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STRUCTURE OF THE SEMIGROUP OF SEMIGROUP EXTENSIONS

BY
R. O. FULP AND J. W. STEPP

Abstract. Let B denote a compact semigroup with identity and G a compact abelian group. Let Ext (B, G) denote the semigroup of extensions of G by B. We show that Ext (B, G) is always a union of groups. We show that it is a semilattice whenever B is. In case B is also an abelian inverse semigroup with its subspace of idempotent elements totally disconnected, we obtain a determination of the maximal groups of a commutative version of Ext (B, G) in terms of the extension functor of discrete abelian groups.

This paper is a continuation of our paper [3] in which we consider the notion of an extension of a compact abelian group G by a compact semigroup B with identity. In that paper we show that the collection (of equivalence classes) of extensions of G by B is a commutative semigroup with identity under the usual "Baer sum" of extensions. As in [3] let us denote this semigroup of extensions by Ext (B, G). In our previous paper we obtained, among other results, some theorems relating to the problem as to when a particular extension in Ext (B, G) is an idempotent. We also characterized those extensions which belong to the maximal subgroup of Ext (B, G) containing some particular idempotent. In this paper we are interested in the same kind of problem from a more global point of view, e.g., we are interested in the semigroup structure of Ext (B, G). In particular we show that Ext (B, G) is always a union of groups. We show that Ext (B, G) is a semilattice whenever B is a semilattice but that it may or may not be isomorphic to B. In case B is a compact abelian inverse semigroup with its subspace of idempotents totally disconnected we are able to obtain a fairly complete description of the maximal subgroups of a commutative version of Ext (B, G) in terms of the groups Ext (H, K) where $K^\wedge$ is a maximal subgroup of B and $H^\wedge$ is a quotient group of G (here the "hat" denotes the Prontrjagin dual). Since H and K are necessarily discrete this computes the structure of Ext in terms of the usual extension functor of discrete abelian group theory. For pertinent comments regarding the historical development of Ext see [3] or [6].

1. **Preliminaries.** In this section we give the basic definitions and state explicit results from [3] which will be needed in this paper. Let B denote a compact (Hausdorff) semigroup with identity 1_B and let G denote a compact abelian group with

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identity 1. We say that (S, τ) is an extension of G by B iff S is a compact semigroup with identity 1, G is a closed subgroup of both $H_S(1)$ and the center of S and τ is a continuous homomorphism from S onto B subject to the condition that for s_1 and s_2 in S, $\tau(s_1)=\tau(s_2)$ iff $s_1G=s_2G$. We sometimes indicate that (S, τ) is an extension of G by B by writing $G \hookrightarrow S \overset{\tau}{\twoheadrightarrow} B$. Two extensions (S_1, τ_1) and (S_2, τ_2) of G by B are *equivalent* iff there exists an iseomorphism $\psi\colon S_1 \to S_2$ such that the restriction $\psi|G$ is the identity map on G and

$$\begin{array}{ccccc} G & \hookrightarrow & S_1 & \overset{\tau_1}{\twoheadrightarrow} & B \\ \| & & \downarrow \psi & & \| \\ G & \hookrightarrow & S_2 & \overset{\tau_2}{\twoheadrightarrow} & B \end{array}$$

is a commutative diagram. Let Ext (B, G) denote the collection of equivalence classes of extensions of G by B.

We shall briefly describe an operation on Ext (B, G) which is called the *Baer sum*. Let $E_1\colon G \hookrightarrow S_1 \overset{\tau_1}{\twoheadrightarrow} B$ and $E_2\colon G \hookrightarrow S_2 \overset{\tau_2}{\twoheadrightarrow} B$ denote extensions of G by B. Define

$$D = \{(s_1, s_2) \in S_1 \times S_2 \mid \tau_1(s_1) = \tau_2(s_2)\} \quad \text{and} \quad G_0 = \{(g, g^{-1}) \mid g \in G\}.$$

Then G_0 is a closed central subgroup of D and consequently a compact quotient semigroup D/G_0 may be constructed (see [3] or [5]). For $(a, b) \in D$, let $[a, b]$ denote the congruence class of D/G_0 which contains (a, b). Let $\bar{G}=\{[g, 1] \mid g \in G\}$ and $\bar{\tau}\colon D/G_0 \to B$ the map defined by $\bar{\tau}([a, b])=\tau_1(a)=\tau_2(b)$. It follows from [3] that $\bar{G}$ is iseomorphic to G and that $\bar{G} \hookrightarrow D/G_0 \overset{\bar{\tau}}{\twoheadrightarrow} B$ is an extension of $\bar{G}$ by B. We identify $\bar{G}$ with G and thus obtain an extension $G=\bar{G} \hookrightarrow D/G_0 \overset{\bar{\tau}}{\twoheadrightarrow} B$ of G by B. This extension is called *the Baer sum of* E_1 *and* E_2. It follows from [3] that the Baer sum induces an operation $+$ on Ext (B, G) such that (Ext (B, G), $+$) is a commutative semigroup with identity. The following theorems are proven in [3].

THEOREM 1.1. *Let* $E\colon G \hookrightarrow S \overset{\tau}{\twoheadrightarrow} B$ *denote an extension of* G *by* B *where* G *is a compact abelian group and* B *is a compact semigroup with identity. Then* E *represents an idempotent in the semigroup* Ext (B, G) *iff there is a continuous homomorphism* $\sigma\colon B \to S$ *such that* $\tau\sigma$ *is the identity map on* B.

If G is a subgroup of a semigroup S and $s \in S$ let $G_s=\{g \in G \mid gs=s\}$.

THEOREM 1.2. *Assume that* $E\colon G \hookrightarrow S \overset{\tau}{\twoheadrightarrow} B$ *represents an idempotent in the semigroup* Ext (B, G). *Then the extension* $E^*\colon G \hookrightarrow S^* \overset{\tau^*}{\twoheadrightarrow} B$ *represents a member of the maximal subgroup of* Ext (B, G) *determined by* E *iff whenever* $(s, t) \in S \times S^*$ *such that* $\tau(s)=\tau^*(t)$, *it follows that* $G_s=G_t$.

2. **Structure of** Ext (B, G). In this section we prove that Ext (B, G) is always a union of groups. We also show that if B is a semilattice (commutative idempotent semigroup), then Ext (B, G) is a semilattice. Examples are given which show that the semilattice Ext (B, G) may or may not be isomorphic to B.

THEOREM 2.1. *Let G be a compact abelian group and B a compact semigroup with identity. Then* Ext (B, G) *is a union of groups.*

Proof. Let $E: G \hookrightarrow S \xrightarrow{\tau}\!\!\!\!\twoheadrightarrow B$ denote an extension of G by B. We construct an idempotent extension $\bar{E}: G \hookrightarrow \bar{S} \xrightarrow{\bar{\tau}}\!\!\!\!\twoheadrightarrow B$ such that whenever $s \in S$, $\bar{s} \in \bar{S}$, and $\tau(s) = \bar{\tau}(\bar{s})$, it follows that $G_s = G_{\bar{s}}$. It will then follow from Theorem 1.2 that E represents a member of the subgroup of Ext (B, G) determined by $\bar{E}$.

Let $N = \{(g, g) \mid g \in G\}$; then N is a closed central subgroup of $G \times S$ and thus we may form the quotient semigroup $(G \times S)/N$ which we denote by S^*. Let η denote the natural map from $G \times S$ into S^* and, for $(g, s) \in G \times S$, let $[g, s]$ denote $\eta((g, s))$. Observe that $\eta|(G \times \{1\})$ is an iseomorphism onto $\eta(G \times \{1\})$ and thus G is iseomorphic to $\eta(G \times \{1\})$. Also for each $g \in G$, $[g, 1] = [g, 1][g^{-1}, g^{-1}] = [1, g^{-1}]$. Let $\tau^*: S^* \to B$ be the function defined by $\tau^*([g, s]) = \tau(s)$. Then τ^* is well defined and by the techniques of [3] one sees that $E^*: \eta(G \times \{1\}) \hookrightarrow S^* \xrightarrow{\tau^*}\!\!\!\!\twoheadrightarrow B$ is an extension of $G \cong \eta(G \times \{1\})$ by B. Let

$$\bar{D} = \{(s, [g, t]) \in S \times S^* \mid \tau(s) = \tau^*([g, t])\} \quad \text{and} \quad \bar{G}_0 = \{(g, [g^{-1}, 1]) \mid g \in G\}.$$

Then $\bar{G}_0$ is a closed central subgroup of $\bar{D}$. Let $\bar{S}$ denote the quotient semigroup $\bar{D}/\bar{G}_0$ and $\bar{G} = \{[g, [1, 1]] \mid g \in G\}$. Define $\bar{\tau}: \bar{S} \twoheadrightarrow B$ by $\bar{\tau}([s, [g, t]]) = \tau(s) = \tau^*([g, t]) = \tau(t)$. Then it follows from [3] that $G = \bar{G} \hookrightarrow \bar{S} \xrightarrow{\bar{\tau}}\!\!\!\!\twoheadrightarrow B$ is an extension of $\bar{G}$ by B (it is actually the Baer sum of E and E^*). We now show that $E + E^*: G \hookrightarrow \bar{S} \xrightarrow{\bar{\tau}}\!\!\!\!\twoheadrightarrow B$ represents an idempotent of Ext (B, G).

Let $\sigma: B \to \bar{S}$ be the map defined by $\sigma(b) = [s, [1, s]]$ where $\tau(s) = b$. We show that σ is well defined and continuous (it is clearly a homomorphism). Assume that $b = \tau(s_1) = \tau(s_2)$ for $s_1 \in S$, $s_2 \in S$. Then $s_1 = s_2 g$ for some $g \in G$ and

$$\begin{aligned}
[s_1, [1, s_1]] &= [s_2 g, [1, s_2 g]] \\
&= [s_2, [g, 1][1, s_2][1, g]] = [s_2, [1, s_2]][1, [g, 1]][1, [1, g]] \\
&= [s_2, [1, s_2]][1, [1, g^{-1}]][1, [1, g]] = [s_2, [1, s_2]].
\end{aligned}$$

Thus σ is a well-defined homomorphism. To show that σ is continuous assume that $\{b_\alpha\}_{\alpha \in A}$ is a net in B which converges to some $b \in B$. For each $\alpha \in A$, let $s_\alpha \in S$ such that $\tau(s_\alpha) = b_\alpha$. Since S is compact $\{s_\alpha\}_{\alpha \in A}$ clusters to some $s \in S$. Thus $\{(1, s_\alpha)\}_{\alpha \in A}$ clusters to $(1, s)$ and $\{[1, s_\alpha]\}_{\alpha \in A}$ to $[1, s]$. It follows that $\{[s_\alpha, [1, s_\alpha]]\}_{\alpha \in A}$ clusters to $[s, [1, s]]$. Since τ is continuous, $\tau(s) = b$. Thus we have that $\{\sigma(b_\alpha)\}_{\alpha \in A}$ clusters to $\sigma(b)$ and σ is continuous.

It follows that $\bar{E} = E + E^*: G \hookrightarrow \bar{S} \xrightarrow{\bar{\tau}}\!\!\!\!\twoheadrightarrow B$ represents an idempotent in Ext (B, G). We show that E belongs to the maximal subgroup of Ext (B, G) containing $\bar{E}$ by use of Theorem 2.1. Assume that $s \in S$ and $\bar{s} \in \bar{S}$ such that $\tau(s) = \bar{\tau}(\bar{s})$. Choose $t_1 \in S$, $t_2 \in S$, and $g \in G$ such that $\bar{s} = [t_1, [g, t_2]]$. Since $\tau(s) = \bar{\tau}(\bar{s}) = \tau(t_1) = \tau(t_2)$, there exists $h \in G$ such that $sh = t_1$. Thus for each $x \in G_s$,

$$\begin{aligned}
\bar{s}x &= [t_1, [g, t_2]][x, [1, 1]] = [shx, [g, t_2]] \\
&= [sh, [g, t_2]] = [t_1, [g, t_2]] = \bar{s}
\end{aligned}$$

and $x \in G_{\hat{s}}$. It follows that $G_s \subseteq G_{\hat{s}}$. If, on the other hand, $y \in G_{\hat{s}}$, then

$$[y, [1, 1]][t_1, [g, t_2]] = [t_1, [g, t_2]]$$

and there exists $l \in G$ such that $yt_1 = t_1 l$ and $[g, t_2] = [g, t_2][l^{-1}, 1] = [gl^{-1}, t_2]$. Thus there exists $k \in G$ such that $gk = gl^{-1}$ and $t_2 k = t_2$. It follows that $k = l^{-1}$ and $l \in G_{t_2}$. Since $\tau(t_1) = \tau(t_2)$, $G_{t_1} = G_{t_2}$ and $lt_1 = t_1$. Thus $yt_1 = t_1 l = t_1$ and $y \in G = G_{sh} = G_s$. We have that $y \in G_{\hat{s}}$ implies $y \in G_s$ and hence $G_{\hat{s}} \subseteq G_s$. The theorem follows.

LEMMA 2.2. *Let G be a compact abelian group, B a compact semigroup with identity, and (S, τ) an extension of G by B. Assume that e is an idempotent of B and that B_e is a closed subgroup of B with identity e. Then $\tau^{-1}(B_e)$ is a compact subgroup of S and, if f is the identity of $\tau^{-1}(B_e)$,*

$$Gf \hookrightarrow \tau^{-1}(B_e) \xrightarrow{\tau_e} B_e$$

is an extension of Gf by B_e where $\tau_e = \tau | \tau^{-1}(B_e)$.

Proof. Obviously $\tau^{-1}(B_e)$ is a closed subsemigroup of S. Since S is compact, so is $\tau^{-1}(B_e)$. Let f be an idempotent in $\tau^{-1}(B_e)$ (see [5]). Necessarily, $\tau(f) = e$. We show that f is an identity of $\tau^{-1}(B_e)$. Let $s \in \tau^{-1}(B_e)$, then $\tau(s) = \tau(s)e = \tau(sf)$ and $sf = sg$ for some $g \in G$. Thus

$$sf = sf^2 = sgf = sfg = sg^2.$$

It follows that $sf = sg^n$ for each positive integer n. The net $\{g^n\}_{n>0}$ clusters to the identity of 1 of G (which is also the identity of S). Thus, $sf = s$. Similarly $fs = s$ and f is an identity of $\tau^{-1}(B_e)$. Let $s \in \tau^{-1}(B_e)$. To see that s has a right inverse observe that $\tau(s)$ has an inverse in B_e. Thus for some $t \in \tau^{-1}(B_e)$, $\tau(st) = \tau(s)\tau(t) = e = \tau(f)$ and $stg = f$ for some $g \in G$. Thus $\tau^{-1}(B_e)$ is a compact subgroup of S. It is straightforward to show that $Gf \hookrightarrow \tau^{-1}(B_e) \xrightarrow{\tau_e} B_e$ is an extension of Gf by B_e.

COROLLARY 2.3. *If $G \hookrightarrow S \xrightarrow{\tau} B$ is an extension of G by B and B is a union of groups, then S is a union of groups.*

LEMMA 2.4. *Let $G \hookrightarrow S \xrightarrow{\tau} B$ denote an extension of G by B. If B is a semilattice, then the set $E(S)$ of idempotents of S is a subsemigroup of S.*

Proof. Assume e and f belong to $E(S)$. Then $\tau(ef) = \tau(e)\tau(f) = \tau(f)\tau(e) = \tau(fe)$ and there exists $g \in G$ such that $ef = feg$. Thus $f(ef) = f(feg) = feg = ef$ and $fef = ef$. Multiply by e to get $efef = ef$. Thus $ef \in E(S)$. The lemma follows.

THEOREM 2.5. *Let G denote a compact abelian group and B a compact semilattice with identity. Then* Ext (B, G) *is a semilattice.*

Proof. Let $E\colon G \hookrightarrow S \xrightarrow{\tau} B$ denote an arbitrary extension of G by B. We show that E represents an idempotent of Ext (B, G). By Lemma 2.4, $E(S)$ is a subsemigroup of S. By Lemma 2.2, $\tau^{-1}(b)$ is a group for each $b \in B$. It follows that $\tau | E(S)$ is an iseomorphism from $E(S)$ onto B. Let σ denote the inverse of $\tau|E(S)$. Then $\tau\sigma$ is the identity map on B. By Theorem 1.1, it follows that E represents an idempotent of Ext (B, G). By [3], Ext (B, G) is commutative and thus Ext (B, G) is a semilattice.

The following lemma is needed in order to derive some of the properties of a very interesting example.

LEMMA 2.6. *Let* $E_1: G \hookrightarrow S_1 \xrightarrow{\tau_1} B$ *and* $E_2: G \hookrightarrow S_2 \xrightarrow{\tau_2} B$ *denote extensions of* G *by* B *and let* $E_1+E_2: G \hookrightarrow D/G_0 \xrightarrow{\tau} B$ *denote the Baer sum of* E_1 *and* E_2. *If* $(s_1, s_2) \in D$, *then* $G_{[s_1,s_2]}=G_{s_1}G_{s_2}$.

Proof. Let $h \in G_{[s_1,s_2]}$, then $[h, 1][s_1, s_2]=[s_1, s_2]$ and thus $hs_1=s_1g$ and $s_2=s_2g$ for some $g \in G$. It follows that $h=(g^{-1}h)g \in G_{s_1}G_{s_2}$. Hence $G_{[s_1,s_2]} \subseteq G_{s_1}G_{s_2}$. If, on the other hand, $k \in G_{s_1}G_{s_2}$ then $k=k_1k_2$ with $k_1s_1=s_1$ and $k_2s_2=s_2$. Thus

$$[k, 1][s_1, s_2] = [k_1k_2, 1][s_1, s_2] = [k_1, k_2][s_1, s_2] = [s_1, s_2]$$

and $[k, 1] \in G_{[s_1,s_2]}$. It follows that $G_{[s_1,s_2]}=G_{s_1}G_{s_2}$.

THEOREM 2.7. *Let* $B=\{0, 1\}$ *under the usual real number multiplication. Let* G *denote any compact abelian group. Let* $\mathscr{S}(G)$ *denote the collection of all closed subgroups of* G *with an operation* $*$ *defined on* $\mathscr{S}(G)$ *by*

$$G_1 * G_2 = G_1G_2 = \{g_1g_2 \mid g \in G_1, g_2 \in G\}.$$

Then $\mathscr{S}(G)$ *is a semigroup and* Ext $(B, G) \cong \mathscr{S}(G)$.

Proof. Define ϕ: Ext $(B, G) \to \mathscr{S}(G)$ by $\phi([E])=G_s$ where E is the extension $G \hookrightarrow S \xrightarrow{\tau} B$ and $s \in S$ such that $\tau(s)=0$. It is a consequence of Theorem 1.2 that ϕ is well defined and injective. That ϕ is a homomorphism follows from Lemma 2.6. We show that ϕ is surjective. Let C be a closed subgroup of G. Let ρ denote the closed congruence on $B \times G$ defined by $(b, g)\ \rho\ (b_1, g_1)$ iff either $(b, g)=(b_1, g_1)$ or $b=b_1=0$ and $gg^{-1} \in C$. Let $\bar{\tau}: (B \times G)/\rho \to B$ denote the map defined by $\bar{\tau}([b, g])=b$ and let $\bar{G}=\{[g, 1] \mid g \in G\}$. It is easy to show that $\bar{G}$ is iseomorphic to G and that $G=\bar{G} \hookrightarrow (B \times G)/\rho \xrightarrow{\bar{\tau}} B$ is an extension of $\bar{G}$ by B. Moreover if $s \in (B \times G)/\rho$ with $\bar{\tau}(s)=0$ then $\bar{G}_s=C$. Thus ϕ is surjective and the theorem follows.

REMARK. It is clear from the last theorem even though Ext (B, G) is a semilattice it may fail to be isomorphic to B. In fact $B=\{0, 1\}$ is a chain, but, in general, Ext $(\{0, 1\}, G)$ is not. Note, however, that if $G=Z_p$ is the cyclic group whose order is some prime p, then

$$\text{Ext}\,(\{0, 1\}, Z_p) = \{0, 1\}.$$

In general we have the

EXAMPLE. Let B be a compact chain with identity and let Z_p denote the integers modulo the prime p. Then Ext (B, Z_p) is a chain.

The phenomenon of this example will be examined in more detail in a subsequent publication.

3. **The maximal groups of** Ext (B, G). In this section we are concerned with the problem of determining the maximal groups of Ext (B, G) in terms of Ext (H, K) where H is a maximal group of B and K is a quotient group of G. We are able to find a partial solution to this problem for a fairly large class of semigroups B, but in doing so we restrict ourselves to the commutative case. Thus we consider only extensions $G \hookrightarrow S \twoheadrightarrow B$ in which all three semigroups are commutative. We obtain in this way a functor Ext which is generally different from the extension functor we have been discussing. The two functors do not even agree at pairs (B, G) where both G and B are abelian groups. Of course practically all the theorems we have discussed hold for this modified Ext. Throughout the remainder of this paper we consider this "abelian" version of Ext exclusively.

Our determination of the maximal groups of Ext (B, G) is in terms of a new kind of product group whose definition is similar to that of a direct or inverse limit group. Before giving a precise formulation of the concept we prove some lemmas which are necessary to the proof of the main result of this section.

LEMMA 3.1. *Assume that* $[\{S_e\}_{e\in E}; \{\pi_{ef}\}_{f\leq e}]$ *is an inverse system of compact abelian groups indexed by a compact totally disconnected semilattice E with identity. Let* $S=\bigcup_{e\in E} S_e$ *denote the semigroup with operation defined by*

$$ab = \pi_{e,ef}(a)\pi_{f,ef}(b)$$

for $a \in S_e$ *and* $b \in S_f$. *Then there is at most one compact semigroup topology on S which agrees with the topology on E and with the topology on* S_e *for each* $e \in E$. *Moreover, in order that there exist at least one such topology it is necessary and sufficient that the canonical mapping from* S_e *into* inv lim $[\{S_f\}_{f\in E(e)}; \{\pi_{ff'}\}]$ *defined by*

$$x \mapsto \langle \pi_{ef}(x)\rangle_{f\in E(e)}$$

be an iseomorphism (here $E(e)$ *denotes the set of all generating idempotents of E which lie in eE).*

The proof of this lemma may be found in [4]. It is the case that the canonical mapping defined in the lemma is always a continuous homomorphism.

The following lemma is admittedly somewhat technical, but is precisely what is needed for the proof of our theorem.

LEMMA 3.2. *Let G denote a compact abelian group and B a compact abelian inverse semigroup with identity such that the subspace* $E(B)$ *of idempotents of B is totally disconnected. Let* $G \hookrightarrow T \overset{\pi}{\twoheadrightarrow} B$ *denote an idempotent extension of G by B and* σ *the continuous homomorphism from B into T such that* $\pi\sigma$ *is the identity on B. For each* $f \in E(B)$ *let* H_f *denote the kernel of the map from G onto* $\sigma(f)G$ *defined by* $x \mapsto \sigma(f)x$. *Assume that* $[\{S_f\}_{f\in E(B)}; \{\pi_{ef}\}_{f\leq e}]$ *is an inverse system of compact abelian groups such that for each* $f \in E(B)$ *there is an extension*

$$G/H_f \hookrightarrow S_f \overset{\tau_f}{\twoheadrightarrow} B_f$$

of G/H_f by B_f where B_f denotes the maximal subgroup of B containing f. Finally assume that for $f \leqq e$ the diagram

$$\begin{array}{ccccc} G/H_e & \hookrightarrow & S_e & \xrightarrow{\tau_e} & B_e \\ \downarrow{\scriptstyle \eta_{ef}} & & \downarrow{\scriptstyle \pi_{ef}} & & \downarrow{\scriptstyle m_{ef}} \\ G/H_e & \hookrightarrow & S_f & \xrightarrow{\tau_f} & B_f \end{array}$$

is commutative where $\eta_{ef}(gH_e)=gH_f$ and $m_{ef}(x)=xf$. Then there exists one and only one compact semigroup topology on the semigroup $S=\bigcup_{e\in E(B)} S_e$ which agrees with the topology on $E(B)$ and with the topology on S_e for each $e \in E(B)$. If S is endowed with this topology, $G \hookrightarrow S \xrightarrow{\tau} B$ is an extension of G by B where $\tau(x)=\tau_e(x)$ for $x \in S_e$.

Proof. In order to show that $S=\bigcup_{e\in E(B)} S_e$ has a compact semigroup topology with the desired properties, we verify the hypothesis of Lemma 3.1. Let $e \in E(B)$. It is a consequence of Lemma 3.1 (applied to B) that the mapping m from B_e into inv lim $[\{B_f\}_{f\in E(e)}; \{m_{ff'}\}]$ defined by

$$x \mapsto \langle m_{ef}(x)\rangle_{f\in E(e)}$$

is an iseomorphism. Consider the mapping η from G/H_e into

$$\text{inv lim } [\{G/H_f\}_{f\in E(e)}; \{\eta_{ff'}\}]$$

defined by

$$x \mapsto \langle \eta_{ef}(x)\rangle_{f\in E(e)}.$$

It is clear that η is a continuous homomorphism. We show that it is an iseomorphism. Assume that $g \in G$ such that $\eta(gH_e)=0$, then $g \in \bigcap_{f\in E(e)} H_f$ and $\sigma(f)g=\sigma(f)$ for each $f \in E(e)$. By [2] there exists a net $\{f_\alpha\}$ in $E(e)$ converging upward to e. Thus $\sigma(f_\alpha)g=\sigma(f_\alpha)$ for each α and $\sigma(e)g=\sigma(e)$. It follows that $g \in H_e$ and consequently that the kernel of η is trivial. We show that η is surjective. Let $\{g_f\}_{f\in E(e)}$ denote a family of elements of G such that $\langle g_fH_f\rangle_{f\in E(e)} \in \text{inv lim }\{G/H_f\}$. Choose a net $\{f_\alpha\}_{\alpha\in A}$ of idempotents in $E(e)$ which converges upward to e (see [2]). Since G is compact there is a subnet of $\{g_{f_\alpha}\}_{\alpha\in A}$ which converges to $g_0 \in G$. Redefine the notation so that $g_0=\lim g_{f_\alpha}$. Let f denote any member of $E(e)$. Choose $\beta \in A$ so that $f_\beta \geqq f$. Then for $\alpha \geqq \beta$, $f_\alpha \geqq f_\beta \geqq f$ and $g_fH_f=\eta_{f_\alpha f}(g_{f_\alpha}H_{f_\alpha})=g_{f_\alpha}H_f$. Thus $g_f^{-1}g_{f_\alpha} \in H_f$ for each $\alpha \geqq \beta$ and since H_f is closed, $g_f^{-1}g_0 \in H_f$. It follows that $g_0H_f=g_fH_f$ for arbitrary f in $E(e)$; consequently $\eta(g_0H_e)=\langle g_fH_f\rangle_{f\in E(e)}$. It is immediate that η is an iseomorphism. Observe that the extensions $G/H_f \hookrightarrow S_f \xrightarrow{\tau_f} B_f$ naturally induce an exact sequence

$$\text{inv lim }\{G/H_f\} \xrightarrow{\varepsilon} \text{inv lim }\{S_f\} \xrightarrow{\delta} \text{inv lim }\{B_f\}$$

with ε injective and δ surjective. This follows from the fact that the dual system of groups

$$B_f^\wedge \rightarrowtail S_f^\wedge \twoheadrightarrow (G/H_f)^\wedge$$

is a direct system of discrete abelian groups and the direct limit functor preserves exactness (we also need that the dual of a direct limit of discrete abelian groups is the inverse limit of their duals). Thus we obtain a commutative diagram

$$\begin{array}{ccccc} G/H_e & \longrightarrow & S_e & \xrightarrow{\tau_e} & B_e \\ \eta\downarrow & & \pi\downarrow & & m\downarrow \\ \operatorname{inv\,lim}\{G/H_f\} & \rightarrowtail & \operatorname{inv\,lim}\{S_f\} & \twoheadrightarrow & \operatorname{inv\,lim}\{B_f\} \end{array}$$

where π is the map defined by $x \mapsto \langle \pi_{ef}(x)\rangle_{f\in E(e)}$. It follows from the five lemma that π is injective and surjective. By Lemma 3.1, $S=\bigcup_{e\in E(e)} S_e$ may be given a unique compact semigroup topology which agrees with the topologies on $E(B)$ and on the various S_e.

Let $\tau\colon S\to B$ denote the mapping defined by $\tau(x)=\tau_e(x)$. It follows from the commutativity of

$$\begin{array}{ccc} S_e & \xrightarrow{\tau_e} & B_e \\ \pi_{ef}\downarrow & & \downarrow m_{ef} \\ S_f & \xrightarrow{\tau_f} & B_f \end{array}$$

that τ is a homomorphism. By the definition of the operation on $S=\bigcup_{e\in E(B)} S_e$ and the fact that the diagram

$$\begin{array}{ccccc} G & \hookrightarrow & S_1 & \xrightarrow{\tau_1} & B_1 \\ \eta_{1e}\downarrow & & \pi_{1e}\downarrow & & m_{1e}\downarrow \\ G/H_e & \hookrightarrow & S_e & \xrightarrow{\tau_e} & B_e \end{array}$$

is commutative, we have that $\tau(x)=\tau(y)$ for x and y in S iff $xG=yG$. Thus if we can show that τ is continuous it will follow that $G\hookrightarrow S\overset{\tau}{\twoheadrightarrow} B$ is an extension of G by B. Let $\nu\colon B^{\wedge}\to S^{\wedge}$ denote the mapping from the character semigroup of B into the character semigroup of S defined by $\nu(X)s=X(\tau_e(s))$ for $X\in B^{\wedge}$ and $s\in S_e$. Observe that $\nu(X)$ is continuous since $\nu(X)$ is also given by

$$\begin{aligned} \nu(X)(s) &= X(\tau_e(s\bar{e})), && s\in s_X, \\ &= 0, && s\notin s_X, \end{aligned}$$

where $s_X=\{x\in B \mid X(x)\neq 0\}$ and where $\bar{e}$ is the least idempotent in s_X. It follows easily that $\nu(X)\in S^{\wedge}$ and that ν is a homomorphism from the discrete semigroup $B^{\wedge}$ into the discrete semigroup $S^{\wedge}$. It follows that $\nu^{\wedge}\colon S^{\wedge\wedge}\to B^{\wedge\wedge}$ is a continuous homomorphism (since ν is). But $\tau=\psi_B^{-1}\cdot\nu^{\wedge}\cdot\psi_S$ where $\psi_S\colon S\to S^{\wedge\wedge}$ and $\psi_B\colon B\to B^{\wedge\wedge}$ are the natural iseomorphisms (see [1], [2], and [4]). The lemma now follows.

We now have the lemmas necessary to the proof of the main theorem of this section, but we must first develop considerable notation prior to the statement of our result.

Assume that E is a semilattice and that $\{G_e\}_{e\in E}$ and $\{B_e\}_{e\in E}$ are families of groups indexed by E. Assume also that for $f\leqq e$ in E there exist homomorphisms $\phi_{ef}: G_e \to G_f$ and $\theta_{ef}: B_e \to B_f$ such that $[\{G_e\}_{e\in E}; \{\phi_{ef}\}_{f\leqq e}]$ and $[\{B_e\}_{e\in E}; \{\theta_{ef}\}_{f\leqq e}]$ are inverse systems of groups. Let $P_{\phi\theta}$ denote the subset of $\prod_{e\in E} \operatorname{Ext}(B_e, G_e)$ consisting of all those $\mathscr{E}=\{\mathscr{E}_e\}_{e\in E}$ for which there exist extensions $E_e: G_e \hookrightarrow S_e \xrightarrow{\tau_e} B_e$ and homomorphisms $\mu_{ef}: S_e \to S_f$, $f\leqq e$ in E, such that the following conditions hold:

(i) $\mathscr{E}_f=[E_f]$ for each $f\in E$,

(ii) $[\{S_e\}_{e\in E}; \{\mu_{ef}\}_{f\leqq e}]$ is an inverse system of groups, and

(iii) for $f\leqq e$ the diagram

$$\begin{array}{ccccc} G_e & \hookrightarrow & S_e & \xrightarrow{\tau_e} & B_e \\ {\scriptstyle \phi_{ef}}\downarrow & & \downarrow{\scriptstyle \mu_{ef}} & & \downarrow{\scriptstyle \theta_{ef}} \\ G_f & \hookrightarrow & S_f & \xrightarrow{\tau_f} & B_f \end{array}$$

is commutative. The subset $P_{\phi\theta}$ is a subgroup of $\prod_{e\in E} \operatorname{Ext}(B_e, G_e)$. The family $\mathscr{E}=\{\mathscr{E}_e\}_{e\in E}$ is called an *inverse family of extensions* and the families $\{E_f\}_{f\in E}$ and $\{\mu_{ef}\}_{f\leqq e}$ are called *representatives* of $\mathscr{E}$. If $\mathscr{E}'$ is also in $P_{\phi\theta}$ and there exist representatives $\{E_f\}_{f\in E}$, $\{\mu_{ef}\}_{f\leqq e}$ of $\mathscr{E}$ and $\{E'_f\}_{f\in E}$, $\{\mu'_{ef}\}_{f\leqq e}$ of $\mathscr{E}'$ such that for some family of isomorphisms $\{\psi_e\}_{e\in E}$ the diagrams

$$\begin{array}{rccccc} E_e: & G_e & \hookrightarrow & S_e & \xrightarrow{\tau_e} & B_e \\ & \| & & \downarrow{\scriptstyle \psi_e} & & \| \\ E'_e: & G_e & \hookrightarrow & S'_e & \xrightarrow{\tau'_e} & B_e \end{array} \qquad \begin{array}{ccc} S_e & \xrightarrow{\mu_{ef}} & S_f \\ {\scriptstyle \psi_e}\downarrow & & \downarrow{\scriptstyle \psi_f} \\ S'_e & \xrightarrow{\mu'_{ef}} & S'_f \end{array}$$

are commutative, then we say that $\mathscr{E}$ and $\mathscr{E}'$ are *congruent*. If $\mathscr{E}_1$ and $\mathscr{E}'_1$ are *congruent* and $\mathscr{E}_2$ and $\mathscr{E}'_2$ are congruent then so are $\mathscr{E}_1+\mathscr{E}_2$ and $\mathscr{E}'_1+\mathscr{E}'_2$. It is easily seen that the collection of congruence classes of $P_{\phi\theta}$ is a group which we shall denote by $\operatorname{Prod}_{\phi\theta}\{\operatorname{Ext}(B_e, G_e)\} = \operatorname{Prod}\{\operatorname{Ext}(B_e, G_e)\}$. We may now state our theorem.

Let G denote a compact abelian group, B a compact abelian inverse semigroup with identity such that the maximal idempotent subsemigroup $E(B)$ of B is totally disconnected, and let

$$E_0: G \hookrightarrow T \underset{\sigma}{\overset{\pi}{\rightleftarrows}} B$$

denote an idempotent extension of G by B. For each $e\in E(B)$, let B_e denote the maximal subgroup of B containing e and, for $f\leqq e$ in $E(B)$, let $m_{ef}: B_e \to B_f$ denote the continuous homomorphism defined by $m_{ef}(x)=xf$. For each $f\in E(B)$ let H_f denote the kernel of the continuous homomorphism from $G\subseteq T$ onto $\sigma(f)G\subseteq T$ defined by $x\mapsto \sigma(f)x$. Finally, for $f\leqq e$, let $\eta_{ef}: G/H_e \to G/H_f$ denote the continuous homomorphism defined by $\eta_{ef}(gH_e)=gH_f$.

THEOREM 3.3. *The maximal subgroup of* Ext (B, G) *containing the idempotent extension is isomorphic to the group* Prod $[\{\text{Ext}\,(B_e, G/H_e)\}; \{\eta_{ef}\}_{f\leqq e}, \{m_{ef}\}_{f\leqq e}]$.

Proof. Let $H(E_0)$ denote the maximal subgroup of Ext (B, G) containing E_0. We wish to show that there is a well-defined homomorphism

$$\Gamma\colon H(E_0) \to \text{Prod}\,\{\text{Ext}\,(B_e, G/H_e)\}$$

such that if $E\colon G \hookrightarrow S \overset{\tau}{\twoheadrightarrow} B$ represents some element of $H(E_0)$ then $\Gamma([E]) = \{\mathscr{E}_f\}_{f\in E(B)}$ where, for each $f \in E(B)$, $\mathscr{E}_f$ is represented by the extension $G/H_f \cong \delta(f)G \hookrightarrow \tau^{-1}(B_f) \overset{\tau_f}{\twoheadrightarrow} B_f$ with $\delta(f)$ the idempotent of $\tau^{-1}(B_f)$ and $\tau_f = \tau|\tau^{-1}(B_f)$. First note that if $G \hookrightarrow S \overset{\tau}{\twoheadrightarrow} B$ is any extension of G by B it follows from Lemma 2.2 that, for each $f \in E(B)$, $\tau^{-1}(B_f)$ is a maximal subgroup of S and that $\delta(f)G \hookrightarrow \tau^{-1}(B_f) \overset{\tau_f}{\twoheadrightarrow} B_f$ is an extension of $\delta(f)G$ by B_f. We show that $\delta(f)G$ is iseomorphic to G/H_f whenever $[E] \in H(E_0)$. By Theorem 1.2 it follows that $[E] \in H(E_0)$ iff whenever $s \in S$ and $t \in T$ such that $\tau(s) = \pi(t)$ then $G_s = G_t$. If $s \in S$ and $\tau(s) = f$ then $G_s = G_{\delta(f)}$. Similarly if $t \in T$ and $\pi(t) = f$ then $G_t = G_{\delta(f)}$. Since the kernel of the mapping from $G \subseteq T$ onto $\sigma(f)G \subseteq T$ is H_f it follows easily that $[E] \in H(E_0)$ iff, for each f, the kernel of the mapping from $G \subseteq S$ onto $\delta(f)G \subseteq S$ is H_f. Thus $\delta(f)G \cong G/H_f$. For each $f \in E(B)$, let E_f denote the extension $E_f\colon G/H_f = \delta(f)G \hookrightarrow \tau^{-1}(B_f) \overset{\tau_f}{\twoheadrightarrow} B_f$ and let $\mathscr{E}_f = [E_f]$. We wish to show that $\mathscr{E} = \{\mathscr{E}_f\}_{f\in E(B)}$ belongs to Prod $\{\text{Ext}\,(B_e, G/H_e)\}$. First observe that for $f \leqq e$ in $E(B)$ one has a commutative diagram

$$\begin{array}{ccccc}
\delta(e)G & \hookrightarrow & \tau^{-1}(B_e) & \twoheadrightarrow & B_e \\
\downarrow{\scriptstyle \alpha_{ef}} & & \downarrow{\scriptstyle \mu_{ef}} & & \downarrow{\scriptstyle m_{ef}} \\
\delta(f)G & \hookrightarrow & \tau^{-1}(B_f) & \longrightarrow & B_f
\end{array}$$

where the maps α_{ef} and μ_{ef} denote multiplication by $\delta(f)$ in S. Also note that if $\delta(e)G$ is identified with G/H_e and $\delta(f)G$ with G/H_f then α_{ef} becomes identified with η_{ef}. It is clear that $\{\mu_{ef}\}_{f\leqq e}$ is an inverse system of maps. Thus $\mathscr{E}$ is a member of Prod $\{\text{Ext}\,(B_e, G/H_e)\}$. We define $\Gamma([E]) = \mathscr{E}$. It is routine to check that this definition of Γ is independent of the choice of a representative of $[E]$. Similarly, while it is a tedious computation, it is easy to verify that if E_1 and E_2 are extensions of G by B such that $[E_1]$ and $[E_2]$ are in $H(E_0)$ then

$$\Gamma([E_1]+[E_2]) = \Gamma([E_1])+\Gamma([E_2]).$$

We now show that Γ is surjective. Let $\mathscr{E} = \{\mathscr{E}_f\}_{f\in E(B)}$ denote any member of Prod $\{\text{Ext}\,(B_e, G/H_e)\}$. Let $\{E_f\}_{f\in E(B)}, \{\mu_{ef}\}_{f\leqq e}$ denote representatives of $\mathscr{E}$. For each f let $G/H_f \hookrightarrow S_f \overset{\tau_f}{\twoheadrightarrow} B_f$ denote the extension E_f. Since $\{\mu_{ef}\}_{f\leqq e}$ is an inverse system of maps there is a unique semigroup structure defined on the disjoint union $S = \bigcup_{f\in E(B)} S_f$ given by $ab = \mu_{e,ef}(a)\mu_{f,ef}(b)$ for $a \in S_e$ and $b \in S_f$. By Lemma 3.2 there exists a unique compact semigroup topology on S which agrees with the topologies already given on $E(B)$ and the various S_e. By that same lemma

$E\colon G \hookrightarrow S \overset{\tau}{\twoheadrightarrow} B$ is an extension of G by B with τ defined by $\tau(x)=\tau_e(x)$ for $x \in S_e$. It is clear that for each $f \in E(B)$, $\tau^{-1}(B_f)=S_f$ and if $\delta(f)$ is the idempotent of $\tau^{-1}(B_f)$, $\delta(f)G=G/H_f$. Thus $\Gamma([E])=\mathscr{E}$ and Γ is surjective. To see that Γ is injective assume that $E_1\colon G \hookrightarrow S_1 \overset{\tau_1}{\twoheadrightarrow} B$ and $E_2\colon G \hookrightarrow S_2 \overset{\tau_2}{\twoheadrightarrow} B$ represent members of $H(E_0)$ such that $\Gamma([E_1])=\Gamma([E_2])$. Then there exists a family of iseomorphisms $\{\psi_e\}_{e\in E(B)}$ for which the diagrams

$$\begin{array}{ccccc} \delta_1(f)G & \hookrightarrow & \tau_1^{-1}(B_f) & \overset{\tau_{1f}}{\twoheadrightarrow} & B_f \\ \| & & \downarrow{\scriptstyle \psi_f} & & \| \\ \delta_2(f)G & \hookrightarrow & \tau_2^{-1}(B_f) & \overset{\tau_{2f}}{\twoheadrightarrow} & B_f \end{array} \qquad \begin{array}{ccc} \tau_1^{-1}(B_e) & \longrightarrow & \tau_1^{-1}(B_f) \\ {\scriptstyle \tau_e}\downarrow & & \downarrow{\scriptstyle \psi_f} \\ \tau_2^{-1}(B_e) & \longrightarrow & \tau_2^{-1}(B_f) \end{array}$$

are commutative. Due to the fact that the second diagram is commutative it easily follows that $\psi=\bigcup_e \psi_e$ is an algebraic isomorphism from S_1 onto S_2. As in Lemma 3.2 one may apply the theory of characters to show that ψ is the dual of an isomorphism between discrete semigroups and thus conclude that ψ is continuous. It follows that E_1 and E_2 are congruent and that Γ is injective. The theorem follows.

COROLLARY. *The maximal group of* Ext (B, G) *containing E is isomorphic to* Prod $\{\text{Ext}\,((G/H_f)^\wedge, B_f^\wedge)\}$ *and thus is determined completely by the extension functor for discrete abelian groups.*

Proof. Observe that Prod $\{\text{Ext}\,(B_f, G/H_f)\} \cong$ Prod $\{\text{Ext}\,((G/H_f)^\wedge, B_f^\wedge)\}$.

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NORTH CAROLINA STATE UNIVERSITY,
RALEIGH, NORTH CAROLINA 27607
UNIVERSITY OF HOUSTON,
HOUSTON, TEXAS 77004