^t\tilde{c}^{-1}\tilde{h}$.

In the special case where $a = c$, the symmetric tensor G of type (0, 2) in Theorem 7.2 is the diagonal lift of the nonsingular symmetric M-tensor c to TM, namely

$$\begin{bmatrix} c + \Gamma^t c\Gamma & \Gamma^t c \\ c\Gamma & c \end{bmatrix},$$

which is characterized by the conditions that for any two vectors $X = [X^i]$, $Y = [Y^i]$ in M,

$$G(X^H, Y^H) = G(X^V, Y^V) = c_{ij}X^iY^j, \qquad G(X^H, Y^V) = 0.$$

The following are two well-known examples of Riemannian metrics G whose associated M-tensor is everywhere nonsingular. Both of them are diagonal lifts.

(a) Let $g = [g_{ij}]$ be a Riemannian metric on M and $\Gamma^i_j = \{{}^{\,i}_{jk}\}y^k$, where $\{{}^{\,i}_{jk}\}$ is the Christoffel symbol of g_{ij}. Then $\Gamma = [\Gamma^i_j]$ is an M-connection on TM. Using this Γ and putting $a = c = g$ in (7.1), we get the component matrix of the original Sasaki metric (2.5) on TM.

(b) Let M be a Finslerian manifold and F: ${}^sTM \to R$ the fundamental function defining the Finsler metric. It is easy to see that $g_{ij} = \partial^2(\frac{1}{2}F^2)/\partial y^i \partial y^j$ are the components of a symmetric M-tensor of type (0, 2) on ${}^sTM$ (which is everywhere nonsingular by definition of the fundamental function). If we define the functions $\{{}^{\,i}_{jk}\}$ and Γ^i_j on ${}^sTU$ by

$$\left\{ {i \atop jk} \right\} = \frac{1}{2}\, g^{ih}(\partial_j g_{kh} + \partial_k g_{jh} - \partial_h g_{jk}),$$

$$\Gamma^i_j = \frac{1}{2}\, \frac{\partial}{\partial y^j}\left(\left\{ {i \atop hk} \right\} y^h y^k\right),$$

then we can prove that $\Gamma = [\Gamma^i_j]$ is an M-connection on ${}^sTM$. Using this Γ and putting $a = c = g = [g_{ij}]$ in (7.1), we get the Riemannian metric on the slit tangent bundle of a Finslerian manifold considered by Yano and Davies in [1] and [2].

8. **Symmetric (0, 2)-tensors with zero associated M-tensor.** First, we can easily prove

THEOREM 8.1. *The following conditions on a symmetric tensor G of type* (0, 2) *on TM are equivalent*:

(i) *The M-tensor $A(G)$ is everywhere zero,*

(ii) *V is null with respect to G,*

(iii) *$V_\sigma \subset V_\sigma^\perp$ for all $\sigma \in TM$.*

In case G is a (pseudo)-Riemannian metric, then $\dim V_\sigma^\perp = n$ *and so* (iii) *means that $V_\sigma = V_\sigma^\perp$ for all $\sigma \in TM$.*

Now, putting $c = 0$ in Theorem 5.1, we get

THEOREM 8.2. *The most general symmetric tensor G of type* (0, 2) *on TM whose associated M-tensor $A(G)$ is zero has a component matrix of the form*

$$\begin{bmatrix} a + h^t\Gamma + \Gamma^t h & h^t \\ h & 0 \end{bmatrix}, \tag{8.1}$$

where Γ *is an* M*-connection and* a *(symmetric),* h *are* M*-tensors of type* $(0, 2)$ *on* TM.

Relative to the M*-connection* Γ, *the frame component matrix of* G *is*

$$\begin{bmatrix} a & h^t \\ h & 0 \end{bmatrix}. \tag{8.2}$$

The M-tensor h in (8.1) and (8.2) is independent of the Γ chosen, and we can prove that there is a bijection between the set of symmetric tensors of type $(0, 2)$ on TM whose associated M-tensor is zero and the set of ordered pairs $(h, [a]_h)$, where h is an M-tensor of type $(0, 2)$ and $[a]_h$ is the equivalence class of the following equivalence relation in the set of symmetric M-tensors of type $(0, 2)$:

$\tilde{a} \sim a$ if there exists an M-tensor T of type $(1, 1)$ such that $\tilde{a} = a - h^tT - T^th$.

If G is a (pseudo-)Riemannian metric on TM, we can further simplify the component matrix (8.1). In fact, we have

THEOREM 8.3. *The most general (psuedo-)Riemannian metric* G *on* TM *whose associated* M*-tensor is zero has a component matrix of the form*

$$G = \begin{bmatrix} h^t\Gamma + \Gamma^t h & h^t \\ h & 0 \end{bmatrix}, \tag{8.3}$$

where Γ *is an* M*-connection and* h *is a nonsingular* M*-tensor of type* $(0, 2)$ *on* TM. *Relative to* Γ, *the frame component matrix of* G *is*

$$\begin{bmatrix} 0 & h^t \\ h & 0 \end{bmatrix}. \tag{8.4}$$

For a given G, h *is uniquely determined, whereas* Γ *is not.* Γ *is characterized by the property that* H_Γ *is null with respect to* G. *If* $\tilde{\Gamma}$ *is any* M*-connection, then* (8.3) *with* Γ *replaced by* $\tilde{\Gamma}$ *represents the same metric* G *iff* $\tilde{\Gamma} = \Gamma + (h^t)^{-1}d$ *for some skew-symmetric* M*-tensor* d *of type* $(0, 2)$ *on* TM.

PROOF. From Theorem 8.2, we know that G is of the form (8.1), with h everywhere nonsingular. Since a is symmetric, we may write $a + h^t\Gamma + \Gamma^t h = (h^t\Gamma + \frac{1}{2}a) + (h^t\Gamma + \frac{1}{2}a)^t$. As h is everywhere nonsingular, there exists a unique Γ^* satisfying $h^t\Gamma^* = h^t\Gamma + \frac{1}{2}a$, and this Γ^* is an M-connection because $\frac{1}{2}(h^t)^{-1}a$ is an M-tensor of type $(1, 1)$. On renaming Γ^* as Γ, we get the desired form (8.3) for G. Clearly, h is uniquely determined and independent of the choice of Γ. Now, let $\tilde{\Gamma}$ be any M-connection such that

$$h^t\tilde{\Gamma} + \tilde{\Gamma}^t h = h^t\Gamma + \Gamma^t h, \quad \text{i.e.,} \qquad h^t(\tilde{\Gamma} - \Gamma) + (\tilde{\Gamma} - \Gamma)^t h = 0.$$

This means that $d \equiv h^t(\tilde{\Gamma} - \Gamma)$ is a skew-symmetric M-tensor of type $(0, 2)$. Hence, $\tilde{\Gamma} = \Gamma + (h^t)^{-1}d$ as required. □

It follows from Theorem 8.3 above that there is a bijection between the set of such (pseudo-)Riemannian metrics on TM and the set of ordered pairs $(h, [\Gamma]_h)$ where h is a nonsingular M-tensor of type (0, 2) and $[\Gamma]_h$ is the equivalence class of the following equivalence relation in the set of M-connections on TM:

$\tilde{\Gamma} \sim \Gamma$ if $\tilde{\Gamma} = \Gamma + (h^t)^{-1}d$ for some skew-symmetric M-tensor d of type (0, 2).

We note that (8.3) is just the component matrix of the symmetric lift of the M-tensor h. Furthermore, from (8.4), we easily see that the metric of Theorem 8.3 has signature n, i.e., its canonical form has n positive and n negative signs. negative signs.

An important example of the metric G in Theorem 8.3 is the horizontal lift g^H of a (pseudo-)Riemannian metric g on M to TM considered in Yano and Ishihara [2]. This can be obtained from (8.3) by putting $h = g$ and $\Gamma = [\Gamma^i_k] = [y^j\gamma^i_{jk}]$, where γ^i_{jk} are the components of a linear connection γ on M. The component matrix of g^H is therefore

$$\begin{bmatrix} g\Gamma + \Gamma^t g & g \\ g & 0 \end{bmatrix}. \tag{8.5}$$

In the special case where γ is a metric connection of g,

$$\nabla_k g_{ij} \equiv \partial_k g_{ij} - \gamma^a_{ki} g_{aj} - \gamma^a_{kj} g_{ia} = 0$$

and so $g\Gamma + \Gamma^t g = \partial g$. Thus in this case, (8.5) becomes

$$\begin{bmatrix} \partial g & g \\ g & 0 \end{bmatrix},$$

which is the component matrix of the complete lift g^C of g to TM considered by Yano and Kobayashi in [1].

We can now easily prove

THEOREM 8.4. *Let g be a (pseudo-)Riemannian metric and γ a linear connection on M. Then γ is a metric connection of g iff the horizontal distribution associated with the M-connection $\Gamma^i_k = y^j\gamma^i_{jk}$ on TM is null with respect to the complete lift g^C on TM.*

9. Structure of a tensor of type (1, 1) **on** TM. In the remaining sections, we shall study the structure of a tensor F of type (1, 1) on TM. Following Theorem 4.2, we suppose that Γ is an M-connection and express F on TU as

$$F = a^i_j D_i \otimes \omega^j + f^i_j D_i \otimes \omega^{\bar{j}} + b^i_j D_{\bar{i}} \otimes \omega^j + c^i_j D_{\bar{i}} \otimes \omega^{\bar{j}},$$

where $[D_\beta]$ and $[\omega^\alpha]$ are the frame and coframe on TU adapted to Γ. Then $a = [a^i_j]$, $f = [f^i_j]$, $b = [b^i_j]$ and $c = [c^i_j]$ are M-tensors of type (1, 1) and f is the M-tensor $A(F)$ associated with F which is independent of the choice of Γ.

Relative to the M-connection Γ, the frame component matrix of F is $[\begin{smallmatrix} a & f \\ b & c \end{smallmatrix}]$. Using the relation (4.7) between the frame components and the ordinary components of F, we can prove

THEOREM 9.1. *The most general tensor F of type* (1, 1) *on TM has a component matrix of the form*

$$F = \begin{bmatrix} a + f\Gamma & f \\ b + c\Gamma - \Gamma a - \Gamma f \Gamma & c - \Gamma f \end{bmatrix}, \tag{9.1}$$

where $f = A(F)$ is the associated M-tensor of F, Γ is an M-connection and a, b, c are M-tensors of type (1, 1) *on TM. Relative to the M-connection Γ, the frame component matrix of F is*

$$\begin{bmatrix} a & f \\ b & c \end{bmatrix}. \tag{9.2}$$

Let $\tilde{\Gamma}$ be another M-connection, and

$$\begin{bmatrix} \tilde{a} & \tilde{f} \\ \tilde{b} & \tilde{c} \end{bmatrix}$$

the frame component matrix of F relative to $\tilde{\Gamma}$. Then we have

$$\tilde{f} = f, \tag{9.3}$$

$$\begin{aligned} \tilde{a} &= a - fT, \\ \tilde{c} &= Tf + c, \\ \tilde{b} &= Ta - TfT + b - cT, \end{aligned} \tag{9.4}$$

where $T = \tilde{\Gamma} - \Gamma$. Now let f be an M-tensor of type (1, 1) on TM, $\mathbb{S}$ the set of ordered triples (a, c, b) of M-tensors of type (1, 1) on TM, and $\sim$ the relation in $\mathbb{S}$ defined by $(\tilde{a}, \tilde{c}, \tilde{b}) \sim (a, c, b)$ if (9.4) is satisfied for some M-tensor T of type (1, 1). Then $\sim$ is an equivalence relation and there is a bijection between the set of tensors of type (1, 1) on TM and the set of ordered pairs $(f, [a, c, b]_f)$, where $[a, c, b]_f$ is the equivalence class containing (a, c, b).

10. **Two classes of tensors of type** (1, 1) **on** TM. As in the case of a tensor of type (0, 2) on TM, we shall now confine our attention to a tensor of type (1, 1) on TM whose associated M-tensor has full rank everywhere or is zero. We first consider the geometric meaning of these assumptions.

Let F be a tensor of type (1, 1) on TM. Then F assigns to each point $\sigma \in TM$ a linear transformation F_σ in $T_\sigma(TM)$. This F_σ maps the vertical n-plane V_σ at σ onto the subspace $F_\sigma(V_\sigma)$. To find $F_\sigma(V_\sigma)$ we use the component matrix (9.1) of F and the component matrix $[\begin{smallmatrix} 0 \\ I \end{smallmatrix}]$ of the n vectors

$D_{\bar{j}} = \partial/\partial y^j$ which span V_σ. Since

$$\begin{bmatrix} a + f\Gamma & f \\ b + c\Gamma - \Gamma a - \Gamma f\Gamma & c - \Gamma f \end{bmatrix}\begin{bmatrix} 0 \\ I \end{bmatrix} = \begin{bmatrix} f \\ c - \Gamma f \end{bmatrix},$$

we see that $F_\sigma(V_\sigma)$ is spanned by the n vectors $[{}^{\;f}_{c-\Gamma f}]_\sigma$. Thus, dim $F_\sigma(V_\sigma)$ in general varies from point to point.

Suppose $f = A(F)$ is of full rank everywhere. We then have

$$\begin{bmatrix} f \\ c - \Gamma f \end{bmatrix} = \begin{bmatrix} I \\ cf^{-1} - \Gamma \end{bmatrix} f.$$

Since cf^{-1} is an M-tensor of type (1, 1), $\tilde{\Gamma} = \Gamma - cf^{-1}$ is an M-connection. Therefore, $F_\sigma(V_\sigma)$ is spanned by the n vectors $[{}^{\;\;I}_{-\tilde{\Gamma}(\sigma)}]$ and so, by (4.3), $F_\sigma(V_\sigma) = H_{\tilde{\Gamma}}(\sigma)$. Hence, the assignment $\sigma \to F_\sigma(V_\sigma)$ is a horizontal distribution. Conversely, if this is true, then the matrix

$$\begin{bmatrix} f & 0 \\ c - \Gamma f & I \end{bmatrix}.$$

must have rank $2n$ and so f must have rank n.

On the other hand, if $f = A(F) = 0$, then $F_\sigma(V_\sigma)$ is spanned by the n vectors $[{}^{\;\;0}_{c(\sigma)}]$ and so $F_\sigma(V_\sigma) \subset V_\sigma$, i.e., V_σ is stable under F_σ. Conversely, it is easy to see that if V_σ is stable under F_σ for every σ, then $f = 0$.

We summarize our discussion so far in

THEOREM 10.1. *Let F be a tensor of type* (1, 1) *on TM and $A(F)$ its associated M-tensor. Then $A(F)$ has full rank everywhere iff the assignment $\sigma \to F_\sigma(V_\sigma)$ is a horizontal distribution $F(V)$ on TM. And $A(F) = 0$ iff V is stable under F.*

We now consider the case where $A(F)$ has full rank everywhere. In this case we have the horizontal distribution $F(V)$ and so it is natural to consider the frame component of F relative to the (unique) M-connection Γ such that $H_\Gamma = F(V)$. Now putting $\tilde{\Gamma} = \Gamma$ in $\tilde{\Gamma} = \Gamma - cf^{-1}$, we get $c = 0$. Therefore, the frame component matrix of F relative to this Γ is of the form $[{}^{a}_{b}\,{}^{f}_{0}]$, where a, b are M-tensors of type (1, 1). The component matrix of F can be obtained from Theorem 9.1. Thus, we have

THEOREM 10.2. *The most general tensor F of type* (1, 1) *on TM whose associated M-tensor $f = A(F)$ has full rank everywhere has a component matrix of the form*

$$\begin{bmatrix} a + f\Gamma & f \\ b - \Gamma a - \Gamma f\Gamma & -\Gamma f \end{bmatrix}, \tag{10.1}$$

where Γ is the M-connection such that $H_\Gamma = F(V)$ and a, b are M-tensors of type (1, 1) *on TM. Relative to the M-connection Γ, the frame component matrix* *of F is*

$$\begin{bmatrix} a & f \\ b & 0 \end{bmatrix}. \tag{10.2}$$

We remark that Theorem 10.2 can also be proved by using the fact that f is of rank n to simplify the component matrix in (9.1).

It follows from (10.1) that when F is given, first $f = A(F)$, then Γ, and then a, and finally b are uniquely determined. Therefore, there is a bijection between the set of tensors of type (1, 1) on TM whose associated M-tensor is everywhere nonsingular and the set of ordered quadruples (f, Γ, a, b) where f (nonsingular), a, b are M-tensors of type (1, 1) and Γ is an M-connection.

For the case $A(F) = 0$, we put $f = 0$ in Theorem 9.1, and obtain

THEOREM 10.3. *The most general tensor F of type* (1, 1) *on TM whose associated M-tensor $A(F)$ is zero has a component matrix of the form*

$$\begin{bmatrix} a & 0 \\ b + c\Gamma - \Gamma a & c \end{bmatrix}, \tag{10.3}$$

where Γ is an M-connection and a, b, c are M-tensors of type (1, 1) *on TM. Relative to the M-connection Γ, the frame component matrix of F is*

$$\begin{bmatrix} a & 0 \\ b & c \end{bmatrix}. \tag{10.4}$$

The M-tensors a and c in (10.4) are independent of the Γ chosen, and we can prove (cf. (9.4)) that there is a bijection between the set of such tensors on TM and the set of ordered triples $(a, c, [b]_{(a,c)})$, where a, c are M-tensors of type (1, 1) on TM and $[b]_{(a,c)}$ is the equivalence class of the following equivalence relation in the set of M-tensors of type (1, 1):

$\tilde{b} \sim b$ if there exists an M-tensor T of type (1, 1) on TM such that $\tilde{b} = b + Ta - cT$.

11. **(1, 1)-tensors F on TM satisfying $F^2 = \lambda E$.** Let F be a tensor of type (1, 1) on TM satisfying the condition $F^2 = \lambda E$, where λ is a real number and E is the "identity" tensor of type (1, 1) on TM with component matrix $\left[\begin{smallmatrix} I & 0 \\ 0 & I \end{smallmatrix}\right]$. If $\lambda = 1$, then F is an almost product structure on TM. If $\lambda = -1$, then F is an almost complex structure on TM. If $\lambda = 0$ and F is of rank n, then F is an almost tangent structure on TM (Clark and Bruckheimer [1]). In this section, we shall study such tensors F whose associated M-tensors have rank n everywhere or are zero, so that the results of §10 apply.

We first prove

THEOREM 11.1. (a) *Let F be any tensor of type* (1, 1) *on TM whose associated M-tensor f is everywhere nonsingular and which satisfies the conditon $F^2 = \lambda E$. Then, there exists a unique M-connection Γ relative to which the frame* *component matrix of F is*

$$\begin{bmatrix} 0 & f \\ \lambda f^{-1} & 0 \end{bmatrix}. \tag{11.1}$$

(b) *If* Γ *is any M-connection, f any nonsingular M-tensor of type* (1, 1) *on TM and* λ *any real number, then* (11.1) *is the frame component matrix* (*relative to* Γ) *of a tensor F of type* (1, 1) *on TM which satisfies the conditions that* $A(F) = f$ *and* $F^2 = \lambda E$.

Proof. (a) By Theorem 10.2, there is an M-connection Γ relative to which the frame component matrix of F satisfies $F^2 = \lambda E$, then

$$\begin{bmatrix} a & f \\ b & 0 \end{bmatrix}\begin{bmatrix} a & f \\ b & 0 \end{bmatrix} = \lambda\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix},$$

i.e.,

$$\begin{bmatrix} a^2 + fb & af \\ ba & bf \end{bmatrix} = \lambda\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}.$$

Therefore, $a = 0$, $b = \lambda f^{-1}$ and hence we have (11.1). To prove the uniqueness of Γ, we observe from Theorem 10.2 that if the frame component matrix of F relative to Γ is (11.1), then the component matrix of F is

$$F = \begin{bmatrix} f\Gamma & f \\ \lambda f^{-1} - \Gamma f\Gamma & -\Gamma f \end{bmatrix}.$$

Therefore, since f is everywhere nonsingular, Γ is uniquely determined. Hence (a) is completely proved. The proof of (b) is straightforward. □

Since linear connections exist on M and consequently M-connections exist on TM, if we take $f = +I$ or $-I$ in Theorem 11.1 (b), we have

Corollary 11.2. *There always exists on TM a tensor F of type* (1,1) *whose associated M-tensor is everywhere nonsingular and which satisfies the condition* $F^2 = \lambda E$.

In particular, if $\lambda = -1$, $f = -I$ and Γ is an $h(1)$ M-connection on ${}^sTM$ (see Wong and Mok [1] for definition), we have the almost complex structure considered in Yano and Ishihara [1, §4].

As an example of a class of almost product structures on TM, we prove the following theorem on a pair of complementary horizontal distributions on TM.

Theorem 11.3. *Let* Γ^0, Γ *be two M-connections on TM. Then* H_{Γ^0} *and* H_Γ *are complementary iff the M-tensor* $S = \Gamma - \Gamma^0$ *of type* (1, 1) *on TM has full rank everywhere. In this case, the almost product structure F determined by* H_{Γ^0}

and H_Γ *is such that* $A(F) = 2S^{-1}$, *and, with respect to the M-connection* $\tilde{\Gamma} = \frac{1}{2}(\Gamma^0 + \Gamma)$, *the frame component matrix of F is*

$$\begin{bmatrix} 0 & 2S^{-1} \\ \frac{1}{2}S & 0 \end{bmatrix}, \tag{11.2}$$

which is (11.1) *with* $f = 2S^{-1}$ *and* $\lambda = 1$.

PROOF. It suffices to see what happens in TU. In TU, H_{Γ^0} and H_Γ are spanned by the vectors

$$[D_j^0] = \begin{bmatrix} I \\ -\Gamma^0 \end{bmatrix} \quad \text{and} \quad [D_j] = \begin{bmatrix} I \\ -\Gamma \end{bmatrix}$$

respectively. Therefore, a necessary and sufficient condition for H_{Γ^0} and H_Γ to be complementary is that the matrix

$$\begin{bmatrix} I & I \\ -\Gamma^0 & -\Gamma \end{bmatrix}$$

is of rank $2n$ everywhere. But this matrix has the same rank as the matrix

$$\begin{bmatrix} 0 & I \\ \Gamma - \Gamma^0 & \Gamma \end{bmatrix},$$

and hence is of rank $2n$ everywhere iff the M-tensor $S = \Gamma - \Gamma^0$ is of rank n everywhere. Suppose now that H_{Γ^0}, H_Γ are complementary. Then the almost product structure F determined by (H_{Γ^0}, H_Γ) is the "reflection" about H_{Γ^0} along H_Γ, and is therefore determined by $F(D_i^0) = D_i^0$, $F(D_i) = -D_i$. Applying F to the relation $D_i - D_i^0 = -S_i^j D_{\bar{j}}$ (where $S = [S_i^j]$ and $[D_{\bar{j}}] = [{0 \atop I}]$), we get

$$F(D_{\bar{j}}) = 2(S^{-1})_j^i D_i^0 - D_{\bar{j}}.$$

From this, it follows that the frame component matrix of F relative to Γ^0 is

$$\begin{bmatrix} I & 2S^{-1} \\ 0 & -I \end{bmatrix}.$$

Since $A(F) = 2S^{-1}$ is of full rank everywhere, there is (by Theorem 11.1) a unique M-connection $\tilde{\Gamma}$ relative to which the frame component matrix of F is of the form (11.1). Since (11.2) is of the form (11.1), it must be the frame component matrix of F relative to $\tilde{\Gamma}$. To find $\tilde{\Gamma}$, we use $(9.4)_1$, namely, $\tilde{a} = a - fT$, and substitute in it $\tilde{a} = 0$, $a = I$, $f = 2S^{-1}$ and $T = \tilde{\Gamma} - \Gamma^0$. Then we get $\tilde{\Gamma} - \Gamma^0 = \frac{1}{2}S = \frac{1}{2}(\Gamma - \Gamma^0)$, and therefore, $\tilde{\Gamma} = \frac{1}{2}(\Gamma^0 + \Gamma)$. □

The proof of the next theorem is similar to that of Theorem 11.1.

THEOREM 11.4. (a) *Let F be any tensor of type* (1, 1) *on TM whose associated*

M-tensor $A(F)$ is zero and which satisfies the condition $F^2 = \lambda E$. Relative to any M-connection Γ, the frame component matrix of F is of the form

$$\begin{bmatrix} a & 0 \\ b & c \end{bmatrix}, \tag{11.3}$$

where a, b, c are M-tensors of type (1, 1) *satisfying the conditions*

$$a^2 = c^2 = \lambda I, \qquad ba + cb = 0, \tag{11.4}$$

and a and c do not depend on Γ but b does.

(b) *Conversely, if a, b, c are M-tensors of type* (1, 1) *on TM satisfying conditions* (11.4), *and Γ is any M-connection on TM, then* (11.3) *is the frame component matrix of a tensor F of type* (1, 1) *on TM satisfying the condition $A(F) = 0$ and $F^2 = \lambda E$.*

We now prove

COROLLARY 11.5. *There exists on TM a tensor F of type* (1, 1) *whose associated M-tensor is zero and which satisfies the condition $F^2 = \lambda E$ iff there exists on M a tensor α of type* (1, 1) *which satisfies the condition $\alpha^2 = \lambda I$. In particular, if* dim M = *odd, there does not exist on TM any almost complex structure whose associated M-tensor is zero.*

PROOF. The "necessity" follows immediately from Theorem 11.4(a) and Theorem 3.2(e). To prove the "sufficiency" we put $a = c = \alpha$ and $b = 0$ in Theorem 11.4(b) and obtain a tensor F of type (1, 1) on TM whose frame component matrix is $\left[\begin{smallmatrix} \alpha & 0 \\ 0 & \alpha \end{smallmatrix}\right]$. (This F is precisely the α^H described in Example 1 below.) The last assertion in the corollary follows from the fact that if n is odd, there does not exist any real $n \times n$ matrix A with $A^2 = -I$. □

Let us look at some examples of the tensor F in Theorem 11.4.

EXAMPLE 1. Let h be an almost complex structure on M and h^H its horizontal lift (Yano and Ishihara [2]). Then h^H is an almost complex structure on TM which can be obtained from Theorem 11.4 by putting $a = c = h$, $b = 0$ and $\Gamma = [\Gamma^i_k] = [y^j \gamma^i_{jk}]$, where γ^i_{jk} are the components of a linear connection γ on M. The component matrix of h^H is therefore (cf. Theorem 10.3)

$$\begin{bmatrix} h & 0 \\ -\Gamma h + h\Gamma & h \end{bmatrix}. \tag{11.5}$$

In case γ is a linear connection with respect to which h is parallel,

$$\nabla_k h^i_j \equiv \partial_k h^i_j + \gamma^i_{ka} h^a_j - \gamma^a_{kj} h^i_a = 0$$

and so $-\Gamma h + h\Gamma = \partial h$. Therefore, in this case, (11.5) becomes

$$\begin{bmatrix} h & 0 \\ \partial h & h \end{bmatrix},$$

which is the component matrix of the complete lift h^C of h to TM as considered in Yano and Kobayashi [1].

EXAMPLE 2. Let Γ be an M-connection on TM. Then the "reflection" about H_Γ along the vertical distribution V is an almost product structure on TM whose frame component matrix relative to Γ and component matrix are respectively

$$\begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}, \quad \begin{bmatrix} I & 0 \\ -2\Gamma & -I \end{bmatrix}.$$

This almost product structure has been studied in some detail by Grifone [1, § I-4] under the name of "nonhomogeneous connection".

EXAMPLE 3. Let b be an M-tensor of type (1, 1) on TM which is everywhere of full rank, and F the tensor of type (1, 1) on TM whose component matrix is $\left[\begin{smallmatrix} 0 & 0 \\ b & 0 \end{smallmatrix}\right]$ (which by Theorem 10.2 is also the frame component matrix of F relative to an arbitrary M-connection on TM). Since $F^2 = 0$, F is an almost tangent structure on TM. If $b = I$, we have the almost tangent structure considered in Clark and Goel [1, Example 1-1].

12. Metrics compatible with an almost product or almost complex structure. Let G be a nonzero symmetric (resp. skew-symmetric) tensor of type (0, 2) and F a nonzero tensor of type (1, 1) on TM. Then, at each point σ of TM, G induces an inner product (resp. exterior product) G_σ in $T_\sigma(TM)$ while F induces a linear transformation F_σ in $T_\sigma(TM)$. We say that G is *compatible* with F (or G and F are *compatible*) if $G_\sigma(F_\sigma X, F_\sigma Y) = G_\sigma(X, Y)$ for all $X, Y \in T_\sigma(TM)$ and all $\sigma \in TM$. In terms of component matrices or frame component matrices, G is compatible with F iff $F^tGF = G$.

Suppose now that G is compatible with F and F satisfies the condition $F^2 = \lambda E$, where λ is a real number, and E the "identity tensor" on TM. Then from

$$F^tGF = G \Rightarrow F^t(F^tGF)F = F^tGF = G$$

and $F^2 = \lambda E$, it follows that $\lambda^2 G = G$ so that $\lambda = \pm 1$. Thus, if G is compatible with an F satisfying $F^2 = \lambda E$, then F must be an almost product or almost complex structure. For convenience, we shall refer to a tensor F of type (1, 1) on TM satisfying the condition $F^2 = \lambda E$, where $\lambda = \pm 1$, as a λ-*structure* on TM, and a nonsingular symmetric tensor of type (0, 2) on TM as a *metric* on TM.

In this section we determine all the metrics G and λ-structures F on TM whose associated M-tensors are everywhere nonsingular or zero and which are compatible with each other. Thus we shall always assume that $\lambda = \pm 1$. For convenience, we shall consider the following three cases separately:

Case 1. $A(F)$ everywhere nonsingular,

Case 2a. $A(F) = 0$ and $A(G)$ everywhere nonsingular,
Case 2b. $A(F) = 0$ and $A(G) = 0$.

Case 1. $f - A(F)$ *everywhere nonsingular*. Since F satisfies $F^2 = \lambda E$, the frame component matrix of F relative to a suitably chosen M-connection Γ is

$$\begin{bmatrix} 0 & f \\ \lambda f^{-1} & 0 \end{bmatrix}$$

(cf. Theorem 11.1). Let G be a metric whose frame component matrix relative to Γ is

$$\begin{bmatrix} a & h^t \\ h & c \end{bmatrix}.$$

Then the condition for G to be compatible with F, namely,

$$\begin{bmatrix} a & h^t \\ h & c \end{bmatrix} = \begin{bmatrix} 0 & \lambda(f^t)^{-1} \\ f^t & 0 \end{bmatrix} \begin{bmatrix} a & h^t \\ h & c \end{bmatrix} \begin{bmatrix} 0 & f \\ \lambda f^{-1} & 0 \end{bmatrix}$$
$$= \begin{bmatrix} (f^t)^{-1} c f^{-1} & \lambda (f^t)^{-1} h f \\ \lambda f^t h^t f^{-1} & f^t a f \end{bmatrix},$$

is that

$$a = (f^t)^{-1} c f^{-1}, \qquad c = f^t a f, \qquad h = \lambda f^t h^t f^{-1},$$

which is equivalent to

$$a = (f^t)^{-1} c f^{-1}, \qquad d \equiv hf \ \text{ satisfies } \ d^t = \lambda d.$$

Hence we have proved

THEOREM 12.1. *A metric G on TM is compatible with a λ-structure F on TM whose associated M-tensor $f = A(F)$ is everywhere nonsingular iff there exists an M-connection relative to which the frame component matrices of F and G are respectively*

$$\begin{bmatrix} 0 & f \\ \lambda f^{-1} & 0 \end{bmatrix},$$

$$\begin{bmatrix} (f^t)^{-1} c f^{-1} & \lambda (f^t)^{-1} d \\ d f^{-1} & c \end{bmatrix} = \begin{bmatrix} 0 & \lambda (f^t)^{-1} \\ I & 0 \end{bmatrix} \begin{bmatrix} c & \lambda d \\ d & c \end{bmatrix} \begin{bmatrix} 0 & I \\ \lambda f^{-1} & 0 \end{bmatrix},$$

where $c = A(G)$ and d is a symmetric or skew-symmetric M-tensor of type (0, 2) *according as $\lambda = 1$ or $\lambda = -1$ such that*

$$\begin{bmatrix} c & \lambda d \\ d & c \end{bmatrix}$$

is nonsingular.

In particular, if $f = I$ and $\lambda = -1$, then the most general metric compatible with the almost complex structure

$$\begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$$

has a frame component matrix of the form

$$\begin{bmatrix} c & -d \\ d & c \end{bmatrix},$$

which is therefore the sum of the diagonal lift of c and the symmetric lift of d relative to the M-connection Γ (cf. §5). On putting $d = 0$, we get the known result that the Sasaki metric is compatible with the almost complex structure

$$\begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$$

(cf. Tachibana and Okumura [1] and Yano and Davies [1]).

Case 2a. $A(F) = 0$ *and* $A(G)$ *everywhere nonsingular.* In this case, there is a unique M-connection Γ relative to which G has a frame component matrix of the form $\left[\begin{smallmatrix} a & 0 \\ 0 & c \end{smallmatrix}\right]$, where $c = A(G)$ is nonsingular and a is a nonsingular symmetric M-tensor of type (0, 2) (cf. Theorem 7.2). At the same time, relative to any M-connection (and in particular, relative to Γ), the frame component matrix of F satisfying $F^2 = \lambda E$ is of the form

$$\begin{bmatrix} \alpha & 0 \\ \beta & \gamma \end{bmatrix},$$

where α, β, γ are M-tensors of type (1, 1) satisfying

(12.1) $$\alpha^2 = \lambda I, \qquad \gamma^2 = \lambda I, \qquad \beta\alpha + \gamma\beta = 0$$

(cf. Theorem 11.4). The condition for the compatibility of G and F is

$$\begin{bmatrix} a & 0 \\ 0 & c \end{bmatrix} = \begin{bmatrix} \alpha^t & \beta^t \\ 0 & \gamma^t \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & c \end{bmatrix} \begin{bmatrix} \alpha & 0 \\ \beta & \gamma \end{bmatrix} = \begin{bmatrix} \alpha^t a\alpha + \beta^t c\beta & \beta^t c\gamma \\ \gamma^t c\beta & \gamma^t c\gamma \end{bmatrix}.$$

Since c and γ are both nonsingular, this condition is easily seen to be equivalent to $\beta = 0$ and

(12.2) $$a = \alpha^t a\alpha, \qquad c = \gamma^t c\gamma.$$

Multiplying the two equations in (12.2) on the right by α and γ respectively and making use of (12.1), we get

$$a\alpha = \lambda(a\alpha)^t, \qquad c\gamma = \lambda(c\gamma)^t.$$

From these it follows that $d_1 \equiv \lambda a\alpha$, $d_2 \equiv \lambda c\gamma$ are M-tensors of type (0, 2) such that

(12.3) $$d_1^t = \lambda d_1, \quad d_2^t = \lambda d_2; \qquad a = d_1\alpha, \quad c = d_2\gamma.$$

On account of the last two equations, the two equations in (12.2) become

(12.4) $$d_1 = \alpha^t d_1 \alpha, \qquad d_2 = \gamma^t d_2 \gamma.$$

Finally, it can easily be verified that when a, c are defined as in (12.3), then (12.3) and (12.4) imply that $a^t = a$ and $c^t = c$. Hence we have proved

THEOREM 12.2. *A metric G on TM with nonsingular $A(G)$ is compatible with a λ-structure F with $A(F) = 0$ iff there exists an M-connection relative to which the frame component matrices of F and G are respectively*

$$\begin{bmatrix} \alpha & 0 \\ 0 & \gamma \end{bmatrix}, \qquad \begin{bmatrix} d_1\alpha & 0 \\ 0 & d_2\gamma \end{bmatrix},$$

where α, γ are M-tensors of type (1, 1) *such that $\alpha^2 = \lambda I$, $\gamma^2 = \lambda I$, and d_1 and d_2 are nonsingular M-tensors of type* (0, 2) *which are symmetric if $\lambda = 1$ and skew-symmetric if $\lambda = -1$ and which satisfy the conditions*

$$\alpha^t d_1 \alpha = d_1, \qquad \gamma^t d_2 \gamma = d_2.$$

We recall (Corollary 11.5) that if dim M is odd, then there does not exist on TM any almost complex structure F with $A(F) = 0$. In this case, the problem of finding the metrics on TM compatible with F does not arise. Thus, when dim M is odd Theorem 12.2 has a meaning only for $\lambda = 1$, i.e., for an almost product structure.

Case 2b. $A(F) = 0$ *and* $A(G) = 0$. In this case, we can take

$$\begin{bmatrix} 0 & h^t \\ h & 0 \end{bmatrix}$$

as the frame component matrix of G relative to some suitably chosen M-connection Γ (cf. Theorem 8.3) and

$$\begin{bmatrix} \alpha & 0 \\ \beta & \gamma \end{bmatrix}$$

as the frame component matrix of F relative to the same M-connection Γ, where the M-tensors α, β, γ are such that

$$\alpha^2 = \lambda I, \qquad \gamma^2 = \lambda I, \qquad \beta\alpha + \gamma\beta = 0.$$

The condition for the compatibility of G and F is

$$\begin{bmatrix} 0 & h^t \\ h & 0 \end{bmatrix} = \begin{bmatrix} \alpha^t & \beta^t \\ 0 & \gamma^t \end{bmatrix} \begin{bmatrix} 0 & h^t \\ h & 0 \end{bmatrix} \begin{bmatrix} \alpha & 0 \\ \beta & \gamma \end{bmatrix}$$

$$= \begin{bmatrix} \beta^t h\alpha + \alpha^t h^t \beta & \alpha^t h^t \gamma \\ \gamma^t h\alpha & 0 \end{bmatrix}.$$

On account of (12.1) and putting $d \equiv \beta^t h\alpha$, we can easily see that d is an M-tensor of type (0, 2) and the above condition of compatibility is equivalent to

(12.5) $$d^t = -d, \qquad \gamma^t = \lambda h\alpha h^{-1}, \qquad \beta^t = \lambda d\alpha h^{-1}.$$

The γ given by (12.5) is such that $(\gamma^2)^t = (\lambda h\alpha h^{-1})(\lambda h\alpha h^{-1}) = \lambda I$, and therefore satisfies $(12.1)_2$. On the other hand, using (12.5) and $(12.1)_1$, we see that $(12.1)_3$ is equivalent to

$$0 = \alpha^t\beta^t + \beta^t\gamma^t = \alpha^t\lambda d\alpha h^{-1} + (\lambda d\alpha h^{-1})(\lambda h\alpha h^{-1})$$
$$= \lambda(\alpha^t d\alpha + d)h^{-1},$$

i.e.,

$$\alpha^t d\alpha + d = 0.$$

On account of this and $(12.5)_1$, $(12.5)_3$ becomes

$$\beta = \lambda(h^{-1})^t\alpha^t(-d) = \lambda(h^{-1})^t\lambda d\alpha = (h^{-1})^t d\alpha.$$

Hence we have proved

THEOREM 12.3. *A metric G on TM with $A(G) = 0$ is compatible with a λ-structure F on TM with $A(F) = 0$ iff there exists an M-connection relative to which the frame component matrices of F and G are respectively*

$$\begin{bmatrix} \alpha & 0 \\ (h^{-1})^t d\alpha & \lambda(h\alpha h^{-1})^t \end{bmatrix}, \qquad \begin{bmatrix} 0 & h^t \\ h & 0 \end{bmatrix},$$

where h is an M-tensor of type (0, 2) which is everywhere nonsingular, α is an M-tensor of type (1, 1) such that $\alpha^2 = \lambda I$, and d is a skew-symmetric M-tensor of type (0, 2) such that $\alpha^t d\alpha = -d$.

As in Theorem 12.2, if dim M is odd, then Theorem 12.3 has a meaning only for $\lambda = 1$, i.e., for an almost product structure.

In Theorems 12.2 and 12.3, there appear the M-tensors d_1, d_2 and d of type (0, 2) satisfying certain conditions. The question naturally arises whether these M-tensors exist. We answer this question in the affirmative by showing that with a given M-tensor α of type (1, 1) on TM satisfying $\alpha^2 = \lambda I$ ($\lambda = \pm 1$), there always exist symmetric and skew-symmetric M-tensors of type (0, 2) satisfying $\alpha^t d\alpha = d$ or $\alpha^t d\alpha = -d$. In fact, let d^0 be any symmetric or skew-symmetric tensor of type (0, 2) on M, and put $d = \frac{1}{2}(d^0 \pm \alpha^t d^0\alpha)$.

Then

$$\alpha^t d\alpha = \tfrac{1}{2}(\alpha^t d^0 \alpha \pm d^0) = \pm d.$$

Thus, d is an M-tensor of type (0, 2) on TM having the required properties.

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