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STRUCTURED NONCOMMUTATIVE MULTIDIMENSIONAL LINEAR SYSTEMS*

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Abstract. We introduce a class of multidimensional linear systems with evolution along a free semigroup. The transfer function for such a system is a formal power series in noncommuting indeterminates. Standard system-theoretic properties (the operations of cascade/parallel connection and inversion, controllability, observability, Kalman decomposition, state-space similarity theorem, minimal state-space realizations, Hankel operators, realization theory) are developed for this class of systems. We also draw out the connections with the much earlier studied theory of rational and recognizable formal power series. Applications include linear-fractional models for classical discrete-time systems with structured, time-varying uncertainty, dimensionless formulas in robust control, multiscale systems and automata theory, and the theory of formal languages.

Key words. multidimensional linear systems, free semigroup, controllability, observability, minimality, realization, formal power series, noncommuting indeterminates

AMS subject classifications. 93B10, 13F25, 47A56, 93B28

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1. Introduction. This paper considers extensions of standard system-theoretic ideas for classical, discrete-time, input/state/output linear systems to the case of certain types of generalized i/s/o systems having evolution along a free semigroup (in place of evolution along the nonnegative integers, as in the classical case). One can introduce formal frequency-domain techniques and arrive at a transfer function for such a system which is a formal power series in noncommuting variables; such objects have occurred in the context of the theory of formal languages and automata theory as well as in connection with realization theory for bilinear systems in the work of Schützenberger and Fliess (see [37, 38, 39, 20, 21, 22, 23] and the book [15] for a good survey).

We first review those aspects of the classical theory which we here generalize to the setting of systems evolving on a free semigroup; this material can be found in many books on linear system and control theory (see, e.g., [32, 16]). By a classical, discrete-time, i/s/o linear system (referred to here simply as a *linear system* for short) we mean a system Σ of linear equations of the form

$$\begin{aligned} x(n+1) &= Ax(n) + Bu(n), \\ y(n) &= Cx(n) + Du(n) \end{aligned} \tag{1.1}$$

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(where n takes values in the integers $\mathbb{Z}$), with $x(n)$ taking values in the *state-space* $\mathcal{H}$, $u(n)$ taking values in the *input-space* $\mathcal{U}$, and $y(n)$ taking values in the *output-space* $\mathcal{Y}$, where here we assume that $\mathcal{H}$, $\mathcal{U}$, and $\mathcal{Y}$ are *finite-dimensional* linear spaces over the field of complex numbers $\mathbb{C}$. It is convenient to identify the operator

$$U = \begin{bmatrix} A & B \\ C & D \end{bmatrix} : \begin{bmatrix} \mathcal{H} \\ \mathcal{U} \end{bmatrix} \to \begin{bmatrix} \mathcal{H} \\ \mathcal{Y} \end{bmatrix}$$

as the *connection matrix* or *colligation* of the system Σ. Given such a system Σ, if one initializes the state $x(0)$ at time 0 and feeds in an input string $\{u(n)\}_{n\in\mathbb{Z}_+}$, one can use the system equations (1.1) to uniquely determine the state $x(n)$ for all future times $n > 0$ and the output $y(n)$ for the present and all future times $n \geq 0$; the result is

$$\begin{aligned} x(n) &= A^n x(0) + \sum_{k=0}^{n-1} A^{n-1-k} B u(k), \\ y(n) &= CA^n x(0) + \sum_{k=0}^{n-1} CA^{n-1-k} B u(k) + Du(n). \end{aligned} \tag{1.2}$$

Application of the *Z-transform*

$$\{x(n)\}_{n\in\mathbb{Z}_+} \mapsto \sum_{n=0}^{\infty} x(n) z^n$$

to the system equations (1.1) converts the expressions (1.2) to the so-called frequency-domain formulas

$$\begin{aligned} \widehat{x}(z) &= (I - zA)^{-1} x(0) + z(I - zA)^{-1} B \widehat{u}(z), \\ \widehat{y}(z) &= C(I - zA)^{-1} x(0) + T_\Sigma(z) \widehat{u}(z), \end{aligned} \tag{1.3}$$

where

$$T_\Sigma(z) = D + zC(I - zA)^{-1} B \tag{1.4}$$

is a rational $\mathcal{L}(\mathcal{U}, \mathcal{Y})$-valued function analytic at the origin called the *transfer function* of the system Σ. Standard system-theoretic ideas in this context are *controllability* and *observability*. The system is said to be *controllable* if for every h in the state-space $\mathcal{H}$ there is an $N < 0$ and an input string $\{u(n)\}_{n=N,N+1,\dots,-1}$ so that h is achievable as $h = x(0)$ if the system is run with initialization $x(N) = 0$ and input string $\{u(n)\}_{n=N,N+1,\dots,-1}$. It works out that the system Σ is controllable if and only if the *controllability operator*

$$\mathcal{C} = \begin{bmatrix} B & AB & A^2B & \cdots \end{bmatrix} : \ell_{\text{fin}}(\mathbb{Z}_-, \mathcal{U}) \to \mathcal{H}$$

has full rank (equal to $\dim \mathcal{H}$).[1] Here $\ell_{\text{fin}}(\mathbb{Z}_-, \mathcal{U})$ denotes the linear space of all $\mathcal{U}$-valued summable sequences on $\mathbb{Z}_-$ with finite support. Similarly, the system is said to be *observable* if the state-vector $h \in \mathcal{H}$ can be uniquely recovered from the output string $\{y(n)\}_{n\geq 0}$ generated by running the system with initial condition $x(0) = h$ and

[1] By the Cayley–Hamilton theorem, it suffices to consider only the finite matrix $\mathcal{C}_n = \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix}$, where $n = \dim \mathcal{H}$ in place of $\mathcal{C}$.

zero input string $u(n) = 0$ for $n \geq 0$; this in turn is equivalent to the *observability operator*

$$\mathcal{O} = \mathrm{col}_{n\geq 0}[CA^n] \colon \mathcal{H} \to \ell(\mathbb{Z}_+, \mathcal{Y})$$

being injective.[2] Here and in what follows, we often use the following notation. If $\mathcal{H}_i$, $\widetilde{\mathcal{H}}_j$, $\mathcal{U}$, and $\mathcal{Y}$ are finite-dimensional linear spaces (for each index i in an index set S and index j in an index set $\widetilde{S}$), and if we are given linear operators $B_j \colon \mathcal{U} \to \widetilde{\mathcal{H}}_j$ and $C_i \colon \mathcal{H}_i \to \mathcal{Y}$, then $\mathrm{col}_{j\in\widetilde{S}} B_j$ denotes the block-operator *column* matrix representing a linear operator from $\mathcal{U}$ into $\oplus_{j\in\widetilde{S}}\widetilde{\mathcal{H}}_j$ given by

$$\mathrm{col}_{j\in\widetilde{S}} B_j \colon u \to \oplus_{j\in\widetilde{S}} B_j u, \tag{1.5}$$

while $\mathrm{row}_{i\in S}\, C_i$ denotes the block-operator *row* matrix representing a linear operator from $\oplus_{i\in S}\mathcal{H}_i$ into $\mathcal{Y}$ given by

$$\mathrm{row}_{i\in S}\, C_i \colon \ \oplus_{i\in S}\, h_i \mapsto \sum_{i\in S} C_i h_i. \tag{1.6}$$

We say that the system $\Sigma = (U \colon (\mathcal{H}\oplus\mathcal{U}) \to (\mathcal{H}\oplus\mathcal{Y}))$ is a *realization* of the $\mathcal{L}(\mathcal{U},\mathcal{Y})$-valued function $T(z)$ if $T(z) = T_\Sigma(z)$. There is a theory of *minimality* of a realization Σ of a given matrix-valued function $T(z)$: we say that the realization $\Sigma = (U \colon (\mathcal{H}\oplus\mathcal{U}) \to (\mathcal{H}\oplus\mathcal{Y}))$ of $T(z)$ is a *minimal realization* if, whenever $\Sigma' = (U' \colon (\mathcal{H}'\oplus\mathcal{U}) \to (\mathcal{H}'\oplus\mathcal{Y}))$ is another realization of $T(z)$, it is the case that $\dim\mathcal{H} \leq \dim\mathcal{H}'$. It is well known that Σ is a minimal realization of $T_\Sigma(z)$ if and only if Σ is both controllable and observable; moreover, given a realization $\Sigma' = (U' \colon (\mathcal{H}'\oplus\mathcal{U}) \to (\mathcal{H}'\oplus\mathcal{Y}))$ of $T(z)$ which is not controllable and/or not observable, the *Kalman decomposition* of the system leads to a procedure for cutting down the system to a controllable and observable (and therefore minimal) realization $\Sigma'_{c/o} = (U_{c/o} \colon \mathcal{H}'_{c/o}\oplus\mathcal{U} \to \mathcal{H}'_{c/o}\oplus\mathcal{Y})$ for $T(z)$ ($T_{\Sigma'_{c/o}}(z) = T_\Sigma(z) = T(z)$). Moreover, the *Hankel operator* $\mathbb{H} = \mathcal{O}\cdot\mathcal{C} \colon \ell_{\mathrm{fin}}(\mathbb{Z}_-,\mathcal{U}) \to \ell(\mathbb{Z}_+,\mathcal{Y})$, the map of a past input signal to the future output signal generated by the system (under the assumption that the state is initialized to be zero sufficiently far in the past and if the input string is taken to be zero on the present and future), plays a prominent role in realization theory, since $\mathbb{H} = \mathbb{H}^T$ is also completely determined by the Taylor coefficients of the transfer function $T(z)$ of Σ. Indeed, a given $\mathcal{L}(\mathcal{U},\mathcal{Y})$-valued function $T(z)$ analytic at the origin can be realized as the transfer function $T(z) = T_\Sigma(z)$ for some finite-dimensional system Σ (1.1) if and only if the Hankel operator $\mathbb{H}^T$ constructed from $T(z)$ has finite rank; in this case there is a canonical construction (the *shift realization*) of a minimal realization $\Sigma_{\mathbb{H}^T} = (U_{\mathbb{H}^T} \colon \mathcal{H}_{\mathbb{H}^T}\oplus\mathcal{U} \to \mathcal{H}_{\mathbb{H}^T}\oplus\mathcal{Y})$ for $T(z)$ with $\dim\mathcal{H}_{\mathbb{H}^T} = \mathrm{rank}\,\mathbb{H}^T$.

The purpose of this paper is to extend these ideas to various classes of systems with evolution along a free semigroup rather than along $\mathbb{Z}_+$ or $\mathbb{Z}$. We consider three main classes of such systems, which we refer to as (1) *noncommutative Fornasini–Marchesini systems*, (2) *noncommutative Givone–Roesser systems*, and (3) *noncommutative full-structured systems*. In all these examples, application of a formal Z-transform to the system equations, under the assumption that the state-vector is initialized to be zero, gives rise to the input-output map for the system being given by

[2] Again by the Cayley–Hamilton theorem, for the present classical case one can replace $\mathcal{O}$ by the finite matrix $\mathcal{O}_n = \mathrm{col}_{j=0,1,\dots,n-1}[CA^j]$, where n is the dimension of the state-space $\mathcal{H}$.

multiplication by a formal power series in noncommuting indeterminates (the *transfer function* of the system) of the form

$$T_\Sigma(z) = D + C(I - Z(z)A)^{-1}Z(z)B, \tag{1.7}$$

where $Z(z)$ is a linear pencil in noncommuting indeterminates $z = (z_1, \dots, z_d)$. The particular form of the linear pencil $Z(z)$ is determined by the particular form of the state equations. For the reader's convenience, section 2 states the main results in explicit, concrete form for these particular classes of examples. In section 3 the added formalism is introduced to describe a general structured noncommutative multidimensional linear system (SNMLS; see Definition 3.7 below). In section 4 we show that such standard system-theoretic operations as cascade connection, parallel connection, and system inversion can be carried out in this context. With the formalism from section 3 in hand, unified proofs are given of the results on controllability, observability, Kalman decomposition, state-space similarity, minimality of realizations, Hankel operators, and construction of minimal realizations in sections 5, 6, 7, 8, 9, 10, and 11, respectively. The final section 12 makes connections of our framework with the theory of recognizable formal power series presented in [15], developed in the work of Schützenberger [37, 38, 39] and Fliess [20, 21].

In applications it is sometimes convenient to view the indeterminates as noncommuting variables, and a formal power series $T(z) = \sum_{w \in \mathcal{F}_d} T_w z^w$ (where $\mathcal{F}_d$ is the set of all words in the d letters $1, 2, \dots, d$ and where $z^w = z_{i_N} \cdots z_{i_1}$ if $w = i_N \cdots i_1$) as a function $\delta \mapsto T(\delta) = \sum_{w \in \mathcal{F}_d} T_w \otimes \delta^w$ defined on some domain of noncommuting operator-tuples $\delta = (\delta_1, \dots, \delta_d)$ (where $\delta^w = \delta_{i_N} \cdots \delta_{i_1}$, multiplication here given by operator composition); this calculus of operator-substitution is important for several of the applications listed below.

Now we mention several areas for applications of the results of this paper.

1. *Robust control theory.* Formal power series and their realizations appear prominently in the theory of robust control of classical 1-D (one-dimensional) systems subject to structured possibly time-varying uncertainty (see [33, 13, 12, 10, 11]). A commonly used model for structured uncertainty in a classical linear, finite-dimensional, feedback-control system is a so-called linear-fractional model, whereby the uncertainty is assumed to have a certain block structure which then enters the nominal plant through a feedback loop. In the case where one considers time-varying uncertainty, the time-varying input-output operator for the disturbed plant can be identified with the evaluation of the transfer function $T_\Sigma(z)$ at $z = \delta$, where $\delta = (\delta_1, \dots, \delta_d)$ is a d-tuple of time-varying operators on ℓ^2 parametrizing the time-varying structured uncertainty. Questions concerning minimality, realizability, and reduction which we explore here have direct relevance for this application. In a companion paper [5], we impose an energy balance law on an SNMLS to define the notion of a *conservative* SNMLS and obtain a realization theorem for this class of noncommutative systems; such conservative (or more generally dissipative) SNMLSs are directly relevant to the robust H^∞-control problems discussed in [33]. In the followup paper [6], we make more explicit the connections of this paper and [5] with linear-fractional models for structured uncertainty and μ-analysis in the presence of structured time-varying uncertainty. Conservative SNMLSs of noncommutative Fornasini–Marchesini type appear also in [7] and [8] in connection with other kinds of problems from multivariable operator theory. Recent closely related work of Alpay and Kalyuzhnyĭ-Verbovetzkiĭ [1] uses the state-space similarity theorem for noncommutative Givone–Roesser systems to develop a realization theory for noncommutative rational J-unitary formal

power series, including connections with noncommutative formal reproducing kernel Pontryagin spaces.

2. *Dimensionless linear matrix inequalities.* As pointed out in [28, 30], many formulas occurring in engineering involving matrix quantities have the same form independent of the size of the matrices. This motivates the study of rational functions in noncommuting variables and the study of noncommutative positivity domains associated with such rational expressions. Realizations such as (1.7) are exactly what is needed to convert (numerically unmanageable) rational matrix inequalities into (highly manageable) linear matrix inequalities (see [30]). Here one substitutes d-tuples of symmetric matrices of variable common size for the indeterminates in the noncommutative rational expression.

3. *Wavelet analysis/multiscale systems.* There have been some attempts in the literature (see [14, 2]) to attach a system evolution to multiresolution structure and multiscale modeling. We expect the setting and results of this paper to have some connections with the work in [14, 2], but details remain to be worked out.

4. *Automata and the theory of formal languages.* It has been known for some time (see [15] and the references there) that formal power series in noncommuting variables, particularly, recognizable and rational formal power series (see section 12 below), arise naturally in connection with the theory of automata and formal languages. In this context, the coefficients of the formal power series may come from a semiring (a ring without subtraction such as the nonnegative integers or the nonnegative rational numbers) rather than operators between two Hilbert spaces, and the free semigroup may be only a monoid. Roughly, a formal power series is said to be recognizable if the support set of its coefficients is recognizable. A subset of a free semigroup (or, more generally, of a monoid) is said to be *recognizable*, in turn, if it can be identified with the set of successful paths (from an initial state to a final state) generated by a finite automaton. Recognizability of a formal power series turns out to be equivalent to existence of a certain type of realization (see section 12 below). Many of the familiar results (e.g., realization through a Hankel-matrix construction, equivalence of minimality of realization with simultaneous controllability and observability, and a state-space similarity theorem) have been worked out in this automaton context. Further details can be found in [15, 19, 32]. Our results give a broader perspective in which to view recognizable formal power series.

5. *Commutative multidimensional system theory.* We view the "noncommutative Fornasini–Marchesini systems" introduced here as noncommutative analogues of the (commutative) Fornasini–Marchesini systems introduced by Fornasini and Marchesini [24] in the multidimensional system theory literature, while the "noncommutative Givone–Roesser systems" are noncommutative analogues of the (commutative) multidimensional Givone–Roesser systems appearing in [26, 27, 36]. In what we call the *commutative case* (evolution along an integer lattice rather than along a free semigroup), the theory of controllability, observability, state-space similarity, and reduction to and construction of a minimal realization of a transfer function is problematic (see, e.g., [31, 25]). By the results here, however, the situation in the noncommutative case is much more like the classical 1-D case. A possible direction for future work is the application of the noncommutative theory as a vehicle for deeper understanding of the commutative case; indeed, the realization theorem in [24] for commutative Fornasini–Marchesini systems is based on the noncommutative realization theorem from [20].

In other directions the commutative theory is ahead of the noncommutative theory. We mention the recent work of Ambrozie and Timotin [3] and of the first author and Bolotnikov [4], which studies classes of functions with a realization similar to the type of realizations discussed here (see (3.19)) in a commutative (and conservative) setting but with resolvent containing a certain polynomial in the frequency variables rather than just a linear term. In particular, [4] contains a realization result which generalizes the commutative analogue of the main result of [5]. A nonlinear analogue of the realization results of [4] would probably demand a nonlinear version of the Taylor functional calculus (see [41] for a start in this direction). Results on minimality, controllability, and observability obtained in the present paper for this case of higher-degree polynomial in the resolvent of the realization could be obtained by first finding an equivalent system representation having a linear resolvent (or first-order system equations), or, more directly, by developing a more coordinate-free behavioral framework for noncommutative system theory (see [35] for the commutative case).

2. Three classes of examples of structured noncommutative multidimensional linear systems.

In this section we introduce and state the main results for the three main examples of SNMLSs. Here the reader can understand the examples and statements of all the main results without having to confront the added formalism of the general definition involving an "admissible graph" (see Definition 3.7 below).

2.1. Noncommutative Fornasini–Marchesini systems.

For d a positive integer, let $\mathcal{F}_d$ be the free semigroup generated by the set of d letters $\{1, 2, \ldots, d\}$. Elements of $\mathcal{F}_d$ are words w of the form $w = i_N i_{N-1} \cdots i_1$, where $i_k \in \{1, 2, \ldots, d\}$ for each $k = 1, \ldots, N$. We include the empty word $\emptyset$ as an element of $\mathcal{F}_d$. The semigroup operation is concatenation: $w \cdot w' = i_N i_{N-1} \cdots i_1 i'_{N'} i'_{N'-1} \cdots i'_1$ if $w = i_N i_{N-1} \cdots i_1$ and $w' = i'_{N'} i'_{N'-1} \cdots i'_1$; the empty word $\emptyset$ serves as the identity element of the semigroup $\mathcal{F}_d$. A Fornasini–Marchesini connection matrix U^{FM} is a matrix of the form

$$U^{FM} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A_1 & B_1 \\ \vdots & \vdots \\ A_d & B_d \\ C & D \end{bmatrix} : \begin{bmatrix} \mathcal{H} \\ \mathcal{U} \end{bmatrix} \to \begin{bmatrix} \oplus_{i=1}^d \mathcal{H} \\ \mathcal{Y} \end{bmatrix}.$$

The associated system equations are

$$\Sigma^{FM} \colon \begin{cases} x(1w) = A_1 x(w) + B_1 u(w), \\ \qquad \vdots \\ x(dw) = A_d x(w) + B_d u(w), \\ y(w) \;\;\, = Cx(w) + Du(w) \quad \text{for } w \in \mathcal{F}_d, \end{cases} \tag{2.1}$$

where the state $x(w)$ takes values in the state-space $\mathcal{H}$ and consists of only one component, $u(w)$ takes values in the input-space $\mathcal{U}$, and $y(w)$ takes values in the output-space $\mathcal{Y}$. We consider this type of system as a noncommutative analogue of the (commutative) multidimensional linear systems studied by Fornasini and Marchesini (see, e.g., [24]). We let $z = (z_1, \ldots, z_d)$ be a collection of d formal noncommuting variables and consider the formal noncommutative multivariable Z-transform

$$\{x(w)\}_{w \in \mathcal{F}_d} \mapsto \widehat{x}(z) := \sum_{w \in \mathcal{F}_d} x(w) z^w, \tag{2.2}$$

where $z^w = z_{i_N} z_{i_{N-1}} \cdots z_{i_1}$ if $w = i_N i_{N-1} \cdots i_1$. Then, as will be seen in more generality in section 3 (see Example 3.8 and formula (3.20) below), application of the formal Z-transform to the system (2.1) on $\mathcal{T}_{\rm future}$ leads to the representation

$$\widehat{x}(z) = (I - (Z_{\rm row}(z) \otimes I_{\mathcal{H}})A)^{-1} x(\emptyset) + (I - (Z_{\rm row}(z) \otimes I_{\mathcal{H}})A)^{-1}(Z_{\rm row}(z) \otimes I_{\mathcal{H}}) \cdot B\widehat{u}(z), \tag{2.3}$$
$$\widehat{y}(z) = C(I - (Z_{\rm row}(z) \otimes I_{\mathcal{H}})A)^{-1} x(\emptyset) + T_{\Sigma^{FM}}(z)\widehat{u}(z),$$

where the formal power series $T_{\Sigma}^{FM}(z)$ (the *transfer function* of the noncommutative Fornasini–Marchesini system Σ^{FM}) is given by

$$\begin{aligned} T_{\Sigma^{FM}}(z) &= D + C(I - (Z_{\rm row}(z) \otimes I_{\mathcal{H}})A)^{-1}(Z_{\rm row}(z) \otimes I_{\mathcal{H}})B \\ &= D + C(I - z_1 A_1 - \cdots - z_d A_d)^{-1}(z_1 B_1 + \cdots + z_d B_d) \\ &= D + \sum_{v \in \mathcal{F}_d} \sum_{j=1}^{d} C A^v B_j z^v z_j, \end{aligned} \tag{2.4}$$

where we have used the conventions

$$Z_{\rm row}(z) \otimes I_{\mathcal{H}} = \begin{bmatrix} z_1 I_{\mathcal{H}} & \cdots & z_d I_{\mathcal{H}} \end{bmatrix},$$
$$A^v = A_{i_N} A_{i_{N-1}} \cdots A_{i_1} \text{ if } v = i_N i_{N-1} \cdots i_1.$$

We now also consider the associated backward system equations

$$\Sigma^{FM}_{\rm past}\colon \begin{cases} x(w) = \sum_{i=1}^d A_i x(wi) + \sum_{i=1}^d B_i u(wi), \\ y(w) = Cx(w) + Du(w) \quad \text{for } w \in \mathcal{F}_d. \end{cases} \tag{2.5}$$

We view the system as running on both the present and future $\mathcal{T}_{\rm future} := \mathcal{F}_d$ and on the past $\mathcal{T}_{\rm past} := \mathcal{F}_d \setminus \emptyset$ (where we think of the two appearances of $\mathcal{F}_d$ here as two distinct copies of $\mathcal{F}_d$). The forward equations (2.1) apply for $w \in \mathcal{T}_{future}$, while the backward equations (2.5) apply for $wi \in \mathcal{T}_{past}$. The noncommutative Fornasini–Marchesini system Σ^{FM} is said to be *FM-controllable* if any state-vector $h \in \mathcal{H}$ can be achieved as $h = x(\emptyset)$ by running the system on the past $\mathcal{T}_{\rm past}$ with state-initialization equal to zero on all locations $w \in \mathcal{T}_{\rm past}$ of sufficiently long length with some input string $\{u(w)\}_{w \in \mathcal{T}_{\rm past}}$ having finite support on the past; this condition turns out to be equivalent to the *Fornasini–Marchesini controllability matrix* $\mathcal{C}^{FM}$ given by

$$\mathcal{C}^{FM} = \text{row}_{N=1,2,\dots}\, \text{row}_{i_1,i_2,\dots,i_N \in \{1,\dots,d\}} [A_{i_N} A_{i_{N-1}} \cdots A_{i_2} B_{i_1}] \tag{2.6}$$

having full rank, i.e., having $\operatorname{im} \mathcal{C}^{FM} = \mathcal{H}$. This fact amounts to the specialization of the analysis in section 5 to Example 3.8; a direct analysis can be found in [34].

Dually, we say that the noncommutative Fornasini–Marchesini system Σ^{FM} is *FM-observable* if the state-vector $h \in \mathcal{H}$ can be uniquely recovered from the present and future output string $\{y_i(w)\}_{w \in \mathcal{T}_{\rm future}}$ generated by running the forward system equations (2.1) of Σ^{FM} with the state initialized by $x(\emptyset) = h$ and with zero input string on the future ($u(w) = 0$ for $w \in \mathcal{T}_{\rm future} = \mathcal{F}_d$). In terms of the system operators, FM-observability of Σ^{FM} is equivalent to the *Fornasini–Marchesini observability operator* $\mathcal{O}^{FM}$ being injective, where

$$\mathcal{O}^{FM} = \text{col}_{N=0,1,2,\dots}\, \text{col}_{i_1,i_2,\dots,i_N \in \{1,\dots,d\}} [C A_{i_N} A_{i_{N-1}} \cdots A_{i_1}]. \tag{2.7}$$

(Here and elsewhere we interpret $A_{i_N}A_{i_{N-1}}\cdots A_{i_1}$ to be equal to the identity operator $I_{\mathcal{H}}$ in case $N=0$.) This fact follows from specializing the results of section 6 to Example 3.8 below; again a direct discussion can be found in [34].

The Hankel operator $\mathbb{H}^{FM}$ of the noncommutative Fornasini–Marchesini system Σ^{FM} is the composition $\mathbb{H}^{FM}=\mathcal{O}^{FM}\mathcal{C}^{FM}\colon \ell_{\text{fin}}(\mathcal{T}_{\text{past}},\mathcal{U})\to\ell(\mathcal{T}_{\text{future}},\mathcal{Y})$; the Hankel operator has the same physical interpretation as in the classical case; $\mathbb{H}^{FM}$ maps a past input to the corresponding future output of a given system trajectory, under the assumption that the state has been initialized to zero in the distant past. Matrix entries of $\mathbb{H}^{FM}$ are given by

$$\mathbb{H}^{FM}_{i_Ni_{N-1}\cdots i_1;\,i'_{N'}i'_{N'-1}\cdots i'_1}=CA_{i_N}A_{i_{N-1}}\cdots A_{i_1}A_{i'_{N'}}A_{i'_{N'-1}}\cdots A_{i'_2}B_{i'_1}, \tag{2.8}$$

where $N=0,1,2,\ldots,$ $N'=1,2,\ldots,$ and $i_k,i'_{k'}\in\{1,\ldots,d\}$ for all k,k'. From the factorization $\mathbb{H}^{FM}=\mathcal{O}^{FM}\mathcal{C}^{FM}$ we see that $\mathbb{H}^{FM}$ has finite rank for any (finite-dimensional) noncommutative Fornasini–Marchesini system. The matrix entries of $\mathbb{H}^{FM}$ can also be expressed directly in terms of the Taylor coefficients (sometimes also called Markov parameters) of the transfer function $T_{\Sigma^{FM}}(z)=\sum_{w\in\mathcal{F}_d}T_wz^w$:

$$\mathbb{H}^{FM}_{v,w}=T_{vw}. \tag{2.9}$$

This type of Hankel operator is obtained by specializing the Hankel operator discussed in section 10 to Example 3.8 below; an explicit discussion of this (Fornasini–Marchesini) case is given in [34].

Given a formal power series $T(z)=\sum_{w\in\mathcal{F}_d}T_wz^w$ in d noncommuting variables $z=(z_1,\ldots,z_d)$ (where $z^w=z_{i_N}\cdots z_{i_1}$ if $w=i_N\cdots i_1$ and where $z^{\emptyset}=1$) with operator-valued coefficients $T_w\in\mathcal{L}(\mathcal{U},\mathcal{Y})$, we say that the noncommutative Fornasini–Marchesini system Σ^{FM} is a (noncommutative Fornasini–Marchesini) realization of $T(z)$ if $T(z)=T_{\Sigma^{FM}}(z)$. A given (noncommutative Fornasini–Marchesini) realization Σ^{FM} of $T(z)$ with state-space $\mathcal{H}$ is said to be *FM-minimal* if, whenever $\Sigma^{FM\prime}$ is another noncommutative Fornasini–Marchesini realization of $T(z)$ with state-space $\mathcal{H}'$, then $\dim\mathcal{H}\le\dim\mathcal{H}'$. Two noncommutative Fornasini–Marchesini systems Σ^{FM} and $\Sigma^{FM\prime}$ with the same input- and output-spaces and connection matrices

$$U^{FM}=\begin{bmatrix}A & B\\ C & D\end{bmatrix}=\begin{bmatrix}A_1 & B_1\\ \vdots & \vdots\\ A_d & B_d\\ C & D\end{bmatrix}\colon\begin{bmatrix}\mathcal{H}\\ \mathcal{U}\end{bmatrix}\to\begin{bmatrix}\oplus_{i=1}^d\mathcal{H}\\ \mathcal{Y}\end{bmatrix},$$

$$U^{FM\prime}=\begin{bmatrix}A' & B'\\ C' & D'\end{bmatrix}=\begin{bmatrix}A'_1 & B'_1\\ \vdots & \vdots\\ A'_d & B'_d\\ C' & D'\end{bmatrix}\colon\begin{bmatrix}\mathcal{H}'\\ \mathcal{U}\end{bmatrix}\to\begin{bmatrix}\oplus_{i=1}^d\mathcal{H}'\\ \mathcal{Y}\end{bmatrix}$$

are said to be *FM-similar* if there is a bijective linear operator $\Gamma\colon\mathcal{H}\to\mathcal{H}'$ such that

$$\begin{bmatrix}\Gamma & & & \\ & \ddots & & \\ & & \Gamma & \\ & & & I_{\mathcal{Y}}\end{bmatrix}\begin{bmatrix}A_1 & B_1\\ \vdots & \vdots\\ A_d & B_d\\ C & D\end{bmatrix}=\begin{bmatrix}A'_1 & B'_1\\ \vdots & \vdots\\ A'_d & B'_d\\ C' & D'\end{bmatrix}\begin{bmatrix}\Gamma & 0\\ 0 & I_{\mathcal{U}}\end{bmatrix}.$$

The following theorem summarizes the results of Theorems 8.2, 9.1, and 11.1 when specialized to the case of noncommutative Fornasini–Marchesini systems (Example 3.8).

THEOREM 2.1.

(1) *Suppose that* Σ^{FM} *and* $\Sigma^{FM\prime}$ *are two noncommutative Fornasini–Marchesini systems which are both FM-controllable and FM-observable. Then* Σ^{FM} *and* $\Sigma^{FM\prime}$ *are FM-similar if and only if they realize the same transfer function:*

$$T_{\Sigma^{FM}}(z) = T_{\Sigma^{FM\prime}}(z).$$

(2) *The noncommutative Fornasini–Marchesini system* Σ^{FM} *is an FM-minimal realization of its transfer function* $T_{\Sigma^{FM}}(z)$ *if and only if* Σ^{FM} *is both FM-controllable and FM-observable.*

(3) *Suppose that* $T(z) = \sum_{w \in \mathcal{F}_d} T_w z^w$ *is a formal power series in* d *noncommuting variables* $z = (z_1, \dots, z_d)$ *with matrix coefficients* $T_w \in \mathcal{L}(\mathcal{U}, \mathcal{Y})$*. Then* $T(z)$ *can be realized as the transfer function* $T(z) = T_{\Sigma^{FM}}(z)$ *of a finite-dimensional noncommutative Fornasini–Marchesini system* Σ^{FM} *if and only if the associated Hankel matrix*

$$\mathbb{H}^T = [T_{vw}]_{v \in \mathcal{F}_d, w \in \mathcal{F}_d \setminus \{\emptyset\}}$$

has finite rank. In this case there is a canonical construction (shift realization) of a minimal realization with state-space $\mathcal{H}$ *having* $\dim \mathcal{H} = \operatorname{rank} \mathbb{H}^T$*.*

2.2. Noncommutative Givone–Roesser systems. Just as was done above for the case of noncommutative Fornasini–Marchesini systems, the domain evolution for a *noncommutative Givone–Roesser system* which we discuss now is the free semigroup $\mathcal{F}_d$ on the set of d letters $\{1, 2, \dots, d\}$ (for d a positive integer). We take the associated Givone–Roesser connection matrix U^{GR}, however, to have the form

$$U^{GR} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A_{11} & \dots & A_{1d} & B_1 \\ \vdots & & \vdots & \vdots \\ A_{d1} & \dots & A_{dd} & B_d \\ C_1 & \dots & C_d & D \end{bmatrix} : \begin{bmatrix} \mathcal{H}_1 \\ \vdots \\ \mathcal{H}_d \\ \mathcal{U} \end{bmatrix} \to \begin{bmatrix} \mathcal{H}_1 \\ \vdots \\ \mathcal{H}_d \\ \mathcal{Y} \end{bmatrix}$$

for *auxiliary state-spaces* $\mathcal{H}_1, \dots, \mathcal{H}_d$, an input-space $\mathcal{U}$, and an output-space $\mathcal{Y}$ (all finite-dimensional linear spaces for our discussion here). The associated system equations then are

$$\Sigma^{GR}\colon \begin{cases} x_1(1w) = A_{11}x_1(w) + \dots + A_{1d}x_d(w) + B_1 u(w), \\ \qquad \vdots \\ x_d(dw) = A_{d1}x_1(w) + \dots + A_{dd}x_d(w) + B_d u(w), \\ y(w) \quad = C_1 x_1(w) + \dots + C_d x_d(w) + D u(w), \end{cases} \quad \text{for } w \in \mathcal{F}_d, \tag{2.10}$$

where the state $x(w) = \operatorname{col}_{j=1,\dots,d} x_j(w)$ at position $w \in \mathcal{F}_d$ consists of d components $x_1(w), \dots, x_d(w)$ with $x_j(w)$ taking values in the auxiliary state-space $\mathcal{H}_j$ for $j = 1, \dots, d$; $u(w)$ takes values in the input-space $\mathcal{U}$; and $y(w)$ takes values in the output-space $\mathcal{Y}$. In case $i, j \in \{1, \dots, d\}$ with $i \neq j$ we set $x_i(jw) = 0$. We consider this type of system as a noncommutative analogue of the (commutative) multidimensional linear systems introduced by Givone and Roesser (see, e.g., [26, 27, 36]). If we apply the noncommutative formal Z-transform (2.2) to the system equations (2.10) and solve, we get

$$\begin{aligned}\widehat{x}(z) &= (I-(Z_{\text{diag}}(z)\otimes I_{\mathcal{H}})A)^{-1}x(\emptyset) + (I-(Z_{\text{diag}}(z)\otimes I_{\mathcal{H}})A)^{-1}(Z_{\text{diag}}(z)\otimes I_{\mathcal{H}})\\ &\quad \cdot B\widehat{u}(z),\\ \widehat{y}(z) &= C(I-(Z_{\text{diag}}(z)\otimes I_{\mathcal{H}})A)^{-1}x(\emptyset) + T_{\Sigma^{GR}}(z)\widehat{u}(z),\end{aligned} \tag{2.11}$$

where the formal power series $T_{\Sigma^{GR}}(z)$ (the *transfer function* of the noncommutative Givone–Roesser system Σ^{GR}) is given by

$$\begin{aligned} T_{\Sigma^{GR}}(z) &= D + C(I-(Z_{\text{diag}}(z)\otimes I_{\mathcal{H}})A)^{-1}(Z_{\text{diag}}(z)\otimes I_{\mathcal{H}})B\\ &= D + \begin{bmatrix} C_1 & \cdots & C_d\end{bmatrix}\left(\begin{bmatrix} I_{\mathcal{H}_1} & & \\ & \ddots & \\ & & I_{\mathcal{H}_d}\end{bmatrix} - \begin{bmatrix} z_1A_{11} & \cdots & z_1A_{1d}\\ \vdots & & \vdots\\ z_dA_{d1} & \cdots & z_dA_{dd}\end{bmatrix}\right)^{-1}\begin{bmatrix} z_1B_1\\ \vdots \\ z_dB_d\end{bmatrix}\\ &= D + \sum_{N=1}^{\infty}\sum_{i_1,\dots,i_N\in\{1,\dots,d\}} C_{i_N}A_{i_N,i_{N-1}}A_{i_{N-1},i_{N-2}}\cdots A_{i_2,i_1}B_{i_1}z_{i_N}z_{i_{N-1}}\cdots z_{i_2}z_{i_1},\end{aligned} \tag{2.12}$$

where we have used the convention

$$Z_{\text{diag}}(z)\otimes I_{\mathcal{H}} = \begin{bmatrix} z_1I_{\mathcal{H}_1} & & \\ & \ddots & \\ & & z_dI_{\mathcal{H}_d}\end{bmatrix}.$$

We now also consider the associated backward system equations

$$\Sigma^{GR}_{\text{past}}: \begin{cases} x_1(w) = \sum_{i=1}^d A_{1i}x_i(w1) + B_1u(w1),\\ \quad\vdots\\ x_d(w) = \sum_{i=1}^d A_{di}x_i(wd) + B_du(wd),\\ y(w) \;\;= \sum_{i=1}^d C_ix(w) + Du(w) \quad \text{for } w\in\mathcal{F}_d.\end{cases} \tag{2.13}$$

We follow the same convention as explained above for noncommutative Fornasini–Marchesini systems and view the system Σ^{GR} as running on both the present and future $\mathcal{T}_{\text{future}} := \mathcal{F}_d$ and on the past $\mathcal{T}_{\text{past}} := \mathcal{F}_d\setminus\emptyset$, with the forward equations (2.10) applying for $w\in\mathcal{T}_{future}$ and the backward equations (2.13) applying for $wi\in\mathcal{T}_{past}$. The noncommutative Givone–Roesser system Σ^{GR} is said to be *GR-controllable* if, for each $i\in\{1,\dots,d\}$, any state-vector $h_i\in\mathcal{H}_i$ can be achieved as the ith component $h_i = x_i(\emptyset)$ of the state-vector $x(\emptyset)$ at the empty-set location by running the system on the past $\mathcal{T}_{\text{past}}$ with state-initialization equal to zero on all locations $w\in\mathcal{T}_{\text{past}}$ of sufficiently long length with some input string $\{u(w)\}_{w\in\mathcal{T}_{\text{past}}}$ having finite support on the past; this condition turns out to be equivalent to the ith *Givone–Roesser controllability matrix* $\mathcal{C}_i^{GR}$ given by

$$\mathcal{C}_i^{GR} = \text{row}_{N=0,1,\dots}\,\text{row}_{i_1,i_2,\dots,i_N\in\{1,\dots,d\}}[A_{i,i_N}A_{i_N,i_{N-1}}\cdots A_{i_2,i_1}B_{i_1}] \tag{2.14}$$

(where the $N=0$ term is to be interpreted as simply B_i) having full rank, i.e., having $\text{im}\,\mathcal{C}_i^{GR} = \mathcal{H}_i$ for each $i=1,\dots,d$. This fact follows by specializing the analysis in section 5 to Example 3.9 below; a direct discussion is in [34].

Dually, we say that the noncommutative Givone–Roesser system Σ^{GR} is *GR-observable* if, for each $i = 1, \dots, d$, the state-vector $h_i \in \mathcal{H}_i$ can be uniquely recovered from the present and future output string $\{y(w)\}_{w \in \mathcal{T}_{\text{future}}}$ generated by running the forward system equations (2.10) of Σ^{GR} with the state initialized by $x_i(\emptyset) = h_i$ and $x_{i'}(\emptyset) = 0$ for $i' \neq i$, and with zero input string on the future ($u(w) = 0$ for $w \in \mathcal{T}_{\text{future}} = \mathcal{F}_d$). In terms of the system operators, GR-observability of Σ^{GR} is equivalent to the ith *Givone–Roesser observability operator* $\mathcal{O}_i^{GR}$ being injective for each $i = 1, \dots, d$, where

$$\mathcal{O}_i^{GR} = \operatorname{col}_{N=0,1,2,\dots} \operatorname{col}_{i_1, i_2, \dots, i_N \in \{1,\dots,d\}} [C_{i_N} A_{i_N, i_{N-1}} A_{i_{N-1}, i_{N-2}} \cdots A_{i_1, i}]. \tag{2.15}$$

Here the $N = 0$ term is to be interpreted as simply C_i. All these matters follow upon specialization of the analysis in section 6 to Example 3.9 below; again, a direct discussion is in [34].

There are d Hankel operators $\mathbb{H}^{GR,1}, \dots, \mathbb{H}^{GR,d}$ for a noncommutative Givone–Roesser system Σ^{GR}; namely, for each $i = 1, \dots, d$,

$$\mathbb{H}^{GR,i} = \mathcal{O}_i^{GR} \mathcal{C}_i^{GR} \colon \ell_{\text{fin}}(\mathcal{T}_{\text{past}}, \mathcal{U}) \to \ell(\mathcal{T}_{\text{future}}, \mathcal{Y}).$$

Each Hankel operator $\mathbb{H}^{GR,i}$ again has a physical interpretation as mapping a past input to the corresponding future output of a given system trajectory under the assumption that the state has been initialized to zero in the distant past, but where the observations are taken only with respect to the ith component $x_i(\emptyset)$ of the state at position $\emptyset$. Matrix entries of $\mathbb{H}^{GR,i}$ are given by

$$\begin{aligned} &\mathbb{H}^{GR,i}_{i_N i_{N-1} \cdots i_1;\, i'_{N'} i'_{N'-1} \cdots i'_1} \\ &\quad = C_{i_N} A_{i_N, i_{N-1}} A_{i_{N-1}, i_{N-2}} \cdots A_{i_1, i} A_{i, i'_{N'}} A_{i'_{N'}, i'_{N'-1}} \cdots A_{i'_2, i'_1} B_{i'_1}, \end{aligned} \tag{2.16}$$

where $N' = 0, 1, 2, \dots$, $N = 0, 1, 2, \dots$, and $i_k, i'_{k'} \in \{1, \dots, d\}$ for all k, k'. Some small values of N and N' in formula (2.16) require special interpretation; for example, for case $N = 0$ and $N' = 0$ we interpret (2.16) as giving

$$\mathbb{H}^{GR,i}_{\emptyset;\emptyset} = C_i B_i.$$

From the factorization $\mathbb{H}^{GR,i} = \mathcal{O}_i^{GR} \mathcal{C}_i^{GR}$ we see that $\mathbb{H}^{GR,i}$ has finite rank for each $i = 1, \dots, d$ for any (finite-dimensional) noncommutative Givone–Roesser system. The matrix entries of $\mathbb{H}^{GR,i}$ can also be expressed directly in terms of the Taylor coefficients (sometimes also called Markov parameters) of the transfer function $T_{\Sigma^{GR}}(z) = \sum_{w \in \mathcal{F}_d} T_w z^w$: indeed,

$$\mathbb{H}^{GR,i}_{v,w} = T_{viw} \quad \text{for } v, w \in \mathcal{F}_d,\ i \in \{1, \dots, d\}. \tag{2.17}$$

These details amount to the specialization of section 10 to Example 3.9 below, and also can be found (in explicit form) in [34].

Given a formal power series $T(z) = \sum_{w \in \mathcal{F}_d} T_w z^w$ in d noncommuting variables $z = (z_1, \dots, z_d)$ (where $z^w = z_{i_N} \cdots z_{i_1}$ if $w = i_N \cdots i_1$ and where $z^\emptyset = 1$) with operator-valued coefficients $T_w \in \mathcal{L}(\mathcal{U}, \mathcal{Y})$, we say that the noncommutative Givone–Roesser system Σ^{GR} is a (noncommutative Givone–Roesser) realization of $T(z)$ if $T(z) = T_{\Sigma^{GR}}(z)$. A given (noncommutative Givone–Roesser) realization Σ^{GR} of $T(z)$ with auxiliary state-spaces $\mathcal{H}_1, \dots, \mathcal{H}_d$ is said to be *GR-minimal* if, whenever $\Sigma^{GR\prime}$

is another noncommutative Givone–Roesser realization of $T(z)$ with auxiliary state-spaces $\mathcal{H}'_1, \dots, \mathcal{H}'_d$, then $\dim \mathcal{H}_i \le \dim \mathcal{H}'_i$ for each $i = 1, \dots, d$. Two noncommutative Givone–Roesser systems Σ^{GR} and $\Sigma^{GR\prime}$ with the same input- and output-spaces and connection matrices

$$U^{GR} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A_{11} & \dots & A_{1d} & B_1 \\ \vdots & & \vdots & \vdots \\ A_{d1} & \dots & A_{dd} & B_d \\ C_1 & \dots & C_d & D \end{bmatrix} : \begin{bmatrix} \mathcal{H}_1 \\ \vdots \\ \mathcal{H}_d \\ \mathcal{U} \end{bmatrix} \to \begin{bmatrix} \mathcal{H}_1 \\ \vdots \\ \mathcal{H}_d \\ \mathcal{Y} \end{bmatrix},$$

$$U^{GR\prime} = \begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix} = \begin{bmatrix} A'_{11} & \dots & A'_{1d} & B'_1 \\ \vdots & & \vdots & \vdots \\ A'_{d1} & \dots & A'_{dd} & B'_d \\ C'_1 & \dots & C'_d & D' \end{bmatrix} : \begin{bmatrix} \mathcal{H}'_1 \\ \vdots \\ \mathcal{H}'_d \\ \mathcal{U} \end{bmatrix} \to \begin{bmatrix} \mathcal{H}'_1 \\ \vdots \\ \mathcal{H}'_d \\ \mathcal{Y} \end{bmatrix}$$

are said to be *GR-similar* if, for each $i = 1, \dots, d$, there is a bijective linear operator $\Gamma_i \colon \mathcal{H}_i \to \mathcal{H}'_i$ such that

$$\begin{bmatrix} \Gamma_1 & & & \\ & \ddots & & \\ & & \Gamma_d & \\ & & & I_{\mathcal{Y}} \end{bmatrix} \begin{bmatrix} A_{11} & \cdots A_{1d} & B_1 \\ \vdots & \vdots & \vdots \\ A_{d1} & \cdots A_{dd} & B_d \\ C_1 & \cdots C_d & D \end{bmatrix}$$
$$= \begin{bmatrix} A'_{11} & \cdots & A'_{1d} & B'_1 \\ \vdots & & \vdots & \vdots \\ A'_{d1} & \cdots & A'_{dd} & B'_d \\ C'_1 & \cdots & C'_d & D' \end{bmatrix} \begin{bmatrix} \Gamma_1 & & & \\ & \ddots & & \\ & & \Gamma_d & \\ & & & I_{\mathcal{U}} \end{bmatrix}.$$

The following theorem summarizes the results of Theorems 8.2, 9.1, and 11.1 when specialized to the case of noncommutative Givone–Roesser systems (Example 3.9).

THEOREM 2.2.

(1) *Suppose that Σ^{GR} and $\Sigma^{GR\prime}$ are two noncommutative Givone–Roesser systems which are both GR-controllable and GR-observable. Then Σ^{GR} and $\Sigma^{GR\prime}$ are GR-similar if and only if they realize the same transfer function:*

$$T_{\Sigma^{GR}}(z) = T_{\Sigma^{GR\prime}}(z).$$

(2) *The noncommutative Givone–Roesser system Σ^{GR} is a GR-minimal realization of its transfer function $T_{\Sigma^{GR}}(z)$ if and only if Σ^{GR} is both GR-controllable and GR-observable.*

(3) *Suppose that $T(z) = \sum_{w \in \mathcal{F}_d} T_w z^w$ is a formal power series in d noncommuting variables $z = (z_1, \dots, z_d)$ with matrix coefficients $T_w \in \mathcal{L}(\mathcal{U}, \mathcal{Y})$. Then $T(z)$ can be realized as the transfer function $T(z) = T_{\Sigma^{GR}}(z)$ of a finite-dimensional noncommutative Givone–Roesser system Σ^{GR} if and only if the associated Hankel matrices*

$$\mathbb{H}^{T,i} = [T_{viw}]_{v \in \mathcal{F}_d, w \in \mathcal{F}_d}$$

have finite rank for $i = 1, \dots, d$. In this case there is a canonical construction (shift realization) of a minimal realization with auxiliary state-space $\mathcal{H}_i$ having $\dim \mathcal{H}_i = \operatorname{rank} \mathbb{H}^{T,i}$ for $i = 1, \dots, d$.

In fact, it can be shown that a formal power series $T(z) = \sum_{w \in \mathcal{F}_d} T_w z^w$ in d noncommuting variables $z = (z_1, \dots, z_d)$ has an FM-realization if and only if it has a GR-realization if and only if it is *rational* in the sense of Schützenberger (see [15]). One of the points of part (3) in Theorems 2.1 and 2.2 is that they identify the precise Hankel matrices with rank(s) equal to the state-space dimension(s) in a minimal realization of Fornasini–Marchesini or Givone–Roesser type. We discuss these connections between various types of noncommutative realizations in section 12.

2.3. Noncommutative full-structured systems. Our last concrete example of a structured noncommutative system is what we call a "full-structured" system. For this case it is convenient to assume that the evolution of the system takes place on the free semigroup generated by a certain Cartesian product set. Denote by $\mathcal{F}_{n,m}$ the free semigroup generated by the set $E = \{1, \dots, n\} \times \{1, \dots, m\}$. Thus elements of $\mathcal{F}_{n,m}$ are words w of the form $(i_N, j_N)(i_{N-1}, j_{N-1}) \cdots (i_1, j_1)$, where $i_k \in \{1, \dots, n\}$ and $j_k \in \{1, \dots, m\}$ for all $k = 1, \dots, N$. Again we let $\emptyset$ denote the empty word which serves as the identity for the semigroup $\mathcal{F}_{n,m}$. By a *full-structured connection matrix* U^{full} we mean a block-operator matrix of the form

$$U^{\text{full}} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A_{11} & \cdots & A_{1n} & B_1 \\ \vdots & & \vdots & \vdots \\ A_{m1} & \cdots & A_{mn} & B_m \\ C_1 & \cdots & C_n & D \end{bmatrix} : \begin{bmatrix} \oplus_{i=1}^n \mathcal{H} \\ \mathcal{U} \end{bmatrix} \to \begin{bmatrix} \oplus_{j=1}^m \mathcal{H} \\ \mathcal{Y} \end{bmatrix},$$

where $\mathcal{H}$ (the *state-space*), $\mathcal{U}$ (the *input-space*), and $\mathcal{Y}$ (the *output-space*) are finite-dimensional linear spaces. The associated system equations are

(2.18)
$$\Sigma^{\text{full}}: \begin{cases} x_1((1,j)\cdot w) &= A_{j1}x_1(w) + \cdots + A_{jn}x_n(w) + B_j u(w) \quad \text{for } j = 1, \dots, m, \\ & \vdots \\ x_n((n,j)\cdot w) &= A_{j1}x_1(w) + \cdots + A_{jn}x_n(w) + B_j u(w) \quad \text{for } j = 1, \dots, m, \\ x_i((i',j)\cdot w) &= 0 \quad \text{if } i \neq i', \\ y(w) &= C_1 x_1(w) + \cdots + C_n x_n(w) + D u(w). \end{cases}$$

Here the state-vector $x(w) = \text{col}_{i=1,\dots,n}\, x_i(w) \in \oplus_{i=1}^n \mathcal{H}$ consists of n components $x_i(w)$ for $i = 1, \dots, n$ with each $x_i(w)$ in the auxiliary state-space $\mathcal{H}$, while $u(w)$ assumes values in the input-space $\mathcal{U}$ and $y(w)$ assumes values in the output-space $\mathcal{Y}$. Note that the state trajectory $\{x(w)\}_{w \in \mathcal{F}_{n,m}}$ incorporates some redundancy; namely, *if* $\{x(w)\}_{w \in \mathcal{F}_{n,m}} = \{\text{col}_{i=1,\dots,n}[x_i(w)]\}_{w \in \mathcal{F}_{n,m}}$ *is the state trajectory satisfying the state-update equation in* (2.18) *for some choice of input signal* $\{u(w)\}_{w \in \mathcal{F}_{n,m}}$, *then, for each fixed* $j \in \{1, \dots, m\}$ *and* $w \in \mathcal{F}_{n,m}$,

$$x_i((i,j)\cdot w) \text{ is independent of } i \in \{1, \dots, n\}. \tag{2.19}$$

We shall work with the redundant form (2.18) of the system equations rather than rewriting them in a more economical form.

We let $z = (z_{11}, \dots, z_{1m}; z_{21}, \dots, z_{2m}; \dots; z_{n1}, \dots, z_{nm})$ be a collection of nm noncommuting variables indexed by $\{1, \dots, n\} \times \{1, \dots, m\}$. Application of the noncommutative Z-transform (2.2) (with respect to $\mathcal{F}_{n,m}$ rather than with respect to $\mathcal{F}_d$)

converts the system equations to

$$
\begin{aligned}
\widehat{x}(z) &= (I - (Z_{\text{full}}(z) \otimes I_{\mathcal{H}})A)^{-1} x(\emptyset) + (I - (Z_{\text{full}}(z) \otimes I_{\mathcal{H}})A)^{-1} (Z_{\text{full}}(z) \otimes I_{\mathcal{H}}) \cdot B\widehat{u}(z), \\
\widehat{y}(z) &= C(I - (Z_{\text{full}}(z) \otimes I_{\mathcal{H}})A)^{-1} x(\emptyset) + T_{\Sigma^{\text{full}}}(z)\widehat{u}(z),
\end{aligned} \tag{2.20}
$$

where $T_{\Sigma^{\text{full}}}(z)$ (the *transfer function* of the noncommutative full-structured system Σ^{full}) is given by

$$
\begin{aligned}
T_{\Sigma^{\text{full}}}(z) &= D + C(I - (Z_{\text{full}}(z) \otimes I_{\mathcal{H}})A)^{-1} (Z_{\text{full}}(z) \otimes I_{\mathcal{H}})B \\
&= D + \begin{bmatrix} C_1 & \cdots & C_n \end{bmatrix} \\
&\cdot \left(\begin{bmatrix} I_{\mathcal{H}} & & \\ & \ddots & \\ & & I_{\mathcal{H}} \end{bmatrix} - \begin{bmatrix} \sum_{j=1}^m z_{1j} A_{j1} & \cdots & \sum_{j=1}^m z_{1j} A_{jn} \\ \vdots & & \vdots \\ \sum_{j=1}^m z_{nj} A_{j1} & \cdots & \sum_{j=1}^m z_{nj} A_{jn} \end{bmatrix} \right)^{-1} \begin{bmatrix} \sum_{j=1}^m z_{1j} B_j \\ \vdots \\ \sum_{j=1}^m z_{nj} B_j \end{bmatrix} \\
&= D + \sum_{N=1}^{\infty} \sum_{i_1,\dots,i_N \in \{1,\dots,n\}} \sum_{j_1,\dots,j_N \in \{1,\dots,m\}} C_{i_N} A_{j_N, i_{N-1}} A_{j_{N-1}, i_{N-2}} \cdots A_{j_2, i_1} B_{j_1} \\
&\qquad \cdot z_{i_N, j_N} z_{i_{N-1}, j_{N-1}} \cdots z_{i_2, j_2} z_{i_1, j_1},
\end{aligned} \tag{2.21}
$$

and where $Z_{\text{full}}(z) \otimes I_{\mathcal{H}}$ is given by

$$
Z_{\text{full}}(z) \otimes I_{\mathcal{H}} = \begin{bmatrix} z_{1,1} I_{\mathcal{H}} & \cdots & z_{1,m} I_{\mathcal{H}} \\ \vdots & & \vdots \\ z_{n,1} I_{\mathcal{H}} & \cdots & z_{n,m} I_{\mathcal{H}} \end{bmatrix}.
$$

The backward full-structured system equations have the form

$$
\Sigma^{\text{full}}_{past} \colon \begin{cases} x_1(w) &= \sum_{j=1}^m \sum_{i'=1}^n A_{j,i'} x_{i'}(w \cdot (1,j)) + \sum_{j=1}^m B_j u(w \cdot (1,j)), \\ &\vdots \\ x_n(w) &= \sum_{j=1}^m \sum_{i'=1}^n A_{j,i'} x_{i'}(w \cdot (n,j)) + \sum_{j=1}^m B_j u(w \cdot (n,j)), \\ y(w) &= \sum_{i=1}^n C_i x_i(w) + Du(w), \end{cases} \tag{2.22}
$$

and are to be interpreted as the evolution of the system on the past $\mathcal{T}_{\text{past}} = \mathcal{F}_{n,m} \setminus \{\emptyset\}$. The noncommutative full-structured system Σ^{full} is said to be *full-controllable* if, for each $i \in \{1,\dots,n\}$, any state-vector $h \in \mathcal{H}$ can be achieved as the ith component $h_i = x_i(\emptyset)$ (for some, or equivalently for any $i \in \{1,\dots,n\}$) of the state-vector $x(\emptyset)$ at the empty-set location by running the system on the past $\mathcal{T}_{\text{past}}$ with state-initialization equal to zero on all locations $w \in \mathcal{T}_{\text{past}}$ of sufficiently long length with some input string $\{u(w)\}_{w \in \mathcal{T}_{\text{past}}}$ having finite support on the past; this condition turns out to be equivalent to the *full-structured controllability matrix* $\mathcal{C}_1^{\text{full}}$ given by

$$
\begin{aligned}
\mathcal{C}_1^{\text{full}} &= \operatorname{row}_{N=1,2,\dots} \operatorname{row}_{(1,j_N)(i_{N-1},j_{N-1})\cdots(i_1,j_1)\colon i_1,i_2,\dots,i_{N-1} \in \{1,\dots,n\}; j_1,\dots,j_N \in \{1,\dots,m\}} \\
&\qquad [A_{j_N, i_{N-1}} A_{j_{N-1}, i_{N-2}} \cdots A_{j_2, i_1} B_{j_1}]
\end{aligned} \tag{2.23}
$$

having full rank, i.e., having $\operatorname{im} \mathcal{C}_1^{\text{full}} = \mathcal{H}$. These facts amount to the specialization of the results of section 5 to Example 3.10 below.

Dually, we say that the noncommutative full-structured system $\Sigma^{\rm full}$ is *full-observable* if the state-vector $h \in \mathcal{H}$ can be uniquely recovered from the n-tuple of present and future output strings $\{y_i(w)\}_{i=1,\dots,n;w\in\mathcal{T}_{\rm future}}$ (with $\mathcal{T}_{\rm future} = \mathcal{F}_{n,m}$). Here, for each $i = 1,\dots,n$, the ith output string $\{y_i(w)\}_{w\in\mathcal{F}_{n,m}}$ is generated by running the forward system equations (2.18) of $\Sigma^{\rm full}$ with the state initialized by $x_i(\emptyset) = h_i$ and $x_{i'}(\emptyset) = 0$ for $i' \neq i$ and with zero input string on the future ($u(w) = 0$ for $w \in \mathcal{T}_{\rm future} = \mathcal{F}_{n,m}$). In terms of the system operators, full-observability of $\Sigma^{\rm full}$ is equivalent to the *full observability operator* $\mathcal{O}^{\rm full}\colon \mathcal{H} \to \oplus_{i=1}^n \ell(\mathcal{F}_{n,m},\mathcal{Y})$ being injective, where

$$\mathcal{O}^{\rm full} = \operatorname{col}_{i=1,\dots,n} \operatorname{col}_{N=0,1,2,\dots} \operatorname{col}_{(i_N,j_N)\cdots(i_1,j_1)\colon i_1,\dots,i_N\in\{1,\dots,n\};j_1,\dots,j_N\in\{1,\dots,m\}} [C_{i_N} A_{j_N,i_{N-1}} A_{j_{N-1},i_{N-2}} \cdots A_{j_1,i}]. \tag{2.24}$$

Here the $N = 0$ term is to be interpreted as simply C_i. These matters amount to specialization of the results of section 6 to Example 3.10 below.

We define the Hankel operator $\mathbb{H}^{\rm full}$ for a noncommutative full-structured system $\Sigma^{\rm full}$ as the composition $\mathbb{H}^{\rm full} = \mathcal{O}^{\rm full}\mathcal{C}_1^{\rm full}\colon \ell_{\rm fin}(\mathcal{T}^1_{\rm past},\mathcal{U}) \to \oplus_{i=1}^n \ell(\mathcal{T}_{\rm future},\mathcal{Y})$; here $\mathcal{T}^1_{\rm past}$ denotes a certain subset of the past $\mathcal{T}_{\rm past}$, namely, the set of all nonempty words $(i_1,j_1)\cdot(i_2,j_2)\cdots(i_N,j_N)$ for which the leading letter (i_1,j_1) has first component i_1 equal to 1. Again the Hankel operator $\mathbb{H}^{\rm full}$ has a physical interpretation as mapping a past input to the corresponding future output of a given system trajectory (in this case an n-tuple of future outputs) under the assumption that the state has been initialized to zero in the distant past. Matrix entries of $\mathbb{H}^{\rm full}$ are given by

$$\begin{aligned}&\mathbb{H}^{\rm full}_{i,(i_N,j_N)\cdots(i_1,j_1);(i'_{N'},j'_{N'}),(i'_{N'-1},j'_{N'-1})\cdots(i'_1,j'_1)}\\ &\quad = C_{i_N} A_{j_N,i_{N-1}} A_{j_{N-1},i_{N-2}} \cdots A_{j_1,i} A_{j'_{N'},i'_{N'-1}} \cdots A_{j'_2,i'_1} B_{j'_1},\end{aligned} \tag{2.25}$$

where $N = 0,1,2,\dots$, $N' = 1,2,\dots$, and $i_k, i'_{k'} \in \{1,\dots,n\}$ and $j_k, j'_{k'} \in \{1,\dots,m\}$ for all k and k'; some small values of N and N' in formula (2.25) require special interpretation; for example, for case $N = 0$ and $N' = 1$ we interpret (2.25) as giving

$$\mathbb{H}^{\rm full}_{i,\emptyset;(1,j)} = C_i B_j.$$

From the factorization $\mathbb{H}^{\rm full} = \mathcal{O}^{\rm full}\mathcal{C}_1^{\rm full}$ we see that $\mathbb{H}^{\rm full}$ has finite rank equal to the dimension of the state-space in a minimal realization for any (finite-dimensional) noncommutative full-structured system. The matrix entries of $\mathbb{H}^{\rm full}$ can also be expressed directly in terms of the Taylor coefficients of the transfer function $T_{\Sigma^{\rm full}}(z) = \sum_{w\in\mathcal{F}_{n,m}} T_w z^w$: indeed

$$\mathbb{H}^{\rm full}_{i,v;(1,j_N)w'} = T_{v\cdot(i,j_N)\cdot w'} \quad \text{for } v, w' \in \mathcal{F}_{n,m},\ i \in \{1,\dots,n\}. \tag{2.26}$$

These results all fall out of specializing the results of section 10 to Example 3.10 below.

Given a formal power series $T(z) = \sum_{w\in\mathcal{F}_{n,m}} T_w z^w$ in $n\cdot m$ noncommuting variables $z = (z_{11},\dots,z_{1m};\cdots;z_{n1},\cdots,z_{nm})$ (where $z^w = z_{i_N,j_N}\cdots z_{i_1,j_1}$ if $w = (i_N,j_N)\cdots(i_1,j_1)$ and where $z^\emptyset = 1$) with $\mathcal{L}(\mathcal{U},\mathcal{Y})$-valued coefficients T_w, we say that the noncommutative full-structured system $\Sigma^{\rm full}$ is a (noncommutative full) realization of $T(z)$ if $T(z) = T_{\Sigma^{\rm full}}(z)$. A given (noncommutative full) realization $\Sigma^{\rm full}$ of $T(z)$ with auxiliary state-space $\mathcal{H}$ is said to be *full-minimal* if, whenever $\Sigma^{\rm full\,\prime}$ is another noncommutative full realization of $T(z)$ with auxiliary state-space $\mathcal{H}$, then

$\dim \mathcal{H} \leq \dim \mathcal{H}'$. Two noncommutative full-structured systems $\Sigma^{\rm full}$ and $\Sigma^{\rm full\prime}$ with the same input- and output-spaces and connection matrices

$$U^{\rm full} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A_{11} & \cdots & A_{1n} & B_1 \\ \vdots & & \vdots & \vdots \\ A_{m1} & \cdots & A_{mn} & B_m \\ C_1 & \cdots & C_n & D \end{bmatrix} : \begin{bmatrix} \oplus_{i=1}^n \mathcal{H} \\ \mathcal{U} \end{bmatrix} \to \begin{bmatrix} \oplus_{j=1}^m \mathcal{H} \\ \mathcal{Y} \end{bmatrix},$$

$$U^{\rm full\prime} = \begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix} = \begin{bmatrix} A'_{11} & \cdots & A'_{1n} & B'_1 \\ \vdots & & \vdots & \vdots \\ A'_{m1} & \cdots & A'_{mn} & B'_m \\ C'_1 & \cdots & C'_n & D' \end{bmatrix} : \begin{bmatrix} \oplus_{i=1}^n \mathcal{H}' \\ \mathcal{U} \end{bmatrix} \to \begin{bmatrix} \oplus_{j=1}^m \mathcal{H}' \\ \mathcal{Y} \end{bmatrix}$$

are said to be *full-similar* if there is a bijective linear operator $\Gamma \colon \mathcal{H} \to \mathcal{H}'$ such that

$$\begin{bmatrix} \Gamma & & & \\ & \ddots & & \\ & & \Gamma & \\ & & & I_{\mathcal{Y}} \end{bmatrix} \begin{bmatrix} A_{11} & \cdots A_{1n} & B_1 \\ \vdots & \vdots & \vdots \\ A_{m1} & \cdots A_{mn} & B_m \\ C_1 & \cdots C_n & D \end{bmatrix} = \begin{bmatrix} A'_{11} & \cdots & A'_{1n} & B'_1 \\ \vdots & & \vdots & \vdots \\ A'_{m1} & \cdots & A'_{mn} & B'_m \\ C'_1 & \cdots & C'_n & D' \end{bmatrix} \begin{bmatrix} \Gamma & & & \\ & \ddots & & \\ & & \Gamma & \\ & & & I_{\mathcal{U}} \end{bmatrix}.$$

The following theorem summarizes the results of Theorems 8.2, 9.1, and 11.1 when specialized to the case of noncommutative full-structured systems (Example 3.10).

THEOREM 2.3.

(1) *Suppose that Σ^{full} and $\Sigma^{full\prime}$ are two noncommutative full-structured systems which are both full-controllable and full-observable. Then Σ^{full} and $\Sigma^{full\prime}$ are full-similar if and only if they realize the same transfer function:*

$$T_{\Sigma^{full}}(z) = T_{\Sigma^{full\prime}}(z).$$

(2) *The noncommutative full-structured system Σ^{full} is a full-minimal realization of its transfer function $T_{\Sigma^{full}}(z)$ if and only if Σ^{full} is both full-controllable and full-observable.*

(3) *Suppose that $T(z) = \sum_{w \in \mathcal{F}_{n,m}} T_w z^w$ is a formal power series in $n \cdot m$ noncommuting variables $z = (z_{11}, \ldots, z_{1m}; \ldots; z_{n1}, \ldots, z_{nm})$ with matrix coefficients $T_w \in \mathcal{L}(\mathcal{U}, \mathcal{Y})$. Then $T(z)$ can be realized as the transfer function $T(z) = T_{\Sigma^{full}}(z)$ of a finite-dimensional noncommutative full-structured system Σ^{full} if and only if the associated Hankel matrix*

$$\mathbb{H}^T = [T_{v \cdot (i, i_N) \cdot w}]_{i \in \{1, \ldots, n\}, v \in \mathcal{F}_{n,m}; (1, j_N) \cdot w \in \mathcal{F}_{n,m} \setminus \{\emptyset\}}$$

has finite rank for $i = 1, \ldots, n$. In this case there is a canonical construction (shift realization) of a minimal realization with state-space $\mathcal{H}$ having $\dim \mathcal{H} = \operatorname{rank} \mathbb{H}^T$.

3. Structured noncommutative multidimensional linear systems: Definition and basic properties. Our general notion of structured noncommutative multidimensional linear system (SNMLS) will be associated with a graph G. As is standard, a graph G consists of a set of vertices V together with a set of edges E. Each edge $e \in E$ connects a source vertex $\mathbf{s}(e)$ (where $\mathbf{s}\colon E \to V$ is the *source map*) to a range vertex $\mathbf{r}(e)$ (where $\mathbf{r}\colon E \to V$ is the *range map*). We assume throughout that V and E are *finite* sets. For our application to SNMLSs, we require a few additional properties, encoded in the following definition of an *admissible graph*.

DEFINITION 3.1. *We say that the graph $G = (V,\ E,\ \mathbf{s}\colon E \to V,\ \mathbf{r}\colon E \to V)$ is an admissible graph if*

(1) *the set of vertices V of G has a disjoint partitioning $V = S \dot{\cup} R$ into two subsets S and R such that each edge e of G has source vertex $\mathbf{s}(e) \in S$ and range vertex $\mathbf{r}(e) \in R$;*
(2) *for a given $s \in S$ and $r \in R$ there is at most one edge $e \in E$ connecting s to r (i.e., at most one edge e with $\mathbf{s}(e) = s$ and $\mathbf{r}(e) = r$);*
(3) *each pathwise-connected component G_k of G is a nondegenerate complete bipartite graph; i.e., the vertices of G_k have a partitioning $V(G_k) = S_k \dot{\cup} R_k$ (with $S_k \subset S$, $R_k \subset R$ and both $S_k \neq \emptyset$ and $R_k \neq \emptyset$) such that for each pair (s, r) with $s \in S_k$ and $r \in R_k$ there is exactly one edge $e \in E$ with $\mathbf{s}(e) = s$ and $\mathbf{r}(e) = r$.*

In other words, conditions (1) and (2) say that G is a *bipartite* graph. Thus admissible graphs amount to bipartite graphs having connected path components which are complete bipartite subgraphs. Thus the set of edges E can be identified with a subset of the Cartesian product $S \times R$, where S and R are called the source vertices and range vertices, respectively.

Admissible graphs G have the following intrinsic characterization.

THEOREM 3.2. *Suppose that we are given finite disjoint sets S, R, and E together with mappings $\mathbf{s}\colon E \to S$ and $\mathbf{r}\colon E \to R$. Associated with these data is a graph G defined as follows: the vertex set of G is $V := S \cup R$, and there exists an edge connecting v to v' if and only if there is an $e \in E$ either with $v = \mathbf{s}(e), v' = \mathbf{r}(e)$ or with $v' = \mathbf{s}(e), v = \mathbf{r}(e)$. Then G is admissible in the sense of Definition* 3.1 *if and only if the following conditions hold:*

(1) *$\mathbf{s}\colon E \to S$ is surjective.*
(2) *$\mathbf{r}\colon E \to R$ is surjective.*
(3) *The map $\mathbf{s} \times \mathbf{r}\colon E \to S \times R$ given by*

$$\mathbf{s} \times \mathbf{r}\colon e \mapsto (\mathbf{s}(e), \mathbf{r}(e))$$

is injective.

(4) *Whenever e_1, e_2, and e_3 are elements of E with $\mathbf{r}(e_1) = \mathbf{r}(e_2)$ and $\mathbf{s}(e_1) = \mathbf{s}(e_3)$, then there is an edge e_4 in E, with $\mathbf{s}(e_4) = \mathbf{s}(e_2)$, and $\mathbf{r}(e_4) = \mathbf{r}(e_3)$.*

Proof. Let G be an admissible graph with pathwise-connected components equal to the subgraphs $G_1, \dots, G_K$. Since each G_k is a complete bipartite graph by assumption, we have that the vertex set $V(G_k)$ has a disjoint partitioning $V(G_k) = S_k \dot{\cup} R_k$ for nonempty subsets $S_k \subset S$ and $R_k \subset R$, and the edge set $E(G_k)$ of G_k can be identified with the Cartesian product $S_k \times R_k$ (with $\mathbf{s}(s, r) = s$ and $\mathbf{r}(s, r) = r$ for $s \in S_k$ and $r \in R_k$). As $\mathbf{s}$ maps $E(G_k)$ onto S_k and $\mathbf{r}$ maps $E(G_k)$ onto R_k for each $k = 1, \dots, K$, we see that $\mathbf{s}$ maps E onto S and $\mathbf{r}$ maps E onto R. Condition (2) in Definition 3.1 says that $\mathbf{s} \times \mathbf{r}$ is injective on E. Finally, suppose that $e_1, e_2, e_3 \in E$ as in condition (4). Then $\mathbf{r}(e_1) = \mathbf{r}(e_2) = r$ implies that $\mathbf{s}(e_1)$ and $\mathbf{s}(e_2)$ are in the same pathwise-connected component S_i of G. On the other hand, $\mathbf{s}(e_1) = \mathbf{s}(e_3)$ implies that $\mathbf{s}(e_3)$ is also in S_i and $\mathbf{r}(e_3) \in R_i$. The assumption that the pathwise-connected component G_i is a complete bipartite graph implies that there is an edge e_4 connecting $\mathbf{s}(e_2)$ to $\mathbf{r}(e_3)$.

Conversely, suppose that G arises from source vertex function $\mathbf{s}\colon E \to S$ and range vertex function $\mathbf{r}\colon E \to R$ satisfying conditions (1)–(4) as in the statement of the theorem. By definition, the vertex set V is partitioned into two disjoint subsets S and R such that each edge of G connects an element of S with an element of R or vice versa; i.e., Definition 3.1(1) holds. Condition (3) in Theorem 3.2 gives Definition 3.1(2). Suppose that $s \in S$ and $r \in R$ are in the same pathwise-connected component of the graph G. By the bipartite structure of G, this means that there is a path $e_1 e_2 \cdots e_{2N-1}$ (necessarily of odd length) connecting s to r:

$$\mathbf{s}(e_1) = s,\, \mathbf{r}(e_1) = \mathbf{s}(e_2),\, \mathbf{r}(e_2) = \mathbf{s}(e_3),\, \dots,\, \mathbf{r}(e_{2N-2}) = \mathbf{s}(e_{2N-1}),\, \mathbf{r}(e_{2N-1}) = r.$$

Without loss of generality we may suppose that we have chosen the shortest such path. If $N > 1$, we may use condition (4) to produce a shorter path connecting s to r. Hence it must be the case that $N = 1$ and the path consists of a single edge $e \in E$ connecting s to r and hence (s, r). Thus if $s \in S$ and $r \in R$ are connected by a path of G, then they are connected by a path of length 1. Condition (1) in the theorem implies that every $s \in S$ is connected to some $r \in R$. We conclude that each pathwise-connected component of G is a complete bipartite graph; i.e., Definition 3.1(3) is satisfied, and the theorem follows. $\square$

If e is an edge in the admissible graph G, then we have the notation $\mathbf{s}(e)$ for the source vertex of e, and $\mathbf{r}(e)$ for the range vertex of e. Conversely, given an $s \in S$ and a $r \in R$, there is an edge e connecting s to r (i.e., $e \in E$ with $\mathbf{s}(e) = s$ and $\mathbf{r}(e) = r$); exactly one s and r are in the same path-connected component p of G. For v any vertex of G (either a source vertex or a range vertex) we shall let $[v]$ denote the path-connected component containing v. Thus s and r are in the same path-connected component exactly when $[s] = [r]$. When this is the case, by the admissibility axioms the edge e connecting s to r is unique. We shall denote this edge by $e_{s,r}$:

$$e_{s,r} \text{ determined by } \mathbf{s}(e_{s,r}) = s \text{ and } \mathbf{r}(e_{s,r}) = r. \tag{3.1}$$

Note that $e_{s,r}$ is defined for $s \in S$ and $r \in R$ exactly when $[s] = [r]$.

We associate with each admissible graph G a linear form in noncommuting indeterminates $z = (z_e : e \in E)$ indexed by the edge set E of G, as follows. For each $e \in E$, define a matrix $I_{G,e} = [I_{G,e;s,r}]_{s \in S, r \in R}$ (with rows indexed by S and columns indexed by R) with matrix entries given by

$$I_{G,e;s,r} = \begin{cases} 1 & \text{if } (s,r) = (\mathbf{s}(e), \mathbf{r}(e)), \\ 0 & \text{otherwise.} \end{cases} \tag{3.2}$$

We then define the *structure matrix* $Z_G(z)$ associated with each admissible graph G to be the linear form in the noncommuting indeterminates $z = (z_e : e \in E)$ given by

$$Z_G(z) = \sum_{e \in E} I_{G,e} z_e.$$

We are now ready to give examples of admissible graphs with their associated structure matrices in connection with certain well-known noncommutative multidimensional linear models. We refer to [34] for further details on the motivation and construction of these models.

Example 3.3 (noncommutative Fornasini–Marchesini structure matrix). In this case, we take the admissible graph G^{FM} to be a complete bipartite graph having only one source vertex. Thus we take $S^{FM} = \{1\}$, and $R^{FM} = E^{FM} = \{1, \dots, d\}$ with $\mathbf{s}^{FM}(i) = 1$, $\mathbf{r}^{FM}(i) = i$, i.e., $n = 1, m = d$. Thus we have

$$I_{G^{FM},i} = \begin{bmatrix} 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \end{bmatrix},$$

where 1 is located in the ith slot. Thus, the structure matrix for the noncommutative Fornasini–Marchesini case is simply given by

$$Z_{G^{FM}}(z) = \sum_{i=1}^{d} I_{G^{FM},i} z_i = \begin{bmatrix} z_1 & \cdots & z_d \end{bmatrix} =: Z_{\text{row}}(z).$$

Example 3.4 (noncommutative Givone–Roesser structure matrix). In this case, we take the admissible graph G^{GR} to have d path-connected components, with each path-connected component containing only one source and one range vertex. Thus, we take $S^{GR} = R^{GR} = E^{GR} = \{1, \dots, d\}$ with $\mathbf{s}^{GR}(i) = i$, $\mathbf{r}^{GR}(i) = i$, and thus $n = d = m$. We then have

$$I_{G^{GR},i} = \begin{bmatrix} 0 & \cdots & 0 & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & 1 & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \end{bmatrix},$$

where 1 is located at the (i, i)th entry. Therefore, the structure matrix for the noncommutative Givone–Roesser case has the diagonal form

$$Z_{G^{GR}}(z) = \sum_{i=1}^{d} z_i I_{G^{GR},i} = \begin{bmatrix} z_1 & & \\ & \ddots & \\ & & z_d \end{bmatrix} := Z_{\text{diag}}(z).$$

Example 3.5 (full matrix block structure matrix). In this case, we take G^{full} to be a general finite, complete bipartite graph. Thus we take $S = \{1, \dots, n\}$, $R = \{1, \dots, m\}$, and $E = \{(i, j) : i \in S, j \in R\}$ with $\mathbf{s}^{\text{full}}(i, j) = i$, $\mathbf{r}^{\text{full}}(i, j) = j$, where $d = nm$. Then we have

$$I_{G^{\text{full}},(i,j)} = \begin{bmatrix} 0 & \cdots & 0 & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & 1 & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \end{bmatrix},$$

where 1 is located at the (i, j)th entry. Thus the structure matrix for this case has the full-block structure

$$Z_{G^{\text{full}}}(z) = \begin{bmatrix} z_{1,1} & \cdots & z_{1,m} \\ \vdots & & \vdots \\ z_{n,1} & \cdots & z_{n,m} \end{bmatrix} =: Z_{\text{full}}(z).$$

Note that Example 3.3 amounts to the special case of this example where $n = 1$.

Example 3.6 (the general structure matrix). Suppose that the admissible graph G has path-connected components G_k with source vertices $S_k = \{(k,1),\dots,(k,n_k)\}$, range vertices $R_k = \{(k,1),\dots,(k,m_k)\}$, and edge sets $E_k = \{(k,i,j) : 1 \le i \le n_k, 1 \le j \le m_k\}$ for $k = 1,\dots,K$. Define a graph G to have source vertex set

$$S = \cup_{k=1}^K S_k = \{(k,i) : 1 \le k \le K, 1 \le i \le n_k\},$$

range vertex set

$$R = \cup_{k=1}^K R_k = \{(k,j) : 1 \le k \le K, 1 \le j \le m_k\},$$

and edge set

$$E = \cup_{k=1}^K E_k = \{(k,i,j) : 1 \le k \le K, 1 \le i \le n_k, 1 \le j \le m_k\},$$

with $\mathbf{s}(k,i,j) = (k,i)$, $\mathbf{r}(k,i,j) = (k,j)$ for $(k,i,j) \in E$. Then the associated structure matrix $Z_G(z)$ is given by

$$Z_G(z) = \begin{bmatrix} Z_{\mathrm{full},1}(z^1) & & \\ & \ddots & \\ & & Z_{\mathrm{full},K}(z^K) \end{bmatrix},$$

where we let z^k denote the $(n_k \cdot m_k)$-tuple of variables $z^k = (z_{k,i,j} \colon 1 \le i \le n_k; 1 \le j \le m_k)$ and where

$$Z_{\mathrm{full},k}(z^k) = \begin{bmatrix} z_{k,1,1} & \cdots & z_{k,1,m_k} \\ \vdots & & \vdots \\ z_{k,n_k,1} & \cdots & z_{k,n_k,m_k} \end{bmatrix}$$

is as in Example 3.5 for $k = 1,\dots,K$. By the definition of an admissible graph as a graph with path-connected components equal to complete bipartite graphs, we see that this example amounts to the general case.

To define an SNMLS, in addition to an admissible graph we require a collection of finite-dimensional linear spaces $\mathcal{H}_p$ indexed by each path-connected component p of G. We often abbreviate the whole collection simply by

$$\mathcal{H} = \{\mathcal{H}_p : p \in P(G)\},$$

where $P(G)$ denotes the set of path-connected components of G. In general, for $v \in V$ we use the notation $[v]$ to denote the path-connected component of G containing v (whether v be in S or in R). Thus, for each $s \in S$ and $r \in R$ we have associated finite-dimensional linear spaces $\mathcal{H}_{[s]}$ and $\mathcal{H}_{[r]}$, which are distinct only for s and r in distinct path-connected components of G. In addition, we need a *connection matrix* or *colligation*

(3.3)
$$U = \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} [A_{r,s}]_{r\in R, s\in S} & \mathrm{col}_{r\in R}[B_r] \\ \mathrm{row}_{s\in S}[C_s] & D \end{bmatrix} : \begin{bmatrix} \mathrm{col}_{s\in S}\, \mathcal{H}_{[s]} \\ \mathcal{U} \end{bmatrix} \to \begin{bmatrix} \mathrm{col}_{r\in R}\, \mathcal{H}_{[r]} \\ \mathcal{Y} \end{bmatrix},$$

where $\mathcal{U}$ and $\mathcal{Y}$ are linear spaces, here taken also to be finite-dimensional, called the input-space and output-space, respectively.

We now introduce our notion SNMLS.

DEFINITION 3.7. *By an* SNMLS*, we mean a collection of objects*

$$\Sigma = (G,\ \mathcal{H},\ U), \tag{3.4}$$

where

(1) *G is an admissible graph (called the* structure graph *of Σ),*

(2) *$\mathcal{H} = \{\mathcal{H}_p : p \in P(G)\}$ is a collection of finite-dimensional spaces $\mathcal{H}_p$ (called the* auxiliary state-spaces *of Σ), and*

(3) *U is a matrix of the form* (3.3) *(called the* connection matrix *or* colligation *of Σ).*

With any SNMLS we associate an i/s/o linear system with evolution along a free semigroup as follows. We denote by $\mathcal{F}_E$ the free semigroup generated by the edge set E. An element of $\mathcal{F}_E$ is then a word w of the form $w = e_N \cdots e_1$, where each e_r is an edge of G for $r = 1, \ldots, N$. We denote the empty word (consisting of no letters) by $\emptyset$. The semigroup operation is concatenation: if $w = e_N \cdots e_1$ and $w' = e'_{N'} \cdots e'_1$, then ww' is defined to be

$$ww' = e_N \cdots e_1 e'_{N'} \cdots e'_1.$$

Note that the empty word $\emptyset$ acts as the identity element for this semigroup. Equivalently, we may view $\mathcal{F}_E$ as a homogeneous tree of degree $\#E + 1$ (where $\#E$ is the number of edges of G) with root $\emptyset$; this point of view appears in the "multiscale system theory" in [14].

If $\Sigma = (G,\ \mathcal{H},\ U)$ is an SNMLS, we associate the system equations (with evolution along $\mathcal{F}_E$)

$$\Sigma\colon \left\{ \begin{array}{rcl} x_{\mathbf{s}(e)}(ew) & = & \Sigma_{s\in S} A_{\mathbf{r}(e),s} x_s(w) + B_{\mathbf{r}(e)} u(w), \\ x_{s'}(ew) & = & 0 \quad \text{if } s' \neq \mathbf{s}(e), \\ y(w) & = & \Sigma_{s\in S} C_s x_s(w) + Du(w). \end{array} \right. \tag{3.5}$$

Here the *state-vector* $x(w)$ at position w (for $w \in \mathcal{F}_E$) has the form of a column vector

$$x(w) = \mathrm{col}_{s\in S}\, x_s(w),$$

with column entries indexed by the source vertices $s \in S$ and with column entry $x_s(w) \in \mathcal{H}_{[s]}$ (thus $x(w) \in \oplus_{s\in S}\mathcal{H}_{[s]}$), while $u(w) \in \mathcal{U}$ denotes the *input* at position w and $y(w) \in \mathcal{Y}$ denotes the *output* at position w. Just as in the classical case, if we specify an initial condition $x(\emptyset) \in \oplus_{s\in S}\mathcal{H}_{[s]}$ and feed in an input string $\{u(w)\}_{w\in\mathcal{F}_E}$, then (3.5) enables us to recursively compute $x(w)$ for all $w \in \mathcal{F}_E \setminus \{\emptyset\}$ and $y(w)$ for all $w \in \mathcal{F}_E$.

As these systems include the full-structured case discussed in section 2.3 as a special case (see Example 3.10 below) where some redundancy occurs in the state-vector of a system trajectory (see (2.18)), in general some redundancy in the state-vector occurs for trajectories of a general SNMLS Σ as well. Indeed, the analogue of (2.19) for this more general setting is the following: *if $\{x(w)\}_{w\in\mathcal{F}_E} = \{\mathrm{col}_{s\in S}[x_s(w)]\}_{w\in\mathcal{F}_E}$ is the state trajectory solving the state-update equation in* (3.5) *for some choice of input signal $\{u(w)\}_{w\in\mathcal{F}_E}$, then necessarily, for each fixed $r \in R$ and $w \in \mathcal{F}_E$,*

$$x_s(e_{s,r}w) \textit{ is independent of } s \textit{ for all } s \textit{ with } [s] = [r]. \tag{3.6}$$

It will be convenient for purposes of the matrix manipulations to come that we maintain the form (3.5) of the system equations rather than rewriting them in a more economical form.

The solution of these recursions can be made more explicit as follows. Note first of all that a consequence of the system equations is that

$$x(ew) \in \mathcal{H}_{\mathbf{s}(e)} := \mathrm{col}_{s \in S}[\boldsymbol{\delta}_{s,\mathbf{s}(e)} \mathcal{H}_{[\mathbf{s}(e)]}] \quad \text{for all } e \in E \text{ and } w \in \mathcal{F}_E$$

(where $\boldsymbol{\delta}_{s,s'}$ is the Kronecker delta function). Given $x(\emptyset)$ and $\{u(w)\}_{w \in \mathcal{F}_E}$, we can solve the system equations (3.5) or (3.10) uniquely for $\{x(w)\}_{w \in \mathcal{F}_E \setminus \{\emptyset\}}$ and $\{y(w)\}_{w \in \mathcal{F}_E}$ as follows:

$$\begin{aligned} x_{\mathbf{s}(e_N)}(e_N \cdots e_1) &= \sum_{s \in S} A_{\mathbf{r}(e_N),\mathbf{s}(e_{N-1})} A_{\mathbf{r}(e_{N-1}),\mathbf{s}(e_{N-2})} \cdots A_{\mathbf{r}(e_1),s} x_s(\emptyset) \\ &\quad + \sum_{r=1}^{N} A_{\mathbf{r}(e_N),\mathbf{s}(e_{N-1})} \cdots A_{\mathbf{r}(e_{r+1}),\mathbf{s}(e_r)} B_{\mathbf{r}(e_r)} u(e_{r-1} \cdots e_1), \end{aligned} \tag{3.7}$$

where we interpret $u(e_{r-1} \cdots e_1)$ to be $u(\emptyset)$ when $r = 1$, and where we set

$$x_s(e_N e_{N-1} \cdots e_1) = 0 \quad \text{if } s \neq \mathbf{s}(e_N).$$

Also,

$$\begin{aligned} y(e_N \cdots e_1) &= \sum_{s \in S} C_{\mathbf{s}(e_N)} A_{\mathbf{r}(e_N),\mathbf{s}(e_{N-1})} A_{\mathbf{r}(e_{N-1}),\mathbf{s}(e_{N-2})} \cdots A_{\mathbf{r}(e_1),s} x_s(\emptyset) \\ &\quad + \sum_{r=1}^{N} C_{\mathbf{s}(e_N)} A_{\mathbf{r}(e_N),\mathbf{s}(e_{N-1})} \cdots A_{\mathbf{r}(e_{r+1}),\mathbf{s}(e_r)} B_{\mathbf{r}(e_r)} u(e_{r-1} \cdots e_1) + D u(e_N \cdots e_1). \end{aligned} \tag{3.8}$$

This formula must be interpreted appropriately for special cases. As examples, for the particular cases $r = 1$ and $r = N$ we have the interpretations

$$\begin{aligned} &A_{\mathbf{r}(e_N),\mathbf{s}(e_{N-1})} \cdots A_{\mathbf{r}(e_{r+1}),\mathbf{s}(e_r)} B_{\mathbf{r}(e_r)} u(e_{r-1} \cdots e_1)|_{r=1} \\ &\qquad = A_{\mathbf{r}(e_N),\mathbf{s}(e_{N-1})} \cdots A_{\mathbf{r}(e_2),\mathbf{s}(e_1)} B_{\mathbf{r}(e_1)} u(\emptyset), \\ &A_{\mathbf{r}(e_N),\mathbf{s}(e_{N-1})} \cdots A_{\mathbf{r}(e_{r+1}),\mathbf{s}(e_r)} B_{\mathbf{r}(e_r)} u(e_{r-1} \cdots e_1)|_{r=N} = B_{\mathbf{r}(e_N)} u(e_{N-1} \cdots e_1). \end{aligned}$$

If we set

$$\Delta_e = i_{\mathbf{s}(e)} A_{\mathbf{r}(e),\cdot} \colon \oplus_{s \in S} \mathcal{H}_{[s]} \to \oplus_{s \in S} \mathcal{H}_{[s]},$$

where i_s denotes the natural injection $h \mapsto \mathrm{col}_{s' \in S}[\delta_{s',s} h]$ of $\mathcal{H}_{[s]}$ into $\oplus_{s' \in S} \mathcal{H}_{[s']}$, and if we use our assumption that $x_{s'}(ew) = 0$ if $s' \neq \mathbf{s}(e)$, then (3.7) and (3.8) can be rewritten as

$$\begin{aligned} x(w) &= \Delta^w x(\emptyset) + \sum_{w',w'' \in \mathcal{F}_E, e \in E \colon w'ew''=w} \Delta^{w'} i_{\mathbf{s}(e)} B_{\mathbf{r}(e)} u(w''), \\ y(w) &= C\Delta^w x(\emptyset) + \sum_{w',w'' \in \mathcal{F}_E, e \in E \colon w'ew''=w} C\Delta^{w'} i_{\mathbf{s}(e)} B_{\mathbf{r}(e)} u(w'') + Du(w), \end{aligned} \tag{3.9}$$

where we use the noncommutative functional calculus

$$\Delta^v = \Delta_{e_N}\Delta_{e_{N-1}}\cdots\Delta_{e_1} \quad \text{if } v = e_N e_{N-1}\cdots e_1 \in \mathcal{F}_E, \qquad \Delta^{\emptyset} = I_{\mathcal{H}}.$$

The system equations (3.5) can also be written more compactly in operator-theoretic form as

$$\Sigma\colon \left\{ \begin{array}{rcl} x(ew) & = & I_{\Sigma,e}Ax(w) + I_{\Sigma,e}Bu(w), \\ y(w) & = & Cx(w) + Du(w), \end{array} \right. \tag{3.10}$$

where $I_{\Sigma;e}$ is a higher-multiplicity version of the coefficient matrices $I_{G,e}$ appearing in (3.2):

$$I_{\Sigma;e}\colon \oplus_{r\in R}\,\mathcal{H}_{[r]} \to \oplus_{s\in S}\mathcal{H}_{[s]}$$

with matrix entries $[I_{\Sigma;e}]_{s\in S, r\in R}$ given by

$$[I_{\Sigma;e}]_{s,r} = \begin{cases} I_{\mathcal{H}_{[\mathbf{s}(e)]}} = I_{\mathcal{H}_{[\mathbf{r}(e)]}} & \text{if } s = \mathbf{s}(e) \text{and } r = \mathbf{r}(e), \\ 0 & \text{otherwise.} \end{cases} \tag{3.11}$$

Also, just as in the classical case, it is convenient to introduce "frequency-domain" notation for explicit representation of system trajectories. For any linear space $\mathcal{H}$, we define the formal noncommutative Z-transform of a sequence of $\mathcal{H}$-valued functions as a formal power series in several noncommuting indeterminates $z = (z_e : e \in E)$ as follows:

$$\{h(w)\}_{w\in\mathcal{F}_E} \mapsto \widehat{h}(z) = \sum_{w\in\mathcal{F}_E} h(w)z^w, \tag{3.12}$$

where $z^{\emptyset} = 1$, $z^w = z_{e_N}z_{e_{N-1}}\cdots z_{e_1}$ if $w = e_N e_{N-1}\cdots e_1$. Then, applying the Z-transform to (3.10) gives

$$\sum_{w\in\mathcal{F}_E} x(ew)z^w = I_{\Sigma,e}A\widehat{x}(z) + I_{\Sigma,e}B\widehat{u}(z). \tag{3.13}$$

Multiply (3.13) on the left by z_e to get

$$\sum_{w\in\mathcal{F}_E} x(ew)z^{ew} = z_e I_{\Sigma,e}A\widehat{x}(z) + z_e I_{\Sigma,e}B\widehat{u}(z). \tag{3.14}$$

Summing (3.14) over all edges $e \in E$, we get

$$\sum_{e\in E}\sum_{w\in\mathcal{F}_E} x(ew)z^{ew} = Z_{\Sigma}(z)A\widehat{x}(z) + Z_{\Sigma}(z)B\widehat{u}(z), \tag{3.15}$$

where we have set

$$Z_{\Sigma}(z) = \sum_{e\in E} z_e I_{\Sigma,e}. \tag{3.16}$$

Note that the definition of the formal Z-transform yields

$$\sum_{e\in E}\sum_{w\in\mathcal{F}_E} x(ew)z^{ew} = \widehat{x}(z) - x(\emptyset).$$

Thus (3.15) becomes

$$\widehat{x}(z) = x(\emptyset) + Z_\Sigma(z)A\widehat{x}(z) + Z_\Sigma(z)B\widehat{u}(z). \tag{3.17}$$

Solving (3.17) for $\widehat{x}(z)$, we obtain

$$\widehat{x}(z) = (I - Z_\Sigma(z)A)^{-1}\, x(\emptyset) + (I - Z_\Sigma(z)A)^{-1}\, Z_\Sigma(z)B\widehat{u}(z). \tag{3.18}$$

Substitution of (3.17) into the formal Z-transform of the output equation of (3.10) then gives

$$\begin{aligned} \widehat{y}(z) &= C\widehat{x}(z) + D\widehat{u}(z) \\ &= C\,(I - Z_\Sigma(z)A)^{-1}\, x(\emptyset) + T_\Sigma(z)\widehat{u}(z), \end{aligned} \tag{3.19}$$

where we have set

$$T_\Sigma(z) = D + C(I - Z_\Sigma(z)A)^{-1}Z_\Sigma(z)B \tag{3.20}$$

equal to the *transfer function* of the SNMLS Σ, where the inverse is taken in the algebra $\mathcal{L}(\oplus_{s\in S}\mathcal{H}_{[s]})\langle\langle z\rangle\rangle$ of formal power series with operator coefficients in the noncommuting variables $z = (z_e : e \in E)$. We can write $T_\Sigma(z)$ explicitly as a formal power series in the form

$$\begin{aligned} T_\Sigma(z) = T_\emptyset + \sum_{N=1}^{\infty} \sum_{e_1,\dots,e_N\in E} & C_{\mathbf{s}(e_N)} \\ & \cdot A_{\mathbf{r}(e_N),\mathbf{s}(e_{N-1})} \cdots A_{\mathbf{r}(e_2),\mathbf{s}(e_1)} B_{\mathbf{r}(e_1)} z_{e_N} z_{e_{N-1}} \cdots z_{e_2} z_{e_1}. \end{aligned} \tag{3.21}$$

Example 3.8 (noncommutative Fornasini–Marchesini system). Here we continue Example 3.3. As the structure graph G is connected in this case, we assume that we are given a single finite-dimensional linear space $\mathcal{H}$ together with an input-space $\mathcal{U}$ and an output-space $\mathcal{Y}$. Then the structure matrix (3.16) $Z_{FM}(z)$ is the row matrix

$$Z_{\Sigma^{FM}}(z) = \sum_{j=1}^{d} z_j I_{\Sigma^{FM},j} = \begin{bmatrix} z_1 I_{\mathcal{H}} & \dots & z_d I_{\mathcal{H}} \end{bmatrix} =: Z_{\mathrm{row}}(z) \otimes I_{\mathcal{H}},$$

where

$$I_{\Sigma^{FM},j} = \begin{bmatrix} 0 & \cdots & 0 & I_{\mathcal{H}} & 0 & \cdots & 0 \end{bmatrix}$$

(with nonzero entry in the jth column), and the connection matrix U^{FM} has the form

$$U^{FM} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} \mathrm{col}_{j=1,\dots,d}[A_j] & \mathrm{col}_{j=1,\dots,d}[B_j] \\ C & D \end{bmatrix} : \begin{bmatrix} \mathcal{H} \\ \mathcal{U} \end{bmatrix} \to \begin{bmatrix} \oplus_{j=1}^d \mathcal{H} \\ \mathcal{Y} \end{bmatrix}. \tag{3.22}$$

Thus, $I_{\Sigma^{FM},j}A = A_j$, $I_{\Sigma^{FM},j}B = B_j$, and therefore the associated noncommutative Fornasini–Marchesini system is given by

$$\Sigma^{FM}: \begin{cases} x(1w) = A_1 x(w) + B_1 u(w), \\ \qquad\vdots \\ x(dw) = A_d x(w) + B_d u(w), \\ y(w) \;\;= Cx(w) + Du(w), \end{cases} \tag{3.23}$$

i.e., we are in the setting of the noncommutative Fornasini–Marchesini systems discussed in section 2.1. Since in this case $Z_{\Sigma^{FM}}(z)A = \sum_{i=1}^{d} z_i A_i$ and similarly $Z_{\Sigma^{FM}}(z)B = \sum_{i=1}^{d} z_i B_i$, the transfer function $T_{\Sigma^{FM}}(z)$ in (3.20) for the noncommutative Fornasini–Marchesini system has the form given in (2.4).

We remark that any SNMLS can be embedded into a noncommutative Fornasini–Marchesini system having a certain special form as follows. Given a general SNMLS $\Sigma = (G, \mathcal{H}, U)$, we associate a Fornasini–Marchesini system

$$\Sigma^{FM} = (G^{FM}, \mathcal{H}^{FM}, U^{FM})$$

as follows. We let G^{FM} be the unique Fornasini–Marchesini graph having the same edge set as G: $E^{FM} = E$. Thus we take the source-vertex set S^{FM} to be $S^{FM} = \{1\}$, and the range-vertex set R^{FM} to be $R^{FM} = E$, with associated source and range vertex maps $\mathbf{s}^{FM}$ and $\mathbf{r}^{FM}$ given by $\mathbf{s}^{FM}(e) = 1$ and $\mathbf{r}^{FM}(e) = e$ for $e \in E$. We let $\mathcal{H}^{FM} = \oplus_{s \in S}\mathcal{H}$, and we define the connection matrix $U^{FM} = \left[\begin{smallmatrix} A^{FM} & B^{FM} \\ C^{FM} & D^{FM}\end{smallmatrix}\right]$ by

$$\begin{bmatrix} A^{FM} & B^{FM} \\ C^{FM} & D^{FM} \end{bmatrix} = \begin{bmatrix} \mathrm{col}_{e \in E}[A_e^{FM}] & \mathrm{col}_{e \in E}[B_e^{FM}] \\ C & D \end{bmatrix} : \begin{bmatrix} \mathcal{H}^{FM} \\ \mathcal{U} \end{bmatrix} \to \begin{bmatrix} \oplus_{e \in E}\mathcal{H}^{FM} \\ \mathcal{Y} \end{bmatrix}$$

and by

$$\begin{aligned} A_e^{FM} &= i_{\mathbf{s}(e)} A_{\mathbf{r}(e),\cdot} \colon \mathcal{H}^{FM} \to \mathcal{H}^{FM}, \\ B_e^{FM} &= i_{\mathbf{s}(e)} B_{\mathbf{r}(e)} \colon \mathcal{U} \to \mathcal{H}^{FM}, \\ C^{FM} &= C \colon \mathcal{H}^{FM} \to \mathcal{Y}, \\ D^{FM} &= D \colon \mathcal{U} \to \mathcal{Y}, \end{aligned}$$

where $i_{\mathbf{s}(e)} \colon \mathcal{H}_{[s]} \to \mathrm{col}_{s' \in S}\, \mathcal{H}_{[s']}$ is the natural injection $h \mapsto \mathrm{col}_{s' \in S}\, \delta_{s',s} h$. A consequence of formula (3.9) is that Σ *and* Σ^{FM} *associated in this way have the same system trajectories.*

Example 3.9 (noncommutative Givone–Roesser system). Here we continue Example 3.4. In this case the structure graph G has d connected components, so we assume that we give d auxiliary state-spaces $\mathcal{H}_1, \dots, \mathcal{H}_d$. The structure matrix (3.16) then has the diagonal form

$$Z_{\Sigma^{GR}}(z) = \sum_{j=1}^{d} I_{\Sigma^{GR},j} z_j = \begin{bmatrix} z_1 I_{\mathcal{H}_1} & & \\ & \ddots & \\ & & z_d I_{\mathcal{H}_d} \end{bmatrix} =: Z_{\mathrm{diag}}(z) \otimes I_{\mathcal{H}},$$

where $I_{\Sigma^{GR},j}$ is a $d \times d$ matrix with zero entries except at the (j,j)th entry, where $[I_{\Sigma^{GR},j}]_{j,j} = I_{\mathcal{H}_j}$, and the connecting matrix U^{GR} is of the form

(3.24)
$$U^{GR} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} [A_{j,i}]_{j,i=1,\dots,d} & \mathrm{col}_{j=1,\dots,d}[B_j] \\ \mathrm{row}_{i=1,\dots,d}[C_i] & D \end{bmatrix} : \begin{bmatrix} \oplus_{i=1}^{d}\mathcal{H}_i \\ \mathcal{U} \end{bmatrix} \to \begin{bmatrix} \oplus_{j=1}^{d}\mathcal{H}_j \\ \mathcal{Y} \end{bmatrix}.$$

Thus,

$$(3.25) \qquad I_{\Sigma^{GR},i}A = \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & & \vdots \\ A_{i,1} & \cdots & A_{i,d} \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{bmatrix} \qquad \text{and} \qquad I_{\Sigma^{GR},i}B = \begin{bmatrix} 0 \\ \vdots \\ B_i \\ \vdots \\ 0 \end{bmatrix}$$

(where the nonzero row is row i in both expressions), and therefore the noncommutative Givone–Roesser system is given by

$$\Sigma^{GR}\colon \begin{cases} x_i(iw) &= \sum_{i' \in S} A_{i,i'} x_{i'}(w) + B_i u(w) \quad \text{for } e \in E, \\ x_{i''}(iw) &= 0 \quad \text{if } i'' \neq i, \\ y(w) &= \sum_{i'=1}^{d} C_{i'} x_{i'}(w) + Du(w), \end{cases} \tag{3.26}$$

as stated in section 2.2. Here $x_i(iw) \in \mathcal{H}_i$ for $i = 1, \dots, d$. The transfer function $T_{\Sigma^{GR}}(z)$ for the noncommutative Givone–Roesser system then has the form as given in (2.12).

Example 3.10 (noncommutative full-structured system). Here we continue Example 3.5. We assume that the structure matrix G has the form G^{full}, as in Example 3.5. As the structure graph G^{full} has only one connected component, we need specify only one auxiliary state-space $\mathcal{H}$ for an SNMLS $\Sigma = (G^{\text{full}}, \mathcal{H}, U)$ with structure graph G^{full}. The structure matrix (3.16) is the full-block operator matrix with each matrix entry containing one of the variables

$$Z_{\Sigma^{\text{full}}}(z) = \sum_{i=1}^{n} \sum_{j=1}^{m} I_{\Sigma^{\text{full}},(i,j)} z_{i,j} = \begin{bmatrix} z_{1,1} I_{\mathcal{H}} & \dots & z_{1,m} I_{\mathcal{H}} \\ \vdots & & \vdots \\ z_{n,1} I_{\mathcal{H}} & \dots & z_{n,m} I_{\mathcal{H}} \end{bmatrix} =: Z_{\text{full}}(z) \otimes I_{\mathcal{H}},$$

where $I_{\Sigma^{\text{full}},(i,j)}$ is an $n \times m$ matrix with zero entries except at the (i,j)th entry, where $[I_{\Sigma^{\text{full}},(i,j)}]_{i,j} = I_{\mathcal{H}}$. The connecting operator U^{full} in this case is given by

$$U^{\text{full}} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \colon \begin{bmatrix} \oplus_1^n \mathcal{H} \\ \mathcal{U} \end{bmatrix} \to \begin{bmatrix} \oplus_1^m \mathcal{H} \\ \mathcal{Y} \end{bmatrix},$$

where

$$A = \begin{bmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & & \vdots \\ A_{m,1} & \cdots & A_{m,n} \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ \vdots \\ B_m \end{bmatrix}, \quad C = \begin{bmatrix} C_1 & \cdots & C_n \end{bmatrix},$$

and the system equations (3.5) assume the form

$$\Sigma^{\text{full}}\colon \begin{cases} x_i((i,j) \cdot w) &= \sum_{i'=1}^{n} A_{j,i'} x_{i'}(w) + B_j u(w), \\ x_{i''}((i,j) \cdot w) &= 0 \quad \text{if } i'' \neq i, \\ y(w) &= \sum_{i'=1}^{n} C_{i'} x_{i'}(w) + Du(w), \end{cases} \tag{3.27}$$

and we are in the setting of the noncommutative full-structured systems discussed in section 2.3. The transfer function $T_{\Sigma^{\text{full}}}(z)$ for the full-block operator matrix case then has the form as in (2.21).

Example 3.11 (the general SNMLS system). Here we continue Example 3.6. Suppose that the admissible graph G is the union of complete bipartite graphs G_k with source-vertex set $S_k = \{(k,i)\colon 1 \leq i \leq n_k\}$, range-vertex set $R_k = \{(k,j)\colon 1 \leq j \leq m_k\}$, and edge set $E_k = \{(k,i,j)\colon 1 \leq i \leq n_k; 1 \leq j \leq m_k\}$ for $k = 1, \dots, K$. Note that $k = 1, \dots, K$ labels the set P of path-connected components of G. Let $\mathcal{H} = \{\mathcal{H}_k\colon k = 1, \dots, K\}$ denote a specification of a finite-dimensional linear space for each path-connected component $k = 1, \dots, K$, and suppose that $\Sigma = (G, \mathcal{H}, U)$ is an SNMLS with structure graph G. Then the connection matrix U has the form

$$U = \begin{bmatrix} [A_{k',k}] & [B_{k'}] \\ [C_k] & D \end{bmatrix} \colon \begin{bmatrix} \oplus_{k=1}^{K} [\oplus_{i=1}^{n_k} \mathcal{H}_k] \\ \mathcal{U} \end{bmatrix} \to \begin{bmatrix} \oplus_{k'=1}^{K} [\oplus_{j=1}^{m_{k'}} \mathcal{H}_{k'}] \\ \mathcal{Y} \end{bmatrix},$$

where each $A_{k',k}$, $B_{k'}$, and C_k in turn has the form

$$\begin{aligned}
&A_{k',k} = [A_{k',k;j,i}]_{j=1,\dots,m_{k'};i=1,\dots,n_k}, \quad \text{where } A_{k',k;j,i}\colon \mathcal{H}_k \to \mathcal{H}_{k'},\\
&B_{k'} = \mathrm{col}_{j=1,\dots,m_{k'}}[B_{k',j}], \quad \text{where } B_{k',j}\colon \mathcal{U} \to \mathcal{H}_{k'},\\
&C_k = \mathrm{row}_{i=1,\dots,n_k}[C_{k,i}], \quad \text{where } C_{k,i}\colon \mathcal{H}_k \to \mathcal{Y}.
\end{aligned}$$

The structure matrix $Z_\Sigma(z)$ has the block-diagonal form

$$Z_\Sigma(z) = \begin{bmatrix} Z_{\mathrm{full},1}(z^1)\otimes I_{\mathcal{H}_1} & & \\ & \ddots & \\ & & Z_{\mathrm{full},K}(z^K)\otimes I_{\mathcal{H}_K} \end{bmatrix},$$

where z^k is the collection of variables $z^k = \{z_{k,i,j} : i = 1,\dots,n_k; j = 1,\dots,m_k\}$ and each $Z_{\mathrm{full},k}(z^k)\otimes I_{\mathcal{H}_k}$ is a full-block structure matrix (of block size $n_k \times m_k$), as in Example 3.10. While the structure matrix splits as the direct sum, the system trajectories for the whole system Σ in general can be quite complicated since there is no corresponding splitting for the A matrix generating the system dynamics.

If one substitutes general noncommuting operators $\delta = (\delta_{k,i,j}\colon k = 1,\dots,K; i = 1,\dots,n_k; j = 1,\dots,m_k)$ for the noncommuting formal variables $z_{k,i,j}$, then $Z_\Sigma(\delta)$ is the most general structure matrix coming up in μ-synthesis analysis (see [33]). Part of the advantage of the notion of SNMLS introduced here is the setting thereby given for proving results in the theory of μ-synthesis in a unified way for a general structure. We refer to [6] for further details.

4. System operations: Cascade/parallel connection and inversion. Suppose that we are given two SNMLSs

$$\Sigma'' = (G, \mathcal{H}'', U''), \qquad \Sigma' = (G, \mathcal{H}', U')$$

with the same structure graph G and with connection matrices

$$\begin{aligned}
U'' &= \begin{bmatrix} A'' & B'' \\ C'' & D'' \end{bmatrix} : \begin{bmatrix} \mathrm{col}_{s\in S}\,\mathcal{H}''_{[s]} \\ \mathcal{U}'' \end{bmatrix} \to \begin{bmatrix} \mathrm{col}_{r\in R}\,\mathcal{H}''_{[r]} \\ \mathcal{Y}'' \end{bmatrix},\\
U' &= \begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix} : \begin{bmatrix} \mathrm{col}_{s\in S}\,\mathcal{H}'_{[s]} \\ \mathcal{U}' \end{bmatrix} \to \begin{bmatrix} \mathrm{col}_{r\in R}\,\mathcal{H}'_{[r]} \\ \mathcal{Y}' \end{bmatrix},
\end{aligned}$$

with the property that the output-space for U' coincides with the input-space for U'':

$$\mathcal{Y}' = \mathcal{U}''.$$

We then define the *cascade connection* $\Sigma = \Sigma''\circ\Sigma'$ of Σ'' with Σ' to be the SNMLS $\Sigma = (G, \mathcal{H}, U)$ with auxiliary state-spaces $\mathcal{H}_p$ given by $\mathcal{H}_p = \left[\begin{smallmatrix} \mathcal{H}''_p \\ \mathcal{H}'_p \end{smallmatrix}\right]$ and with colligation U given by

$$U = \begin{bmatrix} A & B \\ C & D \end{bmatrix} := \begin{bmatrix} A'' & B''C' & B''D' \\ 0 & A' & B' \\ C'' & D''C' & D''D' \end{bmatrix} : \begin{bmatrix} \mathrm{col}_{s\in S}\,\mathcal{H}''_{[s]} \\ \mathrm{col}_{s\in S}\,\mathcal{H}'_{[s]} \\ \mathcal{U}' \end{bmatrix} \to \begin{bmatrix} \mathrm{col}_{r\in R}\,\mathcal{H}''_{[r]} \\ \mathrm{col}_{r\in R}\,\mathcal{H}'_{[r]} \\ \mathcal{Y}'' \end{bmatrix}.$$

Here we have identified the space $\mathrm{col}_{s\in S}\left[\begin{smallmatrix} \mathcal{H}''_{[s]} \\ \mathcal{H}'_{[s]} \end{smallmatrix}\right]$ with $\left[\begin{smallmatrix} \mathrm{col}_{s\in S}\,\mathcal{H}''_{[s]} \\ \mathrm{col}_{s\in S}\,\mathcal{H}'_{[s]} \end{smallmatrix}\right]$ as well as $\mathrm{col}_{r\in R}\left[\begin{smallmatrix} \mathcal{H}''_{[r]} \\ \mathcal{H}'_{[r]} \end{smallmatrix}\right]$ with $\left[\begin{smallmatrix} \mathrm{col}_{r\in R}\,\mathcal{H}''_{[r]} \\ \mathrm{col}_{r\in R}\,\mathcal{H}'_{[r]} \end{smallmatrix}\right]$ in the natural way. In more detail, the colligation coefficients A, B,

C, D are given by

$$A_{r,s} = \begin{bmatrix} A''_{r,s} & B''_r C'_s \\ 0 & A'_{r,s} \end{bmatrix} : \begin{bmatrix} \mathcal{H}''_{[s]} \\ \mathcal{H}'_{[s]} \end{bmatrix} \to \begin{bmatrix} \mathcal{H}''_{[r]} \\ \mathcal{H}'_{[r]} \end{bmatrix}, \qquad B_r = \begin{bmatrix} B''_r D' \\ B'_r \end{bmatrix} : \mathcal{U}' \to \begin{bmatrix} \mathcal{H}''_{[r]} \\ \mathcal{H}'_{[r]} \end{bmatrix},$$

$$C_s = \begin{bmatrix} C''_s & D'' C'_s \end{bmatrix} : \begin{bmatrix} \mathcal{H}''_{[s]} \\ \mathcal{H}'_{[s]} \end{bmatrix} \to \mathcal{Y}'', \qquad D = D'' D' : \mathcal{U}' \to \mathcal{Y}''.$$

We note that the cascade connection $\Sigma = \Sigma'' \circ \Sigma'$ has the following interpretation. Suppose that we are given an initial condition $x'(\emptyset) = x'_0$ and an input string $\{u'(w)\}_{w \in \mathcal{F}_E}$ to generate a trajectory $\{u'(w), x'(w), y'(w)\}_{w \in \mathcal{F}_E}$ of Σ' via the system equations

$$\Sigma' \colon \left\{ \begin{array}{rcl} x'_{\mathbf{s}(e)}(ew) & = & \Sigma_{s \in S} A'_{\mathbf{r}(e),s} x'_s(w) + B'_{\mathbf{r}(e)} u'(w), \\ x'_{s'}(ew) & = & 0 \quad \text{if } s' \neq \mathbf{s}(e), \\ y'(w) & = & \Sigma_{s \in S} C'_s x'_s(w) + D' u'(w). \end{array} \right. \tag{4.1}$$

We then let $x''(\emptyset) = x''_0 \in \mathcal{H}''$ be arbitrary and set $u''(w) = y'(w)$ to generate a system trajectory $\{u''(w), x''(w), y''(w)\}_{w \in \mathcal{F}_E}$ of Σ'', via the system equations

$$\Sigma'' \colon \left\{ \begin{array}{rcl} x''_{\mathbf{s}(e)}(ew) & = & \Sigma_{s \in S} A''_{\mathbf{r}(e),s} x''_s(w) + B''_{\mathbf{r}(e)} u''(w), \\ x''_{s'}(ew) & = & 0 \quad \text{if } s' \neq \mathbf{s}(e), \\ y''(w) & = & \Sigma_{s \in S} C''_s x''_s(w) + D'' u''(w). \end{array} \right. \tag{4.2}$$

The resulting triple $\left\{u'(w), \left[\begin{smallmatrix} x''(w) \\ x'(w) \end{smallmatrix}\right], y''(w)\right\}_{w \in \mathcal{F}_E}$ then is a system trajectory of $\Sigma = \Sigma'' \circ \Sigma'$, and every system trajectory of $\Sigma'' \circ \Sigma'$ arises in this way.

The main result concerning cascade connection is that this is the state-space operation corresponding to multiplication of the corresponding transfer functions.

THEOREM 4.1. *Let Σ'' and Σ' be SNMLSs for which the cascade connection $\Sigma := \Sigma'' \circ \Sigma'$ is defined as above. Then the transfer function $T_\Sigma(z)$ for Σ is the product of the transfer functions $T_{\Sigma''}(z)$ and $T_{\Sigma'}(z)$ for Σ'' and Σ':*

$$T_{\Sigma'' \circ \Sigma'}(z) = T_{\Sigma''}(z) \cdot T_{\Sigma'}(z). \tag{4.3}$$

Proof. We have seen (see (3.19)) that the transfer function $T_\Sigma(z)$ is characterized by the property that

$$\widehat{y}(z) = T_\Sigma(z) \widehat{u}(z)$$

whenever $\{u(w), x(w), y(w)\}_{w \in \mathcal{F}_E}$ is a trajectory of Σ with $x(\emptyset) = 0$. By the interpretation for the cascade connection $\Sigma'' \circ \Sigma'$ given in the preceding paragraph, we know that $\{u(w), x(w), y(w)\}_{w \in \mathcal{F}_E}$ has the form $\left\{u'(w), \left[\begin{smallmatrix} x''(w) \\ x'(w) \end{smallmatrix}\right], y''(w)\right\}_{w \in \mathcal{F}_E}$, where

$$\{u'(w), x'(w), y'(w)\}_{w \in \mathcal{F}_E}$$

is a trajectory of Σ' with $x'(\emptyset) = 0$, where $\{u''(w), x''(w), y''(w)\}_{w \in \mathcal{F}_E}$ is a trajectory of Σ'' with $x''(\emptyset) = 0$, and where we impose the interconnection law $y'(w) = u''(w)$. It therefore follows that

$$\begin{aligned} \widehat{y}(z) = \widehat{y''}(z) &= T_{\Sigma''}(z) \widehat{y'}(z) \\ &= T_{\Sigma''}(z) \left(T_{\Sigma'}(z) \widehat{u'}(z) \right) \\ &= \left(T_{\Sigma''}(z) T_{\Sigma'}(z) \right) \widehat{u}(z), \end{aligned}$$

and we conclude that it must be the case that $T_\Sigma(z) = T_{\Sigma''}(z)T_{\Sigma'}(z)$, as asserted. Of course the result can also be verified by direct computation using the formula (3.20) for the transfer function in terms of A, B, C, D. $\square$

We next define the *parallel connection* of two SNMLSs as follows. We suppose that we are given two SNMLSs

$$\Sigma'' = (G, \mathcal{H}'', U''), \qquad \Sigma' = (G, \mathcal{H}', U')$$

with the same structure graph G and with the same input-space $\mathcal{U}$ and the same output-space $\mathcal{Y}$:

$$U'' = \begin{bmatrix} A'' & B'' \\ C'' & D'' \end{bmatrix} : \begin{bmatrix} \mathrm{col}_{s\in S}\, \mathcal{H}''_{[s]} \\ \mathcal{U} \end{bmatrix} \to \begin{bmatrix} \mathrm{col}_{r\in R}\, \mathcal{H}''_{[r]} \\ \mathcal{Y} \end{bmatrix},$$
$$U' = \begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix} : \begin{bmatrix} \mathrm{col}_{s\in S}\, \mathcal{H}'_{[s]} \\ \mathcal{U} \end{bmatrix} \to \begin{bmatrix} \mathrm{col}_{r\in R}\, \mathcal{H}'_{[r]} \\ \mathcal{Y} \end{bmatrix}.$$

We then define the *parallel sum* $\Sigma = \Sigma''[+]\Sigma'$ of Σ'' and Σ' to be $\Sigma = (G, \mathcal{H}, U)$ with auxiliary state-spaces $\mathcal{H}_p$ again equal to the direct sums $\mathcal{H}_p = \left[\begin{smallmatrix} \mathcal{H}''_p \\ \mathcal{H}'_p \end{smallmatrix}\right]$ and with connection matrix U given by

$$U = \begin{bmatrix} A'' & 0 & B'' \\ 0 & A' & B' \\ C'' & C' & D'' + D' \end{bmatrix} : \begin{bmatrix} \mathrm{col}_{s\in S}\, \mathcal{H}''_{[s]} \\ \mathrm{col}_{s\in S}\, \mathcal{H}'_{[s]} \\ \mathcal{U} \end{bmatrix} \to \begin{bmatrix} \mathrm{col}_{r\in R}\, \mathcal{H}''_{[r]} \\ \mathrm{col}_{r\in R}\, \mathcal{H}'_{[r]} \\ \mathcal{Y} \end{bmatrix}.$$

Here again we identify $\mathrm{col}_{s\in S} \left[\begin{smallmatrix} \mathcal{H}''_{[s]} \\ \mathcal{H}'_{[s]} \end{smallmatrix}\right]$ with $\left[\begin{smallmatrix} \mathrm{col}_{s\in S}\, \mathcal{H}''_{[s]} \\ \mathrm{col}_{s\in S}\, \mathcal{H}'_{[s]} \end{smallmatrix}\right]$ and $\mathrm{col}_{r\in R} \left[\begin{smallmatrix} \mathcal{H}''_{[r]} \\ \mathcal{H}'_{[r]} \end{smallmatrix}\right]$ with $\left[\begin{smallmatrix} \mathrm{col}_{r\in R}\, \mathcal{H}''_{[r]} \\ \mathrm{col}_{r\in R}\, \mathcal{H}'_{[r]} \end{smallmatrix}\right]$ in the natural way. In this case the physical interpretation is that we feed an initial state $x'(\emptyset) = x'_0 \in \mathrm{col}_{s\in S}\, \mathcal{H}'_{[s]}$ and an input string $\{u(w)\}_{w\in\mathcal{F}_E}$ into Σ' to generate a trajectory $\{u(w), x'(w), y'(w)\}_{w\in\mathcal{F}_E}$ of Σ' along with an initial state $x''(\emptyset) = x''_0 \in \mathrm{col}_{s\in S}\, \mathcal{H}''_{[s]}$ and the same input string $(u(w))_{w\in\mathcal{F}_E}$ to generate a trajectory $\{u(w), x''(w), y''(w)\}$ of Σ''. We then set $y(w) = y'(w) + y''(w)$. Then $\left\{u(w), \left[\begin{smallmatrix} x''(w) \\ x'(w) \end{smallmatrix}\right], y(w)\right\}_{w\in\mathcal{F}_E}$ is a system trajectory of $\Sigma = \Sigma''[+]\Sigma'$, and every trajectory of $\Sigma''[+]\Sigma'$ is of this form. With this system interpretation, the following result follows easily along the same lines as the proof of Theorem 4.1.

THEOREM 4.2. *Suppose that Σ'' and Σ' are two SNMLSs for which the parallel sum $\Sigma := \Sigma''[+]\Sigma'$ is defined as above. Then the transfer function $T_\Sigma(z)$ for Σ is the sum of the transfer functions $T_{\Sigma''}(z)$ and $T_{\Sigma'}(z)$ for Σ'' and Σ':*

$$T_{\Sigma''[+]\Sigma'}(z) = T_{\Sigma''}(z) + T_{\Sigma'}(z). \tag{4.4}$$

Our final system operation is inversion. We suppose that we are given an SNMLS $\Sigma = (G, \mathcal{H}, U)$ for which the colligation

$$U = \begin{bmatrix} A & B \\ C & D \end{bmatrix} : \begin{bmatrix} \mathrm{col}_{s\in S}\, \mathcal{H}_{[s]} \\ \mathcal{U} \end{bmatrix} \to \begin{bmatrix} \mathrm{col}_{r\in R}\, \mathcal{H}_{[r]} \\ \mathcal{Y} \end{bmatrix}$$

is such that the feedthrough operator $D\colon \mathcal{U} \to \mathcal{Y}$ is invertible. We then define the inverse colligation

$$\Sigma^\times = (G, \mathcal{H}, U^\times)$$

with the same structure graph G and auxiliary state-spaces $\mathcal{H} = \{\mathcal{H}_p : p \in P(G)\}$ but with colligation $U^\times$ given by

$$U^\times = \begin{bmatrix} A^\times & B^\times \\ C^\times & D^\times \end{bmatrix} = \begin{bmatrix} A - BD^{-1}C & BD^{-1} \\ -D^{-1}C & D^{-1} \end{bmatrix} : \begin{bmatrix} \mathrm{col}_{s \in S} \mathcal{H}_{[s]} \\ \mathcal{Y} \end{bmatrix} \to \begin{bmatrix} \mathrm{col}_{r \in R} \mathcal{H}_{[r]} \\ \mathcal{U} \end{bmatrix}.$$

The point here is that $\{y(w), x(w), u(w)\}_{w \in \mathcal{F}_E}$ is a system trajectory of $U^\times$ if and only if $\{u(w), x(w), y(w)\}_{w \in \mathcal{F}_E}$ is a system trajectory of U; i.e., system-inversion amounts to interchange of inputs and outputs. If we then work with system trajectories having $x(\emptyset) = 0$, we see that $\widehat{y}(z) = T_\Sigma(z)\widehat{u}(z)$ is equivalent to $\widehat{u}(z) = T_{\Sigma^\times}(z)\widehat{y}(z)$. Of course it is also possible to verify the formal power series identities

$$T_{\Sigma^\times}(z) \cdot T_\Sigma(z) = I_{\mathcal{U}}, \qquad T_\Sigma(z) \cdot T_{\Sigma^\times}(z) = I_{\mathcal{Y}}$$

directly by use of the explicit formula (3.20) for the transfer function. In any case, we record this observation in the following theorem.

THEOREM 4.3. *Suppose that $\Sigma = (G, \mathcal{H}, U)$ is an SNMLS with colligation*

$$U = \begin{bmatrix} A & B \\ C & D \end{bmatrix} : \begin{bmatrix} \mathrm{col}_{s \in S} \mathcal{H}_{[s]} \\ \mathcal{U} \end{bmatrix} \to \begin{bmatrix} \mathrm{col}_{r \in R} \mathcal{H}_{[r]} \\ \mathcal{Y} \end{bmatrix}$$

having invertible feedthrough operator $D : \mathcal{U} \to \mathcal{Y}$. Then

$$T_\Sigma(z) = D + C(I - Z_\Sigma(z)A)^{-1} Z_\Sigma(z) B$$

is invertible in the space $\mathcal{L}(\mathcal{U}, \mathcal{Y})\langle\langle z \rangle\rangle$ (formal power series in the noncommuting variables $z = (z_e)_{e \in E}$ with coefficients in the space $\mathcal{L}(\mathcal{U}, \mathcal{Y})$ of operators from $\mathcal{U}$ to $\mathcal{Y}$), with inverse $T_\Sigma^{-1}(z) \in \mathcal{L}(\mathcal{Y}, \mathcal{U})\langle\langle z \rangle\rangle$ given by

$$T_\Sigma^{-1}(z) = T_{\Sigma^\times}(z) := D^{-1} - D^{-1}C(I - Z_\Sigma(z)[A - BD^{-1}C])^{-1} Z_\Sigma(z) BD^{-1}. \tag{4.5}$$

Remark 4.4. For the classical case, there exists a converse to Theorem 4.1; i.e., given Σ, it is possible to describe geometrically all possible nontrivial decompositions of Σ as $\Sigma = \Sigma'' \circ \Sigma'$ (see, e.g., [9]). These results can also be extended to more general linear-fractional decompositions (see [29] and [18]). Presumably such results can also be worked out for SNMLSs, but we leave this project to another occasion.

5. Reachability and controllability. The building blocks for reachability and controllability operators are certain operators $\Psi_w : \mathcal{U} \to \mathcal{H}_s$ associated with any word w,

$$\Psi_w = A_{\mathbf{r}(e_N), \mathbf{s}(e_{N-1})} \cdots A_{\mathbf{r}(e_2), \mathbf{s}(e_1)} B_{\mathbf{r}(e_1)} \quad \text{if } w = e_N \cdots e_1. \tag{5.1}$$

Note that the word $w = e_N e_{N-1} \cdots e_2 e_1$ can be written, for each $r = 1, 2, \dots, N$, as the concatenation

$$w = w'_r w''_{r-1},$$

where we have set

$$w'_r = e_N e_{N-1} \cdots e_r \text{ for } r = 1, \dots, N, \quad w''_{r-1} = e_{r-1} \cdots e_1 \text{ for } r = 2, \dots, N, \quad w''_0 = \emptyset.$$

From formula (3.7), we see that the $\mathbf{s}(e_N)$th component of the state trajectory at location $w = e_N \cdots e_1$ for Σ generated by input string $\{u(v)\}_{v\in\mathcal{F}_E}$ with zero initial condition $x(\emptyset) = 0$ is given by

$$\begin{aligned} x_{\mathbf{s}(e_N)}(w) &= \sum_{r=1}^{N} A_{\mathbf{r}(e_N),\mathbf{s}(e_{N-1})} \cdots A_{\mathbf{r}(e_{r+1}),\mathbf{s}(e_r)} B_{\mathbf{r}(e_r)} u(w''_{r-1}) \\ &= \sum_{r=1}^{N} \Psi_{w'_r} u(w''_{r-1}). \end{aligned}$$

Just as in the classical case, the indexing is a little more natural if we consider controllability operators instead. Up to this point we have been considering the system evolution only on the "future time" $\mathcal{T}_{\text{future}} := \mathcal{F}_E$. We now define the "past time" $\mathcal{T}_{\text{past}}$ to be a second copy of $\mathcal{F}_E$ but with the empty word deleted: $\mathcal{T}_{\text{past}} := \mathcal{F}_E \setminus \{\emptyset\}$. We emphasize that $\mathcal{T}_{\text{future}}$ and $\mathcal{T}_{\text{past}}$ are considered to be disjoint sets; given a nonempty word w in $\mathcal{F}_E$, we will specify in the particular context whether it is to be considered as an element of $\mathcal{T}_{\text{future}}$ or of $\mathcal{T}_{\text{past}}$.

Let us now introduce the system evolution on the past, which is given by

$$\text{(5.2)} \quad \Sigma_{\text{past}}\colon \begin{cases} x_s(w) = \sum_{e\colon \mathbf{s}(e)=s} \sum_{s'\in S} A_{\mathbf{r}(e),s'} x_{s'}(we) + \sum_{e\colon \mathbf{s}(e)=s} B_{\mathbf{r}(e)} u(we), \\ y(w) \ = \sum_{s\in S} C_s x_s(w) + Du(w), \end{cases}$$

or, in aggregate form,

$$\text{(5.3)} \qquad \Sigma_{\text{past}}\colon \begin{cases} x(w) = \sum_{e\in E} I_{\Sigma,e} A x(we) + \sum_{e\in E} I_{\Sigma,e} B u(we), \\ y(w) = Cx(w) + Du(w). \end{cases}$$

This evolution can actually be derived from the forward evolution by doing the change of "time" variable $w''_{r-1} \mapsto w'_r$ along each finite path w (where the initial segment w''_{r-1} is viewed as a point in the future $\mathcal{T}_{\text{future}}$, while the corresponding final segment w'_r is viewed as a position in the past $\mathcal{T}_{\text{past}}$), and then taking a span over paths as was done above. In this way, the span of all vectors generated at some finite position in the future from zero initial condition on the state at $\emptyset$ over all possible input strings on $\mathcal{T}_{\text{future}}$ is transformed into the set of all possible states achieved at time $\emptyset$ (the final point for the past) over all possible finitely supported input strings on the past with zero state initialization in the distant past.

More precisely, fix a finite word $w = e_N \cdots e_1$, and assume that we run the system in the past $\mathcal{T}_{\text{past}}$ using the system equations (5.2) or (5.3) under the assumption that $x(v) = 0$ for all $v \in \mathcal{T}_{\text{past}}$ with $|v| \geq N$, where N is an arbitrary length, and that $u(v) = 0$ for all $v \in \mathcal{T}_{\text{past}}$ except for those of the form $v = w'_r = e_N \cdots e_r$ for some r with $1 \leq r \leq N$. Then the $\mathbf{s}(e_N)$th component of the resulting state trajectory $x(\cdot)$ at the location $\emptyset$ is

$$\begin{aligned} x_{\mathbf{s}(e_N)}(\emptyset) &= \sum_{r=1}^{N} A_{\mathbf{r}(e_N),\mathbf{s}(e_{N-1})} \cdots A_{\mathbf{r}(e_{r+1}),\mathbf{s}(e_r)} B_{\mathbf{r}(e_r)} u(w'_r) \\ &= \sum_{r=1}^{N} \Psi_{w'_r} u(w'_r). \end{aligned}$$

Then the linear space $\boldsymbol{\mathcal{C}}_w$ consisting of all vectors $x_s \in \mathcal{H}_s$ achievable as $x_s(\emptyset)$ when the system is run with state set equal to zero in the distant past and with input taken to be equal to zero except along some left segment of the word w is characterized as

$$\boldsymbol{\mathcal{C}}_w = \operatorname{im} \mathcal{C}_w,$$

where the *controllability operator* associated with the word w is given by

$$\mathcal{C}_w = \underset{r=1,\dots,N}{\operatorname{row}} \Psi_{w'_r} : \ell_{\text{fin}}(\mathcal{T}^w_{\text{past}}, \mathcal{U}) \to \mathcal{H}_{[s]}, \tag{5.4}$$

where $\mathcal{T}^w_{\text{past}} = \{w'_r : r = 1, \dots, N\} \subset \mathcal{T}_{\text{past}}$.

More generally, we denote by $\mathcal{F}_E^{\infty_R}$ the set of all nonempty words which have a beginning on the left but are infinite to the right:

$$\mathcal{F}_E^{\infty_R} = \{e_1 e_2 \cdots e_N \cdots : e_j \in E \text{ for } j = 1, 2, 3, \dots\}.$$

Fix an infinite word $w = e_1 e_2 \cdots e_N \cdots \in \mathcal{F}_E^{\infty_R}$. Set $w^N = e_1 e_2 \cdots e_N$ equal to the finite word obtained as the truncation of w after N letters, and define

$$\mathcal{C}_w = \operatorname{row}_{N=1,2,3,\dots} \Psi_{w^N} : \ell_{\text{fin}}(\mathcal{T}^w_{\text{past}}, \mathcal{U}) \to \mathcal{H}_{[\mathbf{s}(LL[w])]},$$

where $LL[w]$ (for w a finite or infinite word) denotes the *leading letter* of w,

$$LL[e_1 e_2 \cdots e_N \cdots] = e_1,$$

and where $\mathcal{T}^w_{\text{past}} = \cup\{w^N : N = 1, 2, 3, \dots\}$. Then the image of $\mathcal{C}_w$ (as an operator on $\ell_{\text{fin}}(\mathcal{T}^w_{\text{past}}, \mathcal{U})$) is the linear space of all possible states $x_{\mathbf{s}(e_1)} \in \mathcal{H}_{[\mathbf{s}(e_1)]}$ ($e_1 = LL[w]$) arising in the form $x_{\mathbf{s}(e_1)}(\emptyset)$ from a system trajectory (5.2) under the assumptions that $x(w) = 0$ for all words $w \in \mathcal{T}_{\text{past}}$ of sufficiently large length and that the input string $\{u(w)\}_{w \in \mathcal{T}_{\text{past}}}$ is supported on $w^1, \dots, w^N$ for some finite N.

It is natural to initialize the state to be zero in the far past but to allow input strings of arbitrary finite support. Given $s \in S$, we define the controllability operator $\mathcal{C}_s$ as the block row matrix

$$\mathcal{C}_s = \underset{w \in \mathcal{T}_{\text{past}} \text{ with } \mathbf{s}(LL[w]) = s}{\operatorname{row}} \Psi_w : \ell_{\text{fin}}(\mathcal{T}^s_{\text{past}}, \mathcal{U}) \to \mathcal{H}_{[s]}. \tag{5.5}$$

Here we set

$$\mathcal{T}^s_{\text{past}} = \bigcup_{w \in \mathcal{T}_{\text{past}} \text{ with } s = \mathbf{s}(LL[w])} \mathcal{T}^w_{\text{past}}. \tag{5.6}$$

If we define $\boldsymbol{\mathcal{C}}_s$ to be the linear space of all vectors $x_s \in \mathcal{H}_s$ achievable as $x_s = x_s(\emptyset)$ when we run the system on $\mathcal{T}_{\text{past}}$ with an input string of finite support and with state initialization set equal to zero at all positions $v \in \mathcal{T}_{\text{past}}$ with $|v|$ sufficiently large, then we have

$$\boldsymbol{\mathcal{C}}_s = \operatorname{im} \mathcal{C}_s.$$

Remark 5.1. More generally, we may define an apparently more general controllability operator as follows. For $p \in P$ (the set of path-connected components of the structure graph G associated with the SNMLS Σ (see Definition 3.7)), set

$\mathcal{T}^p_{\rm past} = \bigcup_{s:[s]=p} \mathcal{T}^s_{\rm past}$. We define the controllability operator $\mathcal{C}_p$ as the block row matrix

$$\mathcal{C}_p = \operatorname*{row}_{s:[s]=p} \mathcal{C}_s \colon \ell_{\rm fin}(\mathcal{T}^p_{\rm past}, \mathcal{U}) \to \mathcal{H}_p. \tag{5.7}$$

Then the image of $\mathcal{C}_p$ consists of the linear span of all vectors $x_p \in \mathcal{H}_p$ expressible as $x_s(\emptyset)$ (for some s with $[s] = p$) when the system Σ is run over the past $\mathcal{T}^p_{\rm past}$ with some input string on $\mathcal{T}^p_{\rm past}$ of finite support and with state-vector initialized to be zero at all positions v sufficiently far in the past.

Note, however, from the formula (5.1) for Ψ_w that Ψ_w is independent of the value of $\mathbf{s}(LL[w])$; i.e., if $w = e_N e_{N-1} \cdots e_2 e_1$ and $w' = e'_N e_{N-1} \cdots e_2 e_1$, then $\Psi_{w'} = \Psi_w$, as long as $\mathbf{r}(e'_N) = \mathbf{r}(e_N)$. Thus $\operatorname{im} \mathcal{C}_s = \operatorname{im} \mathcal{C}_p$ for any $s \in S$ with $[s] = p$.

It will be convenient to make this invariance property more explicit. We define a bijection $w \mapsto w^{\wedge s}$ from $\mathcal{T}^{s'}_{\rm past}$ to $\mathcal{T}^s_{\rm past}$ by

$$w^{\wedge s} = e_{s,\mathbf{r}(e_N)} e_{N-1} \cdots e_1 \quad \text{if } w = e_N e_{N-1} \cdots e_1. \tag{5.8}$$

Note that $e_{s,\mathbf{r}(e_N)}$ is well defined as (3.1) whenever it is the case that $[s] = [\mathbf{s}(e_N)](= [\mathbf{r}(e_N)])$. As observed in the previous paragraph, the controllability-operator building blocks Ψ_w given by (5.1) are invariant under this transformation:

$$\text{for } s, s' \in S \text{ with } [s] = [s'], \quad \Psi_w = \Psi_{w^{\wedge s'}} \text{ for } w \in \mathcal{T}^s_{\rm past}. \tag{5.9}$$

For each of the three choices of controllability operator $\mathcal{C}_w$, $\mathcal{C}_s$, and $\mathcal{C}_p$ (where $\mathcal{C}_p$ is as in Remark 5.1), we have a corresponding notion of controllability, namely, the system Σ is X-controllable (where $X = \mathcal{F}_E^{\infty_R}$ (the set of words which are infinite to the right), $X = S$ or $X = P$) if the operator $\mathcal{C}_x$ is surjective for all $x \in X$. A consequence of Remark 5.1, however, is that S-controllability and P-controllability are equivalent. The notion of controllability most convenient for our purposes here is the weakest of these, namely P-controllability (or equivalently, S-controllability). We therefore make the following definition.

Definition 5.2. *We say that the SNMLS* Σ *is* structured-controllable *or simply* controllable *if the operator*

$$\mathcal{C}_p \colon \ell_{\mathit{fin}}(\mathcal{T}^p_{\mathit{past}}, \mathcal{U}) \to \mathcal{H}_p$$

given by (5.7) *is surjective for each path-connected component* p *of the admissible graph* G *associated with* Σ, *or equivalently (by Remark* 5.1*), if the operator*

$$\mathcal{C}_s \colon \ell_{\mathit{fin}}(\mathcal{T}^s_{\mathit{past}}, \mathcal{U}) \to \mathcal{H}_{[s]}$$

given by (5.5) *is surjective for each* $s \in S$ *(or equivalently, for some* s *with* $[s] = p$ *for each* $p \in P$*).*

6. Observability.

Analogously, we have a dual array of observability operators, but with one additional parameter (roughly due to the fact that $\mathcal{T}_{\rm future}$ includes the empty word $\emptyset$, while $\mathcal{T}_{\rm past}$ does not), namely $\mathcal{O}_{s,w}$ for each $s \in S$ and infinite word $w = e_1 e_2 \cdots e_N \cdots \in \mathcal{F}_E^{\infty_R}$, $\mathcal{O}_s$ for each $s \in S$, and $\mathcal{O}_p$ for each $p \in P$. For $w = e_1 e_2 \cdots e_N \cdots \in \mathcal{F}_E^{\infty_R}$ and $s \in S$, we define $\mathcal{O}_{s,w}$ as the block-operator column matrix

$$\mathcal{O}_{s,w} = \operatorname*{col}_{N=0,1,2,\dots} [C_{\mathbf{s}(e_N)} A_{\mathbf{r}(e_N),\mathbf{s}(e_{N-1})} \cdots A_{\mathbf{r}(e_1),s}] \colon \mathcal{H}_{[s]} \to \ell(\mathcal{T}^w_{\rm future}, \mathcal{Y}), \tag{6.1}$$

where we interpret the formula for the case $N = 0$ to mean

$$[\mathcal{O}_{s,w}]_0 = C_s \tag{6.2}$$

and where we put $\mathcal{T}^w_{\text{future}} = \{(w^N)^\top = e_N e_{N-1} \cdots e_1 \colon N = 0, 1, 2, \dots\} \subset \mathcal{T}_{\text{future}}$. For any $s \in S$, we define an associated observability operator $\mathcal{O}_s$ as the column matrix

$$\mathcal{O}_s = \operatorname*{col}_{v = e_N e_{N-1} \cdots e_1 \in \mathcal{T}_{\text{future}}} [C_{\mathbf{s}(e_N)} A_{\mathbf{r}(e_N), \mathbf{s}(e_{N-1})} \cdots A_{\mathbf{r}(e_1), s}] \colon \mathcal{H}_{[s]} \to \ell(\mathcal{T}_{\text{future}}, \mathcal{Y}), \tag{6.3}$$

with again the interpretation (6.2) for the case $v = \emptyset$ column entry. Finally, for path-connected component $p \in P$ we define an associated observability operator $\mathcal{O}_p$ by

$$\mathcal{O}_p = \operatorname*{col}_{s \in S \colon [s] = p} \mathcal{O}_s \colon \mathcal{H}_p \to \operatorname*{col}_{s \in S \colon [s] = p} \ell(\mathcal{T}_{\text{future}}, \mathcal{Y}). \tag{6.4}$$

Clearly, for each infinite word $w \in \mathcal{F}_E^{\infty_R}$, index $s \in S$, and path-connected component $p \in P$ with $[s] = p$, we have the subspace inclusions

$$\ker \mathcal{O}_p \subset \ker \mathcal{O}_s \subset \ker \mathcal{O}_{s,w}. \tag{6.5}$$

For each of the cases $X = S \times \mathcal{F}_E^{\infty_R}$, $X = S$, and $X = P$, we have a notion of X-observability: Σ is X-observable if the operator $\mathcal{O}_x$ is injective for all $x \in X$. By the set of inclusions (6.5) we see that we have the chain of implications: $S \times \mathcal{F}_E^{\infty_R}$-observability implies S-observability, which in turn implies P-observability. Note that each of these observability notions has a system-theoretic interpretation, as follows:

1. $S \times \mathcal{F}_E^{\infty_R}$-observability means that, for each fixed infinite word $w \in \mathcal{F}_E^{\infty_R}$, an initial state $x_s \in \mathcal{H}_{[s]}$ is uniquely determined from the observations $y((w^N)^\top)$ (for $N = 0, 1, 2, \dots$) obtained by letting the system drift with initial condition $x_s(\emptyset) = x_s$ and $x_{s'}(\emptyset) = 0$ for $s' \neq s$ and with zero input string $u(w) = 0$ for all $w \in \mathcal{F}_E$.
2. S-observability means again that, for each $s \in S$, one can detect an initial state $x_s \in \mathcal{H}_{[s]}$ by the same experiment, but with additional observations, namely $y(v)$ for all $v \in \mathcal{F}_E$.
3. P-observability means again that one can detect an initial state $x_p \in \mathcal{H}_p$ but one must do the experiment described above for S-observability with initial condition $x_s(\emptyset) = x_p$ and $x_{s'}(\emptyset) = 0$ for $s' \neq s$ for each $s \in S$ with $[s] = p$.

For our notion of observability here, we take the weakest of these notions and make the following definition.

DEFINITION 6.1. *We say that the SNMLS* $\Sigma = (G, \mathcal{H}, U)$ *is* structured-observable *(or simply* observable*) if the operator* $\mathcal{O}_p \colon \mathcal{H}_p \to col_{s \in S \colon [s] = p}\, \ell(\mathcal{T}_{future}, \mathcal{Y})$ *given by* (6.4) *is injective for each* $p \in P$.

7. Kalman decomposition.

In this section we obtain a Kalman-type decomposition for SNMLSs; for a good summary of these results for the classical case, we refer to [16].

Let $\Sigma = (G, \mathcal{H}, U)$ be an SNMLS as in Definition 3.7. For each $p \in P$ (the set of path-connected components of the admissible graph G), we let $\mathcal{C}_p$ be the controllability operator defined by (5.7) and $\mathcal{O}_p$ be the observability operator defined by (6.4).[3] From

[3]As it is only the images $\operatorname{im} \mathcal{C}_p$ of the controllability operators $\mathcal{C}_p$ which enter in here, by Remark 5.1 without loss of generality one can in all the discussion below replace $\mathcal{C}_p$ with $\mathcal{C}_{s_p}$ for any fixed choice of $s_p \in S$ with $[s_p] = p$.

the definitions we see that

$$A_{r,s}\colon \operatorname{im}\mathcal{C}_{[s]} \to \operatorname{im}\mathcal{C}_{[r]}, \tag{7.1}$$
$$A_{r,s}\colon \ker\mathcal{O}_{[s]} \to \ker\mathcal{O}_{[r]}, \tag{7.2}$$
$$\ker\mathcal{O}_{[s]} \subset \ker C_s, \tag{7.3}$$
$$\operatorname{im} B_r \subset \operatorname{im}\mathcal{C}_{[r]} \tag{7.4}$$

for all $r \in R$ and $s \in S$. We introduce a direct-sum decomposition

$$\mathcal{H}_p = \mathcal{H}_{p,c/o} \oplus \mathcal{H}_{p,c/no} \oplus \mathcal{H}_{p,nc/o} \oplus \mathcal{H}_{p,nc/no} \tag{7.5}$$

according to the following recipe:

1. Set $\mathcal{H}_{p,c/no} = \operatorname{im}\mathcal{C}_p \cap \ker\mathcal{O}_p$.
2. Choose $\mathcal{H}_{p,c/o}$ so that $\mathcal{H}_{p,c/no} \oplus \mathcal{H}_{p,c/o} = \operatorname{im}\mathcal{C}_p$.
3. Choose $\mathcal{H}_{p,nc/no}$ such that $\mathcal{H}_{p,c/no} \oplus \mathcal{H}_{p,nc/no} = \ker\mathcal{O}_p$.
4. Choose $\mathcal{H}_{p,nc/o}$ such that $\mathcal{H}_p = \mathcal{H}_{p,c/o} \oplus \mathcal{H}_{p,c/no} \oplus \mathcal{H}_{p,nc/o} \oplus \mathcal{H}_{p,nc/no}$.

Fix an $r \in R$ and an $s \in S$. Note that $A_{r,s}\colon \mathcal{H}_{[s]} \to \mathcal{H}_{[r]}$, $B_r\colon \mathcal{U} \to \mathcal{H}_{[r]}$, and $C_s\colon \mathcal{H}_{[s]} \to \mathcal{Y}$, while $\mathcal{H}_{[s]}$, and $\mathcal{H}_{[r]}$ have the direct-sum decompositions

$$\begin{aligned}
\mathcal{H}_{[s]} &= \mathcal{H}_{[s],c/o} \oplus \mathcal{H}_{[s],c/no} \oplus \mathcal{H}_{[s],nc/o} \oplus \mathcal{H}_{[s],nc/no},\\
\mathcal{H}_{[r]} &= \mathcal{H}_{[r],c/o} \oplus \mathcal{H}_{[r],c/no} \oplus \mathcal{H}_{[r],nc/o} \oplus \mathcal{H}_{[r],nc/no}.
\end{aligned}$$

We may therefore represent $A_{r,s}$, B_r, and C_s as matrices with respect to these direct-sum decompositions of $\mathcal{H}_{[s]}$ and $\mathcal{H}_{[r]}$:

$$\begin{aligned}
A_{r,s} &= \begin{bmatrix} A_{r,s;c/o,c/o} & A_{r,s;c/o,c/no} & A_{r,s;c/o,nc/o} & A_{r,s;c/o,nc/no} \\ A_{r,s;c/no,c/o} & A_{r,s;c/no,c/no} & A_{r,s;c/no,nc/o} & A_{r,s;c/no,nc/no} \\ A_{r,s;nc/o,c/o} & A_{r,s;nc/o,c/no} & A_{r,s;nc/o,nc/o} & A_{r,s;nc/o,nc/no} \\ A_{r,s;nc/no,c/o} & A_{r,s;nc/no,c/no} & A_{r,s;nc/no,nc/o} & A_{r,s;nc/no,nc/no} \end{bmatrix},\\
B_r &= \begin{bmatrix} B_{r,c/o} \\ B_{r,c/no} \\ B_{r,nc/o} \\ B_{r,nc/no} \end{bmatrix}, \qquad C_s = \begin{bmatrix} C_{s,c/o} & C_{s,c/no} & C_{s,nc/o} & C_{s,nc/no} \end{bmatrix}.
\end{aligned}$$

From (7.1) we see that

$$A_{r,s;nc/o,c/o} = 0, \quad A_{r,s;nc/o,c/no} = 0, \quad A_{r,s;nc/no,c/o} = 0, \quad A_{r,s;nc/no,c/no} = 0.$$

From (7.2) we see that

$$A_{r,s;c/o,c/no} = 0, \quad A_{r,s;c/o,nc/no} = 0, \quad A_{r,s;nc/o,c/no} = 0, \quad A_{r,s;nc/o,nc/no} = 0.$$

From (7.4) we see that

$$B_{r,nc/o} = 0, \qquad B_{r,nc/no} = 0.$$

From (7.3) we see that

$$C_{s,c/no} = 0, \qquad C_{s,nc/no} = 0.$$

We are therefore left with

$$A_{r,s} = \begin{bmatrix} A_{r,s;c/o,c/o} & 0 & A_{r,s;c/o,nc/o} & 0 \\ A_{r,s;c/no,c/o} & A_{r,s;c/no,c/no} & A_{r,s;c/no,nc/o} & A_{r,s;c/no,nc/no} \\ 0 & 0 & A_{r,s;nc/o,nc/o} & 0 \\ 0 & 0 & A_{r,s;nc/no,nc/o} & A_{r,s;nc/no,nc/no} \end{bmatrix},$$

$$B_r = \begin{bmatrix} B_{r,c/o} \\ B_{r,c/no} \\ 0 \\ 0 \end{bmatrix}, \qquad C_s = \begin{bmatrix} C_{s,c/o} & 0 & C_{s,nc/o} & 0 \end{bmatrix}. \tag{7.6}$$

This analysis leads us to the following result.

THEOREM 7.1. *Let* $\Sigma = (G, \mathcal{H}, U)$ *be an SNMLS. Decompose each* $\mathcal{H}_p$ *as in* (7.5) *with resulting decompositions* (7.6) *for the system matrices* $A_{r,s}$, B_r, *and* C_s.

(1) *Define a reduced SNMLS* $\Sigma_{c/o} = (G, \mathcal{H}_{c/o}, U_{c/o})$ *with auxiliary state-spaces* $(\mathcal{H}_{c/o})_p = \mathcal{H}_{p,c/o}$ *as in* (7.5) *and with connection matrix*

$$U_{c/o} = \begin{bmatrix} A_{c/o} & B_{c/o} \\ C_{c/o} & D_{c/o} \end{bmatrix} : \begin{bmatrix} \oplus_{s \in S} \mathcal{H}_{[s],c/o} \\ \mathcal{U} \end{bmatrix} \to \begin{bmatrix} \oplus_{r \in R} \mathcal{H}_{[r],c/o} \\ \mathcal{Y} \end{bmatrix}$$

given by

$$[A_{c/o}]_{r,s} = A_{r,s;c/o,c/o}, \quad [B_{c/o}]_r = B_{r,c/o}, \quad [C_{c/o}]_s = C_{s,c/o}, \quad D_{c/o} = D$$

determined as in (7.6). *Then the SNMLS* $\Sigma_{c/o}$ *is structured-controllable and structured-observable and has the same transfer function as* Σ:

$$D + C(I - Z_\Sigma(z)A)^{-1} Z_\Sigma(z) B = D_{c/o} + C_{c/o}(I - Z_{\Sigma_{c/o}}(z) A_{c/o})^{-1} Z_{\Sigma_{c/o}}(z) B_{c/o}.$$

(2) *Define a reduced system* $\Sigma_c = (G, \mathcal{H}_c, U_c)$ *with auxiliary state-spaces*

$$(\mathcal{H}_c)_p = \mathcal{H}_{p,c/o} \oplus \mathcal{H}_{p,c/no}$$

with components determined as in (7.5) *and with connection matrix*

$$U_c = \begin{bmatrix} A_c & B_c \\ C_c & D_c \end{bmatrix} : \begin{bmatrix} \oplus_{s \in S} \mathcal{H}_{[s],c} \\ \mathcal{U} \end{bmatrix} \to \begin{bmatrix} \oplus_{r \in R} \mathcal{H}_{[r],c} \\ \mathcal{Y} \end{bmatrix}$$

given by

$$[A_c]_{r,s} = \begin{bmatrix} A_{r,s;c/o,c/o} & 0 \\ A_{r,s;c/no,c/o} & A_{r,s;c/no,c/no} \end{bmatrix}, \qquad [B_c]_r = \begin{bmatrix} B_{r,c/o} \\ B_{r,c/no} \end{bmatrix},$$
$$[C_c]_s = \begin{bmatrix} C_{s,c/o} & 0 \end{bmatrix}, \qquad D_c = D,$$

with matrix entries determined as in (7.6). *Then the SNMLS* Σ_c *is structured-controllable and has the same transfer function as* Σ:

$$D + C(I - Z_\Sigma(z)A)^{-1} Z_\Sigma(z) B = D_c + C_c(I - Z_{\Sigma_c}(z) A_c)^{-1} Z_{\Sigma_c}(z) B_c.$$

(3) *Define a reduced system* $\Sigma_o = (G, \mathcal{H}_o, U_o)$ *with auxiliary state-spaces* $(\mathcal{H}_o)_p = \mathcal{H}_{p,c/o} \oplus \mathcal{H}_{p,nc/o}$ *with components determined as in* (7.5) *and with connection matrix*

$$U_o = \begin{bmatrix} A_o & B_o \\ C_o & D_o \end{bmatrix} : \begin{bmatrix} \oplus_{s \in S} \mathcal{H}_{[s],o} \\ \mathcal{U} \end{bmatrix} \to \begin{bmatrix} \oplus_{r \in R} \mathcal{H}_{[r],o} \\ \mathcal{Y} \end{bmatrix}$$

given by

$$[A_o]_{r,s} = \begin{bmatrix} A_{r,s;c/o,c/o} & A_{r,s;c/o,nc/o} \\ 0 & A_{r,s;nc/o,nc/o} \end{bmatrix}, \qquad [B_o]_r = \begin{bmatrix} B_{r,c/o} \\ 0 \end{bmatrix},$$
$$[C_o]_s = \begin{bmatrix} C_{s,c/o} & C_{s,nc/o} \end{bmatrix}, \qquad D_o = D,$$

with matrix entries determined as in (7.6). *Then the SNMLS Σ_o is structured-observable and has the same transfer function as Σ:*

$$D + C(I - Z_\Sigma(z)A)^{-1} Z_\Sigma(z)B = D_o + C_o(I - Z_{\Sigma_o}(z)A_o)^{-1} Z_{\Sigma_o}(z)B_o.$$

8. State-space similarity theorem. We begin with a definition.

DEFINITION 8.1. *Given two SNMLSs $\Sigma = (G, \mathcal{H}, U)$ and $\Sigma' = (G, \mathcal{H}', U')$ with a common structure graph G and with common input- and output-spaces, so that*

$$U = \begin{bmatrix} A & B \\ C & D \end{bmatrix} : \begin{bmatrix} \oplus_{s\in S}\mathcal{H}_{[s]} \\ \mathcal{U} \end{bmatrix} \to \begin{bmatrix} \oplus_{r\in R}\mathcal{H}_{[r]} \\ \mathcal{Y} \end{bmatrix},$$
$$U' = \begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix} : \begin{bmatrix} \oplus_{s\in S}\mathcal{H}'_{[s]} \\ \mathcal{U} \end{bmatrix} \to \begin{bmatrix} \oplus_{r\in R}\mathcal{H}'_{[r]} \\ \mathcal{Y} \end{bmatrix},$$

we say that Σ and Σ' are similar *(via a state-space similarity) if there is a collection $\Gamma = \{\Gamma_p : p \in P\}$ of bijective linear operators $\Gamma_p : \mathcal{H}_p \to \mathcal{H}'_p$ (for each path-connected component p of G) such that*

$$\begin{bmatrix} (\oplus_{r\in R}\Gamma_{[r]}) & 0 \\ 0 & I_{\mathcal{Y}} \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix} \begin{bmatrix} (\oplus_{s\in S}\Gamma_{[s]}) & 0 \\ 0 & I_{\mathcal{U}} \end{bmatrix}. \tag{8.1}$$

It is an easy computation to see that two systems Σ and Σ' have the same transfer functions if they are similar. On the other hand, Theorem 7.1 is not true in general, since an SNMLS Σ which is not already structured-controllable and structured-observable cannot be similar to its structured-controllable/structured-observable part, as in this case necessarily $\dim \mathcal{H}_{p,c/o} < \dim \mathcal{H}_p$ for some p. The next theorem gives the converse under a controllability/observability hypothesis.

THEOREM 8.2. *Suppose that $\Sigma = (G, \mathcal{H}, U)$ and $\Sigma' = (G, \mathcal{H}', U')$ are two SNMLSs with a common structure graph G and common input- and output-spaces $\mathcal{U}$ and $\mathcal{Y}$. Assume that both Σ and Σ' are structured-controllable and structured-observable. Then Σ and Σ' are similar; i.e., there are bijective linear maps $\Gamma_p : \mathcal{H}_p \to \mathcal{H}'_p$ for each path-connected component p of G such that* (8.1) *holds if and only if Σ and Σ' have the same transfer function*

$$T_\Sigma(z) = T_{\Sigma'}(z).$$

Moreover, in this situation the collection of state-space similarity operators

$$\Gamma_p : \mathcal{H}_{[p]} \to \mathcal{H}'_{[p]}$$

implementing the similarity between Σ and Σ' is unique.

Proof. We have already observed that in general two systems which are similar have the same transfer function. It remains to show the following: *under the assumption that Σ and Σ' are structured-controllable and structured-observable, if*

$T_\Sigma(z) = T_{\Sigma'}(z)$, *then* Σ *and* Σ' *are similar.* From the expression (3.21) for the transfer function, we see that the hypothesis that $T_\Sigma(z) = T_{\Sigma'}(z)$ amounts to the assertion that

$$\begin{aligned} &C_{\mathbf{s}(e_N)} A_{\mathbf{r}(e_N),\mathbf{s}(e_{N-1})} A_{\mathbf{r}(e_{N-1}),\mathbf{s}(e_{N-2})} \cdots A_{\mathbf{r}(e_2),\mathbf{s}(e_1)} B_{\mathbf{r}(e_1)} \\ &\quad = C'_{\mathbf{s}(e_N)} A'_{\mathbf{r}(e_N),\mathbf{s}(e_{N-1})} A'_{\mathbf{r}(e_{N-1}),\mathbf{s}(e_{N-2})} \cdots A'_{\mathbf{r}(e_2),\mathbf{s}(e_1)} B'_{\mathbf{r}(e_1)} \end{aligned} \tag{8.2}$$

for all nonempty words $w = e_N e_{N-1} \cdots e_1 \in \mathcal{F}_E$, with the interpretation

$$C_{\mathbf{s}(e_1)} B_{\mathbf{r}(e_1)} = C'_{\mathbf{s}(e_1)} B'_{\mathbf{r}(e_1)} \tag{8.3}$$

in case $w = e_1$ has length 1 together with

$$D = D' \tag{8.4}$$

corresponding to the case $w = \emptyset$. Recalling the definitions (6.3) and (5.1), we see immediately from (8.2) and (8.3) that

$$[\mathcal{O}_s]_v \mathcal{C}_w = [\mathcal{O}'_s]_v \mathcal{C}'_w \tag{8.5}$$

whenever $s \in S$, $v \in \mathcal{T}_{\text{future}}$, and $w \in \mathcal{T}^s_{\text{past}}$. By the same type of argument as that appearing in Remark 5.1, in fact (8.5) holds for each $s \in S$, $v \in \mathcal{T}_{\text{future}}$, and $w \in \mathcal{T}^{s'}_{\text{past}}$ for any $s' \in S$ in the same path-connected component as s (i.e., with $[s'] = [s]$); indeed, if $w = ew' \in \mathcal{T}^s_{\text{past}}$, there is a unique adjustment $e' \in E$ of e so that $w' = e'w' \in \mathcal{T}^{s'}_{\text{past}}$, $\mathcal{C}_{w'} = \mathcal{C}_w$, and also $\mathcal{C}'_{w'} = \mathcal{C}'_w$. Hence the equality (8.5) with $w \in \mathcal{T}^s_{\text{past}}$ implies the equality (8.5) with $w \in \mathcal{T}^{s'}_{\text{past}}$ for any s' with $[s'] = [s]$ as well.

We attempt to define $\Gamma_p \colon \mathcal{H}_p \to \mathcal{H}'_p$ by

$$\Gamma_p \colon \Psi_w u \mapsto \Psi'_w u \quad \text{for } u \in \mathcal{U} \text{ and } w \in \mathcal{F}_E \text{ with } [\mathbf{r}(LL[w])] = s_p, \tag{8.6}$$

where Ψ_w and Ψ'_w are given by (5.1) and where $s_p \in S$ is any choice of source vertex with $[s_p] = p$. Note that a consequence of Remark 5.1 is that we can always adjust $LL[w]$ to achieve $\mathbf{s}(LL[w]) = s_p$ (for any fixed choice of $s_p \in S$ with $[s_p] = p$) without affecting $\operatorname{im} \Psi_w$ and $\operatorname{im} \Psi'_w$. Explicitly, we have

$$\begin{aligned} \Gamma_p \colon &A_{\mathbf{r}(e_N),\mathbf{s}(e_{N-1})} A_{\mathbf{r}(e_{N-1}),\mathbf{s}(e_{N-2})} \cdots A_{\mathbf{r}(e_2),\mathbf{s}(e_1)} B_{\mathbf{r}(e_1)} u \\ &\quad \mapsto A'_{\mathbf{r}(e_N),\mathbf{s}(e_{N-1})} A'_{\mathbf{r}(e_{N-1}),\mathbf{s}(e_{N-2})} \cdots A'_{\mathbf{r}(e_2),\mathbf{s}(e_1)} B'_{\mathbf{r}(e_1)} u, \end{aligned} \tag{8.7}$$

where $w = e_N e_{N-1} \cdots e_1 \in \mathcal{F}_E$ and where e_N is normalized so that $\mathbf{s}(e_N) = s_p$ with the interpretation

$$\Gamma_p \colon B_{\mathbf{r}(e_1)} u \mapsto B'_{\mathbf{r}(e_1)} u \tag{8.8}$$

in case $w = e_1$ (with $\mathbf{s}(e_1) = s_p$) has length 1. We then extend Γ_p to

$$\mathcal{D}_{s_p} = \operatorname{span}\{\Psi_w u \colon w \in \mathcal{T}^{s_p}_{\text{future}} \text{ with } \mathbf{s}(LL[w]) = s_p,\ u \in \mathcal{U}\} \tag{8.9}$$

by linearity, where we set

$$\mathcal{T}^{s_p}_{\text{future}} = \{w \in \mathcal{F}_E \setminus \{\emptyset\} \colon \mathbf{s}(LL[w]) = s_p\}.$$

We first wish to check that Γ_p is well defined. We must therefore show the following: *given a map* $w \mapsto u_w$ *from* $\mathcal{T}^{s_p}_{\mathit{future}}$ *to* $\mathcal{U}$ *with finite support (so* $u_w = 0$ *for* *all but finitely many words* $w \in \mathcal{T}^{s_p}_{\rm future}$) *such that* $\sum_{w \in \mathcal{T}^{s_p}_{\rm future}} \Psi_w u_w = 0$, *it follows that* $\sum_{w \in \mathcal{T}^{s_p}_{\rm future}} \Psi'_w u_w = 0$. Since $\sum_{w \in \mathcal{T}^{s_p}_{\rm future}} \Psi_w u_w = 0$, we then also have

$$\mathcal{O}_p \cdot \sum_{w \in \mathcal{T}^{s_p}_{\rm future}} \Psi_w u_w = 0. \tag{8.10}$$

From the definition of $\mathcal{O}_p$, equation (8.10) in turn means that

$$\mathcal{O}_s \cdot \sum_{w \in \mathcal{T}^{s_p}_{\rm future}} \Psi_w u_w = 0 \quad \text{for each } s \in S \text{ with } [s] = p. \tag{8.11}$$

From the extended domain of validity of (8.5) explained above, (8.11) immediately implies

$$\mathcal{O}'_s \cdot \sum_{w \in \mathcal{T}^{s_p}_{\rm future}} \Psi'_w u_w = \quad \text{for each } s \in S \text{ with } [s] = p. \tag{8.12}$$

By the assumption that Σ' is structured-observable, we know that $\mathcal{O}'_p$ is injective. Hence we see from (8.12) that

$$\sum_{w \in \mathcal{T}^{s_p}_{\rm future}} \Psi'_w u_w = 0.$$

We conclude that Γ_p is well-defined on its domain $\mathcal{D}_{s_p}$ (see (8.9)), as wanted.

Since Σ by hypothesis is structured-controllable, we see that in fact $\mathcal{D}_{s_p} = \mathcal{H}_p$, and hence Γ_p is defined on all of $\mathcal{H}_p$. Similarly, since Σ' is structured-controllable, we see that $\Gamma_p(\mathcal{H}_p)$ is equal to all of $\mathcal{H}'_p$, i.e., that Γ_p is surjective.

It remains to see that Γ_p is injective; i.e., *given a map* $w \mapsto u_w$ *from* $\mathcal{T}^{s_p}_{future}$ *to* $\mathcal{U}$ *with finite support such that* $\sum_{w \in \mathcal{T}^{s_p}_{future}} \Psi'_w u_w = 0$, *it follows that* $\sum_{w \in \mathcal{T}^{s_p}_{future}} \Psi_w u_w = 0$. This follows by the same argument as in the proof that Γ_p is well defined, with the roles of Σ and Σ' interchanged. We conclude that (8.6) extends by linearity to define a bijective linear transformation from $\mathcal{H}_p$ onto $\mathcal{H}'_p$.

It remains now only to check that $\Gamma = \{\Gamma_p : p \in P\}$ satisfies (8.1). This amounts to verifying

$$\Gamma_{[r]} A_{r,s} = A'_{r,s} \Gamma_{[s]}, \tag{8.13}$$
$$\Gamma_{[r]} B_r = B'_r, \tag{8.14}$$
$$C_s = C'_s \Gamma_{[s]}, \tag{8.15}$$
$$D = D'. \tag{8.16}$$

Note that (8.16) follows immediately from (8.4), while (8.14) follows from (8.8). By the structured-controllability hypothesis on Σ, to show (8.13) and (8.15) it suffices to show

$$\begin{aligned} &\Gamma_{[r]} A_{r,s} A_{\mathbf{r}(e_N), \mathbf{s}(e_{N-1})} A_{\mathbf{r}(e_{N-1}), \mathbf{s}(e_{N-2})} \cdots A_{\mathbf{r}(e_2), \mathbf{s}(e_1)} B_{\mathbf{r}(e_1)} \\ &\quad = A'_{r,s} \Gamma_{[s]} A_{\mathbf{r}(e_N), \mathbf{s}(e_{N-1})} A_{\mathbf{r}(e_{N-1}), \mathbf{s}(e_{N-2})} \cdots A_{\mathbf{r}(e_2), \mathbf{s}(e_1)} B_{\mathbf{r}(e_1)}, \end{aligned} \tag{8.17}$$

$$\begin{aligned} &C_s A_{\mathbf{r}(e_N), \mathbf{s}(e_{N-1})} A_{\mathbf{r}(e_{N-1}), \mathbf{s}(e_{N-2})} \cdots A_{\mathbf{r}(e_2), \mathbf{s}(e_1)} B_{\mathbf{r}(e_1)} \\ &\quad = C'_s \Gamma_{[s]} A_{\mathbf{r}(e_N), \mathbf{s}(e_{N-1})} A_{\mathbf{r}(e_{N-1}), \mathbf{s}(e_{N-2})} \cdots A_{\mathbf{r}(e_2), \mathbf{s}(e_1)} B_{\mathbf{r}(e_1)} \quad \text{if } [s] = [\mathbf{r}(e_N)] \end{aligned} \tag{8.18}$$

for all words $w = e_N e_{N-1} \cdots e_1 \in \mathcal{T}^{s_p}_{\text{future}}$ (with proper interpretation for $N = 1$) for each $p \in P$. Note that (8.17) is an immediate consequence of the definition (8.6) of Γ_p together with the completeness of the path-connected components of G, while (8.18) follows from the definition (8.6) combined with the completeness of the path-connected components of G and the equality of moments (8.2) and (8.3).

As for the last statement in Theorem 8.2, suppose that $\Gamma'_p \colon \mathcal{H}_p \to \mathcal{H}'_p$ is any other linear isomorphism between $\mathcal{H}_p$ and $\mathcal{H}'_p$ so that (8.1) is satisfied. Then a consequence of (8.1) is that necessarily Γ'_p must also satisfy (8.6) (with Γ'_p in place of Γ_p). By the first part of the proof, $\Gamma'_p = \Gamma_p$ for all $p \in P$, and the uniqueness assertion in Theorem 8.2 follows as well. This completes the proof of Theorem 8.2. ∎

9. Minimal state-space realizations. Suppose that we are given an admissible graph G together with a formal power series

$$T(z) = \sum_{w \in \mathcal{F}_E} T_w z^w$$

in the noncommuting variables $z = \{z_e \colon e \in E\}$ (where E is the edge set of G) with coefficients T_w in the space $\mathcal{L}(\mathcal{U}, \mathcal{Y})$ of linear operators between the (finite-dimensional) linear spaces $\mathcal{U}$ and $\mathcal{Y}$. We say that the SNMLS $\Sigma = (G, \mathcal{H}, U)$ (with structure graph equal to G) is a *G-structured realization* for $T(z)$ if $T(z)$ is equal to the transfer function of Σ, i.e., if

$$T_\emptyset = D, \qquad T_{e_N e_{N-1} \cdots e_1} = C_{\mathbf{s}(e_N)} A_{\mathbf{r}(e_N), \mathbf{s}(e_{N-1})} A_{\mathbf{r}(e_{N-1}), \mathbf{s}(e_{N-2})} \cdots A_{\mathbf{r}(e_2), \mathbf{s}(e_1)} B_{\mathbf{r}(e_1)},$$

where the connection matrix U for Σ has the form

$$U = \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} [A_{r,s}] & [B_r] \\ [C_s] & D \end{bmatrix} \colon \begin{bmatrix} \oplus_{s \in S} \mathcal{H}_{[s]} \\ \mathcal{U} \end{bmatrix} \to \begin{bmatrix} \oplus_{r \in R} \mathcal{H}_{[r]} \\ \mathcal{Y} \end{bmatrix}.$$

We say that the SNMLS Σ is a *structured-minimal realization* for $T(z)$ if $\dim \mathcal{H}'_p \geq \dim \mathcal{H}_p$ for each path-connected component p of G whenever $\Sigma' = (G, \mathcal{H}', U')$ is another G-structured realization for $T(z)$. The following theorem establishes the equivalence of structured-minimality with simultaneous structured-controllability and structured-observability for G-structured realizations of a given formal power series $T(z)$.

THEOREM 9.1. *Suppose that $\Sigma = (G, \mathcal{H}, U)$ is a G-structured realization for the formal power series $T(z) = \sum_{w \in \mathcal{F}_E} T_w z^w$. Then Σ is a G-structured minimal realization for $T(z)$ if and only if Σ is both structured-controllable and structured-observable (with structure graph G).*

Proof. Suppose first that $\Sigma = (G, \mathcal{H}, U)$ is a structured-controllable and structured-observable realization of $T(z)$ and that $\Sigma' = (G, \mathcal{H}', U')$ is another structured realization of $T(z)$ (with the same structure graph G). By part (1) of Theorem 7.1, we may cut the realization Σ' down to a structured-controllable and structured-observable realization $\Sigma'_{c/o} = (G, \mathcal{H}'_{c/o}, U'_{c/o})$ for $T(z)$; as part of the construction we have $\dim \mathcal{H}'_p \geq \dim \mathcal{H}'_{p,c/o}$ for each $p \in P$. We now have that $\Sigma = (G, \mathcal{H}, U)$ and $\Sigma'_{c/o} = (G, \mathcal{H}'_{c/o}, U'_{c/o})$ are both structured-controllable and structured-observable realizations of the same formal power series $T(z)$. By the state-space-similarity theorem (Theorem 8.2), it follows that Σ and $\Sigma'_{c/o}$ are similarvia a state-space similarity

$\Gamma = \{\Gamma_p \colon \mathcal{H}_p \to \mathcal{H}'_{p,c/o} \colon p \in P\}$. In particular,

$$\dim \mathcal{H}_p = \dim \mathcal{H}'_{p,c/o} \leq \dim \mathcal{H}'_p.$$

As Σ' was any other G-structured realization of $T(z)$, it follows that Σ is a G-structured minimal realization, as wanted.

Conversely, suppose that Σ is G-structured minimal. By part (1) of Theorem 7.1, we may cut Σ down to a structured-controllable and structured-observable realization $\Sigma_{c/o} = (G, \mathcal{H}_{c/o}, U_{c/o})$ of the same formal power series $T(z)$. By the construction in Theorem 7.1, $\mathcal{H}_{p,c/o} \subset \mathcal{H}_p$. On the other hand, by the assumption that Σ is G-structured minimal, we must also have $\dim \mathcal{H}_p \leq \dim \mathcal{H}_{p,c/o}$, and hence we must have the equality $\mathcal{H}_p = \mathcal{H}_{p,c/o}$ for each $p \in P$. From the construction in Theorem 7.1, this means that the realization Σ is itself structured-controllable and structured-observable. This completes the proof of Theorem 9.1. $\square$

10. Hankel operators. The notion of a Hankel operator $\mathbb{H}$ for a classical (1-D) linear system is the map which maps a past input sequence to the future output sequence, under the assumptions that the state has been initialized to be zero at $-\infty$ (roughly speaking) and that the future input string is set equal to zero. Since the controllability operator $\mathcal{C}$ maps the past history to the state at time zero, also under the assumption that the state has been initialized to be zero at $-\infty$, while the observability operator $\mathcal{O}$ maps a given state at time 0 into the future output sequence (under the assumption that the future input string is set equal to zero), we see immediately from the definitions that the Hankel operator $\mathbb{H}$ has the factorization $\mathbb{H} = \mathcal{O} \cdot \mathcal{C}$. For the case of SNMLSs, we have three notions ($\mathcal{C}_w$ for $w \in \mathcal{F}_E^{\infty_R}$, $\mathcal{C}_s$ for $s \in S$, and $\mathcal{C}_p$ for $p \in P$) of controllability operators which map some version of the past ($\mathcal{T}^w_{\text{past}}$, $\mathcal{T}^s_{\text{past}}$, or $\mathcal{T}^p_{\text{past}}$) to a state at the "present" position $\emptyset$, and three notions of observability operator ($\mathcal{O}_{s,w}$, $\mathcal{O}_s$, and $\mathcal{O}_p$ for $(s,w) \in S \times \mathcal{F}_E^{\infty_R}$, $s \in S$, and $p \in P$) mapping some state at the present position $\emptyset$ to outputs supported on some version of the future ($\mathcal{T}^w_{\text{future}}$, $\mathcal{T}_{\text{future}}$, or $\cup_{s \colon [s]=p} \mathcal{T}_{\text{future}}$. Thus a priori we have nine distinct possible notions of a Hankel operator. However, for purposes of the realization theory to be presented in section 11 below, only some of these are of interest for our purposes here, so we focus on them.

Let $\Sigma = (G, \mathcal{H}, U)$ be an SNMLS as in Definition 3.7. In this section we shall fix a cross section $p \mapsto s_p \in S$ of the map $[\cdot] \colon S \to P$ mapping a source vertex s to its associated path-connected component $[s] \in P$; i.e., for each $p \in P$, we let s_p be a fixed choice of element of S such that $[s_p] = p$. Consider any past input string $\{u(v)\}_{v \in \mathcal{T}^{s_p}_{\text{past}}}$. Run the system with this input string $u(w)$ for $w \in \mathcal{T}^{s_p}_{\text{past}}$ and with the state initialized to be zero in the distant past to generate a state $x(\emptyset)$ with s_pth component x_{s_p} equal to, say, $x_p \in \mathcal{H}_p$. For each $s \in S$ with $[s] = p$, we next run the system with zero inputs $u(w)$ for $w \in \mathcal{T}_{\text{future}}$ and with initial condition $x_s(\emptyset) = x_p$, $x_{s'}(\emptyset) = 0$ for $s' \neq s$. The result is an output sequence $\{y_s(w)\}_{w \in \mathcal{T}_{\text{future}}}$. The resulting composite map defined as taking the input string $\{u(v)\}_{v \in \mathcal{T}^{s_p}_{\text{past}}}$ to the output string $\{y_s(w)\}_{s \colon [s]=p; w \in \mathcal{T}_{\text{future}}}$ we define to be the Hankel operator $\mathbb{H}^p$:

$$\mathbb{H}^p = \mathcal{O}_p \mathcal{C}_{s_p} \colon \ell_{\text{fin}}(\mathcal{T}^{s_p}_{\text{past}}, \mathcal{U}) \to \oplus_{s \colon [s]=p} \ell(\mathcal{T}_{\text{future}}, \mathcal{Y}). \tag{10.1}$$

Explicitly, $\mathbb{H}^p$ is given as a bi-infinite matrix $[\mathbb{H}^p_{(s,w),v}]$ with rows indexed by pairs (s,w) with $s \in S$ with $[s] = p$ and with $w \in \mathcal{T}_{\text{future}}$, and with columns indexed by words $v \in \mathcal{T}^{s_p}_{\text{past}}$. In terms of the connecting operator U for Σ, the matrix entries are given explicitly as

$$\begin{aligned} \mathbb{H}^p_{(s,w),w'} &= C_{\mathbf{s}(e_N)} A_{\mathbf{r}(e_N),\mathbf{s}(e_{N-1})} A_{\mathbf{r}(e_{N-1}),\mathbf{s}(e_{N-2})} \cdots A_{\mathbf{r}(e_2),\mathbf{s}(e_1)} \\ &\qquad \cdot A_{\mathbf{r}(e_1),s} A_{\mathbf{r}(e'_{N'}),\mathbf{s}(e'_{N'-1})} A_{\mathbf{r}(e'_{N'-1}),\mathbf{s}(e'_{N'-2})} \cdots A_{\mathbf{r}(e'_2),\mathbf{s}(e'_1)} B_{\mathbf{s}(e'_1)} \\ &= C_{\mathbf{s}(e_N)} A_{\mathbf{r}(e_N),\mathbf{s}(e_{N-1})} A_{\mathbf{r}(e_{N-1}),\mathbf{s}(e_{N-2})} \cdots A_{\mathbf{r}(e_2),\mathbf{s}(e_1)} A_{\mathbf{r}(e_1),s} \Psi_{w'} \end{aligned} \tag{10.2}$$

if $w = e_N e_{N-1} \cdots e_2 e_1$ and $w' = e'_{N'} e'_{N'-1} \cdots e'_2 e'_1$, where $e'_{N'}$ is constrained to satisfy $\mathbf{s}(e'_{N'}) = s_p$ and where we use (5.1) to define $\Psi_{w'}$. (We leave it to the reader to give the appropriate interpretations for these formulas in case $N = 1$ and/or $N' = 0$.) As explained in the context of Remark 5.1, if we replace w' by w'' of the form

$$w'' = e_{s,\mathbf{r}(e'_{N'})} e'_{N'-1} \cdots e'_2 e'_1$$

for any s with $[s] = [\mathbf{s}(e'_{N'})] = [\mathbf{r}(e'_{N'})]$, then $\Psi_{w''} = \Psi_{w'}$. Since $v \in \mathcal{T}^{s_p}_{\text{past}}$, where $[s_p] = p$, we may therefore rewrite the Hankel matrix entry as a moment of the transfer function $T_\Sigma(z) = \sum_{w \in \mathcal{F}_E} T_w z^w$, namely,

$$\begin{aligned} \mathbb{H}^p_{(s,w),v} &= C_{\mathbf{s}(e_N)} A_{\mathbf{r}(e_N),\mathbf{s}(e_{N-1})} A_{\mathbf{r}(e_{N-1}),\mathbf{s}(e_{N-2})} \cdots A_{\mathbf{r}(e_2),\mathbf{s}(e_1)} \\ &\qquad \cdot A_{\mathbf{r}(e_1),\mathbf{s}(e_{s,\mathbf{r}(e'_{N'})})} A_{\mathbf{r}(e_{s,\mathbf{r}(e'_{N'})}),\mathbf{s}(e'_{N'-1})} A_{\mathbf{r}(e'_{N'-1}),\mathbf{s}(e'_{N'-2})} \cdots A_{\mathbf{r}(e'_2),\mathbf{s}(e'_1)} B_{\mathbf{s}(e'_1)} \\ &= T_{e_N e_{N-1} \cdots e_1 e_{s,\mathbf{r}(e'_{N'})} e'_{N'-1} \cdots e'_2 e'_1}, \end{aligned} \tag{10.3}$$

or, more compactly,

$$\mathbb{H}^p_{(s,w),ev'} = T_{w e_{s,\mathbf{r}(e)} v'} \tag{10.4}$$

for $s \in S$, $w \in \mathcal{T}_{\text{future}}$, and ev' (with $e \in E$ with $\mathbf{s}(e) = s_p$ and $v' \in \mathcal{F}_E$) the generic form of an element in $\mathcal{T}^{s_p}_{\text{past}}$.

From the factorization (10.1) and the definitions, it is easy to see the following result; we shall obtain a converse in section 11 below.

THEOREM 10.1. *Suppose that the SNMLS Σ (see Definition 3.7) is structured-controllable and structured-observable. Then the dimension of the auxiliary state-space $\mathcal{H}_p$ (for a given path-connected component $p \in P$ of the structure graph) is given by*

$$\dim \mathcal{H}_p = \operatorname{rank} \mathbb{H}^p.$$

Proof. By definition, $\mathcal{C}_{s_p}$ is a surjective map to $\mathcal{H}_p$ if Σ is structured-controllable, and $\mathcal{O}_p$ is an injective map if Σ is structured-observable. Hence the result is immediate from the factorization (10.1). □

COROLLARY 10.2. *If $T(z)$ is the transfer function of an SNMLS Σ having structure graph G, then, for each path-connected component $p \in P$, the Hankel operator $\mathbb{H}^p$ formed from G and $T(z)$ according to the formula (10.4) has finite rank.*

We shall obtain a converse of Corollary 10.2 in section 11 below.

11. Realization theory for structured noncommutative linear systems.

Suppose that we are given an admissible graph G together with a formal power series $T(z) = \sum_{v \in \mathcal{F}_E} T_v z^v$ in noncommuting variables $z = (z_e \colon e \in E)$ indexed by the edge set E of G and with coefficients T_v equal to linear operators between the finite-dimensional linear spaces $\mathcal{U}$ and $\mathcal{Y}$. The realization problem associated with the data set $\mathbb{D} := (G, T(z))$ then is the following: *construct a finite-dimensional SNMLS $\Sigma = (G, \mathcal{H}, U)$ having G as its structure graph and $T(z)$ as its transfer function.*

A necessary condition for the problem to have a solution was formulated in Corollary 10.2. The content of the following theorem is the converse. We shall need the following conventions. Let G be an admissible graph. As in section 10, we assume that we have specified a cross section $p \mapsto s_p$ of the map $[\cdot]\colon S \to P$, so $s_p \in S$ with $[s_p] = p$ for each $p \in P$. For $v \in \mathcal{T}^s_{\text{past}}$ (where $\mathcal{T}^s_{\text{past}}$ is defined as in (5.6)), we let δ_v be the Kronecker delta function on $\mathcal{T}^s_{\text{past}}$:

$$\delta_v(v') = \begin{cases} 1 & \text{if } v' = v, \\ 0 & \text{if } v' \neq v, \end{cases} \qquad \text{for } v' \in \mathcal{T}^s_{\text{past}}.$$

Then $\{\delta_v u \colon v \in \mathcal{T}^s_{\text{past}},\, u \in \mathcal{U}\}$ is a spanning set for the linear space $\ell_{\text{fin}}(\mathcal{T}^s_{\text{past}}, \mathcal{U})$. Recall the notation $e_{s,r}$ as in (3.1) for the unique edge connecting $s \in S$ to $r \in R$, defined whenever $[s] = [r]$, and the notation $w^{\wedge s}$ introduced in (5.8).

THEOREM 11.1. *Suppose that we are given the data set $\mathbb{D} = (G, T(z))$ for a realization problem as above. For each path-connected component $p \in P$ of G, associate the Hankel matrix $\mathbb{H}^p$ as in* (10.4)*. Then the realization problem for the data set $\mathbb{D}$ is solvable if and only if*

$$\operatorname{rank} \mathbb{H}^p < \infty \quad \textit{for each } p \in P. \tag{11.1}$$

When the condition (11.1) *holds, a structured-minimal realization of $T(z)$ can be constructed as follows.*

For each $p \in P$, let $\mathcal{H}_p$ be the linear space

$$\mathcal{H}_p = \ell_{\mathit{fin}}(\mathcal{T}^{s_p}_{\mathit{past}}, \mathcal{U}) / \ker \mathbb{H}^p, \tag{11.2}$$

and set $\mathcal{H}$ equal to the collection

$$\mathcal{H} = \{\mathcal{H}_p \colon p \in P\}.$$

For each source vertex $s \in S$ and range vertex $r \in R$ of G, define linear operators $A_{r,s}\colon \mathcal{H}_{[s]} \to \mathcal{H}_{[r]}$, $B\colon \mathcal{U} \to \mathcal{H}_{[r]}$, $C_s\colon \mathcal{H}_{[s]} \to \mathcal{Y}$, and $D\colon \mathcal{U} \to \mathcal{Y}$ by

$$\begin{aligned}
&A_{r,s}\colon \left[\{u(v)\}_{v \in \mathcal{T}^{s_{[s]}}_{\mathit{past}}}\right]_{\mathcal{H}_{[s]}} \mapsto \left[\{u'(v)\}_{v \in \mathcal{T}^{s_{[r]}}_{\mathit{past}}}\right]_{\mathcal{H}_{[r]}}, \ \textit{where} \\
&\qquad u'(v) = \begin{cases} u((v')^{\wedge s_{[s]}}) & \textit{if } v \textit{ has the form } v = e_{s_{[r]},r} v' \textit{ with } v' \in \mathcal{T}^s_{\mathit{past}}, \\ 0 & \textit{otherwise,} \end{cases} \\
&B_r\colon u \mapsto [\delta_{e_{s_{[r]},r}} u]_{\mathcal{H}_{[r]}}, \\
&C_s\colon \left[\{u(v)\}_{v \in \mathcal{T}^{s_{[s]}}_{\mathit{past}}}\right]_{\mathcal{H}_{[s]}} \mapsto \mathbb{H}^{[s]}_{(s,\emptyset),\cdot}\left(\{u(v)\}_{v \in \mathcal{T}^{s_{[s]}}_{\mathit{past}}}\right) \\
&\qquad = \sum_{v \in \mathcal{T}^{s_{[s]}}_{\mathit{past}}} T_{v^{\wedge s}} u(v),
\end{aligned} \tag{11.3}$$

$$D = T_\emptyset.$$

Use (11.3) *to define a connection matrix U by*

$$U = \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} [A_{r,s}] & [B_r] \\ [C_s] & D \end{bmatrix} \colon \begin{bmatrix} \oplus_{s \in S} \mathcal{H}_{[s]} \\ \mathcal{U} \end{bmatrix} \to \begin{bmatrix} \oplus_{r \in R} \mathcal{H}_{[r]} \\ \mathcal{Y} \end{bmatrix}.$$

Then the collection $\Sigma = (G, \mathcal{H}, U)$ *is a structured-minimal SNMLS with structure graph* G *having* $T(z)$ *as its transfer function.*

Proof. We have already observed in Corollary 10.2 the necessity of the condition (11.1) for the realization problem to have a solution. It remains to prove the sufficiency. This follows if we can verify that the formulas (11.2) and (11.3) provide a structured-minimal realization of $T(z)$ (with structure matrix G).

As a preliminary step, we note that the formula for $A_{r,s}$ in (11.3) when specialized to elements of $\mathcal{H}_{[s]}$ of the form $[\delta_v u]_{\mathcal{H}_{[s]}}$ (where $v \in \mathcal{T}^{s_{[s]}}_{\rm past}$) assumes the form

$$A_{r,s}\colon [\delta_v u]_{\mathcal{H}_{[s]}} \mapsto [\delta_{e_{s_{[r]},r}(v^{\wedge s})}]_{\mathcal{H}_{[r]}}. \tag{11.4}$$

Note also that the set $\{[\delta_v u]_{\mathcal{H}_{[s]}}\colon v \in \mathcal{T}^{s_{[s]}}_{\rm past},\, u \in \mathcal{U}\}$ is a spanning set for $\mathcal{H}_{s_{[s]}}$ since $\{\delta_v u\colon v \in \mathcal{T}^{s_{[s]}}_{\rm past},\, u \in \mathcal{U}\}$ is a spanning set for $\ell_{\rm fin}(\mathcal{T}^{s_{[s]}}_{\rm past}, \mathcal{U})$. Similarly, the action of C_s in (11.3) on delta functions can be written as

$$C_s\colon [\delta_v u]_{\mathcal{H}_{[s]}} \mapsto T_{v^{\wedge s}} u \quad \text{for } v \in \mathcal{T}^{s_{[s]}}_{\rm past}. \tag{11.5}$$

The verification proceeds via a number of steps.

Step 1: *Verification that* $A_{r,s}$ *is well defined.* Suppose that $\{u(v)\}_{v \in \mathcal{T}^{s_{[s]}}_{\rm past}}$ represents the zero element of $\mathcal{H}_{[s]}$; thus $\mathbb{H}^{[s]}(\{u(v)\}_{v \in \mathcal{T}^{s_{[s]}}_{\rm past}}) = 0$. Explicitly, this means

$$\sum_{v \in \mathcal{T}^{s_{[s]}}_{\rm past}} T_{wv^{\wedge s'}} u(v) = 0 \quad \text{for all } w \in \mathcal{F}_E \text{ and } s' \in S \text{ with } [s'] = [s]. \tag{11.6}$$

View $\{u(v)\}_{v \in \mathcal{T}^{s_{[s]}}_{\rm past}}$ as equal to $\sum_{v \in \mathcal{T}^{s_{[s]}}_{\rm past}} \delta_v u(v)$, and use the formula (11.4) combined with linearity: the result is

$$A_{r,s}\colon \sum_{v \in \mathcal{T}^{s_{[s]}}_{\rm past}} \delta_v u(v) \mapsto \sum_{v \in \mathcal{T}^{s_{[s]}}_{\rm past}} \delta_{e_{s_{[r]},r} v^{\wedge s}} u(v) \in \ell(\mathcal{T}^{s_{[r]}}_{\rm past}, \mathcal{U}).$$

For the right-hand side of this formula to represent the zero element of $\mathcal{H}_{[r]}$ we need to have $\mathbb{H}^{[r]}(\sum_{v \in \mathcal{T}^{s_{[s]}}_{\rm past}} \delta_{e_{s_{[r]},r} v^{\wedge s}} u(v)) = 0$, which is to say

$$\sum_{v \in \mathcal{T}^{s_{[s]}}_{\rm past}} T_{w'(e_{s_{[r]},r} v^{\wedge s})^{\wedge s''}} u(v) = 0 \quad \text{for all } w' \in \mathcal{F}_E, s'' \in S \text{ with } [s''] = [r]. \tag{11.7}$$

However, it is easily verified that

$$(e_{s_{[r]},r} v^{\wedge s})^{\wedge s''} = e_{s'',r} v^{\wedge s}.$$

Hence the condition (11.7) amounts to the known condition (11.6) for the special case $w = w' e_{s'',r}$ and $s' = s$. We conclude that the formula for $A_{r,s}$ in (11.3), or equivalently the formula (11.4) for $A_{r,s}$ on a spanning subset of $\mathcal{H}_{[r]}$, is well defined.

Step 2: *Verification that* C_s *is well defined.* We again suppose that $\{u(v)\}_{v \in \mathcal{T}^{s_{[s]}}_{\rm past}}$ represents the zero element of $\mathcal{H}_{[s]}$, i.e., that (11.6) holds. Then $C_s(\{u(v)\}_{v \in \mathcal{T}^{[s]}_{\rm past}})$ by definition is the left-hand side of (11.6) for the special case $w = \emptyset$ and $s' = s$. Hence C_s is well defined, as wanted.

Step 3: *Verification that* $T_\Sigma(z) = T(z)$. Let $e \in E$ be an edge of G. Use the formula for B_r in (11.3) and the formula (11.5) for the action of C_s on delta functions to compute

$$\begin{aligned} C_{\mathbf{s}(e)} B_{\mathbf{r}(e)} u &= C_{\mathbf{s}(e)} \left(\left[\delta_{e_{s_{[\mathbf{r}(e)]}, \mathbf{r}(e)}} u \right]_{\mathcal{H}_{[\mathbf{r}(e)]}} \right) = T_{(e_{s_{[\mathbf{r}(e)]}, \mathbf{r}(e)})^{\wedge \mathbf{s}(e)}} u \\ &= T_e u, \end{aligned} \tag{11.8}$$

where the equality $(e_{s_{[\mathbf{r}(e)]}, \mathbf{r}(e)})^{\wedge \mathbf{s}(e)} = e_{\mathbf{s}(e), \mathbf{r}(e)} = e$ follows from the uniqueness condition (3) in the admissibility conditions (see Definition 3.1) for the graph G. Similarly, by using the formula for B_r in (11.3) combined with (11.4), a straightforward induction argument gives that, for any word $w = e_N e_{N-1} \cdots e_2 e_1$ of length at least 2,

$$A_{\mathbf{r}(e_N), \mathbf{s}(e_{N-1})} A_{\mathbf{r}(e_{N-1}), \mathbf{s}(e_{N-2})} \cdots A_{\mathbf{r}(e_2), \mathbf{s}(e_1)} B_{\mathbf{r}(e_1)} u = \left[\delta_{w^{\wedge s_{[\mathbf{r}(e_N)]}}} u \right]_{\mathcal{H}_{[\mathbf{r}(e_N)]}}. \tag{11.9}$$

From the uniqueness axiom in Definition 3.1 we have

$$(w^{\wedge [\mathbf{r}(e_N)]})^{\wedge \mathbf{s}(e_N)} = w \quad \text{if } e_N = LL[w]. \tag{11.10}$$

Applying the formula (11.5) to (11.9) and using (11.10), we get

$$\begin{aligned} C_{\mathbf{s}(e_N)} A_{\mathbf{r}(e_N), \mathbf{s}(e_{N-1})} A_{\mathbf{r}(e_{N-1}), \mathbf{s}(e_{N-2})} \cdots A_{\mathbf{r}(e_2), \mathbf{s}(e_1)} B_{\mathbf{r}(e_1)} u &= T_{(w^{\wedge [\mathbf{r}(e_N)]})^{\wedge \mathbf{s}(e_N)}} u \\ &= T_w u \text{ for } w = e_N e_{N-1} \cdots e_2 e_1. \end{aligned} \tag{11.11}$$

Combining (11.8) and (11) with the definition $D = T_\emptyset$ in (11.3), we see that $T_\Sigma(z) = T(z)$, as wanted.

Step 4: *Verification that* Σ *is structured-controllable.* By formula (11.9) we have

$$\Psi_w u = [\delta_w u]_{\mathcal{H}_p} \quad \text{for } w \in \mathcal{T}_{\text{past}}^{s_p} \text{ and } u \in \mathcal{U}.$$

As the set $\{[\delta_w u]_{\mathcal{H}_p} : w \in \mathcal{T}_{\text{past}}^{s_p} \text{ and } u \in \mathcal{U}\}$ is spanning for the space

$$\mathcal{H}_p = \ell_{\text{fin}}(\mathcal{T}_{\text{past}}^{s_p}, \mathcal{U}) / \ker \mathbb{H}^p,$$

we conclude that Σ is structured-controllable, as wanted.

Step 5: *Verification that* Σ *is structured-observable.* From the various definitions it is easy to verify that

$$\mathcal{O}_s \left(\left[\{u(v)\}_{v \in \mathcal{T}_{\text{past}}^{s_{[s]}}} \right]_{\mathcal{H}_{[s]}} \right) = \mathbb{H}^{[s]}_{(s,\cdot),\cdot} \left(\{u(v)\}_{v \in \mathcal{T}_{\text{past}}^{s_{[s]}}} \right) \in \ell(\mathcal{T}_{\text{future}}, \mathcal{Y})$$

for each source vertex $s \in S$. Since, by definition, $\mathcal{O}_p = \operatorname{col}_{s\colon [s]=p} \mathcal{O}_s$ for each $p \in P$, we can then make the identification

$$\mathcal{O}_p \left(\left[\{u(v)\}_{v \in \mathcal{T}_{\text{past}}^{s_p}} \right]_{\mathcal{H}_p} \right) = \mathbb{H}^p \left(\{u(v)\}_{v \in \mathcal{T}_{\text{past}}^{s_p}} \right) \in \oplus_{s\colon [s]=p} \ell(\mathcal{T}_{\text{future}}, \mathcal{Y}).$$

In this way we see that $[\{u(v)\}_{v \in \mathcal{T}_{\text{past}}^{s_p}}]_{\mathcal{H}_p} \in \ker \mathcal{O}_p$ if and only if $\{u(v)\}_{v \in \mathcal{T}_{\text{past}}^{s_p}} \in \ker \mathbb{H}^p$, i.e., if and only if $[\{u(v)\}_{v \in \mathcal{T}_{\text{past}}^{s_p}}]_{\mathcal{H}_p}$ is the zero equivalence class in $\mathcal{H}_p$. We conclude that Σ is structured-observable as wanted, and the proof of Theorem 11.1 is now complete. $\square$

We now consider the situation where the formal power series $T(z) = \sum_{v \in \mathcal{F}_d} T_v z^v$ is given but the admissible graph G is not specified. By comparing the various Hankel operators involved, we have the following result.

THEOREM 11.2. *Suppose that we are given the formal power series in the d noncommuting variables $z = (z_1, \dots, z_d)$, and let G and G' be two admissible graphs with edge sets E and E' of the same cardinality. Then $T(z)$ has a G-structured realization $\Sigma = \{G, \mathcal{H}, U\}$ if and only if $T(z)$ has a G'-structured realization $\Sigma' = \{G', \mathcal{H}', U'\}$.*

Proof. Let G be any admissible graph with edge set E labeled as $E = \{1, \dots, d\}$, and let G^{FM} be the Fornasini–Marchesini admissible graph with source-vertex set $S = \{1\}$, range-vertex set $R = \{1, \dots, d\}$, and edge set $E = \{1, \dots, d\}$, with $\mathbf{s}(j) = 1$ and $\mathbf{r}(j) = j$ for $j = 1, \dots, d$. We show that $T(z)$ has a G-structured realization $\Sigma = (G, \mathcal{H}, U)$ if and only if $T(z)$ has a G^{FM}-structured realization $\Sigma^{FM} = (G^{FM}, \mathcal{H}^{FM}, U^{FM})$. For s in S define the Hankel operator $\mathbb{H}^s \colon \ell_{\rm fin}(\mathcal{T}^s_{\rm past}, \mathcal{U}) \to \ell(\mathcal{T}_{\rm future}, \mathcal{Y})$ by

$$\mathbb{H}^s \colon \{u(v)\}_{v \in \mathcal{T}^s_{\rm past}} \mapsto \mathbb{H}^{[s]}_{(s,\cdot),\cdot} \left(\{u(v^{\wedge s})\}_{v \in \mathcal{T}^{s_{[s]}}_{\rm past}} \right).$$

As the map $v \mapsto v^{s_{[s]}}$ is a bijection between $\mathcal{T}^s_{\rm past}$ and $\mathcal{T}^{s_{[s]}}_{\rm past}$, we see that $\mathbb{H}^s$ is similar to $\mathbb{H}^{[s]}_{(s,\cdot),\cdot}$. By definition,

$$\mathbb{H}^p = \mathrm{col}_{s\colon [s]=p} \left[\mathbb{H}^p_{(s,\cdot),\cdot} \right]$$

from which we get the estimates

$$\max\nolimits_{s\colon [s]=p} \operatorname{rank} \mathbb{H}^p_{(s,\cdot),\cdot} \le \operatorname{rank} \mathbb{H}^p \le \sum_{s\colon [s]=p} \operatorname{rank} \mathbb{H}^p_{(s,\cdot),\cdot}. \tag{11.12}$$

As we observed above that $\mathbb{H}^s$ and $\mathbb{H}^{[s]}_{(s,\cdot),\cdot}$ have the same rank, we can rewrite (11.12) as

$$\max\nolimits_{s\colon [s]=p} \operatorname{rank} \mathbb{H}^s \le \operatorname{rank} \mathbb{H}^p \le \sum_{s\colon [s]=p} \operatorname{rank} \mathbb{H}^s. \tag{11.13}$$

From the characterization (10.4) of $\mathbb{H}^p$ we see that

$$\mathbb{H}^{FM} = \mathrm{col}_{p \in P}\, \mathrm{col}_{s\colon [s]=p} [\mathbb{H}^s] = \mathrm{col}_{s \in S} [\mathbb{H}^s]. \tag{11.14}$$

By combining (10.4) with the estimates (11.13), we see that $\mathbb{H}^{FM}$ has finite rank if and only if $\mathbb{H}^p$ has finite rank for each $p \in P$.

Now suppose that G and G' are two admissible graphs with the same edge set E and that $T(z)$ is a given formal power series in the noncommuting variables $z = (z_e \colon e \in E)$. By the first part of the proof, realizability of $T(z)$ as the transfer function of an SNMLS with structure graph G and realizability of $T(z)$ as the transfer function of an SNMLS with structure graph G' are each equivalent to realizability of $T(z)$ as the transfer function of a noncommutative Fornasini–Marchesini system with structure graph G^{FM} having edge set E. Hence G-realizability and G'-realizability are equivalent to each other. This completes the proof of Theorem 11.2. $\square$

12. Recognizable and rational formal power series. Formal power series in noncommuting variables of the form arising here have come up in the theory of formal languages as studied in computer science [15]. For the sake of concreteness we index the noncommuting variables simply by $\{1, \dots, d\}$ and work with the semigroup $\mathcal{F}_d$ generated by the concrete set of letters $\{1, \dots, d\}$, as was done in sections 2.1 and 2.2 in the setting of noncommutative Fornasini–Marchesini and Givone–Roesser systems. We specialize the discussion in [15] to the setting here, where we take the scalars to be the field $\mathbb{C}$ of complex numbers rather than a general semiring, i.e., a "ring without subtraction." A formal power series $\sum_{v \in \mathcal{F}_d} T_v z^v$ (with coefficients T_v equal to linear operators acting between the finite-dimensional linear spaces $\mathcal{U}$ and $\mathcal{Y}$) is said to be *recognizable* if there are finite-dimensional linear space $\mathcal{H}$ and operators $A_1, \dots, A_d \colon \mathcal{H} \to \mathcal{H}$, $B \colon \mathcal{U} \to \mathcal{H}$, and $C \colon \mathcal{H} \to \mathcal{Y}$ such that

$$T_v = CA^v B \quad \text{for } v \in \mathcal{F}_d.$$

In terms of the linear systems discussed here, one can view a recognizable series $T(z) = \sum_{v \in \mathcal{F}_d} (CA^v B) z^v$ as the transfer function of a noncommutative Fornasini–Marchesini system

$$\Sigma^{FM} \colon \left\{ \begin{array}{rcl} x(jw) & = & A_j x(w) + B_j u(w) \quad \text{for } j = 1, \dots, d, \\ y(w) & = & Cx(w) + Du(w), \end{array} \right.$$

with the special structure that

$$B_j =: B \text{ is independent of } j \text{ and } D = CB.$$

More economical is to consider the recognizable series as the transfer function of a system of the form

$$\Sigma^{\mathrm{rec}} \colon \left\{ \begin{array}{rcl} x(1w) & = & A_1 x(w) + Bu(1w), \\ & \vdots & \\ x(dw) & = & A_d x(w) + Bu(dw), \\ y(w) & = & Cx(w). \end{array} \right. \tag{12.1}$$

One can check that application of the formal noncommutative Z-transform (2.2) to the system equations Σ^{rec} yields the frequency-domain formulas

$$\begin{aligned} \widehat{x}(z) &= (I - Z_{\mathrm{row}}(z)A)^{-1}(x(\emptyset) - Bu(\emptyset)) + (I - Z_{\mathrm{row}}(z)A)^{-1} B\widehat{u}(z), \\ \widehat{y}(z) &= C(I - Z_{\mathrm{row}}(z)A)^{-1}(x(\emptyset) - Bu(\emptyset)) + T_{\Sigma^{\mathrm{rec}}}(z) \cdot \widehat{u}(z), \end{aligned} \tag{12.2}$$

where the *transfer function* $T_{\Sigma^{\mathrm{rec}}}(z)$ for the recognizable system Σ^{rec} given by

$$T_{\Sigma^{\mathrm{rec}}}(z) = \sum_{v \in \mathcal{F}_d} CA^v B z^v \tag{12.3}$$

has the form of a recognizable formal series. In particular, if the initial condition is given by the input-injection $x(\emptyset) = Bu(\emptyset)$, then multiplication by the transfer function $T_{\Sigma^{\mathrm{rec}}}(z)$ provides the input-output map in the frequency domain $\widehat{y}(z) = T_{\Sigma^{\mathrm{rec}}}(z)\widehat{u}(z)$.

All the results in sections 5, 6, 8, and 11 (notions of controllability and observability, equivalence of controllability and observability with minimality, state-space similarity theorem, realization theorem) have parallels for the case of recognizable systems in place of general SNMLSs; in fact, as surveyed nicely in Chapters 1 and 2 of [15], all these results, with the exception of the identification of a recognizable series $T(z) = \sum_{v \in \mathcal{F}_d} (CA^v B) z^v$ as the transfer function of a noncommutative linear system of the form (12.1), already appear in the literature—even in the more general setting where the scalars are taken to be a general semiring rather than the field $\mathbb{C}$ of complex numbers as is done here (see [37, 38, 17, 39, 20, 21, 22, 23]). We now survey these results from our system-theoretic perspective.

To obtain a physical interpretation for the recognizable controllability operator $\mathcal{C}^{\text{rec}}$ introduced below, it is natural to define the backward system equations giving the evolution on the past $\mathcal{T}^{\text{rec}}_{\text{past}} = \mathcal{F}_d$ to be

$$\Sigma^{\text{rec}}_{\text{backward}}\colon \left\{ \begin{array}{lcl} x(w) & = & \sum_{i=1}^{d} A_i x(wi) + Bu(w), \\ y(w) & = & Cx(w). \end{array} \right. \tag{12.4}$$

If we run the backward system equations on the past and present $\mathcal{T}^{\text{rec}}_{\text{past}} := \mathcal{F}_d$ with the state initialized to be zero sufficiently far in the past and with an input string $\{u(w)\}_{w \in \mathcal{T}^{\text{rec}}_{\text{past}}}$ with finite support on $\mathcal{T}^{\text{rec}}_{\text{past}}$ to compute the state $x(\emptyset)$ at location $\emptyset$, the result is

$$x(\emptyset) = \mathcal{C}^{\text{rec}}(\{u(w)\}_{w \in \mathcal{T}^{\text{rec}}_{\text{past}}}),$$

where the *recognizable controllability operator* $\mathcal{C}^{\text{rec}}$ is given by

$$\mathcal{C}^{\text{rec}} = \text{row}_{w \in \mathcal{T}^{\text{rec}}_{\text{past}}} A^w B, \tag{12.5}$$

where we set $A^w = A_{i_N} A_{i_{N-1}} \cdots A_{i_1}$ if $w = i_N i_{N-1} \cdots i_1 \in \mathcal{T}^{\text{rec}}_{\text{past}}$ (with $A^\emptyset = I_{\mathcal{H}}$). Note that this controllability operator has close to the same form as the Fornasini–Marchesini controllability operator $\mathcal{C}^{FM}$ (2.6); the difference is that a recognizable system has only one input operator B and that the columns of $\mathcal{C}^{\text{rec}}$ are indexed by $\mathcal{T}^{\text{rec}}_{\text{past}}$ which includes the empty word, with $[\mathcal{C}^{\text{rec}}]_\emptyset = B$.

We say that the system Σ^{rec} is *recognizable-controllable* if the image $\operatorname{im} \mathcal{C}^{\text{rec}}$ of the recognizable-controllability operator $\mathcal{C}^{\text{rec}}$ is the whole state-space $\mathcal{H}$.

The observability operator $\mathcal{O}^{\text{rec}}\colon \mathcal{H} \to \ell(\mathcal{T}^{\text{rec}}_{\text{future}}, \mathcal{Y})$ produces the future output $\{y(v)\}_{v \in \mathcal{T}^{\text{rec}}_{\text{future}}}$ generated by the system for a given prescribed initial condition $x(\emptyset) \in \mathcal{H}$ under the assumption that the zero input string $\{u(v)\}_{v \in \mathcal{T}^{\text{rec}}_{\text{future}}}$ is fed into the system; explicitly, we have[4]

$$\mathcal{O}^{\text{rec}} = \underset{v \in \mathcal{F}_d}{\text{row}}\, CA^v. \tag{12.6}$$

Note that $\mathcal{O}^{\text{rec}}$ has exactly the same form as the Fornasini–Marchesini observability operator $\mathcal{O}^{FM}$ from (2.7). We say that the system Σ^{rec} is *recognizable-observable* if the recognizable-observability operator $\mathcal{O}^{\text{rec}}$ is injective on $\mathcal{H}$.

We can now obtain a *recognizable Kalman decomposition* of the state-space $\mathcal{H}$,

$$\mathcal{H} = \mathcal{H}_{c/o} \oplus \mathcal{H}_{c/no} \oplus \mathcal{H}_{nc/o} \oplus \mathcal{H}_{nc/no},$$

[4]Here $\mathcal{T}^{\text{rec}}_{\text{future}}$ is taken to be $\mathcal{F}_d$; the location $\emptyset$ in $\mathcal{T}^{\text{rec}}_{\text{future}}$ is identified with the location $\emptyset$ in $\mathcal{T}^{\text{rec}}_{\text{past}}$ (i.e., both $\mathcal{T}^{\text{rec}}_{\text{future}}$ and $\mathcal{T}^{\text{rec}}_{\text{past}}$ contain the "present"), but a given nonempty word w as an element of the future $\mathcal{T}^{\text{rec}}_{\text{future}}$ is to be considered distinct from the same word w considered as an element of the past $\mathcal{T}^{\text{rec}}_{\text{past}}$.

by the same recipe used in section 7 (by using $\mathcal{C}^{\rm rec}$ in place of $\mathcal{C}_{s_p}$ and $\mathcal{O}^{\rm rec}$ in place of $\mathcal{O}_p$). We then obtain the decompositions

$$A_j = \begin{bmatrix} A_{j;c/o,c/o} & 0 & A_{j;c/o,nc/o} & 0 \\ A_{j;c/no,c/o} & A_{j;c/no,c/no} & A_{j;c/no,nc/o} & A_{j;c/no,nc/no} \\ 0 & 0 & A_{j;nc/o,nc/o} & 0 \\ 0 & 0 & A_{j;nc/no,nc/o} & A_{j;nc/no,nc/no} \end{bmatrix},$$

$$B = \begin{bmatrix} B_{c/o} \\ B_{c/no} \\ 0 \\ 0 \end{bmatrix}, \qquad C = \begin{bmatrix} C_{c/o} & 0 & C_{nc/o} & 0 \end{bmatrix} \tag{12.7}$$

for the system matrices $A_1, \dots, A_d, B, C$ of $\Sigma^{\rm rec}$. It is then easily verified that the reduced recognizable system $\Sigma^{\rm rec}_{c/o}$ with system matrices

$$A_{1;c/o,c/o}, \dots, A_{d;c/o,c/o}, B_{c/o}, C_{c/o}$$

is both recognizable-controllable and recognizable-observable and produces the same transfer function: $T_{\Sigma^{\rm rec}}(z) = T_{\Sigma^{\rm rec}_{c/o}}(z)$. Given two recognizable systems $\Sigma^{\rm rec}$ with system matrices $A_1, \dots, A_d, B, C$ and $\Sigma^{\rm rec\prime}$ with system matrices $A'_1, \dots, A'_d, B', C'$, let us say that $\Sigma^{\rm rec}$ and $\Sigma^{\rm rec\prime}$ are *recognizable-similar* if there is a bijective linear map $\Gamma \colon \mathcal{H} \to \mathcal{H}'$ so that $\Gamma A_j = A'_j \Gamma$ for $j = 1, \dots, d$, $\Gamma B = B'$, and $C' = C\Gamma$. Following the same argument as in section 8, we have the *state-space similarity theorem for recognizable systems*: *given two recognizable systems $\Sigma^{rec} = (A_1 \dots, A_d, B, C)$ and $\Sigma^{rec\prime} = (A'_1, \dots, A'_d, B', C')$ with the same input-space $\mathcal{U}$ and output-space $\mathcal{Y}$, which are both recognizable-controllable and recognizable-observable, then Σ^{rec} and $\Sigma^{rec\prime}$ have the same transfer function*

$$T_{\Sigma^{rec}}(z) = T_{\Sigma^{rec\prime}}(z)$$

if and only if Σ^{rec} and $\Sigma^{rec\prime}$ are recognizable-similar. Furthermore, one can say that the recognizable system $\Sigma^{\rm rec}$ with state-space $\mathcal{H}$ is a *recognizable-minimal realization* for its transfer function $T(z) = T_{\Sigma^{\rm rec}}(z)$ if, whenever $\Sigma^{\rm rec\prime}$ with state-space $\mathcal{H}'$ is any other recognizable realization for the same $T(z)$, then $\dim \mathcal{H} \le \dim \mathcal{H}'$. Following the same line of argument as in section 9, one can show the following: *the recognizable system Σ^{rec} is a recognizable-minimal realization of its transfer function $T_{\Sigma^{rec}}(z)$ if and only if Σ^{rec} is recognizable-controllable and recognizable-observable.*

We next define the *recognizable Hankel operator* by

$$\mathbb{H}^{\rm rec} = \mathcal{O}^{\rm rec} \cdot \mathcal{C}^{\rm rec} \colon \ell_{\rm fin}(\mathcal{T}^{\rm rec}_{\rm past}, \mathcal{U}) \to \ell(\mathcal{T}^{\rm rec}_{\rm future}, \mathcal{Y}). \tag{12.8}$$

The matrix entries of $\mathbb{H}^{\rm rec}$ are then given by

$$\mathbb{H}^{\rm rec}_{w,v} = C A^{wv} B \quad \text{for } w, v \in \mathcal{F}_d \tag{12.9}$$

or directly in terms of the Taylor coefficients of the transfer function $T_{\Sigma^{\rm rec}}(z) = \sum_{v \in \mathcal{F}_d} T_v z^v$ as

$$\mathbb{H}^{\rm rec}_{w,v} = T_{wv} \quad \text{for } w, v \in \mathcal{F}_d. \tag{12.10}$$

In the case that $\Sigma^{\rm rec}$ is both recognizable-controllable and recognizable-observable, we see from the factorization (12.8) that

$$\operatorname{rank} \mathbb{H}^{\rm rec} = \dim \mathcal{H}.$$

In particular, $\operatorname{rank} \mathbb{H}^{\mathrm{rec}} < \infty$, where we now use (12.10) to define $\mathbb{H}^{\mathrm{rec}}$ directly in terms of the formal power series $T(z) = \sum_{v \in \mathcal{F}_d} T_v z^v$, which is a necessary condition for $T(z)$ to have a recognizable realization $T(z) = \sum_{v \in \mathcal{F}_d} (CA^v B) z^v$. For the converse, we have the following realization theorem.

THEOREM 12.1. *Let the formal power series $T(z) = \sum_{v \in \mathcal{F}_d} T_v z^v$ in d noncommuting indeterminates $z = (z_1, \dots, z_d)$, with coefficients T_v equal to linear operators between the linear spaces $\mathcal{U}$ and $\mathcal{Y}$, be given. Then a necessary and sufficient condition for $T(z)$ to be recognizable, i.e., for the existence of a linear space $\mathcal{H}$ and operators $A_1, \dots, A_d$ on $\mathcal{H}$, $B\colon \mathcal{U} \to \mathcal{H}$, and $C\colon \mathcal{H} \to \mathcal{Y}$ with $T_v = CA^v B$ for $v \in \mathcal{F}_d$, is that*

$$\operatorname{rank} \mathbb{H}^{rec} < \infty. \tag{12.11}$$

When this holds, a recognizable-minimal realization $(A_1, \dots, A_d, B, C)$ can be constructed as follows: set

$$\mathcal{H} = \ell_{fin}(\mathcal{T}^{rec}_{past}, \mathcal{U}) / \ker \mathbb{H}^{rec} \tag{12.12}$$

and define operators $A_j\colon \mathcal{H} \to \mathcal{H}$ (for $j = 1, \dots, d$), $B\colon \mathcal{U} \to \mathcal{H}$, and $C\colon \mathcal{H} \to \mathcal{Y}$:

$$A_j\colon [\delta_v]_{\mathcal{H}} \mapsto [\delta_{jv}]_{\mathcal{H}} \quad \text{for } v \in \mathcal{T}^{rec}_{past},$$

$$B\colon u \mapsto [\delta_\emptyset]_{\mathcal{H}}, \tag{12.13}$$

$$C\colon [\{u(v)\}_{v \in \mathcal{T}^{rec}_{past}}]_{\mathcal{H}} \mapsto \sum_{v \in \mathcal{T}^{rec}_{past}} T_v u(v). \tag{12.14}$$

Proof. The proof parallels the ideas in the proof of Theorem 11.1, so we omit the details. The result is also essentially contained in Theorem 1.5 of [15] (without any system-theoretic interpretation using the system equations (12.1) and (12.4)), where it is attributed to [17] and [20]. ☐

Note that the recognizable Hankel $\mathbb{H}^{\mathrm{rec}}$ is almost the same as the Fornasini–Marchesini Hankel $\mathbb{H}^{\mathrm{FM}}$; namely, we have

$$\mathbb{H}^{\mathrm{rec}} = \begin{bmatrix} \operatorname{col}_{v \in \mathcal{F}_d}[T_v] & \mathbb{H}^{FM} \end{bmatrix}. \tag{12.15}$$

In particular, we see that

$$\operatorname{rank} \mathbb{H}^{FM} \le \operatorname{rank} \mathbb{H}^{\mathrm{rec}} \le \dim \mathcal{U} + \operatorname{rank} \mathbb{H}^{FM},$$

and hence $\mathbb{H}^{\mathrm{FM}}$ has finite rank if and only if $\mathbb{H}^{\mathrm{rec}}$ has finite rank. Combining this observation with Theorems 12.1, 11.1, and 11.2, we arrive at the following result.

COROLLARY 12.2. *Let a formal power series $T(z) = \sum_{v \in \mathcal{F}_d} T_v z^v$ in d noncommuting variables $z = (z_1, \dots, z_d)$ and an admissible graph G with edge set E labeled as $E = \{1, \dots, d\}$ be given. Then T has a realization of the form $T(z) = D + C(I - Z_\Sigma(z)A)^{-1} Z_\Sigma(z) B$ for an SNMLS $\Sigma = (G, \mathcal{H}, U)$ if and only if $T(z) = C(I - z_1 A_1 - \cdots - z_d A_d)^{-1} B$ is recognizable.*

A related notion arising in the theory of formal languages, particularly in the work of Schützenberger, is that of rationality. We say that a formal power series $T(z) = \sum_{v \in \mathcal{F}_d} T_v z^v \in \mathbb{C}\langle\langle z \rangle\rangle$ in noncommuting variables $z = (z_1, \dots, z_d)$ with scalar coefficients $T_v \in \mathbb{C}$ is *rational* if it is in the smallest subalgebra of $\mathbb{C}\langle\langle z \rangle\rangle$ which contains the polynomials and is invariant under the operator $R(z) \mapsto R^*(z) = \sum_{n=0}^{\infty} (R(z))^n$ defined on proper formal power series $R(z) = \sum_{v \in \mathcal{F}_d \setminus \{\emptyset\}} R_v z^v$. The demand here that the constant term $R_\emptyset$ vanish guarantees that, for each word w, the w-coefficient of $R(z)^n$ vanishes for all $n \geq N_w$ for some $N_w < \infty$, and hence that the infinite series expression for $R^*(z)$ is convergent in the topology of coefficientwise convergence. The $*$-operation also makes sense in the setting where the scalars are taken from a general semiring K; in case K is a field (as we assume), the $*$-operation $R(z) \mapsto R^*(z)$ can be identified as $R^*(z) = (I - R(z))^{-1}$. In case that $T(z) = \sum_{v \in \mathcal{F}_d} T_v z^v \in \mathcal{L}(\mathcal{U}, \mathcal{Y})\langle\langle z \rangle\rangle$ has coefficients T_v equal to operators between finite-dimensional linear spaces $\mathcal{U}$ and $\mathcal{Y}$, we say that $T(z)$ is rational if each of its matrix entries (with respect to some bases for $\mathcal{U}$ and $\mathcal{Y}$) is rational. In case $\mathcal{U} = \mathcal{Y}$ and $T_\emptyset = 0$, we can define

$$T^*(z) := \sum_{n=0}^{\infty} (T(z))^n = (I - T(z))^{-1} \tag{12.16}$$

just as in the scalar case. The following lemma assures us that $T^*(z)$ is again rational if $T(z)$ is rational. This result is actually a special case of Lemma I.6.3 in [15], but we include a proof for the sake of completeness.

LEMMA 12.3. *Suppose that $T(z) = [T_{ij}(z)]_{i,j=1}^N \in \mathcal{L}(\mathbb{C}^N)\langle\langle z \rangle\rangle$ is a formal power series in the noncommuting variables $z = (z_1, \ldots, z_d)$ with matrix entries $T_{ij}(z) \in \mathbb{C}\langle\langle z \rangle\rangle$ all rational such that $T_\emptyset = [T_{\emptyset,ij}]_{i,j=1}^N = 0$. Then all matrix entries of the formal power series $T^*(z)$ given by* (12.16) *are also rational.*

Proof. If $N = 1$, the result is clear. By induction we assume that the result is true for all $N < N_0$ and seek to prove the result for $N = N_0$. Given $T(z) \in \mathcal{L}(\mathbb{C}^{N_0})\langle\langle z \rangle\rangle$ with $T_\emptyset = 0$, consider a block decomposition of $T(z)$,

$$T(z) = \begin{bmatrix} a(z) & b(z) \\ c(z) & d(z) \end{bmatrix},$$

and a corresponding block decomposition of $T^*(z) = (I_{N_0} - T(z))^{-1}$,

$$(I_{N_0} - T(z))^{-1} = \begin{bmatrix} \alpha(z) & \beta(z) \\ \gamma(z) & \delta(z) \end{bmatrix},$$

where $a(z)$ and $\alpha(z)$ are both of size $K \times K$ for some K with $1 \leq K < N_0$. From the identity

$$(I_{N_0} - T(z))^{-1} = I_{N_0} + T(z)(I_{N_0} - T(z))^{-1}$$

we get the collection of identities

$$\begin{aligned} \alpha(z) &= I_K + a(z)\alpha(z) + b(z)\gamma(z), \\ \beta(z) &= a(z)\beta(z) + b(z)\delta(z), \\ \gamma(z) &= c(z)\alpha(z) + d(z)\gamma(z), \\ \delta(z) &= I_{N_0-K} + c(z)\beta(z) + d(z)\delta(z). \end{aligned} \tag{12.17}$$

We may then solve the second and third equations in (12.17) for $\beta(z)$ and $\gamma(z)$, respectively, to get

$$\beta(z) = (I_K - a(z))^{-1} b(z)\delta(z), \tag{12.18}$$

$$\gamma(z) = (I_{N_0-K} - d(z))^{-1} c(z)\alpha(z). \tag{12.19}$$

By the induction assumption we see immediately from (12.18) and (12.19) that $\beta(z)$ and $\gamma(z)$ are rational. Plugging back into the first and fourth identities in (12.17) then gives

$$\alpha(z) = I_K + a(z)\alpha(z) + b(z)(I_{N_0-K} - d(z))^{-1}c(z)\alpha(z),$$
$$\delta(z) = I_{N_0-K} + c(z)(I_K - a(z))^{-1}b(z)\delta(z) + d(z)\delta(z).$$

We may then solve these equations for $\alpha(z)$ and $\delta(z)$ to get

$$\alpha(z) = \left(I_K - [a(z) + b(z)(I_{N_0-K} - d(z))^{-1}c(z)]\right)^{-1}, \tag{12.20}$$

$$\delta(z) = \left(I_{N_0-K} - [c(z)(I_K - a(z))^{-1}b(z) + d(z)]\right)^{-1}. \tag{12.21}$$

Again as a consequence of the induction assumption, (12.20) and (12.21) imply that $\alpha(z)$ and $\delta(z)$ are rational as well, and the lemma follows. □

The following characterization of rational formal power series can be seen as a corollary of the results of this paper.

COROLLARY 12.4. *Let a formal power series $T(z) = \sum_{v\in\mathcal{F}_d} T_v z^v$ in d noncommuting variables $z = (z_1, \dots, z_d)$ and an admissible graph G with edge set $E = \{1, \dots, d\}$ be given. Then the following are equivalent:*

(1) *$T(z)$ is rational.*

(2) *For each path-connected component p of G, the Hankel operator $\mathbb{H}^p$ given by (10.4) has finite rank.*

(3) *$T(z)$ has a realization $T(z) = D + C(I - Z_\Sigma(z)A)^{-1}Z_\Sigma(z)B$ for an SNMLS $\Sigma = (G, \mathcal{H}, U)$ having structure graph G.*

Proof. We first show that (1) $\Longrightarrow$ (3). Note first that any scalar constant D (considered as a formal power series in noncommuting variables $z = (z_1, \dots, z_d)$) is realizable (with zero auxiliary state-spaces $\mathcal{H}_p$).

We next note that any monomial z_e is realizable for each edge $e = 1, \dots, d$. Indeed, set $\mathcal{H}_{[\mathbf{s}(e)]} = \mathbb{C}$ and $\mathcal{H}_p = \{0\}$ for $p \neq [\mathbf{s}(e)]$ and set

$$A = [A_{r,s}]_{r\in R, s\in S} \quad \text{with } A_{r,s} = 0,$$
$$B = \mathrm{col}_{r\in R}[B_r] \quad \text{with } B_r = \begin{cases} 1 & \text{if } r = \mathbf{r}(e), \\ 0 & \text{otherwise,} \end{cases}$$
$$C = \mathrm{row}_{s\in S}[C_s] \quad \text{with } C_s = \begin{cases} 1 & \text{if } s = \mathbf{s}(e), \\ 0 & \text{otherwise,} \end{cases}$$
$$D = 0.$$

Then the associated transfer function is given by

$$\begin{aligned} T_\Sigma(z) &= D + C(I - Z_\Sigma(z)A)^{-1}Z_\Sigma(z)B \\ &= 0 + CZ_\Sigma(z)B \\ &= \sum_{s\in S}\sum_{r\in R} C_s[Z_\Sigma(z)]_{s,r}B_r \\ &= \sum_{s\in S}\sum_{r\in R}\sum_{e'\in E} C_s I_{\Sigma,e';s,r}B_r z_{e'} \\ &= \sum_{e'\in E} C_{\mathbf{s}(e')}B_{\mathbf{r}(e')}z_{e'} \\ &= z_e. \end{aligned}$$

We conclude that each monomial z_e has a realization as asserted.

By Theorems 4.1, 4.2, and 4.3, products, sums, and inverses of invertible formal power series which are realizable (as the transfer function of an SNMLS Σ with structure graph G) are again realizable. By the inductive definition of rational formal power series given above, we may now conclude that any scalar rational formal power series $T(z)$ has the form of a transfer function $T(z) = T_\Sigma(z)$ for an SNMLS Σ with given admissible graph G as structure graph.

If each scalar entry $[T(z)]_{i,j}$ of a matrix of formal power series is realizable, it is easy to construct a realization (not necessarily minimal) for the formal power series $T(z)$ with matrix coefficients. This concludes the proof of (1) $\implies$ (3).

We next verify (3) $\implies$ (1). Assume that the formal power series $T(z)$ has a realization of the form $T(z) = D + C(I - Z_\Sigma(z)A)^{-1}Z_\Sigma(z)B$ for a finite-dimensional SNMLS $\Sigma = (G, \mathcal{H}, [\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}])$. By Lemma 12.3 it follows that $(I - Z_\Sigma(z)A)^{-1}$ is rational. As products and sums of rational matrix functions are rational, it then follows that $T(z)$ is rational, as wanted.

The equivalence of (2) and (3) is just a restatement of Theorem 11.1. ∎

Remark 12.5. We note that the equivalence (1) $\iff$ (2) between rationality and finiteness of the rank of an associated Hankel operator is known as Kronecker's theorem in the classical case.

Remark 12.6. Combining (1) $\iff$ (3) in Corollary 12.4 with Corollary 12.2, we see that a formal power series is recognizable if and only if it is rational; this result goes back to Schützenberger (see Theorem I.6.1 in [15]).

Remark 12.7. In [22] Fliess gives an alternative system interpretation of a recognizable formal power series in terms of a homogeneous bilinear system with evolution along the nonnegative integers $\mathbb{Z}_+$ but with state-update equation of the form

$$x(n+1) = \left[\sum_{j=0}^{d} u_j(n) A_j \right] x(n),$$

with $A_0, \ldots, A_d$ linear operators on the state-space $\mathcal{H}$ and with $u_0(n), \ldots, u_d(n)$ equal to $d+1$ scalar-valued controls. The input-output operator for the system is obtained as

$$(x_0, (u_0(n), \ldots, u_d(n))_{n \in \mathbb{Z}_+}) \mapsto T_\Sigma(u)x_0,$$

where $T_\Sigma(z)$ is the recognizable formal power series $T_\Sigma(z) = C(I - z_0A_0 - z_1A_1 - \cdots - z_dA_d)^{-1}$ and where $T_\Sigma(u)$ is defined via the substitution

$$z_{i_N} z_{i_{N-1}} \ldots z_{i_0} \mapsto u_{i_N}(N) u_{i_{N-1}}(N-1) \cdots u_{i_0}(0).$$

Multidimensional versions of such bilinear systems, including connections with formal power series in this more general setting, are given in [23]. Sontag [40] used a variation of Fliess's Hankel-matrix construction to solve the following related moment problem connected with an alternative formulation of a bilinear system realization problem: *given operators $T_w \in \mathcal{L}(\mathcal{U}, \mathcal{Y})$ for $w \in \mathcal{F}_E$ ($E = \{1, \ldots, d\}$), find operators $C_1, \ldots, C_d \colon \mathcal{H} \to \mathcal{Y}$, $A_1, \ldots, A_d \colon \mathcal{H} \to \mathcal{H}$, and $B_1, \ldots, B_d \colon \mathcal{U} \to \mathcal{H}$ so that $T_{i_N i_{N-1} \cdots i_2 i_1} = C_{i_N} A_{i_{N-1}} \cdots A_{i_1} B_{i_1}$.*

Our discussion here gives a linear (rather than bilinear) system interpretation for a formal power series, but with evolution along a free semigroup rather than along $\mathbb{Z}_+$ and with a somewhat contrived input-injection for the initial condition on the state required to recover the precise form of a recognizable series. The awkwardness of these various system interpretations for a recognizable formal power series gives some explanation as to why system operations work out well for transfer functions of SNMLSs (see section 4) but not so well for recognizable series—a point discussed in [30].

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