Structure of turbulent line vortices

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A theory is given to explain the observed dependence on Reynolds number of the decay of turbulent line vortices. It is argued that the turbulent vortex has a triple structure. There is an outer region for $r > r_1$ (r_1 is the radius of maximum tangential velocity) with a logarithmic distribution of circulation, and for $r < r_1$ an inner region and viscous core in both of which the motion is close to solid body rotation. It is predicted that $r_1 \sim (\nu\Gamma_1 t^2)^{1/4}$, where Γ_1 is the circulation at r_1. Further, Γ_1/Γ_0 is predicted to be a slowly decreasing function of Γ_0/ν, where Γ_0 is the strength of the vortex. The development of an overshoot of circulation in the outer region is discussed, and the axial velocities produced by growth of a trailing vortex are calculated.

I. INTRODUCTION

There is evidence that the turbulent line vortex is an example of a free turbulent flow which violates the principle of Reynolds number similarity.[1] The large scale properties of the flow appear to depend significantly upon the Reynolds number even though there are no solid boundaries present. Let Γ_0 denote the strength of the vortex, i.e., the circulation around a large circuit, and t denote its age. An appropriate Reynolds number is Γ_0/ν and Reynolds number similarity implies that when initial conditions have been forgotten, all averages of the motion (other than the fine scale structure in which energy is dissipated by the direct action of molecular viscosity) can depend only on Γ_0 and t. Thus, the maximum value v_1 of the mean circumferential velocity should decay asymptotically like

$$v_1 = (0.053/a^{1/2})(\Gamma_0/t)^{1/2}, \quad (1)$$

where a is a constant when $\Gamma_0/\nu \gg 1$. The coefficient in (1) is adopted to conform with Squire's[2] analysis in which it is assumed that the turbulence can be described by a constant eddy viscosity

$$\nu_T = a\Gamma_0. \quad (2)$$

Equation (1) then follows from the well-known formula for the decay of a laminar vortex with ν replaced by ν_T.

Experiments in wind tunnels and free flight on the trailing vortices produced by wings confirm the $t^{-1/2}$ dependence of v_1, and tend to support the assumption that details of the initial conditions are unimportant when the vortex is turbulent. However, the number a is found to vary significantly with Reynolds number, decreasing by a factor 10 when the Reynolds number increases from 2×10^3 to 10^7 (see the data summarized by Owen[3] or Govindaraju and Saffman.[4] The trailing vortices produced by slender winged aircraft have a more rapid decay,[5] but this may be due to interactions between the two trailing vortices as the strengths of the individual vortices appear to decay.)

Similarly, Reynolds number similarity predicts that the radius r_1 at which v_1 occurs should increase asymptotically like

$$r_1 = b(\Gamma_0 t)^{1/2}, \quad (3)$$

where $b^2 = 5.04a$ for constant eddy viscosity. The $t^{1/2}$ dependence is supported by the data, but the number b is also found to vary significantly with Reynolds number, decreasing as the Reynolds number increases.

From (1) and (3), it follows that $\Gamma_1 = 2\pi r_1 v_1$ (the mean circulation at the maximum velocity) satisfies

$$\Gamma_1/\Gamma_0 = 0.33b/a^{1/2}. \quad (4)$$

This ratio is 0.716 for constant eddy viscosity. The experiments suggest that the ratio is indeed constant during the decay of a turbulent vortex, but the value appears to depend weakly on the Reynolds number, decreasing from roughly 0.6 to 0.4 as the Reynolds number increases.

The experiments, especially those in free flight, are difficult and the scatter is somewhat large. Also, it is unclear to what extent a trailing vortex behind a wing represents a decaying line vortex and vice versa. One uncertainty is Γ_0, or the amount of vorticity, shed by the wing, which rolls up tightly. The experiments (see Ref. 4), suggest that only roughly one-half of the vorticity "rolls up," and the rest is in a fairly diffuse cloud whose size is roughly the wing span. The appropriate value of Γ_0 would then be about one-half the circulation at the root of the wing. Also, there appear to be significant axial velocities in trailing vortices which may well affect the decay of the circumferential velocities. Nevertheless, the most reasonable interpretation of the experiments is that there is a significant Reynolds number effect on the decay of a turbulent line vortex, and the present work describes an attempt to understand this unexpected and strange phenomenon.

A theoretical attack has been made by Owen.[3] He postulated a model with mean solid body rotation for $r < r_1$ and a potential vortex for $r > r_1 + \delta$, where $\delta(\ll r_1)$ is the width of a superlayer separating the turbulent vortex from the nonturbulent outer flow. Arguments are then given to show that $b \propto (\nu/\Gamma_0)^{1/4}$. Apart from the obvious drawback that the model gives $\Gamma_1/\Gamma_0=1$, the analysis seems to be algebraically inconsistent [Eq. (4) does not follow from Eq. (1)] and more important, the tangential stress is not continuous so that the model is dynamically unsound. This last point can be seen from consideration of the integral

$$J(t)=\frac{1}{r_1^2}\int_0^\infty \left(1-\frac{\Gamma}{\Gamma_0}\right) r\,dr. \qquad (5)$$

The quantity $J(t)$ is a measure of the angular momentum defect of the flow. It was shown[4] that as an *exact* consequence of the equations of motion, and, in particular, from the requirement that stress is continuous across any surface of discontinuity,

$$J(t)=(A_0/r_1^2)+(2\nu t/r_1^2), \qquad (6)$$

where A_0 is a constant of dimension (length)^2 determined by initial conditions. In particular,

$$J(t)\rightarrow(2/b^2)(\nu/\Gamma_0)\ll 1 \quad \text{as} \quad t\rightarrow\infty, \qquad (7)$$

if the turbulent vortex spreads significantly faster than a laminar one. A small value of J implies an overshoot of circulation, i.e., $\Gamma>\Gamma_0$ for some r, unless the profile is "pathological." For Owen's model, $J(t)\approx\frac{1}{4}$.

II. THE SELF-SIMILAR VORTEX

The rate at which the influence of initial conditions decay is a moot point, and needs to be answered by theory or experiment. For the purpose of the present exploratory study, we shall make the simplifying assumption that a time does exist after which the initial conditions are unimportant and the structure of the vortex is determined by the parameters, ν, Γ_0 and the time t alone. It is hoped that future work will either establish or confound this assumption. The result (6) shows that a necessary condition is $r_1^2\gg A_0$. For trailing vortices, A_0 depends on the (not well understood) mechanism by which the trailing vortex sheet rolls up.

The equation of motion for the mean circulation Γ in an axisymmetric vortex is

$$\frac{\partial\Gamma}{\partial t}=\frac{-2\pi}{r}\frac{\partial}{\partial r}(r^2\tau)+\frac{\nu}{r}\frac{\partial}{\partial r}\left[r^3\frac{\partial}{\partial r}\left(\frac{\Gamma}{r^2}\right)\right], \qquad (8)$$

where

$$\tau=\overline{u_r u_\theta} \qquad (9)$$

is the tangential Reynolds stress. Since Γ_0/ν is dimensionless, a vortex which depends only on the parameters entering into this ratio must be self-similar, with

$$\Gamma=\Gamma_0 f(\eta), \qquad \tau=(\Gamma_0/2\pi t)g(\eta), \qquad (10)$$

where the similarity variable η is defined by

$$\eta=r/(\Gamma_0 t)^{1/2}. \qquad (11)$$

Equation (8) then reduces to

$$\epsilon\frac{d}{d\eta}\left[\eta^3\frac{d}{d\eta}\left(\frac{f}{\eta^2}\right)\right]+\tfrac{1}{2}\eta^2\frac{df}{d\eta}=\frac{d}{d\eta}(g\eta^2), \qquad (12)$$

where we have written $\epsilon=\nu/\Gamma_0$. Boundary conditions are

$$f(\eta)=1+o(\eta^{-m}); \qquad g(\eta)=o(\eta^{-m})$$
$$\text{as } \eta\rightarrow\infty, \quad \text{all } m. \qquad (13)$$

These come from the fact that the vorticity must be exponentially small at infinity, and hence Reynolds stresses are equivalent to a change in the mean pressure. In addition, the flow must be analytic at $r=0$, which implies

$$f=O(\eta^2), \qquad g=O(\eta^2) \qquad \text{as } \eta\rightarrow 0. \qquad (14)$$

In similarity variables, the quantity $J(t)$ becomes

$$J=\frac{1}{\eta_1^2}\int_0^\infty (1-f)\eta\,d\eta, \qquad (15)$$

where $\eta_1=b=r_1/(\Gamma_0 t)^{1/2}$ is a constant for a self-similar flow. It follows immediately on integrating (12) that

$$J=2\epsilon/\eta_1^2, \qquad (16)$$

which is (6) with $A_0=0$ as is necessary for self-similarity.

So far there has been no restriction on the value of ϵ, but, henceforth, we shall restrict our attention to the case of high Reynolds number so that $\epsilon\ll 1$. As is always the case in turbulent motion, there are more unknowns than equations and progress requires ad hoc hypotheses or a closure approximation. In the present problem, another relation between f and g needs to be postulated. Two attempts have been reported in the literature. Govindaraju and Saffman[4] describe the results of an inviscid closure approximation applied to the self-similar vortex, but there is order of magnitude disagreement between theory and experiment and the empirical dependence on Reynolds number is excluded by hypothesis. Donaldson and Sullivan[6] report studies of a decaying vortex using the "invariant modelling" closure approximation. An initial value problem was integrated numerically, but the integration was not carried out for a time long enough for the vortex radius to grow like $t^{1/2}$ and the results are inconclusive. As the closure approximations have not yet been successful (and require very heavy computation), and our aim is to understand general features rather than detailed structure, here, we shall investigate the consequences of some simple ad hoc approximations to the Reynolds stress.

III. THE OUTER VORTEX

Our approach will be motivated by the type of general consideration that has been successful for turbulent boundary layers. First, we introduce a scale r_0, which is the size of the vortex. It is the radius at which the circulation is within 1% of the asymptotic value and the Reynolds stresses are negligible. Or, if we make the assumption that the vortex has a sharp boundary (covered by a viscous "superlayer") and we neglect the convolutions of this boundary and take it to be cylindrical, then r_0 will be its radius. The length r_0 corresponds to the thickness of a turbulent boundary layer.

Between r_0 and r_1, we assume the existence of a defect region in which momentum transport by viscosity is negligible and the Reynolds stress and mean circulation distributions behave like

$$\tau=(u_*^2/2\pi)G(r/r_0),\qquad \Gamma_0-\Gamma=u_*r_0F(r/r_0),\quad (17)$$

where u_* is a characteristic velocity of the turbulence in the region between r_0 and r_1. We shall call this region the outer vortex. Two corollaries of (17), again suggested by the boundary layer, are as follows. First, u_* is characteristic of the turbulent velocities at r_1, i.e., the scale of the turbulence in the outer vortex is set by the turbulence level at the radius of maximum velocity, and we can write

$$u_*^2=2\pi\tau_1.\qquad (18)$$

Second, $r_1\ll r_0$ and moreover

$$r_1/r_0=o(1)\qquad \text{as } \epsilon\rightarrow 0.\qquad (19)$$

If we write

$$\alpha=u_*^2t/\Gamma_0,\qquad \eta_0=r_0/(\Gamma_0t)^{1/2},\qquad (20)$$

where α and η_0 are constants, then (17) is clearly consistent with the requirement of self-similarity, and corresponds to

$$f(\eta)=1-\alpha^{1/2}\eta_0F(\eta/\eta_0),$$
$$g(\eta)=\alpha G(\eta/\eta_0),\qquad g(\eta_1)=\alpha.\qquad (21)$$

It follows from (18) and (19) that Γ is a logarithmic function of r for $r\ll r_0$. This is seen by substituting (21) into (12) and dropping the viscous term. The result is

$$-\tfrac{1}{2}\alpha^{1/2}\eta F'(\eta/\eta_0)=2\alpha G(\eta/\eta_0)+(\eta/\eta_0)G'(\eta/\eta_0).\quad (22)$$

For $\eta\ll\eta_0$, we can replace G by its value at η_1 and drop the last term on the right-hand side, giving

$$F(\eta/\eta_0)\approx-4(\alpha^{1/2}/\eta_0)\log(\eta/\eta_0)+\text{const},\qquad (23)$$

or

$$f(\eta)\approx 4\alpha\log\eta+\text{const}.\qquad (24)$$

Since, by definition, the tangential velocity is a maximum at $\eta=\eta_1$, it follows that

$$f'(\eta)=f(\eta)/\eta=(\Gamma_1/\Gamma_0)\eta^{-1}\qquad \text{when } \eta=\eta_1.\quad (25)$$

Hence, (24) can be written

$$\Gamma/\Gamma_0=f(\eta)\approx(\Gamma_1/\Gamma_0)[\log(\eta/\eta_1)+1]\qquad \text{for } \eta\ll\eta_0,\ (26)$$

and moreover

$$4\alpha\approx\Gamma_1/\Gamma_0 \text{ or } \Gamma_1\approx 8\pi t\tau_1, \text{ or } f_1=4g_1.\qquad (27)$$

The defect law (17) and the logarithmic profile (26) were given by Hoffman and Joubert,[7] although their theoretical argument for (26) is fundamentally different from that given here. The experiments of Hoffman and Joubert and others tend to confirm the defect law and the logarithmic profile. No experimental data have been reported which allow a check on (27) to be performed.

Examples of possible distributions of the Reynolds stress are

$$g(\eta)=\alpha[1-(\eta^2/\eta_0^2)][1-(\eta_1^2/\eta_0^2)]^{-1},\qquad \eta<\eta_0,$$
$$=0,\qquad \eta>\eta_0,\quad (28)$$

or

$$g(\eta)=\alpha\exp(\eta_1^2/\eta^2).\qquad (29)$$

The first is a vortex with a sharp boundary at $\eta=\eta_0$, the second extends to infinity but decays exponentially. For any such distribution, we can integrate (22) or (12) to give the distribution of circulation. Let us, for the sake of example, use (28) since the integrations can then be done in a simple closed form. The result for this particular case is

$$f=1+4\alpha[\log(\eta/\eta_0)+1-(\eta^2/\eta_0^2)][1-(\eta_1^2/\eta_0^2)]^{-1}.\ (30)$$

The condition (25), or the requirement that (30) match (26) as $\eta\rightarrow\eta_1$, leads to the relations

$$f_1=4\alpha[1-(2\eta_1^2/\eta_0^2)][1-(\eta_1^2/\eta_0^2)]^{-1}\approx 4\alpha,\quad (31)$$

and

$$4\alpha=[\log(\eta_0/\eta_1)-(\eta_1/\eta_0)^2]^{-1}[1-(\eta_1^2/\eta_0^2)]^{-1}$$
$$\approx[\log(\eta_0/\eta_1)]^{-1}.\qquad (32)$$

The first relation has already been obtained. The logarithmic dependence in the second relation is quite general and independent of the form of the Reynolds stress distribution; since (12) (with $\epsilon=0$) gives

$$f=1+2g-4\int_\eta^\infty\frac{g}{\eta}\,d\eta.\qquad (33)$$

Hence,

$$4\int_{\eta_1}^\infty\frac{g}{\eta}\,d\eta=1+2g_1-f_1=1-2\alpha.$$

Now,

$$\int_{\eta_1}^\infty\frac{g}{\eta}\,d\eta\approx\alpha\left(\log\frac{\eta_0}{\eta_1}-c\right),$$

where the constant, c depends on the shape of g but is, to leading order, independent of η_1. Thus,

$$4\alpha=\frac{1+(c-2)\alpha}{\log(\eta_0/\eta_1)}, \qquad (34)$$

where c is close to 2 if the distribution of g is roughly parabolic.

A further relation between α and η_0 can be obtained by appeal to an empirical equation such as $dr_0/dt=\lambda u_*$. The value of this equation is that the unknown constant λ can be expected to be independent of Reynolds number whereas u_* is not. For the self-similar vortex, it follows that

$$\eta_0=2\lambda\alpha^{1/2}. \qquad (35)$$

IV. THE INNER VORTEX AND VISCOUS CORE

In the region with $r<r_1$, the natural assumption to make is that the characteristic length scale is r_1. Thus, we write

$$\tau=(u_*^2/2\pi)\tilde{G}(r/r_1), \qquad \Gamma=\Gamma_1\tilde{F}(r/r_1), \qquad (36)$$

where $\tilde{G}(1)=1$, $\tilde{F}(1)=1$. In the similarity variables, these expressions become

$$g(\eta)=\alpha\tilde{G}(\eta/\eta_1), \qquad f(\eta)=(\Gamma_1/\Gamma_0)\tilde{F}(\eta/\eta_1). \qquad (37)$$

Note that the forms (37) and (21), together with the assumption (19) and the further assumption that there is an overlap region, imply that g is constant for $\eta_1\ll\eta\ll\eta_0$, from which the logarithmic law follows. This type of argument is well known for the turbulent boundary layer. We are, of course, assuming somewhat more in this investigation, namely, that the logarithmic region includes η_1.

We now integrate the inviscid form of the equation of motion (12) between η and η_1, giving

$$f(\eta)=2g(\eta)-4\int_\eta^{\eta_1}\frac{g}{\eta}\,d\eta+2g_1, \qquad (38)$$

where we use the relation (27), and the obvious notation $g_1=g(\eta_1)$, etc.

At this stage, we come to the crux of the argument. For kinematic reasons, g must vanish like η^2 as $\eta\rightarrow 0$, and likewise f. Thus, *if viscous effects are completely negligible*, it follows from (38) that

$$g_1=2\int_0^{\eta_1}\frac{g}{\eta}\,d\eta. \qquad (39)$$

In other words, there must be an integral constraint on the Reynolds stress distribution if the principle of Reynolds number similarity is true.

The question now at issue is whether inviscid dynamics can adjust the turbulence so that this constraint is satisfied exactly, or whether the constraint is violated at some order and viscous forces must be brought in to make the circulation zero at the center of the vortex. Suppose, for the sake of argument, that the constraint were violated. Then, the mean velocity at the center would be infinite and clearly violent turbulence could be expected to remove the singularity quickly. Thus, the difference between the two sides of (39) must be small in some sense. It need not be exactly zero because the fluid can always use viscous effects to reduce the circulation to zero, and the experimental results indicate that this in fact happens. It appears that the fluid prefers to use viscous forces rather than constrain the inviscid dynamics to satisfy (39) exactly.

We therefore assume that with the viscosity neglected, the circulation at the origin has the value

$$f_i=2g_1-4\int_0^{\eta_1}\frac{g}{\eta}\,d\eta$$

$$=2\alpha\left(1-2\int_0^1\tilde{G}(\xi)\,\frac{d\xi}{\xi}\right)\ll 1. \qquad (40)$$

The role of the viscous core is to drop the circulation from f_i to zero, and this is easily accomplished by adding the solution

$$f_v(\eta)=-f_i\exp(-\eta^2/4\epsilon). \qquad (41)$$

The composite solution which has the right behavior in this inner vortex is then

$$f=2g-4\int_\eta^{\eta_1}\frac{g}{\eta}\,d\eta+2g_1-f_i\exp\left(\frac{-\eta^2}{4\epsilon}\right). \qquad (42)$$

The width of the viscous core is $\epsilon^{1/2}$ in the dimensionless variables or $(\nu t)^{1/2}$ in dimensional units.

To summarize, the picture we now have is that the vortex has three physically distinct regions: an outer region between r_0 and r_1 where the Reynolds stress varies slowly and the circulation varies logarithmically as $r\rightarrow r_1$; an inner region where the velocity drops down from its maximum at r_1 toward zero which it does not, however, attain exactly; and an inner viscous core of radius $(\nu t)^{1/2}$ in which the velocity is reduced to zero by viscous stresses. The picture supposes that

$$(\nu t)^{1/2}\ll r_1\ll r_0, \qquad \text{or } \epsilon^{1/2}\ll\eta_1\ll\eta_0$$

but the value of η_1 still remains to be determined. Note that when η_1, is known, the dynamics of the outer vortex gives α or Γ_1 through Eq. (32) or (34), and r_0 follows from (35). The problem is to determine η_1.

Now, although f_i is not zero, it is expected that $f_i\ll f_1<1$. A matching argument between the viscous core and the inner vortex gives a relation between f_i and ϵ. The simplest argument is that the mean vorticity in the core should be comparable to that in the inner vortex. The mean vorticity $\bar{\omega}$ is given by

$$\bar{\omega}=(2\pi r)^{-1}\frac{\partial\Gamma}{\partial r}=(2\pi t)^{-1}\frac{f'}{\eta}. \qquad (43)$$

Thus at the origin, it follows from (41) that $2\pi t\bar{\omega}=$

$f_i/2\epsilon$ when $r=0$. In the inner vortex,

$$f'/\eta \sim (\Gamma_1/\Gamma_0)(1/\eta_1^2). \tag{44}$$

To deduce (44), we can either use the expression (26) for f near η_1, or more generally deduce from (37) that

$$f'/\eta = (\Gamma_1/\eta_1^2\Gamma_0)(\bar{F}'(\xi)/\xi) \tag{45}$$

and use the hypothesis that the function of ξ is of order one.

Then, equating the values of the mean vorticity, we find

$$f_i/\epsilon \sim 8\alpha/\eta_1^2. \tag{46}$$

This argument depends on f_i being positive. We believe this to be so, but a general proof has not yet been constructed and so it will be in order to present an alternative argument for f_i which does not depend upon its sign. As $\eta \to 0$, the velocity given by the inviscid solution (38) has either a minimum if $f_i>0$ or a zero if $f_i<0$. Now, the inviscid profile of circulation is parabolic near $\eta=0$ with curvature $O(\Gamma_1/\eta_1^2\Gamma_0)$ from (44) or (45). Thus, the value of η at which the velocity has a minimum or zero is of order $(\Gamma_0\eta_1^2|f_i|/\Gamma_1)^{1/2}$. This value must be the same as that at which viscous effects start to act, namely $\epsilon^{1/2}$, and equating the two values gives (46) with f_i replaced by $|f_i|$.

The problem has now been reduced to the determination of f_i. At this stage, the argument becomes uncertain and a completely satisfactory approach has not yet been found.

V. STRUCTURE OF THE INNER VORTEX

The viscous core is a higher-order effect, in the sense that its function is not to reduce an $O(1)$ circulation to zero, but an $o(1)$ circulation. The dominant contribution or the leading order term in the Reynolds stress satisfies the integral constraint (39). For example, the stress in the inner vortex could be determined from

$$\tilde{G}(\xi)=\xi^2, \qquad (\xi=\eta/\eta_1) \tag{47}$$

to leading order. The constraint (39) is then satisfied. Thus, we require the extent to which the stress departs from a behavior like (47) and this cannot be determined by general considerations, since it is a higher-order effect, but requires a more detailed model of the turbulent motion.

We believe the following result is true:

$$f_i \sim \eta_1^2/8 \ll 1. \tag{48}$$

One argument that leads to (48) is as follows. We copy Squire[2] and assume that in the inner vortex the transfer of momentum by Reynolds stresses can be described by an eddy viscosity ν_T. Squire took ν_T to be constant or rather proportional to Γ_0; but, we shall try to be more general in the first instance and consider the case of a variable ν_T. In particular, we shall assume, as is consistent with modern ideas on shear flow turbulence, that ν_T is to be found as part of the solution and our hypothesis is that

$$\nu_T=\beta\Gamma, \tag{49}$$

where β is a constant independent of the Reynolds number. Then, the dimensionless Reynolds stress is

$$g=-\beta\eta f\frac{d}{d\eta}\left(\frac{f}{\eta^2}\right), \tag{50}$$

and the inviscid equation becomes

$$\frac{d}{d\eta}\left[\beta\eta^3 f\frac{d}{d\eta}\left(\frac{f}{\eta^2}\right)\right]+\tfrac{1}{2}\eta^2\frac{df}{d\eta}=0. \tag{51}$$

Appropriate boundary conditions are that the solution should be continuous at $\eta=\eta_1$ with the logarithmic profile, i.e.,

$$f=4\alpha, \qquad g=\alpha, \tag{52}$$

when $\eta=\eta_1$. Since (51) is a second-order equation, this determines the solution and it is now a matter of mathematics to compute f_i, the value at $\eta=0$ given by (51). Expanding the derivatives, we have

$$\eta ff''-3ff'+\eta f'^2+(\eta^2/2\beta)f'=0 \tag{53}$$

and the boundary condition

$$f_1=4\alpha, \qquad f_1'=(8\alpha/\eta_1)-(\eta_1/4\beta). \tag{54}$$

It is convenient to write

$$f=4\alpha h(\xi), \qquad \xi=\eta/\eta_1. \tag{55}$$

Then,

$$\xi hh''-3hh'+\xi h'^2+(\eta_1^2/8\beta\alpha)\xi^2 h'=0, \tag{56}$$

$$h(1)=1, \qquad h'(1)=2-(\eta_1^2/16\beta\alpha). \tag{57}$$

Now, $\eta_1^2/\alpha \ll 1$, and by inspection

$$h=\xi^2+O(\eta_1^2/8\beta\alpha)$$

is a solution of (56) and (57). Then, writing

$$h=\xi^2+(\eta_1^2/8\beta\alpha)p(\xi),$$

we find that

$$\xi^2p''+\xi p'-4p=-2\xi^2,$$

$$p(1)=0, \qquad p'(1)=-\tfrac{1}{2}.$$

The solution of this equation is actually singular at $\eta=0$, which shows that the hypothesis (49) makes the inviscid solution blow up, but the point we are trying to make is that the correction to the leading order f is $O(\eta_1^2)$ and this determines the order of f_i.

Alternatively, we can follow Squire more closely and use a constant eddy viscosity

$$\nu_T=\beta\Gamma_1. \tag{58}$$

A straightforward calculation then gives

$$f_i = \eta_1^2/24\beta. \qquad (59)$$

The conclusion reached from this model representation of the inner vortex is that

$$f = \alpha[(4\eta^2/\eta_1^2) + O(\eta_1^2/\alpha)],$$
$$g = \alpha[(\eta^2/\eta_1^2) + O(\eta_1^2/\alpha)]. \qquad (60)$$

That is, the mean flow is solid body rotation with relative error $O(\eta_1^2/\alpha)$. A physical interpretation can be given to the error term, which adds plausibility to the estimate. Consider the kinetic energy of the turbulent fluctuations inside r_1. Since the characteristic turbulent velocity is u_*, the turbulent energy is of order $\frac{1}{2}\pi u_*^2 r_1^2$. On the other hand, the kinetic energy of the mean motion is

$$\int_0^{r_1} \frac{\Gamma^2}{4\pi r}\, dr \approx (16\pi)^{-1}\, \Gamma_1^2,$$

since $\Gamma \approx \Gamma_1(r^2/r_1^2)$. The ratio of turbulent energy to mean energy is

$$8\pi^2(u_* r_1^2/\Gamma_1^2) = \tfrac{1}{2}\pi^2(\eta_1^2/\alpha). \qquad (61)$$

VI. THE SIZE OF THE VORTEX

We now combine the results of Secs. III, IV, and V to determine the dependence of r_1 on Reynolds number. The equations to be used are (31), (32), (35), (46), and (48). The numerical coefficients in these equations are, of course, somewhat uncertain.

Combining (48) and (46), we have $\eta_1^4 \sim 64\alpha\epsilon$, or, in dimensional terms

$$r_1 \sim 2(\nu/\Gamma_0)^{1/4}(\Gamma_1/\Gamma_0)^{1/4}(\Gamma_0 t)^{1/2}. \qquad (62)$$

It follows that

$$v_1 = \frac{\Gamma_1}{2\pi r_1} \sim (4\pi)^{-1}\left(\frac{\Gamma_0}{\nu}\right)^{1/4}\left(\frac{\Gamma_1}{\Gamma_0}\right)^{3/4}\left(\frac{\Gamma_0}{t}\right)^{1/2}. \qquad (63)$$

Equations (32) and (35) predict that Γ_1/Γ_0 should be a slowly decreasing function of the Reynolds number. Expanding (30) and using (35) and (61), we obtain

$$\alpha = (4 \log\lambda - \log 4 + \log\alpha + \log\epsilon^{-1})^{-1}. \qquad (64)$$

This transcendental equation cannot be solved in closed form, but it shows clearly that $\alpha[=\frac{1}{4}(\Gamma_1/\Gamma_0)]$ should decrease slowly as ϵ^{-1} increases, and this is in qualitative agreement with experiment. Reasonable quantitative agreement is found by taking $\lambda = 1$. Then, $\epsilon = 10^{-4}$ gives $4\alpha = 0.67$, while $\epsilon = 10^{-7}$ gives $4\alpha = 0.32$. The reported measurements at these Reynolds numbers are 0.6 and 0.4, respectively.

To compare (62) and (63) with experiment, we look at the values of a and b, Eqs. (2) and (3). From (62),

$$b \sim 2(\nu/\Gamma_0)^{1/4}(\Gamma/\Gamma_0)^{1/4}, \qquad (65)$$

and from (63),

$$a \sim 0.44(\nu/\Gamma_0)^{1/2}(\Gamma_0/\Gamma_1)^{3/2}. \qquad (66)$$

The numbers a and b are related by Eq. (4). Owen[3] defined $\Lambda = a^{1/2}(\Gamma_0/\nu)^{1/4}$, and noticed that measured values of Λ vary from about 0.8 in wind tunnel experiments to about 1.2 in free flight. This was attributed by Owen to an aging process in the vortex, but we notice that according to (66), $\Lambda \propto (\Gamma_0/\Gamma_1)^{3/4}$ and the tendency for Λ to increase with Reynolds number is consistent with the fact that Γ_0/Γ_1 also increases.

Thus, the present ideas have led to results in qualitative agreement with observation, and the quantitative agreement is most promising.

VII. THE NONSIMILAR VORTEX AND THE OVERSHOOT OF CIRCULATION

The previous analysis has been based on the assumption that the vortex was self-similar, which required, in particular, that $A_0/r_1^2 \ll 1$, see Eq. (6). Since $r_1^2 \propto (\nu/\Gamma_0)^{1/2}$, the time for the vortex to be self-similar may be very large if the Reynolds number is high. Therefore, it is worth investigating the extent to which the present analysis may apply even if the time is not large enough for the whole vortex to have reached its asymptotic state.

To proceed, we write the results for $r \ll r_0$ in dimensional form. The circulation and Reynolds stress were found to have the following distribution:

$$\Gamma = \Gamma_1(r^2/r_1^2), \qquad \tau = (\Gamma_1/8\pi t)(r^2/r_1^2),$$
$$(\nu t)^{1/2} < r < r_1, \qquad (67)$$

$$\Gamma = \Gamma_1[\log(r/r_1) + 1], \qquad \tau = \Gamma_1/8\pi t,$$
$$r_1 < r \ll r_0, \qquad (68)$$

where

$$r_1 \sim 2(\nu\Gamma_1)^{1/4} t^{1/2}. \qquad (69)$$

The circulation and Reynolds stress satisfy the (inviscid) equation

$$\frac{\partial\Gamma}{\partial t} = -\frac{2\pi}{r}\frac{\partial}{\partial r}(r^2\tau), \qquad (70)$$

if $\partial\Gamma_1/\partial t = 0$.

It is to be noted that the structure is independent of Γ_0 and r_0. Now, the characteristic time of the mean motion is the time for a fluid particle to complete a revolution and this is $2\pi r_1^2/\Gamma_1 \sim (\nu/\Gamma_1)^{1/2} t \ll t$, where t is the age of the vortex. Thus, it is plausible to assume that the inner vortex and logarithmic region will have attained their final similarity form before the outer vortex has settled down to the asymptotic state, because the characteristic time for the outer vortex is just the age t. (The actual time for the inner vortex to attain self-similarity will depend upon the initial state of the vortex. A reasonable estimate is Γ_0/u^2, where u^2 denotes the initial intensity of the turbulence.)

The question we wish to consider now is how long it takes for the overshoot of circulation to develop in the outer vortex. A completely satisfactory answer needs a full understanding of the turbulence but a reasonable estimate can be found as follows. Let us assume the profile for $r>r_1$, and a suitable guess is suggested by (30). We therefore take

$$\Gamma=\Gamma_1\{\log(r/r_0)+\chi(t)[1-(r^2/r_0^2)]\}+\Gamma_0, \quad (71)$$

where $\Gamma=\Gamma_1$ at $r=r_1$ requires

$$\log(r_1/r_0)+\chi(t)+(\Gamma_0/\Gamma_1)=1. \quad (72)$$

The asymptotic state corresponds to $\chi=1$. The value of $J(t)$ is

$$J(t)=\frac{1}{r_1^2}\int_0^\infty\left(1-\frac{\Gamma}{\Gamma_0}\right)r\,dr$$
$$=\tfrac{1}{4}(\Gamma_1/\Gamma_0)(1-\chi)(r_0^2/r_1^2), \quad (73)$$

on using (67) for $r\leq r_1$, (71) for $r_1\leq r\leq r_0$, and (72). In this development, terms of order r_1^2/r_0^2 have been neglected.

The dependence of χ on t can be estimated from the exact result (6) for $J(t)$ which gives (we neglect $\nu t/r_1^2$)

$$\chi=1-(A_0/r_0^2)(\Gamma_0/\Gamma_1). \quad (74)$$

It will be remembered that A_0 is determined by the initial conditions. To determine r_0, we can use (35), with $\lambda=1$, to give

$$r_0^2=4\alpha\Gamma_0 t=\Gamma_1 t. \quad (75)$$

Hence,

$$\chi=1-(A_0/t)(\Gamma_0/\Gamma_1^2)\rightarrow 1 \qquad \text{as } t\rightarrow\infty. \quad (76)$$

For the profile (71), it is easy to see that an overshoot appears when $\chi>\frac{1}{2}$, and for $\chi<\frac{1}{2}$ the circulation increases monotonically to Γ_0. For $\chi>\frac{1}{2}$, the maximum occurs at $r=(r_0^2/2\chi)^{1/2}$. If A_0/r_0^2 is initially large, the overshoot will not occur until r_0 has increased substantially, and the failure to observe an overshoot may be due to the initial conditions producing fairly large values of A_0/r_0^2.

VIII. AXIAL FLOW IN A TRAILING VORTEX

The trailing vortices behind aircraft are observed to have significant axial velocities directed toward the wing. An explanation of this phenomenon for laminar trailing vortices is given by Moore and Saffman.[8] Here, we apply similar ideas to estimate the axial velocity for turbulent vortices.

The trailing vortex is a steady three-dimensional flow, in contrast to the unsteady two-dimensional flows considered above. The identification is made by writing $t=z/U$, where z is the distance behind the wing and U its speed, and assuming that the approximations of slender body theory are valid. Then, the perturbation w to the mean axial velocity satisfies

$$\frac{\partial w}{\partial t}=-(\rho U)^{-1}\frac{\partial p}{\partial t}+\nu\left(\frac{\partial^2 w}{\partial r^2}+r^{-1}\frac{\partial w}{\partial r}\right)-r^{-1}\frac{\partial}{\partial r}\overline{ru'w'}. \quad (77)$$

A relative axial velocity toward the wing corresponds to $w<0$. The last term on the right-hand side of (77) is due to the Reynolds stress produced by radial transfer of axial momentum fluctuations.

For the calculation of the pressure, it is assumed that to leading order there is a balance between the radial pressure gradient and mean centrifugal force; thus,

$$\rho^{-1}\frac{\partial p}{\partial r}=\frac{v^2}{r}=\frac{\Gamma^2}{4\pi^2 r^3}. \quad (78)$$

Integrating (78), we have for $r<r_0$,

$$\rho^{-1}(p-p_\infty)=-\frac{\Gamma_0^2}{8\pi^2 r_0^2}-(4\pi^2)^{-1}\int_r^{r_0}\frac{\Gamma^2}{r_3}\,dr. \quad (79)$$

Here, p_∞ is the pressure at infinity, which is henceforth taken to be zero. Also, we take $\rho=1$ without loss of generality.

The problem now is to integrate (77). In the first instance, neglect the Reynolds stress and viscous stress. The latter will always be negligible outside the viscous core of radius $(\nu t)^{1/2}$. Then,

$$w=-(p/U)+fn(r). \quad (80)$$

We determine $fn(r)$ as follows. Clearly, $w=0$, for $r>r_0$. Next, since $r_0=r_0(t)$, the integral of (79) can be written $p=p(r;\,r_0)$, the dependence on time being incorporated into r_0. (The argument fails if $r_0=$const.) Then, the solution for w is

$$Uw=-[p(r;\,r_0)-p(r,\,r)]$$
$$=\frac{\Gamma_0^2}{8\pi^2}\left(\frac{1}{r_0^2}-\frac{1}{r^2}\right)+\frac{1}{4\pi^2}\int_r^{r_0}\frac{\Gamma^2}{r^3}\,dr, \quad (81)$$

as this satisfies (80) and assures continuity of w across $r=r_0$.

We notice now that $w\rightarrow-\infty$ as $r\rightarrow 0$. To be more precise, let us substitute the similarity solution for Γ into (81). Then,

$$Uw=\frac{\Gamma_0}{8\pi^2 t}\left(\frac{1}{\eta_0^2}-\frac{1}{\eta^2}+2\int_\eta^{\eta_0}\frac{f^2}{\eta^3}\,d\eta\right)\sim-\frac{\Gamma_0}{8\pi^2 t\eta^2} \qquad \text{as } \eta\rightarrow 0. \quad (82)$$

The singularity at $r=0$ will, of course, be removed by viscous effects and, of course, by the action of the Reynolds stress. Nevertheless, the argument convincingly demonstrates that axial flows toward the wing are to be expected in the center of a turbulent trailing vortex.

Unfortunately, it is not possible to proceed further without some knowledge of Reynolds stress, but as the question is important, it seems worthwhile to make the usual type of simplifying approximation and examine the consequences. A simple assumption likely to give reasonable results is that the Reynolds stress is negligible for $r>r_1$ and is given by a constant eddy viscosity $\nu_T=\beta\Gamma_0$ for $r<r_1$. Then, w satisfies the equation, for $r<r_1$,

$$\frac{\partial w}{\partial t}=-U^{-1}\frac{\partial p}{\partial t}+\beta\Gamma_0\left(\frac{\partial^2 w}{\partial r^2}+r^{-1}\frac{\partial w}{\partial r}\right) \quad (83)$$

with boundary conditions $w=w_1$, at $r=r_1$. The value w_1 is given by (81).

We now focus attention on the self-similar vortex, in order to make the calculation simple. In this case,

$$p=-(\Gamma_0/t)P(\xi), \qquad w=(\Gamma_0/Ut)W(\xi), \quad (84)$$

where $\xi=\eta/\eta_1$. We obtain

$$(\beta/\eta_1^2)(W''+\xi^{-1}W')+(W+\tfrac{1}{2}\xi W')=P+\tfrac{1}{2}\xi P'. \quad (85)$$

Here,

$$P(\xi)=P(1)+(4\pi^2\eta_1^2)^{-1}\int_\xi^1\left(\frac{\Gamma}{\Gamma_0}\right)^2\frac{d\xi}{\xi^3}; \quad (86)$$

and

$$P(1)=(8\pi^2\eta_0^2)^{-1}+(4\pi^2)^{-1}\int_{\eta_1}^{\eta_0}\frac{f^2}{\eta^3}\,d\eta \quad (87)$$

is known when the circulation distribution in the outer vortex is known. The boundary condition for (83) gives

$$W(1)=P(1)-(8\pi^2\eta_1^2)^{-1}. \quad (88)$$

Since $\eta_1/\eta_0\ll 1$, we can evaluate (87) with the approximation

$$f=f_1[\log(\eta/\eta_1)+1], \qquad f_1=\Gamma_1/\Gamma_0. \quad (89)$$

Now, we evaluate $P(1)$ and substitute into (88), giving

$$W(1)\approx(8\pi^2\eta_1^2)^{-1}[\tfrac{5}{2}(\Gamma_1/\Gamma_0)^2-1]. \quad (90)$$

Note that $W(1)<0$ if $\Gamma_1/\Gamma_0<(2/5)^{1/2}=0.632$ as is expected to be the case for a turbulent vortex.

A satisfactory approximation to $P(\xi)$ is found by substituting $\Gamma=\Gamma_1\xi^2$, giving

$$P(\xi)=(\Gamma_1/\Gamma_0)^2(8\pi^2\eta_1^2)^{-1}(\tfrac{7}{2}-\xi^2). \quad (91)$$

Equation (85) now becomes

$$(\beta/\eta_1^2)(W''+\xi^{-1}W')+(W+\tfrac{1}{2}\xi W')=(\Gamma_1/\Gamma_0)^2(8\pi^2\eta_1^2)^{-1}(\tfrac{7}{2}-2\xi^2). \quad (92)$$

The solution of (92) can be expressed in terms of confluent hypergeometric functions. A second boundary condition is given, in addition to (90), by the requirement that W is bounded at $\xi=0$. However, since $\eta_1^2\ll 1$, we can solve by expansion in η_1^2, and it is clear that the leading order solution is

$$W=(8\pi^2\eta_1^2)^{-1}[\tfrac{5}{2}(\Gamma_1/\Gamma_0)^2-1]. \quad (93)$$

(This result is consistent with the neglect of Reynolds stresses for $r>r_1$.) Thus, the axial velocity is to leading order uniform across the core and is, in dimensional terms,

$$w=-(\Gamma_0^2/8\pi^2Ur_1^2)[1-\tfrac{5}{2}(\Gamma_1/\Gamma_0)^2]. \quad (94)$$

Remember that according to (62), $r_1^2\propto(\nu/\Gamma_0)^{1/2}$, so the axial velocity is predicted to be Reynolds number dependent. Note that the axial momentum flux in the core is independent of distance behind the wing.

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