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Structures of electromagnetic type on vector bundles

E. REYES, V. CRUCEANU & P.M. GADEA

Abstract

Structures of electromagnetic type on a vector bundle are introduced and studied. The metric case is also defined and studied. The sets of compatible connections are determined and a canonical connection is defined.

1 Introduction

Structures of electromagnetic type (em-structures) and structures of metric electromagnetic type (mem-structures) on a manifold were progressively introduced in [9, 11, 7] (see also [6]) and studied in detail in [5, 7, 8, 13, 14]. In the present paper we define similar structures for the case of a vector bundle $\xi = (E, \pi, M)$, and relate them to product, complex, para-Hermitian, Hermitian, para-Kähler or indefinite Kähler, structures. (In the sequel, by a pseudo-Riemannian metric we shall understand a metric of any signature, and by an indefinite (metric) structure a structure including a pseudo-Riemannian metric.) Then, we determine the set of connections on ξ compatible with those structures and we introduce a canonical connection. Considering an almost para-Hermitian (resp. indefinite Hermitian) structure on the base manifold M and an indefinite Hermitian (resp. para-Hermitian) structure of the bundle ξ , we prove that the corresponding diagonal lift of these structures, with respect to a connection on ξ , are mem-structures on the total space E . Finally, some properties of those mem-structures are established.

We recall the physical origin of the topic ([9, 11]). Let M^4 be a spacetime of general relativity, with gravitational tensor g of signature $-+++$. Let F be the electromagnetic field of type $(0, 2)$, which is skewsymmetric, that is a 2-form. Setting $F(X, Y) = g(JX, Y)$, the tensor field J so defined is the electromagnetic tensor field of type $(1, 1)$ associated to F . We have $g(JX, Y) + g(X, JY) = 0$. The characteristic equation of J is $\det(J - \lambda I) = 0$, which is satisfied by J , and we have

$$J^4 + 2kJ^2 + lI = 0, \quad k = -\frac{1}{4} \text{trace } J^2, \quad l = \det J.$$

If $x \in M^4$, it is said that J_x is of 1^{st} , 2^{nd} , or 3^{rd} class at x if, respectively,

$$l_x \neq 0, \quad l_x = 0, k_x \neq 0, \quad l_x = 0, k_x = 0.$$

It is said that J is of 1st, 2nd, or 3rd class if it is of such class at every x . The characteristic polynomial of the second class is $J^2(J^2 + 2k)$, but the minimal polynomial is $J(J^2 + 2k)$, so that the condition $J(J^2 + 2k) = 0$ characterizes the second class. The field of an electromagnetic plane wave is of 3rd class. The field of a moving electron is of 2nd class. More complicated fields belong to the 1st class. The equation one gets from the minimal polynomial in the 1st class is

$$(1.1) \quad (J^2 - f^2)(J^2 + h^2) = 0.$$

with f, h nowhere-vanishing C^∞ functions on M^4 . Such a tensor field J on a general manifold M determines a G -structure on M .

To handle the nonconstant local cross-section situation corresponding to (1.1), one can use the relationships among G -structures, related sections of an associated bundle and functions of certain kind on M , as follows: Let $(\mathcal{P}, \pi_P, M, H)$ be a principal bundle with group H , $H \times W \rightarrow W$ a left action of H on a manifold W , and $(E = \mathcal{P} \times_H W, \pi_E, M, W)$ the associated bundle. A J -subset S of W with corresponding group G , a subgroup of H , is defined by the conditions: (1) $S \subset \text{fixpoint set of } G$, (2) $h \in H, h(S) \cap S \neq \emptyset \Rightarrow h \in G$. For instance, points are J -subsets with G the corresponding isotropy group. A cross-section K of π_E is a J -section if it can be locally represented as the "product" of a cross-section σ of π_P and a S -valued function \tilde{K} , so that

$$K_x = \sigma_x \cdot \tilde{K}_x = \text{equivalence class of } (\sigma_x, \tilde{K}_x) \text{ in } E.$$

Then \tilde{K} is globally defined, and the σ generate a principal subbundle of \mathcal{P} . K is a constant J -section if and only if \tilde{K} is constant. Different sections can generate the same subbundle, and in fact, every principal subbundle can be generated by a constant J -section.

Now, let \mathcal{P} be the principal bundle of frames over M , so that $H = GL(n, \mathbb{R})$, and let W be a real vector space. If $J \in W$ is given with the conditions stated above, a J -section generates a J -structure with group G , which is a G -structure. The tensor K has in principle *variable components* in adapted frames. This is a slight generalization with respect to the usually considered G -structures, given by tensors with constant components, which here correspond to constant J -sections. Since every J -structure is generated by some constant J -section, this generalization is useless for the study of the J -structure itself; but if the emphasis shifts to the study of variable J -sections, the results are significant, specially with respect to the parallelizability of the tensors.

In the particular case of a $(1, 1)$ tensor field J satisfying $(J^2 - f^2)(J^2 + h^2) = 0$, with characteristic polynomial $(x - p)^{r_1}(x - p)^{r_2}(x^2 + q^2)^s$, $r_1, r_2, s \geq 1$, $r_1 + r_2 + 2s = n = \dim M$, the J -subset consists of matrices of the form

$$\begin{pmatrix} pI_{r_1} & & & \\ & -pI_{r_2} & & \\ & & & -qI_s \\ & & qI_s & \end{pmatrix}$$

and the structural group is $G = GL(r_1, \mathbb{R}) \times GL(r_2, \mathbb{R}) \times GL(s, \mathbb{C})$. It is proved ([7]) that the G -structure defined by J above is also defined by a tensor field, say again J , satisfying $(J^2 - 1)(J^2 + 1) = 0$, that is, the relation $J^4 = 1$ considered in the present paper.

Notice that *the G -structure is exactly the same, not an associated or equivalent one*. In the 4-dimensional case the group reduces to $G = GL(1, \mathbb{R}) \times GL(1, \mathbb{R}) \times GL(1, \mathbb{C})$. It is also proved ([7]) that there exists an adapted Riemannian metric so that the group can be reduced to $G = O(r_1) \times O(r_2) \times U(s)$, and in the 4-dimensional case to $\mathbb{Z}_2 \times \mathbb{Z}_2 \times U(1)$, that is, essentially to the unitary group $U(1)$.

2 Structures of electromagnetic type on a vector bundle

Let $\xi = (E, \pi, M)$ be a C^∞ vector bundle with total space E and projection map π over a connected paracompact base manifold M . The rank of E is the (common) dimension of the fibres. Let $C^\infty(M)$ denote the ring of real functions, $\mathcal{T}_q^p(M)$ the $C^\infty(M)$ -module of (p, q) -tensor fields, and $\mathcal{T}(M)$ the $C^\infty(M)$ -tensor algebra of M . We respectively denote by $\mathcal{T}_q^p(\xi)$ and $\mathcal{T}(\xi)$ the $C^\infty(M)$ -module of tensor fields of type (p, q) and the $C^\infty(M)$ -tensor algebra of the bundle ξ .

We recall that an almost product (resp. almost complex) structure on a manifold M is defined by a tensor field J of type $(1, 1)$ satisfying $J^2 = I$ (resp. $J^2 = -I$). An almost para-Hermitian (resp. indefinite almost Hermitian) structure on M is defined by a couple (J, g) , given by an almost product (resp. almost complex) structure J and a pseudo-Riemannian metric compatible with J in the sense that $g(JX, Y) + g(X, JY) = 0$, $X, Y \in \mathfrak{X}(M)$; that is, as an anti-isometry (resp. isometry). A para-Kähler (resp. indefinite Kähler) manifold is a manifold M endowed with an almost para-Hermitian (resp. indefinite almost Hermitian) structure such that the Levi-Civita connection of g parallelizes J .

Definition 2.1. A *structure of electromagnetic type* on $\xi = (E, \pi, M)$ is an M -endomorphism J of ξ satisfying

$$J^4 = I,$$

with characteristic polynomial $(x - 1)^{r_1}(x + 1)^{r_2}(x^2 + 1)^s$, where r_1, r_2, s are constants greater than or equal to 1 such that $r_1 + r_2 + 2s = \text{rank } E$.

Setting $P = J^2$, we have $P^2 = I$, so P is a product structure on ξ , admitting J as a “square root”. Conversely, if P is a product structure admitting a “square root” J , then J is an em-structure on ξ . Denoting by ξ_1 and ξ_2 respectively the $+1$ and -1 eigen-subbundles of P , it is easy to see that ξ_1 and ξ_2 are invariant by J and that $J_1 = J|_{\xi_1}$ defines a product structure of ξ_1 and $J_2 = J|_{\xi_2}$ a complex structure of ξ_2 . So, one has

$$(2.1) \quad \xi = \xi_1 \oplus \xi_2, \quad J = J_1 \oplus J_2.$$

Conversely, if ξ_1 and ξ_2 are two supplementary subbundles of ξ , J_1 is a product structure of ξ_1 , and J_2 a complex structure of ξ_2 , then $J = J_1 \oplus J_2$ is an em-structure on ξ . Denoting by P_1 and P_2 the projections of ξ on ξ_1 and ξ_2 respectively, we obtain

$$P = P_1 - P_2, \quad J = J_1 \circ P_1 + J_2 \circ P_2.$$

Summing up we have

Proposition 2.1. *An em-structure on the vector bundle $\xi = (E, \pi, M)$ can be defined by each one of the following conditions:*

- (1) *An M -endomorphism J of ξ satisfying $J^4 = I$,*
- (2) *A product structure P of ξ admitting a “square root” J ,*
- (3) *Two supplementary subbundles ξ_1 and ξ_2 of ξ respectively endowed with a product structure and a complex structure.*

Remark 2.1. A product structure P which admits a “square root” is a particular one because rank ξ_2 must be even.

Definition 2.2. *A structure of metric electromagnetic type (mem-structure) on the vector bundle ξ is a pair (J, g) , where J is an em-structure and g a pseudo-Riemannian metric on ξ satisfying the compability condition*

$$(2.2) \quad g(JX, Y) + g(X, JY) = 0, \quad X, Y \in \xi.$$

Denoting by δ_J the derivation defined by J in the tensor algebra $\mathcal{T}(\xi)$, the relation (2.2) can be written as

$$\delta_J g = 0,$$

from which it follows $g(PX, PY) = g(X, Y)$, $X, Y \in \mathfrak{X}(M)$. Therefore, the pair (P, g) is a pseudo-Riemannian product structure of ξ and so the subbundles ξ_1 and ξ_2 are mutually orthogonal with respect to g . Denoting respectively by g_1 and g_2 the restrictions of g to ξ_1 and ξ_2 , from (2.2) we obtain

$$(2.3) \quad \delta_{J_1} g_1 = 0, \quad \delta_{J_2} g_2 = 0,$$

which may be written

$$(2.4) \quad g_1(J_1 X, J_1 X) = -g_1(X, Y), \quad g_2(J_2 X, J_2 Y) = g_2(X, Y), \quad X, Y \in \mathfrak{X}(\xi).$$

Hence (J_1, g_1) is a para-Hermitian structure of ξ_1 and (J_2, g_2) is an indefinite Hermitian structure of ξ_2 . Conversely, if ξ_1 and ξ_2 are two supplementary subbundles of ξ such that ξ_1 is endowed with a para-Hermitian structure (J_1, g_1) and ξ_2 with an indefinite Hermitian structure (J_2, g_2) , then considering J as given by (2.1) and setting

$$g = g_1 \oplus g_2,$$

one obtains a mem-structure on ξ . So we have

Proposition 2.2. *A mem-structure (J, g) on ξ is equivalent to a pair of supplementary subbundles ξ_1 and ξ_2 respectively endowed with a para-Hermitian structure (J_1, g_1) and an indefinite Hermitian structure (J_2, g_2) .*

Remark 2.2. If (J, g) is a mem-structure on ξ , then we have: $\text{rank } \xi_1$ and $\text{rank } \xi_2$ are even; $\text{trace } J_1 = \text{trace } J_2 = 0$; $\text{sign } g_1 = 0$.

Setting for a mem-structure (J, g) on ξ :

$$\Omega(X, Y) = g(JX, Y), \quad \Omega_i(X, Y) = g_i(J_i X, Y), \quad i = 1, 2,$$

it follows that Ω , Ω_1 , and Ω_2 are 2-forms which determine almost symplectic structures of ξ , ξ_1 and ξ_2 , so that

$$\Omega = \Omega_1 \oplus \Omega_2.$$

These 2-forms satisfy

$$(2.5) \quad \delta_J \Omega = 0, \quad \delta_{J_1} \Omega_1 = 0, \quad \delta_{J_2} \Omega_2 = 0.$$

Remark 2.3. The meaning of conditions (2.2), (2.3) and (2.5) is the following: The groups of automorphisms of $\mathfrak{X}(\xi_1)$, $\mathfrak{X}(\xi_2)$, and $\mathfrak{X}(\xi)$ given by

$$\alpha_t = I_1 \cosh t + J_1 \sinh t, \quad \beta_t = I_2 \cos t + J_2 \sin t, \quad \gamma_t = \alpha_t \oplus \beta_t,$$

$t \in \mathbb{R}$, determine actions on the tensor algebras $\mathcal{T}(\xi_1)$, $\mathcal{T}(\xi_2)$, and $\mathcal{T}(\xi)$, which respectively preserve the structures (J_1, g_1, Ω_1) , (J_2, g_2, Ω_2) , and (J, g, Ω) .

3 Compatible connections

3.1 The general case

Definition 3.1. A connection D on the vector bundle ξ is said to be *compatible* with an em-structure J if

$$(3.1) \quad DJ = 0.$$

From this it follows $DP = 0$, hence D preserves the subbundles ξ_1 and ξ_2 , *i.e.*, for $X \in \mathfrak{X}(M)$, $Y_1 \in \mathfrak{X}(\xi_1)$, $Y_2 \in \mathfrak{X}(\xi_2)$, one has $D_X Y_1 \in \mathfrak{X}(\xi_1)$, $D_X Y_2 \in \mathfrak{X}(\xi_2)$. Setting then

$$D_X^1 Y_1 = D_X Y_1, \quad D_X^2 Y_2 = D_X Y_2, \quad X \in \mathfrak{X}(M), \quad Y_1 \in \mathfrak{X}(\xi_1), \quad Y_2 \in \mathfrak{X}(\xi_2),$$

we have that D^1 and D^2 are respectively connections on ξ_1 and ξ_2 , so that

$$(3.2) \quad D_X = D_X^1 \circ P_1 + D_X^2 \circ P_2, \quad D_X^1 J_1 = 0, \quad D_X^2 J_2 = 0, \quad X \in \mathfrak{X}(M).$$

Conversely, if D^1 and D^2 are respectively connections on ξ_1 and ξ_2 , then D given as in (3.2) is a connection on ξ satisfying $DP = 0$. If D_1 and D_2 satisfy the respective conditions in (3.2), then D satisfies (3.1) too. Thus, it follows

Proposition 3.1. *A connection D on ξ is compatible with the em-structure J if and only if there exist two connections D^1 on ξ_1 and D^2 on ξ_2 , respectively compatible with the product structure J_1 and the complex structure J_2 , so that*

$$(3.3) \quad D = D^1 \circ P_1 + D^2 \circ P_2.$$

Consider now on the subbundles ξ_i of ξ , the operators Φ_{J_i} and Ψ_{J_i} given by

$$(3.4) \quad (\Phi_{J_i} D^i)_X = \frac{1}{2}(D_X^i + J_i^{-1} \circ D_X^i \circ J_i), \quad (\Psi_{J_i} \mathcal{A}^i)_X = \frac{1}{2}(\mathcal{A}_X^i + J_i^{-1} \circ \mathcal{A}_X^i \circ J_i),$$

where $X \in \mathfrak{X}(M)$, D^i is a connection on ξ_i , and $\mathcal{A}^i \in \Lambda^1(M) \otimes \mathfrak{X}(\xi_i) \otimes \Lambda^1(\xi_i)$ (now and in the sequel we take $i = 1, 2$). From [1, 13] and Proposition 3.1 we obtain

Proposition 3.2. *The set of connections on ξ compatible with the em-structure J is given by*

$$D_X = \{(\Phi_{J_1} D^{\circ 1})_X + (\Psi_{J_1} \mathcal{A}^1)_X\} \circ P_1 + \{(\Phi_{J_2} D^{\circ 2})_X + (\Psi_{J_2} \mathcal{A}^2)_X\} \circ P_2,$$

where $X \in \mathfrak{X}(M)$ and $D^{\circ i}$ is an arbitrary fixed connection on ξ_i , \mathcal{A}^i denotes any element of $\Lambda^1(M) \otimes \mathfrak{X}(\xi_i) \otimes \Lambda^1(\xi_i)$, and Φ_{J_i}, Ψ_{J_i} are given by (3.4).

Definition 3.2. A connection D on ξ is said to be *compatible with the mem-structure (J, g)* if

$$DJ = 0, \quad Dg = 0,$$

From which it follows: $DP = 0$; $D = D^1 \circ P_1 + D^2 \circ P_2$, where D^i are the restrictions of D to ξ_1 and ξ_2 ; $D^i J_i = 0$; and $D^i g_i = 0$. Conversely, if D^1 and D^2 are connections on ξ_1 and ξ_2 , compatible with the para-Hermitian structure (J_1, g_1) and the indefinite Hermitian structure (J_2, g_2) respectively, then the connection D given by (3.3) is compatible with the mem-structure (J, g) on ξ . So, we have

Proposition 3.3. *A connection D on ξ is compatible with the mem-structure (J, g) on ξ , if and only if there are two connections D^1 and D^2 on the subbundles ξ_1 and ξ_2 , respectively compatible with the para-Hermitian structure (J_1, g_1) and the indefinite Hermitian structure (J_2, g_2) , so that D is given by (3.3).*

Setting then

$$(3.5) \quad (\Phi_{g_i} D^i)_X = \frac{1}{2}(D_X^i + g_i^{-1} \circ D_X^i \circ g_i), \quad (\Psi_{g_i} \mathcal{A}^i)_X = \frac{1}{2}(\mathcal{A}_X^i + g_i^{-1} \circ \mathcal{A}_X^i \circ g_i),$$

we obtain from [1], Prop. 3.3, and (2.4)

Proposition 3.4. *The set of connections on ξ compatible with the mem-structure (J, g) is given by*

$$D_X = \{((\Phi_{g_1} \circ \Phi_{J_1}) D^{\circ 1})_X + ((\Psi_{g_1} \circ \Psi_{J_1}) \mathcal{A}^1)_X\} \circ P_1 \\ + \{((\Phi_{g_2} \circ \Phi_{J_2}) D^{\circ 2})_X + ((\Psi_{g_2} \circ \Psi_{J_2}) \mathcal{A}^2)_X\} \circ P_2,$$

where $D^{\circ i}$ is an arbitrary fixed connection on ξ_i , $\mathcal{A}^i \in \Lambda^1(M) \otimes \mathfrak{X}(\xi_i) \otimes \Lambda^1(\xi_i)$, and $\Phi_{J_i}, \Phi_{g_i}, \Psi_{J_i}, \Psi_{g_i}$ are given by (3.4) and (3.5).

3.2 The case of the tangent bundle

We now consider the case of ξ being the tangent bundle of the manifold M , *i.e.*, $\xi = (TM, \pi, M)$. In this case, for a mem-structure (J, g) on M , the pair (P, g) is a pseudo-Riemannian almost product structure on M , and $(J_1, g_1), (J_2, g_2)$, are respectively a para-Hermitian [4] and an indefinite Hermitian structure [10] on ξ_1 and ξ_2 . If ∇ is a linear connection on M , compatible with P , *i.e.*, $\nabla P = 0$, then its restrictions ∇^1 and ∇^2 to ξ_1 and ξ_2 are connections on these subbundles. If T is the torsion tensor of ∇ , we shall call *torsion tensor* of ∇^i to the tensor fields T^i given by $T^i = P_i \circ T|_{\xi_i}$, or in more detail

$$T^i(X_i, Y_i) = \nabla_{X_i} Y_i - \nabla_{Y_i} X_i - P_i[X_i, Y_i], \quad X_i, Y_i \in \mathfrak{X}(\xi_i).$$

We call *tensors of nonholonomy* of the distributions ξ_1 and ξ_2 to the tensor fields $S^1 = P_2 \circ T|_{\xi_1}$ and $S^2 = P_1 \circ T|_{\xi_2}$, respectively. We obtain

$$S^1(X_1, Y_1) = -P_2[X_1, Y_1], \quad S^2(X_2, Y_2) = -P_1[X_2, Y_2].$$

It follows

Proposition 3.5. *The distribution ξ_1 (resp. ξ_2) is involutive if and only if $S^1 = 0$ (resp. $S^2 = 0$).*

After some computations we obtain from [3, 10, 14]

Proposition 3.6. *For a mem-structure (J, g) on a manifold M , there exists a unique linear connection ∇ with torsion tensor T , satisfying the conditions*

$$(3.6) \quad \nabla P = 0, \quad T(PX, Y) = T(X, PY),$$

$$(3.7) \quad \nabla_{X_i}^i J_i = 0, \quad \nabla_{X_i}^i g_i = 0, \quad T^i(J_i X, I_i Y) = T^i(I_i X, J_i Y).$$

Definition 3.3. We shall call the *canonical connection* associated to the mem-structure (J, g) on the manifold M to the connection given by the conditions (3.6) and (3.7).

Remark 3.1. Notice that this connection slightly differs from that given in Theorem 5.3 in [14].

For the canonical connection we obtain from (3.6):

$$\nabla_{X_2}^1 Y_1 = P_1[X_2, Y_1], \quad \nabla_{X_1}^2 Y_2 = P_2[X_1, Y_2].$$

Denoting by ξ_1^1, ξ_1^2 the eigen-subbundles of J_1 corresponding to $\varepsilon = +1, \varepsilon = -1$, by π_1^1, π_1^2 the projection maps of ξ_1 on ξ_1^1 , and ξ_1^2 and by X_1^i, Y_1^i any elements of $\mathfrak{X}(\xi_1^i)$, we obtain from the first equation in (3.7)

$$\begin{aligned} \nabla_{X_1^2}^1 Y_1^1 &= \pi_1^1 P_1[X_1^2, Y_1^1], & \nabla_{X_1^1}^1 Y_1^2 &= \pi_1^2 P_1[X_1^1, Y_1^2], \\ g_1(\nabla_{X_1^1}^1 Y_1^1, Z_1^2) &= X_1^1 g_1(Y_1^1, Z_1^2) - g_1([X_1^1, Z_1^2], Y_1^1), \\ g_1(\nabla_{X_1^2}^1 Y_1^2, Z_1^1) &= X_1^2 g_1(Y_1^2, Z_1^1) - g_1([X_1^2, Z_1^1], Y_1^2). \end{aligned}$$

From the second equation in (3.7) above it results, exactly as in [14, Th. 5.1], the expression for $\nabla_{X_2}^2 Y_2$.

For J and g we obtain

$$\begin{aligned} (\nabla_{X_1} J)Y_1 &= 0, & (\nabla_{X_2} J)Y_2 &= 0, & (\nabla_{X_1} J)Y_2 &= (\nabla_{X_1}^2 J_2)Y_2, \\ (\nabla_{X_2} J)Y_1 &= (\nabla_{X_1}^1 J_1)Y_1, & (\nabla_{X_1} g)(Y_1, Z_1) &= 0, & (\nabla_{X_2} g)(Y_2, Z_2) &= 0, \\ (\nabla_{X_2} g)(Y_1, Z_1) &= (L_{X_2} g)(Y_1, Z_1), & (\nabla_{X_1} g)(Y_2, Z_2) &= (L_{X_1} g)(Y_2, Z_2), \end{aligned}$$

where L stands for the Lie derivative.

4 Structures of electromagnetic type on the total space of a vector bundle

Let $\xi = (E, \pi, M)$ be a vector bundle and (x^j) , (y^a) , (x^j, y^a) , local coordinates in adapted charts on M , ξ , and E , respectively. We denote by (∂_j) , (e_a) , (∂_j, ∂_a) the corresponding local bases, where $\partial_j = \partial/\partial x^j$, $\partial_a = \partial/\partial y^a$, $j = 1, 2, \dots, m$, $a, b, c = 1, 2, \dots, n$ (see [2]). Setting for each $z = (x, y) \in E$, $V_z E = \text{Ker } \pi_{*z}$, we obtain the *vertical distribution* and so the *vertical subbundle* of TE , denoted by VE . Let $C^{\infty v} = \{f^v = f \circ \pi : f \in C^\infty(M)\}$ be the subring of $C^\infty(E)$ naturally isomorphic to $C^\infty(M)$. Setting for each $\mu \in \Lambda^1(\xi)$, locally given by $\mu(x) = \mu_a(z)e^a$,

$$\gamma(\mu)(z) = \mu_a(x)y^a,$$

we obtain a class of functions on E enjoying the property that every vector field $A \in \mathfrak{X}(E)$ is uniquely determined by its values on those functions. The mapping γ may be extended to tensor fields $S \in \mathcal{T}_1^1(\xi)$ by

$$(\gamma S)(\gamma(\mu)) = \gamma(\mu \circ S), \quad \mu \in \Lambda^1(\xi).$$

If $S(x) = S_b^a(x)e_a \otimes e^b$, then $\gamma S(z) = S_b^a(x)y^b \partial_a$, *i.e.*, γS is a vertical vector field on E . Now, let D be a connection on ξ and $X \in \mathfrak{X}(M)$, $u \in \mathfrak{X}(\xi)$. Setting

$$X^h(\gamma\mu) = \gamma(D_X \mu), \quad u^v(\gamma\mu) = \mu(u) \circ \pi, \quad \mu \in \Lambda^1(\xi),$$

we obtain two vector fields X^h and u^v on E , respectively called the *horizontal lift* of X and the *vertical lift* of u . We have the useful formulas [2]:

$$\begin{aligned} (fX)^h &= f^v X^h, & (fu)^v &= f^v u^v, & [X^h, Y^h] &= [X, Y]^h - \gamma R_{XY}^D, & [u^v, w^v] &= 0, \\ [X^h, u^v] &= (D_X u)^v, & f &\in \mathbb{C}^\infty(M), & X, Y &\in \mathfrak{X}(M), & u, w &\in \mathfrak{X}(\xi). \end{aligned}$$

Now, putting

$$Q(X^h) = X^h, \quad Q(u^v) = -X^v, \quad X \in \mathfrak{X}(M), \quad u \in \mathfrak{X}(\xi),$$

we obtain an almost product Q structure on E whose $+1$ and -1 eigendistributions, are respectively called the *horizontal distribution* HE of the connection D and the *vertical distribution* VE of the bundle.

For $f \in \mathcal{T}_1^1(M)$, $\varphi \in \mathcal{T}_1^1(\xi)$, $g \in \mathcal{T}_2(M)$, $\psi \in \mathcal{T}_2(\xi)$, we define the *horizontal lift* or the *vertical lift* $f^h, \varphi^v, g^h, \psi^v$, respectively by

$$(4.1) \quad \begin{aligned} f^h(X^h) &= f(X)^h, & f^h(u^v) &= 0, & \varphi^v(X^h) &= 0, & \varphi^v(u^v) &= \varphi(u)^v, \\ g^h(X^h, Y^h) &= g(X, Y)^v, & g^h(X^h, u^v) &= g^h(u^v, X^h) = g^h(u^v, w^v) = 0, \\ \psi^v(X^h, Y^h) &= \psi^v(X^h, u^v) = \psi^v(u^v, Y^h) = 0, & \psi^v(u^v, w^v) &= \psi(u, w)^v, \\ & & & & X, Y \in \mathfrak{X}(M), & u, w \in \mathfrak{X}(\xi). \end{aligned}$$

We then define the *diagonal lifts* J and G for the pairs (f, φ) and (g, ψ) by

$$(4.2) \quad J = f^h + \varphi^v, \quad G = g^h + \psi^v.$$

From (4.1) and (4.2) we have

$$J^n(X^h) = (f^n(X))^h, \quad J^n(u^v) = (\varphi^n(u))^v, \quad n \in \mathbb{N}^*.$$

So $J^4 = I$, that is J is an em-structure on E , if and only if $f^4 = I_1$ and $\varphi^4 = I_2$, that is, either f and φ are both em-structures or one is an em-structure and the other an almost product or almost complex structure, or finally f is an almost product (resp. almost complex) and φ is a complex (resp. product) structure on M and ξ respectively. *In the sequel we only consider the last case.*

Hence, let J be an em-structure on the total space E of ξ given by the diagonal lift in the first equation in (4.2) of an almost product (resp. almost complex) structure f on the base manifold M and a complex (resp. product) structure φ on the bundle ξ , that is, which satisfy

$$f^2 = \varepsilon I_1, \quad \varphi^2 = -\varepsilon I_2, \quad \varepsilon = 1 \text{ (resp. } \varepsilon = -1),$$

with respect to a connection D on ξ . For the almost product structure P associated to J , we obtain $P = \varepsilon Q$, that is, P coincides up to the sign with the almost product structure Q above associated to D .

Now, let G be the diagonal lift in the second equation in (4.2), with respect to D , for the pair (g, ψ) of metrics on M and ξ . From (4.2) we obtain

$$\delta_J G = (\delta_f g)^h + (\delta_\varphi \psi)^v,$$

and so $\delta_J G = 0$ if and only if $\delta_f g = 0$ and $\delta_\varphi \psi = 0$. It follows

Proposition 4.1. *The pair (J, G) of diagonal lifts, with respect to a connection D on ξ , of an almost product (resp. almost complex) structure f on M and a complex (resp. product) structure φ of ξ , and the nondegenerate metrics g on M and ψ on ξ , is a mem-structure on the total space E of ξ if and only if the pair (f, g) is an almost para-Hermitian (resp. indefinite almost Hermitian) structure on M . The pair (φ, ψ) is an indefinite Hermitian (resp. para-Hermitian) structure on ξ .*

Denoting by ω and τ the 2-forms associated to the structures (f, g) on M and (φ, ψ) on ξ , and by $\Omega_1, \Omega_2, \Omega$, the 2-forms associated to the structures (f^h, g^h) on HE , (φ^v, ψ^v) on VE and (J, G) on TE , we obtain

$$\Omega_1 = \omega^h, \quad \Omega_2 = \tau^v, \quad \Omega = \omega^h \oplus \tau^v.$$

From the hypotheses of Prop. 4.1 it follows

$$\delta_f g = 0, \quad \delta_f \omega = 0, \quad \delta_\varphi \psi = 0, \quad \delta_\varphi \tau = 0, \quad \delta_J G = 0, \quad \delta_J \Omega = 0.$$

Remark 4.1. The groups of automorphisms of $\mathfrak{X}(M), \mathfrak{X}(\xi), \mathfrak{X}(E)$, given respectively for $\varepsilon = 1$ and $\varepsilon = -1$, by

$$\begin{aligned} \alpha_t &= I_1 \cosh t + f \sinh t, & \beta_t &= I_2 \cos t + \varphi \sin t, & \gamma_t &= \alpha_t^h \oplus \beta_t^h, & t &\in \mathbb{R}, \\ \alpha_t &= I_1 \cos t + f \sin t, & \beta_t &= I_2 \cosh t + \varphi \sinh t, & \gamma_t &= \alpha_t^h \oplus \beta_t^h, & t &\in \mathbb{R}, \end{aligned}$$

determine on the tensor algebras $\mathcal{T}(M), \mathcal{T}(\xi)$, and $\mathcal{T}(E)$, actions which preserve the structures $(f, g, \omega), (\varphi, \psi, \tau)$ and (J, G, Ω) .

For two connections ∇ on M and D on ξ , we define the *horizontal lift* ∇^h on the subbundle HE and the *vertical lift* D^v on the subbundle VE (each one with respect to the connection D), respectively by

$$\nabla_{X^h}^h Y^h = (\nabla_X Y)^h, \quad \nabla_{u^v}^h Y^h = 0, \quad D_{X^h}^v w^v = (D_X w)^v, \quad D_{u^v}^v w^v = 0.$$

Putting them

$$\mathcal{D}_A X = \nabla_A^h HX + D_A^v VX, \quad A, X \in \mathfrak{X}(E),$$

where H and V denote the horizontal and vertical projectors of TE on HE and VE , we obtain a linear connection \mathcal{D} on E , called the *diagonal lift* of the pair (∇, D) with respect to the connection D (see [2]), whose restrictions to the subbundles $\xi_1 = HE$ and $\xi_2 = VE$ are $\mathcal{D}_1 = \nabla^h$ and $\mathcal{D}_2 = D^v$. The nonvanishing components of the torsion and curvature tensors of \mathcal{D} are given by

$$(4.3) \quad \begin{aligned} \mathcal{T}(X^h, Y^h) &= T^\nabla(X, Y)^h + \gamma R_{XY}^D, \\ \mathcal{R}_{X^h Y^h} Z^h &= (R_{XY}^\nabla Z)^h, \quad \mathcal{R}_{X^h Y^h} u^v = (R_{XY}^D u)^v, \end{aligned}$$

where T^∇, R^∇ , and R^D stand for the torsion tensor of ∇ and the curvature tensors of ∇ and D .

For the covariant derivatives, with respect to \mathcal{D} , of the horizontal lift of f and g , and the vertical lift of φ and ψ we obtain

$$\begin{aligned} \mathcal{D}_{X^h} f^h &= (\nabla_X f)^h, & \mathcal{D}_{u^v} f^h &= 0, & \mathcal{D}_{X^h} g^h &= (\nabla_X g)^h, & \mathcal{D}_{u^v} g^h &= 0, \\ \mathcal{D}_{X^h} \varphi^v &= (D_X \varphi)^v, & \mathcal{D}_{u^v} \varphi^v &= 0, & \mathcal{D}_{X^h} \psi^v &= (D_X \psi)^v, & \mathcal{D}_{u^v} \psi^v &= 0. \end{aligned}$$

So, for the diagonal lifts J and G of the pairs (f, φ) and (g, ψ) , it follows

$$(4.4) \quad \begin{aligned} \mathcal{D}_{X^h} J &= (\nabla_X f)^h + (D_X \varphi)^v, & \mathcal{D}_{u^v} J &= 0, \\ \mathcal{D}_{X^h} G &= (\nabla_X g)^h + (D_X \psi)^v, & \mathcal{D}_{u^v} G &= 0. \end{aligned}$$

Hence, $\mathcal{D}J = 0$ if and only if $\nabla f = 0$, $D\varphi = 0$; and $\mathcal{D}G = 0$ if and only if $\nabla g = 0$, $D\psi = 0$. From (4.3) and (4.4) it follows, for $P = J^2$, that $\mathcal{D}P = 0$ and $\mathcal{T} \circ P \times I = \mathcal{T} \circ I \times P$ for any connections ∇ on M and D on ξ . After that we have

$$\begin{aligned} \nabla_{X^h}^h g^h &= (\nabla_X g)^h, & D_{u^v}^v \varphi^v &= 0, & D_{u^v}^v \psi^v &= 0, \\ \nabla_{X^h}^h f^h &= (\nabla_X f)^h, & \mathcal{T}^1(f^h X, I_1 Y) &= (T^\nabla(fX, I_1 Y))^h, & \mathcal{T}^2(\varphi^v X, I_2 Y) &= 0, \end{aligned}$$

where $\mathcal{T}^1 = H \circ \mathcal{T}|_{HE}$ and $\mathcal{T}^2 = V \circ \mathcal{T}|_{VE}$. So we obtain

Proposition 4.2. *The diagonal lift \mathcal{D} on E , for the connections ∇ on M and D on ξ , is the canonical connection associated to the mem-structure (J, G) if and only if*

$$\nabla f = 0, \quad \nabla g = 0, \quad T^\nabla(fX, Y) = T^\nabla(X, fY),$$

i.e., the connection ∇ is the canonical connection [2, 10] associated to the almost para-Hermitian (resp. indefinite almost Hermitian) structure (f, g) on M .

Also from (4.3) and (4.4) we obtain $\mathcal{D}G = 0$ and $\mathcal{T} = 0$ if and only if $\nabla g = 0$, $T^\nabla = 0$, $R^D = 0$ and $D\psi = 0$. Hence we have

Proposition 4.3. *The diagonal lift \mathcal{D} of the pair of connections (∇, D) coincides with the Levi-Civita connection of G if and only if ∇ is the Levi-Civita connection of g , D has vanishing curvature and ψ is covariant constant.*

For the Nijenhuis tensor of J ,

$$N_J(A, B) = [JA, JB] + J^2[A, B] - J[JA, B] - J[A, JB], \quad A, B \in \mathfrak{X}(E),$$

we obtain

$$(4.5) \quad \begin{aligned} N_J(X^h, Y^h) &= N_f(X, Y)^h + \gamma(\varepsilon R_{XY}^D - R_{fXfY}^D + \varphi \circ (R_{fXY}^D + R_{XfY}^D)), \\ N_J(X^h, u^v) &= (D_{fX} \varphi u - \varepsilon D_X u - \varphi \circ (D_{fX} u + D_X \varphi u))^v, & N_J(u^v, w^v) &= 0. \end{aligned}$$

It follows

Proposition 4.4. *The mem-structure J is integrable (i.e., $N_J = 0$, see [8]) if and only if f is a product (resp. a complex) structure in M , the connection D has vanishing curvature and the complex (resp. product) structure φ on ξ is covariant constant.*

For the exterior differential of the 2-form Ω associated to the mem-structure (J, G) we obtain

$$\begin{aligned} d\Omega(X^h, Y^h, Z^h) &= d\omega(X, Y, Z)^v, & 3d\Omega(X^h, Y^h, w^v) &= -\gamma(i_w \tau \circ R_{XY}^D), \\ 3d\Omega(X^h, u^v, w^v) &= D_X \tau(u, w)^v, & d\Omega(u^v, v^v, w^v) &= 0. \end{aligned}$$

Hence

Proposition 4.5. *The almost symplectic structure Ω associated to the mem-structure (J, G) on E is integrable (i.e., $d\Omega = 0$) if and only if the structure (f, g) is almost para-Kähler (resp. indefinite almost Kähler), the connection D has vanishing curvature, and the 2-form τ on ξ is covariant constant.*

Finally we obtain

Proposition 4.6. *For the mem-structure (J, G) on E , the structures J and Ω are simultaneously integrable if and only if the structure (f, g) is a para-Kähler (resp. indefinite Kähler) structure on M , D has vanishing curvature and the pair (φ, ψ) is covariant constant.*

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