KYUNGPOOK Math. J. 56(2016), 95-106
http://dx.doi.org/10.5666/KMJ.2016.56.1.95
pISSN 1225-6951 eISSN 0454-8124

Structures of Pseudo Ideal and Pseudo Atom in a Pseudo Q-Algebra

Young Bae Jun
Department of Mathematics Education, Gyeongsang National University, Jinju 52828, Korea
e-mail : skywine@gmail.com

Hee Sik Kim
Department of Mathematics, Hanyang University, Seoul 04763, Korea
e-mail : heekim@hanyang.ac.kr

Sun Shin Ahn*
Department of Mathematics Education, Dongguk University, Seoul 04620, Korea
e-mail : sunshine@dongguk.edu

Abstract. As a generalization of Q-algebra, the notion of pseudo Q-algebra is introduced, and some of their properties are investigated. The notions of pseudo subalgebra, pseudo ideal, and pseudo atom in a pseudo Q-algebra are introduced. Characterizations of their properties are provided.

1. Introduction

Y. Imai and K. Iséki introduced two classes of abstract algebras: a BCK-algebra and a BCI-algebra ([7,8]). It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. Q. P. Hu and X. Li [5,6] introduced a wide class of abstract algebra: a BCH-algebra. They have shown that the class of BCI-algebra is a proper subclass of the class of BCH-algebra. BCK-algebras have several connections with other areas of investigation, such as: lattice ordered groups, MV-algebras, Wajsberg algebras, and implicative commutative semigroups. J. M. Font et al. [3] have discussed Wajsberg algebras which are term-equivalent to MV-algebras. D. Mundici [13] proved MV-algebras are categorically equivalent to bounded commutative BCK-algebra, and J. Meng [11] proved that implicative com-

* Corresponding Author.
Received December 19, 2014; accepted February 5, 2016.
2010 Mathematics Subject Classification: 06F35, 03G25.
Key words and phrases: pseudo atom, pseudo subalgebra, pseudo ideal, pseudo Q-algebra.

mutative semigroups are equivalent to a class of BCK-algebras. G. Georgescu and A. Iorgulescu [4] introduced the notion of a pseudo BCK-algebra. Y. B. Jun characterized pseudo BCK-algebras. He found conditions for a pseudo BCK-algebra to be $\wedge$-semi-lattice ordered. Y. B. Jun, H.S. Kim, J. Neggers [9] introduced the notion of a pseudo d-algebra as a generalization of the idea of a d-algebra. J. Neggers, S. S. Ahn and H. S. Kim ([14]) introduced a new notion, called a Q-algebra, which is a generalization of $BCH/BCI/BCK$-algebra, and generalized some theorems discussed in a BCI-algebra.

In this paper, we introduce the notion of pseudo Q-algebra as a generalization of Q-algebra and investigate some of their properties. We also define the notions of pseudo subalgebra, pseudo ideal, and pseudo atom in a pseudo Q-algebra and provide characterizations of their properties in a pseudo Q-algebra.

2. Preliminaries

A *Q-algebra* ([14]) is a non-empty set X with a constant 0 and a binary operation "$*$" satisfying axioms:

(I) $x * x = 0$,

(II) $x * 0 = x$,

(III) $(x * y) * z = (x * z) * y$

for all $x, y, z \in X$.

For brevity we also call X a *Q-algebra*. In X we can define a binary relation "$\leq$" by $x \leq y$ if and only if $x * y = 0$.

In a Q-algebra X the following property holds:

(IV) $(x * (x * y)) * y = 0$, for any $x, y \in X$.

A BCK-algebra is a Q-algebra X satisfying the additional axioms:

(V) $((x * y) * (x * z)) * (z * y) = 0$,

(VI) $x * y = 0$ and $y * x = 0$ imply $x = y$,

(VII) $0 * x = 0$,

for all $x, y, z \in X$.

Definition 2.1.([14]) Let $(X; *, 0)$ be a Q-algebra and $\emptyset \neq I \subset X$. I is called a *subalgebra* of X if

(S) $x * y \in I$ whenever $x \in I$ and $y \in I$.

I is called an *ideal* of X if it satisfies:

(Q_0) $0 \in I$,

(Q_1) $x * y \in I$ and $y \in I$ imply $x \in I$.

A Q-algebra X is called a *QS-algebra* ([1]) if it satisfies the following identity:

$$(x * y) * (x * z) = z * y, \qquad \text{for any } x, y, z \in X.$$

Example 2.2.([1]) Let $\mathbb{Z}$ be the set of all integers and let $n\mathbb{Z} := \{nz|z \in \mathbb{Z}\}$, where $n \in \mathbb{Z}$. Then $(\mathbb{Z}; -, 0)$ and $(n\mathbb{Z}; -, 0)$ are both Q-algebras and QS-algebras, where "$-$" is the usual subtraction of integers. Also, $(\mathbb{R}; -, 0)$ and $(\mathbb{C}; -, 0)$ are Q-algebras and QS-algebras where $\mathbb{R}$ is the set of all real numbers, $\mathbb{C}$ is the set of all complex numbers.

Example 2.3. (1) Let $X = \{0, 1, 2\}$ be a set with the table as follows:

$*$	0	1	2
0	0	0	0
1	1	0	0
2	2	0	0

Then X is a Q-algebra, but not a QS/BCI-algebra, since $(2 * 0) * (2 * 1) = 2 \neq 1 = 1 * 0$.
(2) Let $X = \{0, 1, 2\}$ be a set with the table as follows:

$*$	0	1	2
0	0	2	1
1	1	0	2
2	2	1	0

Then X is both a Q-algebra and QS-algebra.
(3) Let $X = \{0, 1, 2\}$ be a set with the table as follows:

$*$	0	1	2
0	0	0	0
1	1	0	0
2	2	1	0

Then X is both a Q-algebra and BCI-algebra , but not a QS-algebra, since $(0 * 1) * (0 * 2) = 0 \neq 1 = 2 * 1$.

3. Pseudo Ideal

In the following, let X denote a pseudo Q-algebra unless otherwise specified.

Definition 3.1. A *pseudo Q-algebra* is a non-empty set X with a constant 0 and two binary operations "$*$" and "$\diamond$" satisfying the following axioms: for any $x, y, z \in X$,

(P1) $x * x = x \diamond x = 0$;

(P2) $x * 0 = x = x \diamond 0$;

(P3) $(x * y) \diamond z = (x \diamond z) * y$.

For brevity, we also call X a pseudo BCH-algebra. In X we can define a binary operation " $\preceq$ " by $x \preceq y$ if and only if $x * y = 0$ if and only if $x \diamond y = 0$. Note that if $(X; *, 0)$ is a Q-algebra, then letting $x \diamond y := x * y$, produces a pseudo Q-algebra $(X; *, \diamond, 0)$. Hence every Q-algebra is a pseudo Q-algebra in a natural way.

Definition 3.2. Let $(X; *, \diamond, 0)$ be a pseudo Q-algebra and let $\emptyset \neq I \subseteq X$. I is called a *pseudo subalgebra* of X if $x * y, x \diamond y \in I$ whenever $x, y \in I$. I is called a *pseudo ideal* of X if it satisfies

(PI1) $0 \in I$;

(PI2) $x * y, x \diamond y \in I$ and $y \in I$ imply $x \in I$ for all $x, y \in X$.

Example 3.3. Let $X := \{0, a, b, c\}$ be a set with the following Cayley tables:

$*$	0	a	b	c
0	0	0	0	0
a	a	0	0	0
b	b	b	0	a
c	c	c	0	0

$\diamond$	0	a	b	c
0	0	0	0	0
a	a	0	0	0
b	b	c	0	c
c	c	c	0	0

Then $(X; *, 0)$ and $(X; \diamond, 0)$ are not Q-algebras, since $(b * a) * c = a \neq 0 = (b * c) * a$ and $(b \diamond a) \diamond c = 0 \neq c = (b \diamond c) \diamond a$. It is easy to check that $(X; *, \diamond, 0)$ is a pseudo Q-algebra. Let $I := \{0, a\}$. Then I is both a pseudo subalgebra of X and a pseudo ideal of X. Let $J := \{0, a, c\}$. Then J is a pseudo subalgebra of X, but it is not a pseudo ideal of X since $b \diamond c = c \in J$ and $b * c = a \in J$, but $b \notin J$.

Proposition 3.4. *Let I be a pseudo ideal of a pseudo Q-algebra X. If $x \in I$ and $y \preceq x$, then $y \in I$.*

Proof. Assume that $x \in I$ and $y \preceq x$. Then $y * x = 0$ and $y \diamond x = 0$. By (PI1) and (PI2), we have $y \in I$. □

Proposition 3.5. *If X is a pseudo Q-algebra satisfying $a * b = a * c$ and $a \diamond b = a \diamond c$ for all $a, b, c \in X$, then $0 * b = 0 * c$ and $0 \diamond b = 0 \diamond c$.*

Proof. For any $a, b, c \in X$, we have

$$0 * b = (a \diamond a) * b = (a * b) \diamond a = (a * c) \diamond a = (a \diamond a) * c = 0 * c$$

and

$$0 \diamond b = (a * a) \diamond b = (a \diamond b) * a = (a \diamond c) * a = (a * a) \diamond c = 0 \diamond c.$$

This concludes the proof. □

Proposition 3.6. *Let $(X; *, \diamond, 0)$ be a pseudo Q-algebra. Then the following hold: for all $x, y, z \in X$.*

(i) $x * (x \diamond y) \preceq y, x \diamond (x * y) \preceq y$,

(ii) $x * y \preceq z \Leftrightarrow x \diamond z \preceq y$,

(iii) $0 * (x * y) = (0 \diamond x) \diamond (0 * y)$,

(iv) $0 \diamond (x \diamond y) = (0 * x) * (0 \diamond y)$,

(v) $0 * x = 0 \diamond x$.

Proof. (i) By (P1) and (P3), we obtain $[x * (x \diamond y)] \diamond y = (x \diamond y) * (x \diamond y) = 0$ and $[x \diamond (x * y)] * y = (x * y) \diamond (x * y) = 0$. Hence $x * (x \diamond y) \preceq y$ and $x * (x \diamond y) \preceq y$.
(ii) $x * y \preceq z \Leftrightarrow (x * y) \diamond z = 0 \Leftrightarrow (x \diamond z) * y = 0 \Leftrightarrow x \diamond z \preceq y$.
(iii) and (iv) For any $x, y \in X$, by (P1) and (P3) we have

$$\begin{aligned}(0 \diamond x) \diamond (0 * y) =&[((x * y) * (x * y)) \diamond x] \diamond (0 * y)\\ =&[((x * y) \diamond x) * (x * y)] \diamond (0 * y)\\ =&[((x \diamond x) * y) * (x * y)] \diamond (0 * y)\\ =&[(0 * y) \diamond (0 * y)] * (x * y)\\ =&0 * (x * y)\end{aligned}$$

and

$$\begin{aligned}(0 * x) * (0 \diamond y) =&[((x \diamond y) * (x \diamond y)) * x] * (0 \diamond y)\\ =&[((x \diamond y) * x) * (x \diamond y)] * (0 \diamond y)\\ =&[((x * x) \diamond y) \diamond (x \diamond y)] * (0 \diamond y)\\ =&[(0 \diamond y) * (0 \diamond y)] \diamond (x \diamond y)\\ =&0 \diamond (x \diamond y).\end{aligned}$$

(v) For any $x \in X$, by (P1) and (P3) we obtain $0*x = (x\diamond x)*x = (x*x)\diamond x = 0\diamond x$. □

Theorem 3.7. *For any pseudo Q-algebra X, the set*

$$K(X) := \{x \in X | 0 \preceq x\}$$

is a pseudo subalgebra of X.

Proof. Let $x, y \in K(X)$. Then $0 \preceq x$ and $0 \preceq y$. Hence $0 * x = 0 \diamond x = 0$ and $0 * y = 0 \diamond y = 0$. Since $0 * (x * y) = (0 \diamond x) \diamond (0 * y) = 0 \diamond 0 = 0$ and $0 \diamond (x \diamond y) = (0 * x) * (0 \diamond y) = 0 * 0 = 0$, we have $x * y, x \diamond y \in K(X)$. Thus $K(X)$ is a pseudo subalgebra of X. □

Theorem 3.8. *If I is a pseudo ideal of a pseudo Q-algebra X, then*

(i) $\forall x, y, z \in X, x, y \in I, z * y \preceq x \Rightarrow z \in I$,

(ii) $\forall a, b, c \in X, a, b \in I, c \diamond b \preceq a \Rightarrow c \in I$.

Proof. (i) Suppose that I is a pseudo ideal of X and let $x, y, z \in X$ be such that $x, y \in I$ and $z * y \preceq x$. Then $(z * y) \diamond x = 0 \in I$. Since $x \in I$ and I is a pseudo ideal

of X, we have $z * y \in I$. Since $y \in I$ and I is a pseudo ideal of X, we obtain $z \in I$. Thus (i) is valid.
(ii) Let $a, b, c \in X$ be such that $a, b \in I$ and $c \diamond b \preceq a$. Then $(c \diamond b) * a = 0 \in I$ and so $c \diamond b \in I$. Since $b \in I$ and I is a pseudo ideal of X, we have $c \in I$. Thus (ii) is true. □

Theorem 3.9. *Let I be a pseudo subalgebra of a pseudo Q-algebra X. Then I is a pseudo ideal of X if and only if $\forall x, y \in X, x \in I, y \in X - I \Rightarrow y * x \in X - I$ and $y \diamond x \in X - I$.*

Proof. Assume that I is a pseudo ideal of X and let $x, y \in X$ be such that $x \in I$ and $y \in X - I$. If $y * x \notin X - I$, then $y * x \in I$. Since I is a pseudo ideal of X, we have $y \in I$. This is a contradiction. Hence $y * x \in X - I$. Now if $y \diamond x \notin X - I$, then $y \diamond x \in I$ and so $y \in I$. This is a contradiction, and therefore $y \diamond x \in X - I$.

Conversely, assume that $\forall x, y \in X, x \in I, y \in X - I \Rightarrow y * x \in X - I$ and $y \diamond x \in X - I$. Since I is a pseudo subalgebra, we have $0 \in I$. Let $x \in I, y \in X$ such that $y * x, y \diamond x \in I$. If $y \notin I$, then $y * x, y \diamond x \in X - I$ by assumption. This is a contradiction. Hence $y \in I$. Thus I is a pseudo ideal of X. □

Proposition 3.10. *Let A be a pseudo ideal of a pseudo Q-algebra X. If B is a pseudo ideal of A, then it is a pseudo ideal of X.*

Proof. Since B is a pseudo ideal of A, we have $0 \in B$. Let $y, x * y, x \diamond y \in B$ for some $x \in X$. If $x \in A$, then $x \in B$ since B is a pseudo ideal of A. If $x \in X - A$, then $y, x * y, x \diamond y \in B \subseteq A$ and so $x \in A$ because A is a pseudo ideal of X. Thus $x \in B$ since B is a pseudo ideal of A. This competes the proof. □

Proposition 3.11. *Let I be a pseudo ideal of a pseudo Q-algebra X. Then*

$$(\forall x \in X)(x \in I \Rightarrow 0 * (0 \diamond x) \in I \ \text{ and } 0 \diamond (0 * x) \in I).$$

Proof. Let $x \in I$. Then

$$0 = (0 \diamond x) * (0 \diamond x) = (0 * (0 \diamond x)) \diamond x$$

and

$$0 = (0 * x) \diamond (0 * x) = (0 \diamond (0 * x)) * x$$

which imply that $0 * (0 \diamond x), 0 \diamond (0 * x) \in I$. This completes the proof. □

Theorem 3.12. *Let I be a pseudo ideal of a pseudo Q-algebra X and let*

$$I^{\#} := \{x \in X | 0 * (0 \diamond x), 0 \diamond (0 * x) \in I\}.$$

Then $I^{\#}$ is a pseudo ideal of X and $I \subseteq I^{\#}$.

Proof. Obviously, $0 \in I^{\#}$. Let $a \in X, y \in I^{\#}$ such that $a * y, a \diamond y \in I^{\#}$. Then

$0*(0\diamond(a*y)), 0\diamond(0*(a*y)), 0*(0\diamond(a\diamond y))\in I$, and $0\diamond(0*(a\diamond y))\in I$. Using Proposition 3.6 (iii) and (iv), we have

$$\begin{aligned}(0*(0\diamond a))*(0\diamond(0*y)) &= 0\diamond((0\diamond a)\diamond(0*y))\\ &= 0\diamond(0*(a*y))\in I\end{aligned}$$

and

$$\begin{aligned}(0\diamond(0*a))\diamond(0*(0\diamond y)) &= 0*((0*a)*(0\diamond y))\\ &= 0*(0\diamond(a\diamond y))\in I.\end{aligned}$$

Since $0*(0\diamond y), 0\diamond(0*y)\in I$, it follows from (PI2) that $0*(0\diamond a), 0\diamond(0*a)\in I$. Hence $a\in I^{\#}$. Thus $I^{\#}$ is a pseudo ideal of X. By Proposition 3.11, we know that $I\subseteq I^{\#}$. This completes the proof. □

Let X be a pseudo Q-algebra. For any non-empty subset S of X, we define

$$G(S) := \{x\in S \mid 0*x = x = 0\diamond x\}.$$

In particular, if $S = X$ then we say that $G(X)$ is the G-part of X.

Proposition 3.13. *If X is a pseudo Q-algebra, then a left cancellation law holds in $G(X)$.*

Proof. Let $x, y, z\in G(X)$ satisfy $x*y = x*z$ and $x\diamond y = x\diamond z$. By Proposition 3.5, we have $0*y = 0*z$ and $0\diamond y = 0\diamond z$. Since $y, z\in G(X)$, it follows that $y = z$. □

Proposition 3.14. *Let X be a pseudo Q-algebra. Then $x\in G(X)$ if and only if $0*x\in G(X)$ and $0\diamond x\in G(X)$.*

Proof. If $x\in G(X)$, then $0*x = x = 0\diamond x$ and $0*(0*x) = 0*x$ and $0\diamond(0\diamond x) = 0\diamond x$. Hence $0*x\in G(X)$ and $0\diamond x\in G(X)$. Conversely, assume that $0*x\in G(X)$ and $0\diamond x\in G(X)$. Then $0*(0*x) = 0*x$ and $0\diamond(0\diamond x) = 0\diamond x$. By applying Proposition 3.13, we obtain $0*x = x = 0\diamond x$. Therefore $x\in G(X)$. □

Proposition 3.15. *Let X be a pseudo Q-algebra. If $x, y\in G(X)$, then $x*y = y\diamond x$.*

Proof. If $x, y\in G(X)$, then $0*x = x = 0\diamond x$ and $0*y = y = 0\diamond y$. Using (P3), we have

$$x*y = (0\diamond x)*(0\diamond y) = (0*(0\diamond y))\diamond x = (0*y)\diamond x = y\diamond x.$$

This completes the proof. □

Theorem 3.16. *If a pseudo Q-algebra X satisfies $x*(x\diamond y) = x*y$ or $x\diamond(x*y) = x\diamond y$ for all $x, y\in X$, then it is a trivial algebra.*

Proof. Assume that X satisfies $x*(x\diamond y) = x*y$ for all $x, y\in X$. Putting $x = y$ in the equation $x*(x\diamond y) = x*y$, we obtain

$$x = x*0 = x*(x\diamond x) = x*x = 0.$$

Concerning the other case, the argument is similar. Hence X is a trivial algebra. □

Theorem 3.17. *Let X be a pseudo Q-algebra such that*

$$x * (y \diamond z) = (x * y) \diamond z \text{ (resp., } x \diamond (y * z) = (x \diamond y) * z) \tag{3.1}$$

for all $x, y, z \in X$. Then X is a group under operation $\diamond$(resp.$*$).

Proof. Putting $x = y = z$ in (3.1) and using (P1) and (P2), we obtain $x = x * 0 = 0 \diamond x$(resp., $x = x \diamond 0 = 0 * x$). This means that 0 is the zero element of X with respect to the operation $\diamond$(resp., $*$). Since $x \diamond x = 0 = x * x$, we know that the inverse of x is itself with respect to the operation $\diamond$(resp., $*$). Hence X is a group under operation $\diamond$(resp., $*$). □

Definition 3.18. A Pseudo Q-algebra X is said to be $\diamond$-*medial* if it satisfies the following identity

$$(x * y) \diamond (z * u) = (x * z) \diamond (y * u), \forall x, y, z, u \in X. \tag{M1}$$

Proposition 3.19. *Every $\diamond$-medial pseudo Q-algebra X satisfies the following identities: for any $x, y \in X$*

(i) $x * y = 0 \diamond (y * x)$,

(ii) $0 \diamond (0 * x) = x$,

(iii) $x \diamond (x * y) = y$.

Proof. (i) For any $x, y \in X$, we have $x * y = (x * y) \diamond 0 = (x * y) \diamond (x * x) = (x * x) \diamond (y * x) = 0 \diamond (y * x)$.
(ii) If we put $y = 0$ in (i), then we have (ii).
(iii) Using (ii), (P1) and (P2), we have $x \diamond (x * y) = (x * 0) \diamond (x * y) = (x * x) \diamond (0 * y) = 0 \diamond (0 * y) = y$. □

4. Pseudo Atom

Definition 4.1. An element a of a pseudo Q-algebra X is called a *pseudo atom* of X if for every $x \in X$, $x \preceq a$ implies $x = a$.

Obviously, 0 is a pseudo atom of X. Let $L(X)$ denote the set of all pseudo atoms of X, we call it the *center* of X.

Theorem 4.2. *Let X be a pseudo Q-algebra. Then the following are equivalent: for all $x, y, z, w, u \in X$*

(i) *w is a pseudo atom,*

(ii) $w = x \diamond (x * w)$ *and* $w = x * (x \diamond w)$

(iii) $(x * y) \diamond (x * w) = w * y$ *and* $(x \diamond y) * (x \diamond w) = w \diamond y$

(iv) $w * (x \diamond y) = y \diamond (x * w)$ *and* $w \diamond (x * y) = y * (x \diamond w)$,

(v) $0 \diamond (y * w) = w * y$ *and* $0 * (y \diamond w) = w \diamond y$,

(vi) $0 \diamond (0 * w) = w$ *and* $0 * (0 \diamond w) = w$,

(vii) $0 \diamond (0 * (w \diamond z)) = w \diamond z$ *and* $0 * (0 \diamond (w * z)) = w * z$,

(viii) $z \diamond (z * (w \diamond u)) = w \diamond u$ *and* $z * (z \diamond (w * u)) = w * u$.

Proof. (i) $\Rightarrow$ (ii): Let w be a pseudo atom of X. Since $x \diamond (x * w) \preceq w$ and $x * (x \diamond w) \preceq w$ by Proposition 3.6 (i), we have $w = x \diamond (x * w)$ and $w = x * (x \diamond w)$.
(ii) $\Rightarrow$ (iii): For every $x \in X$, we obtain $(x * y) \diamond (x * w) = (x \diamond (x * w)) * y = w * y$ and $(x \diamond y) * (x \diamond w) = (x * (x \diamond w)) \diamond y = w \diamond y$ by (P3) and (ii).
(iii) $\Rightarrow$ (iv): Replacing y by $x \diamond y$ in (iii), we get

$$\begin{aligned} w * (x \diamond y) =& (x * (x \diamond y)) \diamond (x * w) \\ =& (x \diamond (x * w)) * (x \diamond y) \\ =& y \diamond (x * w) \end{aligned}$$

and

$$\begin{aligned} w \diamond (x * y) =& (x \diamond (x * y)) * (x \diamond w) \\ =& (x * (x \diamond w)) \diamond (x * y) \\ =& y * (x \diamond w). \end{aligned}$$

(iv) $\Rightarrow$ (v): Put $y := 0$ in (iv). Then $w * (x \diamond 0) = 0 \diamond (x * w)$ and $w \diamond (x * 0) = 0 * (x \diamond w)$. Hence $w * x = 0 \diamond (x * w)$ and $w \diamond x = 0 * (x \diamond w)$ by (P2).
(v) $\Rightarrow$ (vi): Set $y := 0$ in (v). Then $0 \diamond (0 * w) = w * 0 = w$ and $0 * (0 \diamond w) = w \diamond 0 = w$.
(vi) $\Rightarrow$ (vii): For any $w, z \in X$, we have

$$\begin{aligned} 0 \diamond (0 * (w \diamond z)) =& 0 * (0 * (w \diamond z)) \\ =& 0 * (0 \diamond (w \diamond z)) \\ =& 0 * [(0 * w) * (0 \diamond z)] \\ =& (0 \diamond (0 * w)) \diamond (0 * (0 \diamond z)) \\ =& w \diamond z \end{aligned}$$

and

$$\begin{aligned} 0 * (0 \diamond (w * z)) =& 0 * (0 * (w * z)) \\ =& 0 * [(0 \diamond w) \diamond (0 * z)] \\ =& 0 \diamond [(0 \diamond w) \diamond (0 * z)] \\ =& (0 * (0 \diamond w)) * (0 \diamond (0 * z)) \\ =& w * z. \end{aligned}$$

(vii) $\Rightarrow$ (viii): For any $u, w, z \in X$, we have

$$\begin{aligned} w \diamond u =& 0 \diamond (0 * (w \diamond u)) \\ =& 0 \diamond ((z \diamond z) * (w \diamond u)) \\ =& 0 \diamond [((z * (w \diamond u)) \diamond z] \\ =& (0 * (z * (w \diamond u))) * (0 \diamond z) \\ =& (0 \diamond (z * (w \diamond u))) * (0 \diamond z) \\ =& (0 * (0 \diamond z)) \diamond (z * (w \diamond u)) \\ =& (0 * (0 \diamond (z \diamond 0))) \diamond (z * (w \diamond u)) \\ =& (0 \diamond (0 * (z \diamond 0))) \diamond (z * (w \diamond u)) \\ =& (z \diamond 0) \diamond (z * (w \diamond u)) \\ =& z \diamond (z * (w \diamond u)). \end{aligned}$$

By a similar way, we obtain $z * (z \diamond (w * u)) = w * u$.
(viii) $\Rightarrow$ (i): If $z * x = z \diamond x = 0$, then by (viii) we have $x = x * 0 = z * (z \diamond (x * 0)) = z * (z \diamond x) = z * 0 = z$. This shows that z is a pseudo atom of X. This completes the proof. □

Corollary 4.3. *Let X be a pseudo Q-algebra. If a is a pseudo atom of X, then for all x of X, $a * x$ and $a \diamond x$ are pseudo atoms. Hence $L(X)$ is a pseudo subalgebra of X.*

Corollary 4.4. *Let X be a pseudo Q-algebra. For every x of X, there is a pseudo atom a such that $a \preceq x$, i.e., every pseudo Q-algebra is generated by a pseudo atom.*

Proposition 4.5. *A non-zero element $a \in X$ is a pseudo atom of a pseudo Q-algebra X if $\{0, a\}$ is a pseudo ideal of X.*

Proof. Let $x \preceq a$ for any $x \in X$. Then $x * a = x \diamond a = 0 \in \{0, a\}$. Since $x \in \{0, a\}$ is a pseudo ideal of X, we have $x = 0$ or $x = a$. Since $a \neq 0$, we obtain $x = a$. Hence a is a pseudo atom of X. □

Proposition 4.6. *If non-zero element of a pseudo Q-algebra X is a pseudo atom, then any pseudo subalgebra of X is a pseudo ideal of X.*

Proof. Let S be a pseudo subalgebra of X and let $x, y * x, y \diamond x \in S$. By Theorem 4.2, we have $y = x * (x \diamond y) = x * (0 * (y \diamond x))$. Since $0, y \diamond x \in S$ and S is a pseudo subalgebra of X, we obtain $0 * (y \diamond x) \in S$. Hence $y = x * (0 * (y \diamond x)) \in S$. Thus any pseudo subalgebra of X is a pseudo ideal of X. □

For pseudo atom a of a Pseudo Q-algebra X,

$$V(a) := \{x \in X | a \preceq x\}$$

is called a *pseudo branch* of X.

Theorem 4.7. *Let X be a pseudo Q-algebra. Suppose that a and b are pseudo atoms of X. Then the following properties hold:*

(i) *For all* $x \in V(a)$ *and all* $y \in V(b)$, $x * y \in V(a * b)$ *and* $x \diamond y \in V(a \diamond b)$.

(ii) *For all* x *and* $y \in V(a)$, $x \diamond y, x * y \in K(X)$, *where* $K(X) = \{x \in X | 0 \preceq x\}$.

(iii) *If* $a \neq b$, *then for all* $x \in V(a)$ *and* $y \in V(b)$, *we have* $x * y, x \diamond y \in K(X)$.

(iv) *For all* $x \in V(b)$, $a * x = a * b$ *and* $a \diamond x = a \diamond b$.

(v) *If* $a \neq b$, *then* $V(a) \cap V(b) = \emptyset$.

Proof. (i) For all $x \in V(a)$ and all $y \in V(b)$, by Proposition 3.6 and Theorem 4.2 we have

$$\begin{aligned}
(a * b) \diamond (x * y) =& [0 * (0 \diamond (a * b))] \diamond (x * y) \\
=& (0 \diamond (x * y)) * (0 \diamond (a * b)) \\
=& (0 * (x * y)) * (0 \diamond (a * b)) \\
=& ((0 \diamond x) \diamond (0 * y)) * (0 \diamond (a * b)) \\
=& ((0 * (0 \diamond (a * b))) \diamond x) \diamond (0 * y) \\
=& ((a * b) \diamond x) \diamond (0 * y) \\
=& ((a \diamond x) * b) \diamond (0 * y) \\
=& (0 * b) \diamond (0 * y) \\
=& (0 \diamond b) \diamond (0 * y) \\
=& 0 * (b * y) = 0 * 0 = 0
\end{aligned}$$

and

$$\begin{aligned}
(a \diamond b) * (x \diamond y) =& [0 \diamond (0 * (a \diamond b))] * (x \diamond y) \\
=& (0 * (x \diamond y)) \diamond (0 * (a \diamond b)) \\
=& (0 \diamond (x \diamond y)) \diamond (0 \diamond (a \diamond b)) \\
=& ((0 * x) * (0 \diamond y)) \diamond (0 \diamond (a \diamond b)) \\
=& ((0 \diamond (0 \diamond (a \diamond b))) * x) * (0 \diamond y) \\
=& ((0 \diamond (0 * (a \diamond b))) * x) * (0 \diamond y) \\
=& ((a \diamond b) * x) * (0 \diamond y) \\
=& ((a * x) \diamond b) * (0 \diamond y) \\
=& (0 \diamond b) * (0 \diamond y) \\
=& (0 * b) * (0 \diamond y) \\
=& 0 \diamond (b \diamond y) = 0 \diamond 0 = 0.
\end{aligned}$$

Hence $x * y \in V(a * b)$ and $x \diamond y \in V(a \diamond b)$.
(ii) and (iii) are simple consequences of (i).
(iv) For all $x \in V(b)$, by Theorem 4.2 we have $(a*x)\diamond(a*b) = (a\diamond(a*b))*x = b*x = 0$. Moreover, $a * b$ is a pseudo atom by Corollary 4.3. Therefore $a * x = a * b$. Also we

get $(a \diamond x) * (a \diamond b) = (a * (a \diamond b)) * x = b * x = 0$. Moreover, $a \diamond b$ is a pseudo atom by Corollary 4.3. Therefore $a \diamond x = a \diamond b$.
(v) Let $a \neq b$ and $V(a) \cap V(b) \neq \emptyset$. Then there exists $c \in V(a) \cap V(b)$. By (i), we have $0 = c * c = c \diamond c \in V(a * b), V(a \diamond b)$. Hence $a * b = a \diamond b = 0$, which is a contradiction. Thus (v) is true. □

References

[1] S. S. Ahn and H. S. Kim, *On QS-algebras*, J. Chungcheong. Math. Soc., **12**(1999), 33–41.

[2] M. A. Chaudhry, *On BCH-algebras*, Math. Japon., **4**(1991), 665–676.

[3] J. M. Font, A. J. Rodŕgez and A. Torrens, *Wajsbergs Algebras*, Stochastica, **8**(1984), 5–31.

[4] G. Georgescu and A. Iorgulescu, *Pseudo BCK-algebras, in: Proc. DMTS'01: Combinatorics, Computability and Logics*, Springer, London, 2001, 97–114.

[5] Q. P. Hu and X. Li, *On BCH-algebras*, Math. Seminar Notes, **11**(1983), 313–320.

[6] Q. P. Hu and X. Li, *On proper BCH-algebras*, Math. Japon., **30**(1985), 659–661.

[7] K. Iséki, *On BCI-algebras*, Math. Seminar Notes, **8**(1980), 125–130.

[8] K. Iséki and S. Tanaka, An introduction to the theory of BCK-algebras, *Math. Japon.*, **23**(1978), 1–26.

[9] Y. B. Jun, H. S. Kim and J. Neggers, *On pseudo d-algebras*, Inform. Sci., **179**(2009), 1751–1759.

[10] Y. B. Jun, *Characterizations of pseudo BCK-algebras*, Sci. Math. Japon. Online, **7**(2002), 225–230.

[11] J. Meng, *Implicative commutative semigroups are equivalent to a class of BCK-algebras*, Semigroup Forum, **50**(1995), 89–96.

[12] J. Meng and Y. B. Jun, *BCK-algebras*, Kyung Moon Sa, Seoul, 1994.

[13] D. Mundici, *MV-algebras are categorically equivalent to bounded commutative BCK-algebras*, Math. Japon., **31**(1986), 889–894.

[14] J. Neggers, S. S. Ahn and H. S. Kim, *On Q-algebras*, Inter. J. Math. and Math. Sci., **27**(2001), 749–757.

[15] E. H. Roh and S. Y. Kim, *On BH^*-subalgebras of transitive BH^*-algebras*, Far East J. Math. Sci., **1**(1999), 255–263.