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stu Black Holes Unveiled

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Abstract: The general solutions of the radial attractor flow equations for extremal black holes, both for non-BPS with non-vanishing central charge Z and for $Z = 0$, are obtained for the so-called *stu* model, the minimal *rank-3* $\mathcal{N} = 2$ symmetric supergravity in $d = 4$ space-time dimensions.

Comparisons with previous results, as well as the *fake supergravity (first order) formalism* and an analysis of the *BPS bound* all along the non-BPS attractor flows and of the *marginal stability* of corresponding *D*-brane configurations, are given.

Keywords: black holes; attractors; supergravity.

1. Introduction

The physics of black holes (BHs) [1]–[6] has received much attention in the last years. This is also due to the issue of the *Attractor Mechanism* [7]–[10], a general phenomenon which occurs in *extremal* BHs coupled to Maxwell and scalar fields, as it is the case in supersymmetric theories of gravity [11]–[74] (for further developments, see also *e.g.* [75]–[78]).

Supergravity [79] can be obtained as the low-energy (small curvature expansion) limit of *superstrings* [80]–[83] or *M-theory* [84–86]; in such a framework, a certain number of $U(1)$ gauge fields and moduli

fields are coupled to the Einstein-Hilbert action. This is especially the case for theories in $d = 4$ space-time dimensions, and having $\mathcal{N} \geq 2$ *supercharges*, where $4\mathcal{N}$ is the number of supersymmetries. A popular example is the compactification of Type II superstring theory on a Calabi-Yau threefold ($\mathcal{N} = 2$) or on a six-torus ($\mathcal{N} = 8$). The fermionic sector of these theories contains a certain number of spin $1/2$ fermions and \mathcal{N} spin $3/2$ Rarita-Schwinger fields, named *gravitinos* (the gauge fields of local supersymmetry). The vanishing of the supersymmetric variation of the gravitinos determines whether or not a certain number of supersymmetries (*BPS property*) is preserved by the BH background.

In this situation, asymptotically flat charged BH solutions, within a static and spherically symmetric *Ansatz*, can be regarded as a generalization of the famous Schwarzschild BH. However, the presence of additional quantum numbers (such as charges and scalar hair) make their properties change drastically, and new phenomena appear. A novel important feature of electrically (and/or magnetically) charged BHs [87] as well as rotating ones [88] is a somewhat unconventional thermodynamical property named *extremality* [6, 92, 93]. *Extremal* BHs are possibly stable gravitational objects with finite entropy but vanishing temperature, in which case the contribution to the gravitational energy entirely comes from the electromagnetic (charges) and rotational [27] (angular momentum/spin) attributes. *Extremality* also means that the inner (Cauchy) and outer (event) horizons do coincide, thus implying vanishing *surface gravity* (for a recent review see *e.g.* [70], and Refs. therein).

In the regime of *extremality* a particular relation among entropy, charges and spin holds, yielding that the Arnowitt-Deser-Misner (ADM) gravitational mass [89–91] is *not* an independent quantity. Stationary and spherically symmetric BHs in $d = 4$ space-time dimensions and in an environment of scalar fields (typically described by a non-linear sigma model) have scalar hair (*scalar charges*), corresponding to the values of the scalars at (asymptotically flat) spatial infinity. These values may continuously vary, being an arbitrary point in the moduli space of the theory or, in a more geometrical language, a point in the target manifold of the scalar non-linear Lagrangian [7, 94]. Nevertheless, the BH entropy, as given by the Bekenstein-Hawking entropy-area formula [95], is also in this case independent on the scalar charges (“*no scalar hair*”) and it only depends on the asymptotic (generally *dyonic*) BH charges.

This apparent puzzle can be resolved thanks to the aforementioned *Attractor Mechanism*, a fascinating phenomenon that combines extremal BHs, dynamical systems, algebraic geometry and number theory [2]. It was firstly discovered in the context of supergravity; in a few words, in constructing extremal dyonic BHs of $\mathcal{N} = 2$, $d = 4$ *ungauged* supergravity coupled to vector and hypermultiplets (with no $d = 4$ scalar potential), two phenomena occur: the hyperscalars can take arbitrary constant values, while the radial evolution of the vector multiplets’ scalars is described by a dynamical system [8, 9]. Under some mild assumptions, the scalar trajectory flows to a “*fixed point*”, located at the BH event horizon, in the target (moduli) space. The “*fixed point*” (*i.e.* a point of vanishing *phase velocity*) represents the system in equilibrium, and it is the analogue of an *attractor* in the dynamical flow of dissipative systems. In approaching such an *attractor*, the orbits lose practically all memory of initial conditions (*i.e.* of the “*scalar hair*”), even though the dynamics is fully deterministic. The scalars at the BH horizon turn out to depend only on the dyonic (asymptotic) BH charges.

All extremal static, spherically symmetric and asymptotically flat BHs in $d = 4$ have a *Bertotti-Robinson* [96] $AdS_2 \times S^2$ near-horizon geometry, with vanishing scalar curvature and *conformally flat*; in particular, the radius of AdS_2 coincides with the radius of S^2 , and it is proportional to the (square root

of the) BH entropy (in turn proportional, through the Bekenstein-Hawking formula [95], to the area of the event horizon). Non-BPS (*i.e.* non-supersymmetric) (see *e.g.* [10, 26, 41, 43, 57, 59, 60]) extremal BHs exist as well, and they also exhibit an attractor behavior.

A particularly remarkable $\mathcal{N} = 2, d = 4$ ungauged supergravity is the so-called *stu* model, which exhibits the noteworthy *triality symmetry* [15, 39, 41, 50, 97–100]. It has been recently shown to be relevant for the analogy between pure states of multipartite entanglement of qubits in quantum information theory and extremal stringy BHs [3].

The 3 complex scalars coming from the 3 Abelian vector multiplets coupled to the supergravity one span the *rank-3*, completely factorized special Kähler manifold $\frac{G}{H} = \left(\frac{SU(1,1)}{U(1)}\right)^3$, with $dim_{\mathbb{C}} = 3$,

$$G = (SU(1, 1))^3 \sim (SO(2, 1))^3 \sim (SL(2, \mathbb{R}))^3 \sim (Sp(2, \mathbb{R}))^3 \tag{1}$$

being the $d = 4$ *U*-duality group, and $H = (U(1))^3 \sim (SO(2))^3$ its *maximal compact subgroup*. Such a space is nothing but the element $n = 2$ of the sequence of reducible homogeneous symmetric special Kähler manifolds $\frac{SU(1,1)}{U(1)} \otimes \frac{SO(2,n)}{SO(2) \otimes SO(n)}$ (see *e.g.* [26] and Refs. therein). It is here worth pointing out that, with a slight abuse of language, we refer to *U*-duality group as to the *continuous* version, valid for large values of charges, of the string duality group introduced by Hull and Townsend [101].

The *stu* model has 2 non-BPS $Z \neq 0$ *flat* directions, spanning the moduli space $SO(1, 1) \times SO(1, 1)$ (*i.e.* the scalar manifold of the *stu* model in $d = 5$), but *no* non-BPS $Z = 0$ *massless Hessian modes* at all [43] (see also [41] and [40]). In other words, the 6×6 Hessian matrix of the effective BH potential at its non-BPS $Z \neq 0$ critical points has 4 strictly positive and 2 vanishing eigenvalues (these latter correspond to *massless Hessian modes*), whereas at its non-BPS $Z = 0$ critical points all the eigenvalue are strictly positive. After [10], $\frac{1}{2}$ -BPS critical points of V_{BH} in $\mathcal{N} = 2, d = 4$ supergravity are *all* stable, and thus they determine attractors in a strict sense. It is here worth pointing out that the $d = 6$ uplift of the *stu* model is $(1, 0)$ supergravity coupled to $n_T = 1$ tensor multiplet. The BH charge orbits supporting the various classes of non-degenerate attractors have been studied in [26] (see also [50]).

Concerning its stringy origins, the *stu* model can be interpreted *e.g.* as the low-energy limit of Type IIA superstrings compactified on a six-torus T^6 factorized as $T^2 \times T^2 \times T^2$. The $D0 - D2 - D4 - D6$ branes wrapping the various T^2 s determine the 4 magnetic and 4 electric BH charges.

Remarkably, the *stu* model is a sector of *all* $\mathcal{N} > 2, d = 4$ supergravities, as well as of *all* $\mathcal{N} = 2, d = 4$ supergravities based on homogeneous (both *symmetric* [97, 113–115] and *non-symmetric* - see *e.g.* [116, 117] -) scalar manifolds based on cubic geometries. Thus, *stu* model captures the essential features of extremal BHs in all such theories (see *e.g.* the *stu* interpretation of $\mathcal{N} = 8, d = 4$ attractors [24], and the observations in [59]).

Recently, the *stu* model has been object of detailed investigation concerning the integration of the equations of motion of the scalars in the background of a given BH charge configuration, in particular of those ones which support non-supersymmetric flows:

- In [28] the non-BPS $Z \neq 0$ attractor flow was investigated for the first time. The $\frac{1}{2}$ -BPS attractor flow solution was previously studied in [105]-[109], and its most general form is known to be obtained simply by replacing the BH charges with the corresponding, symplectic covariant harmonic

functions in the the most general horizon, critical solution, obtained in [99, 100]. In [28] such a feature has been shown to hold also for non-BPS $Z \neq 0$ attractor flow, for the $D2 - D6$ (*electric*) and $D0 - D2 - D4 - D6$ configurations, for particular cases *without B-fields*.

- In [51] the $D0 - D4$ (*magnetic*) system was considered, and new exact non-BPS $Z \neq 0$ attractor flow solutions were derived, with non-vanishing dynamical axion. Furthermore, the result of [28] was proved to be non-general. Indeed, within the $D0 - D4$ (*magnetic*) configuration (*dual* to the *electric* one, studied in [28]), it was pointed out that non-BPS $Z \neq 0$ attractor flow differ from the $\frac{1}{2}$ -BPS one, because the most general non-BPS $Z \neq 0$ solution *cannot* be obtained by replacing the BH charges with the corresponding harmonic functions in the most general horizon critical solution, as instead it holds for the supersymmetric case [105]-[109], and actually also for the non-BPS $Z = 0$ case (see Sect. 4.). This is actually due to the presence of non-trivial so-called *B-fields* in the non-BPS $Z \neq 0$ attractor flow, as well as to the presence of *flat* directions (spanning a related *moduli space*) *all along* such a flow.
- In [59] the *stu* model was further studied in the $D0 - D4$ (*magnetic*) as well as in the $D0 - D6$ configurations, by fully exploiting the contribution of the *B-fields*, and (as also done in [28]) performing the relevant *U*-duality transformations in order to relate different BH charge configurations supporting the same attractor flow (*i.e.* belonging to the same BH charge orbit of *U*-duality [26]). The *ADM mass* M_{ADM} of the extremal BH was computed in the $\frac{1}{2}$ -BPS and in the non-BPS $Z \neq 0$ cases, and the *marginal stability* [111] of the corresponding physical states was studied, founding that the *marginal bound* [111] was saturated in the non-supersymmetric case, contrarily to the BPS case. The difference between the squared non-BPS $Z \neq 0$ ADM mass $M_{ADM,non-BPS,Z \neq 0}^2$ and $|Z|^2$ was computed at the radial infinity along the non-BPS $Z \neq 0$ attractor flow, showing that the *BPS bound* [112] actually holds also at the infinity, even if dependent on the asymptotical values of the scalars (see Eq. (4.8) of [59]). Moreover, in such a paper the two non-BPS $Z \neq 0$ *flat* directions of the *stu* model [41, 43] were shown to hold also along the whole corresponding attractor flow, as mentioned above.
- The analysis of [59] was further developed in [62], in which the non-BPS $Z \neq 0$ equations of motion of the scalars were solved for the $D2 - D6$ (*electric*) and $D0 - D2 - D4$ supporting BH charge configurations.

The present paper is devoted to a detailed, complete study of the attractor flow equations of the *stu* model, whose fundamental facts are summarized in Sect. 2.. All the classes of *non-degenerate* (*i.e.* with non-vanishing classical Bekenstein-Hawking [95] BH entropy) attractor flow solutions are determined, in their most general form (with all *B-fields* switched on). The main results of our investigation are listed below:

- As mentioned above, the $\frac{1}{2}$ -BPS attractor flow solution is known since [105]-[109], and it is reviewed in Sect. 3.. In Sect. 4. the non-BPS $Z = 0$ attractor flow solution, untreated so far, is determined for the most general supporting BH charge configuration, and its relation to the supersymmetric flow, both at and away from the event horizon radius r_H , is established, consistently with the results of [50].

- Sect. 5. is devoted to the study of the non-BPS $Z \neq 0$ attractor flow solution in full generality. By using suitable U -duality transformations (Subsect. 5.1.), and starting from the $D0 - D6$ configuration (Subsect. 5.2.), the non-BPS $Z \neq 0$ attractor flow supported by the most general $D0 - D2 - D4 - D6$ configuration (with *all* charges switched on) is explicitly derived in Subsect. 5.3.. This completes and generalizes the analyses and the results of [28], [51], [59] and [62]. The above mentioned finding of [59] is confirmed in such a general framework: the moduli space $(SO(1,1))^2$, known to exist at the non-BPS $Z \neq 0$ critical points of V_{BH} [41, 43], is found to be present *all along* the non-BPS attractor flow, *i.e.* for every $r \geq r_H$.
- In Sect. 6. a detailed analysis of particular configurations, namely $D0 - D4$ (*magnetic*, Subsect. 6.1.), its *dual* $D2 - D6$ (*electric*, Subsect. 6.2.), and $D0 - D2 - D4$ (Subsect. 6.3.), is performed.
- The so-called *first order (fake supergravity) formalism*, introduced in [118], has been recently developed in [37] and [42] in order to describe $d = 4$ *extremal* BHs; in general, it is based on a suitably defined real, scalar-dependent, *fake superpotential* \mathcal{W} . It is worth pointing out that the *first order formalism*, as (re)formulated in [37] and [42] for $d = 4$ *extremal* BHs, automatically selects the solutions which do *not* blow up at the BH event horizon. In other words, the (*covariant*) *scalar charges* Σ_i built in terms of the *fake superpotential* \mathcal{W} (see Eq. (20) further below) satisfy by construction all the conditions in order for the *Attractor Mechanism* to hold.

It should be here recalled that for *extremal* BHs the solution converging at the BH event horizon ($r \rightarrow r_H^+$) does *not* depend on the initial, asymptotical values of the scalar fields. See *e.g.* discussions in [33] and [44].

In the framework of *stu* model, we explicitly build up \mathcal{W} in the non-trivial cases represented by the non-BPS attractor flows. For the non-BPS $Z = 0$ attractor flow (Sect. 4.) a manifestly H -invariant \mathcal{W} is determined, in three different (but equivalent) “*polarizations*”. Furthermore, the difference between the squared non-BPS $Z = 0$ *fake superpotential* $\mathcal{W}_{non-BPS,Z=0}^2$ and $|Z|^2$ is computed along the non-BPS $Z = 0$ attractor flow, and the *BPS bound* [112] is found to hold *all along the attractor flow* (*i.e.* not only on the BH event horizon $r = r_H$, but $\forall r \geq r_H$). On the other hand, for non-BPS $Z \neq 0$ attractor flow (Subsects. 5.2. and 5.3.) the fake superpotential is manifestly *not H-invariant*.

As it will be commented in Sect. 7., this is not inconsistent with the treatment of [37] and [42]. Indeed, the *fake superpotential* is *not* unique within the same attractor flow, the various equivalent superpotentials being related through a (possibly scalar-dependent) R -matrix satisfying the conditions (2.28) and (2.29) of [37] (see in general Subsect. 2.2 of [37]). Moreover, in Sect. 6. the difference between the squared non-BPS $Z \neq 0$ *fake superpotential* $\mathcal{W}_{non-BPS,Z \neq 0}^2$ and $|Z|^2$ is computed along the non-BPS $Z \neq 0$ attractor flow in the *magnetic* (Subsect. 6.1.), *electric* (Subsect. 6.2.) and $D0 - D2 - D4$ (Subsect. 6.3.) charge configurations. Analogously to the non-BPS $Z = 0$ case, the *BPS bound* [112] is found to hold *all along the attractor flow* (*i.e.* not only on the BH event horizon $r = r_H$, but $\forall r \geq r_H$). In particular, for the *magnetic* charge configuration at radial infinity the result given by Eq. (4.8) of [59] is recovered. We are grateful to E. G. Gimon for a clarifying discussion about the definition of “gap” above the *BPS bound* as given in [59].

- Within the *first order (fake supergravity) formalism*, for all attractor flows we compute the *covariant scalar charges* as well as the *ADM mass*, studying the issue of *marginal stability* [111]. We thus complete the analysis and the results of [51], [59] and [62]. As expected due to the strict similarity to the $\frac{1}{2}$ -BPS attractor flow (Sect. 3.), also in the non-BPS $Z = 0$ case the *marginal bound* is *not* saturated (Sect. 4.), as instead we confirm to hold in general in the (non-BPS $Z \neq 0$ -supporting branch of the) $D0 - D2 - D4 - D6$ configuration (Subsect. 5.3.).
- Final remarks, comments, and outlook for further developments are given in the concluding Sect. 7..

2. Basics of the *stu* Model

We here recall some basic facts of the above mentioned *stu* model [15, 28, 39–41, 43, 50, 97–100], fixing our notations and conventions. The three complex moduli of the model are defined as

$$z^1 \equiv x^1 - iy^1 \equiv s, \quad z^2 \equiv x^2 - iy^2 \equiv t, \quad z^3 \equiv x^3 - iy^3 \equiv u, \tag{2}$$

with $y^i \in \mathbb{R}_0^+$ [122] (see also the treatment of [47]). In *special coordinates* (see e.g. [119] and Refs. therein) the prepotential determining the relevant special Kähler geometry simply reads

$$f = stu. \tag{3}$$

Due to the aforementioned *triality symmetry* among s , t and u , the expressions of all the relevant geometric quantities acquire quite elegant form. Some formulæ can be found e.g. in the treatments of [28, 50, 59, 99, 100]; for completeness, below we list the expressions of the Kähler potential, contravariant metric tensor, non-vanishing components of the Christoffel symbols of the second kind and of the C -tensor, holomorphic central charge (also named superpotential) and BH effective potential ($i = 1, 2, 3$ throughout):

$$\begin{aligned} K &= -\ln [-i(s - \bar{s})(t - \bar{t})(u - \bar{u})] \Rightarrow \exp(-K) = 8y^1y^2y^3; \\ g^{i\bar{j}} &= -\text{diag}((s - \bar{s})^2, (t - \bar{t})^2, (u - \bar{u})^2); \\ \Gamma_{11}^1 &= -2(s - \bar{s})^{-1}, \quad \Gamma_{22}^2 = -2(t - \bar{t})^{-1}, \quad \Gamma_{33}^3 = -2(u - \bar{u})^{-1}; \\ C_{stu} &= \frac{i}{(s-\bar{s})(t-\bar{t})(u-\bar{u})} = \exp(K); \\ W(s, t, u) &= q_0 + q_1s + q_2t + q_3u + p^0stu - p^1tu - p^2su - p^3st; \\ V_{BH} &= \exp(K) \cdot [|W(s, t, u)|^2 + |W(\bar{s}, t, u)|^2 + |W(s, \bar{t}, u)|^2 + |W(s, t, \bar{u})|^2]. \end{aligned} \tag{4}$$

Thus, the (covariantly holomorphic) *central charge function* for the *stu* model reads (see *e.g.* [119] and Refs. therein)

$$\begin{aligned}
 Z(s, t, u, \bar{s}, \bar{t}, \bar{u}) &\equiv e^{K/2} W(s, t, u) = \\
 &= \frac{1}{\sqrt{-i(s - \bar{s})(t - \bar{t})(u - \bar{u})}} (q_0 + q_1 s + q_2 t + q_3 u + p^0 stu - p^1 tu - p^2 su - p^3 st).
 \end{aligned}
 \tag{5}$$

The definition of the BH charges p^Λ (magnetic) and q_Λ (electric) ($\Lambda = 0, 1, 2, 3$ throughout), the effective 1-dim. (quasi-)geodesic Lagrangian of the *stu* model, and the corresponding Eqs. of motion for the scalars can be found in Subsects. 2.2 and 2.3, as well as in appendix A (treating the case $D0 - D4$ in detail), of [59].

Through the Bekenstein-Hawking entropy-area formula [95], the entropy of an extremal BH in the *stu* model in the Einsteinian approximation reads as follows:

$$S_{BH} = \frac{A_H}{4} = \pi V_{BH}|_{\partial V_{BH}=0} = \pi \sqrt{|\mathcal{I}_4(\Gamma)|},
 \tag{6}$$

where the $(2n_V + 2) \times 1$ vector of BH charges

$$\Gamma \equiv (p^\Lambda, q_\Lambda),
 \tag{7}$$

was introduced, n_V denoting the number of Abelian vector multiplets coupled to the supergravity one (in the case under consideration $n_V = 3$). Furthermore, $\mathcal{I}_4(\Gamma)$ denotes the unique invariant of the *tri-fundamental representation* $(\mathbf{2}, \mathbf{2}, \mathbf{2})$ of the *U*-duality group G , reading as follows (see *e.g.* Eq. (4.10) and Sect. 5 of [50], and Refs. therein):

$$\mathcal{I}_4(\Gamma) = - (p^\Lambda q_\Lambda)^2 + 4 \sum_{i < j} p^i q_i p^j q_j - 4 p^0 q_1 q_2 q_3 + 4 q_0 p^1 p^2 p^3 = -Det(\Psi),
 \tag{8}$$

where $Det(\Psi)$ is the so-called Cayley’s hyperdeterminant [3].

In the next three Sections we will discuss the explicit solutions of the equations of motion of the scalars s, t and u in the dyonic background of an extremal BH of the *stu* model, also named *Attractor Flow Equations*. We will consider only *non-degenerate* attractor flows, *i.e.* those flows determining a regular, non-vanishing area of the horizon in the Einsteinian approximation.

As mentioned above, 3 classes of non-degenerate attractor flows exist in the *stu* model:

- $\frac{1}{2}$ -BPS (Sect. 3.);
- non-BPS $Z = 0$ (Sect. 4.);
- non-BPS $Z \neq 0$ (Sects. 5.) and (6.).

3. The Most General $\frac{1}{2}$ -BPS Attractor Flow

The explicit expression of the attractor flow solution supported by the most general $\frac{1}{2}$ -BPS BH charge configuration in $\mathcal{N} = 2, d = 4$ *ungauged* supergravity coupled to n_V Abelian vector multiplets (and

exhibiting a unique U -invariant \mathcal{I}_4) is known after [105]-[109] (as well as the third of Refs. [111]):

$$\begin{aligned} \exp \left[-4U_{\frac{1}{2}\text{-BPS}}(\tau) \right] &= \mathcal{I}_4(\mathcal{H}(\tau)); \\ z_{\frac{1}{2}\text{-BPS}}^i(\tau) &= \frac{H^i(\tau) + i\partial_{H_i}\mathcal{I}_4^{1/2}(\mathcal{H}(\tau))}{H^0(\tau) + i\partial_{H_0}\mathcal{I}_4^{1/2}(\mathcal{H}(\tau))}, \end{aligned} \tag{9}$$

where $\partial_{H_i} \equiv \frac{\partial}{\partial H_i}$, and the $(2n_V + 2) \times 1 (= 8 \times 1$ in the model under consideration) symplectic vector

$$\mathcal{H}(\tau) \equiv (H^\Lambda(\tau), H_\Lambda(\tau)), \tag{10}$$

was introduced, where $H^\Lambda(\tau)$ and $H_\Lambda(\tau)$ are *harmonic functions* defined as follows ($\tau \equiv (r_H - r)^{-1} \in \mathbb{R}^-$):

$$H^\Lambda(\tau) \equiv p_\infty^\Lambda + p^\Lambda \tau; \tag{11}$$

$$H_\Lambda(\tau) = q_{\Lambda, \infty} + q_\Lambda \tau,$$

such that $\mathcal{H}(\tau)$ can be formally rewritten as

$$\mathcal{H}(\tau) = \Gamma_\infty + \Gamma\tau. \tag{12}$$

The asymptotical constants Γ_∞ must satisfy the following *integrability conditions*:

$$\mathcal{I}_4(\Gamma_\infty) = 1, \quad \langle \Gamma, \Gamma_\infty \rangle = 0, \tag{13}$$

where $\langle \cdot, \cdot \rangle$ is the scalar product defined by the $(2n_V + 2) \times (2n_V + 2)$ symplectic metric. Under such conditions, the flow (9) is the most general solution of the so-called $\frac{1}{2}$ -BPS *stabilization Eqs.* (see e.g. the recent treatment of [28]):

$$\mathcal{H}^T(\tau) = 2e^{K(z(\tau), \bar{z}(\tau))} \text{Im} \left[W(z(\tau), \mathcal{H}(\tau)) \begin{pmatrix} \bar{X}^\Lambda(\bar{z}(\tau)) \\ \bar{F}_\Lambda(\bar{z}(\tau)) \end{pmatrix} \right], \tag{14}$$

obtained from the $\frac{1}{2}$ -BPS *Attractor Eqs.* (see e.g. the treatment in [23], and Refs. therein)

$$\Gamma^T = 2e^{K(z, \bar{z})} \text{Im} \left[W(z, \Gamma) \begin{pmatrix} \bar{X}^\Lambda(\bar{z}) \\ \bar{F}_\Lambda(\bar{z}) \end{pmatrix} \right] \tag{15}$$

by simply replacing Γ with $\mathcal{H}(\tau)$ (see e.g. [106] and Refs. therein). Consistently, Eq. (15) is the *near-horizon* ($\tau \rightarrow -\infty$) *limit* of Eq. (14).

Moreover, the BH charge configurations supporting the $\frac{1}{2}$ -BPS attractors at the BH event horizon satisfy the following constraints, defining the $\frac{1}{2}$ -BPS orbit (see Appendix II of [26])

$$\mathcal{O}_{\frac{1}{2}\text{-BPS}} = \frac{(SU(1,1))^3}{(U(1))^2} \tag{16}$$

of the *tri-fundamental* representation $(2, 2, 2)$ of the U -duality group $(SU(1, 1))^3$ [26, 50]:

$$\begin{aligned}
 \mathcal{I}_4(\Gamma) &> 0; \\
 p^2 p^3 - p^0 q_1 &\geq 0; \\
 p^1 p^3 - p^0 q_2 &\geq 0; \\
 p^1 p^2 - p^0 q_3 &\geq 0.
 \end{aligned}
 \tag{17}$$

Correspondingly, $\mathcal{H}(\tau)$ is constrained as follows along the $\frac{1}{2}$ -BPS attractor flow ($\forall \tau \in \mathbb{R}^-$):

$$\begin{aligned}
 \mathcal{I}_4(\mathcal{H}(\tau)) &> 0; \\
 H^2(\tau) H^3(\tau) - H^0(\tau) H_1(\tau) &\geq 0; \\
 H^1(\tau) H^3(\tau) - H^0(\tau) H_2(\tau) &\geq 0; \\
 H^1(\tau) H^2(\tau) - H^0(\tau) H_3(\tau) &\geq 0.
 \end{aligned}
 \tag{18}$$

In the *near-horizon limit* $\tau \rightarrow -\infty$, Eq. (9) yields the *purely charge-dependent, critical* expressions of the scalars at the BH event horizon, *e.g.* given by Eq. (3.1) of [59]. In the same limit, the constraints (18) consistently yield the constraints (17).

Consistently with the analysis of [47], the general $\frac{1}{2}$ -BPS attractor flow solution (9) of the *stu* model can be *axion-free* only for the configurations $D0 - D6$, $D0 - D4$ (*magnetic*) and $D2 - D6$ (*electric*).

As found in [121] and observed also in [59], an immediate consequence of Eq. (9) is that Γ_∞ satisfies the *$\frac{1}{2}$ -BPS Attractor Eqs.* [106]. This determines a sort of “*Attractor Mechanism at spatial infinity*”, mapping the 6 *real* moduli $(x^1, x^2, x^3, y^1, y^2, y^3)$ into the 8 *real* constants $(p_\infty^1, p_\infty^2, p_\infty^3, q_{1,\infty}, q_{2,\infty}, q_{3,\infty})$, arranged as Γ_∞ and constrained by the 2 *real* conditions (13).

As noticed in [59], the absence of *flat* directions in the $\frac{1}{2}$ -BPS attractor flow (which is a general feature of $\mathcal{N} = 2, d = 4$ *ungauged* supergravity coupled to Abelian vector multiplets, *at least* as far as the metric of the scalar manifold is strictly positive definite $\forall \tau \in \mathbb{R}^-$ [10]) is crucial for the validity of the expression (9).

Now, by exploiting the *first order formalism* [118] for $d = 4$ extremal BHs [37, 42] (see also [70] and [74]), one can compute the relevant BH parameters of the $\frac{1}{2}$ -BPS attractor flow of the *stu* model starting from the expression of the $\frac{1}{2}$ -BPS *fake superpotential* $\mathcal{W}_{\frac{1}{2}\text{-BPS}}$. For instance, the *ADM mass* and *covariant scalar charges* respectively read (see *e.g.* the treatments in [70] and [74]):

$$M_{ADM}(z_\infty, \bar{z}_\infty, \Gamma) = \mathcal{W}(z_\infty, \bar{z}_\infty, \Gamma) \equiv \lim_{\tau \rightarrow 0^-} \mathcal{W}(z(\tau), \bar{z}(\tau), \Gamma); \tag{19}$$

$$\Sigma_i(z_\infty, \bar{z}_\infty, \Gamma) = (\partial_i \mathcal{W})(z_\infty, \bar{z}_\infty, \Gamma) \equiv \lim_{\tau \rightarrow 0^-} (\partial_i \mathcal{W})(z(\tau), \bar{z}(\tau), \Gamma), \tag{20}$$

where the subscript “ ∞ ” denotes the evaluation at the moduli at spatial infinity ($r \rightarrow \infty \Leftrightarrow \tau \rightarrow 0^-$). Notice that Eq. (19) provides, within the considered *first order formalism*, an alternative (eventually

simpler) formula for the computation of M_{ADM} , with respect to the general definition in terms of the warp factor U (see e.g. [10]):

$$M_{ADM} = \lim_{\tau \rightarrow 0^-} \frac{dU(\tau)}{d\tau}. \tag{21}$$

Recalling that for all $\mathcal{N} = 2, d = 4$ ungauged supergravities it holds that $\mathcal{W}_{\frac{1}{2}\text{-BPS}} = |Z|$, Eqs. (5) and (19) yield the following expressions of the ADM mass of the $\frac{1}{2}$ -BPS attractor flow of the stu model:

$$\begin{aligned} M_{ADM, \frac{1}{2}\text{-BPS}}(z_\infty, \bar{z}_\infty, \Gamma) &\equiv \lim_{\tau \rightarrow 0^-} |Z|(z(\tau), \bar{z}(\tau), \Gamma) = \\ &= \frac{|q_0 + q_1 s_\infty + q_2 t_\infty + q_3 u_\infty + p^0 s_\infty t_\infty u_\infty - p^1 t_\infty u_\infty - p^2 s_\infty u_\infty - p^3 s_\infty t_\infty|}{\sqrt{-i(s_\infty - \bar{s}_\infty)(t_\infty - \bar{t}_\infty)(u_\infty - \bar{u}_\infty)}}. \end{aligned} \tag{22}$$

Eq. (22) yields that the marginal bound [111] is not saturated by $\frac{1}{2}$ -BPS states, because $M_{ADM, \frac{1}{2}\text{-BPS}}$ is not equal to the sum of the ADM masses of four $D6$ -branes with appropriate fluxes (for further detail, see the discussion in [59]).

Concerning the (covariant) scalar charges of the $\frac{1}{2}$ -BPS attractor flow of the stu model, they can be straightforwardly computed by using Eqs. (5) and (20):

$$\begin{aligned} \Sigma_{s, \frac{1}{2}\text{-BPS}}(z_\infty, \bar{z}_\infty, \Gamma) &= \\ &= \lim_{\tau \rightarrow 0^-} \frac{e^{K/2}}{2} \left[(\partial_s K) |W| + (\partial_s W) \sqrt{\frac{\bar{W}}{W}} \right] (z(\tau), \bar{z}(\tau), \Gamma) = \\ &= \frac{1}{2\sqrt{-i(s_\infty - \bar{s}_\infty)(t_\infty - \bar{t}_\infty)(u_\infty - \bar{u}_\infty)}} \cdot \\ &\cdot \left[\frac{|q_0 + q_1 s_\infty + q_2 t_\infty + q_3 u_\infty + p^0 s_\infty t_\infty u_\infty - p^1 t_\infty u_\infty - p^2 s_\infty u_\infty - p^3 s_\infty t_\infty|}{(s_\infty - \bar{s}_\infty)} + \right. \\ &\quad \left. + (q_1 + p^0 t_\infty u_\infty - p^2 u_\infty - p^3 t_\infty) \cdot \right. \\ &\quad \left. \sqrt{\frac{q_0 + q_1 \bar{s}_\infty + q_2 \bar{t}_\infty + q_3 \bar{u}_\infty + p^0 \bar{s}_\infty \bar{t}_\infty \bar{u}_\infty - p^1 \bar{t}_\infty \bar{u}_\infty - p^2 \bar{s}_\infty \bar{u}_\infty - p^3 \bar{s}_\infty \bar{t}_\infty}{q_0 + q_1 s_\infty + q_2 t_\infty + q_3 u_\infty + p^0 s_\infty t_\infty u_\infty - p^1 t_\infty u_\infty - p^2 s_\infty u_\infty - p^3 s_\infty t_\infty}} \right]. \end{aligned} \tag{23}$$

$$\begin{aligned}
 & \Sigma_{t, \frac{1}{2}\text{-BPS}}(z_\infty, \bar{z}_\infty, \Gamma) = \\
 & = \lim_{\tau \rightarrow 0^-} \frac{e^{K/2}}{2} \left[(\partial_t K) |W| + (\partial_t W) \sqrt{\frac{\bar{W}}{W}} \right] (z(\tau), \bar{z}(\tau), \Gamma) = \\
 & = \frac{1}{2\sqrt{-i(s_\infty - \bar{s}_\infty)(t_\infty - \bar{t}_\infty)(u_\infty - \bar{u}_\infty)}} \cdot \\
 & \cdot \left[\frac{|q_0 + q_1 s_\infty + q_2 t_\infty + q_3 u_\infty + p^0 s_\infty t_\infty u_\infty - p^1 t_\infty u_\infty - p^2 s_\infty u_\infty - p^3 s_\infty t_\infty|}{(t_\infty - \bar{t}_\infty)} + \right. \\
 & \quad \left. + (q_2 + p^0 s_\infty u_\infty - p^1 u_\infty - p^3 s_\infty) \cdot \right. \\
 & \left. \sqrt{\frac{q_0 + q_1 \bar{s}_\infty + q_2 \bar{t}_\infty + q_3 \bar{u}_\infty + p^0 \bar{s}_\infty \bar{t}_\infty \bar{u}_\infty - p^1 \bar{t}_\infty \bar{u}_\infty - p^2 \bar{s}_\infty \bar{u}_\infty - p^3 \bar{s}_\infty \bar{t}_\infty}{q_0 + q_1 s_\infty + q_2 t_\infty + q_3 u_\infty + p^0 s_\infty t_\infty u_\infty - p^1 t_\infty u_\infty - p^2 s_\infty u_\infty - p^3 s_\infty t_\infty}} \right].
 \end{aligned} \tag{24}$$

$$\begin{aligned}
 & \Sigma_{u, \frac{1}{2}\text{-BPS}}(z_\infty, \bar{z}_\infty, \Gamma) = \\
 & = \lim_{\tau \rightarrow 0^-} \frac{e^{K/2}}{2} \left[(\partial_u K) |W| + (\partial_u W) \sqrt{\frac{\bar{W}}{W}} \right] (z(\tau), \bar{z}(\tau), \Gamma) = \\
 & = \frac{1}{2\sqrt{-i(s_\infty - \bar{s}_\infty)(t_\infty - \bar{t}_\infty)(u_\infty - \bar{u}_\infty)}} \cdot \\
 & \cdot \left[\frac{|q_0 + q_1 s_\infty + q_2 t_\infty + q_3 u_\infty + p^0 s_\infty t_\infty u_\infty - p^1 t_\infty u_\infty - p^2 s_\infty u_\infty - p^3 s_\infty t_\infty|}{(u_\infty - \bar{u}_\infty)} + \right. \\
 & \quad \left. + (q_3 + p^0 s_\infty t_\infty - p^1 t_\infty - p^2 s_\infty) \cdot \right. \\
 & \left. \sqrt{\frac{q_0 + q_1 \bar{s}_\infty + q_2 \bar{t}_\infty + q_3 \bar{u}_\infty + p^0 \bar{s}_\infty \bar{t}_\infty \bar{u}_\infty - p^1 \bar{t}_\infty \bar{u}_\infty - p^2 \bar{s}_\infty \bar{u}_\infty - p^3 \bar{s}_\infty \bar{t}_\infty}{q_0 + q_1 s_\infty + q_2 t_\infty + q_3 u_\infty + p^0 s_\infty t_\infty u_\infty - p^1 t_\infty u_\infty - p^2 s_\infty u_\infty - p^3 s_\infty t_\infty}} \right].
 \end{aligned} \tag{25}$$

4. The Most General Non-BPS $Z = 0$ Attractor Flow

Let us now investigate the non-BPS $Z = 0$ case.

As shortly noticed in [59], in spite of the fact that this attractor flow is non-supersymmetric, it has many common features with the supersymmetric ($\frac{1}{2}$ -BPS) case.

As yielded by the analysis of [50], the non-BPS $Z = 0$ horizon attractor solutions can be obtained from $\frac{1}{2}$ -BPS ones simply by changing the signs of any two imaginary parts of the moduli (*dilatons*) and consistently imposing specific constraints on BH charges. For example, one can choose (without any loss of generality, due to *triatlity symmetry*) to flip the dilatons as follows:

$$y^1 \rightarrow y^1, \quad y^2 \rightarrow -y^2, \quad y^3 \rightarrow -y^3. \tag{26}$$

This yields the following constraints on the BH charge configurations supporting the non-BPS $Z = 0$ attractors at the BH event horizon ($\tau \rightarrow -\infty$) [50]:

$$\begin{aligned} \mathcal{I}_4(\Gamma) &> 0; \\ p^2 p^3 - p^0 q_1 &\leq 0; \\ p^1 p^3 - p^0 q_2 &\geq 0; \\ p^1 p^2 - p^0 q_3 &\geq 0. \end{aligned} \tag{27}$$

The constraints (27) defines the non-BPS $Z = 0$ orbit of the *tri-fundamental* representation $(2, 2, 2)$ of the U -duality group $(SU(1, 1))^3$ (see Appendix II of [26])

$$\mathcal{O}_{non-BPS, Z=0} = \frac{(SU(1, 1))^3}{(U(1))^2}. \tag{28}$$

Notice that such an orbit shares the same coset expression of $\mathcal{O}_{\frac{1}{2}-BPS}$ given by Eq. (16). However, they do *not* coincide, but instead they are two *separated* branches of a *disconnected* manifold, classified by the local value of the function $sgn(|Z|^2 - |D_s Z|^2)$ (see Appendix II of [26]; $D_s Z$ is defined below Eq. (38)).

The same holds *all along the attractor flow*, i.e. $\forall \tau \in \mathbb{R}^-$. Indeed, the most general non-BPS $Z = 0$ attractor flow can be obtained by taking the most general $\frac{1}{2}$ -BPS attractor flow, and flipping any two out of the three dilatons. Thus, by taking Eq. (9) and flipping the dilatons as given by Eq. (26), one achieves the following result:

$$\begin{aligned} exp[-4U_{non-BPS, Z=0}(\tau)] &= \mathcal{I}_4(\mathcal{H}(\tau)); \\ z^1_{non-BPS, Z=0}(\tau) &= \frac{H^\Lambda(\tau)H_\Lambda(\tau) - 2H^1(\tau)H_1(\tau) - i\mathcal{I}_4^{1/2}(\mathcal{H}(\tau))}{2[H^2(\tau)H^3(\tau) - H^0(\tau)H_1(\tau)]} = z^1_{\frac{1}{2}-BPS}(\tau); \\ z^2_{non-BPS, Z=0}(\tau) &= \frac{H^\Lambda(\tau)H_\Lambda(\tau) - 2H^2(\tau)H_2(\tau) + i\mathcal{I}_4^{1/2}(\mathcal{H}(\tau))}{2[H^1(\tau)H^3(\tau) - H^0(\tau)H_2(\tau)]} = \overline{z^2_{\frac{1}{2}-BPS}(\tau)}; \\ z^3_{non-BPS, Z=0}(\tau) &= \frac{H^\Lambda(\tau)H_\Lambda(\tau) - 2H^3(\tau)H_3(\tau) + i\mathcal{I}_4^{1/2}(\mathcal{H}(\tau))}{2[H^2(\tau)H^1(\tau) - H^0(\tau)H_3(\tau)]} = \overline{z^3_{\frac{1}{2}-BPS}(\tau)}. \end{aligned} \tag{29}$$

This is the most general expression of the non-BPS $Z = 0$ attractor flow, in the “*polarization*” given by Eq. (26), which, due to the underlying *triality symmetry* of the *stu* model, does *not* imply any loss of generality.

Consistently with the constraints (27), $\mathcal{H}(\tau)$ is constrained as follows along the non-BPS $Z = 0$

attractor flow ($\forall \tau \in \mathbb{R}^-$):

$$\begin{aligned}
 \mathcal{I}_4(\mathcal{H}(\tau)) &> 0; \\
 H^2(\tau)H^3(\tau) - H^0(\tau)H_1(\tau) &\leq 0; \\
 H^1(\tau)H^3(\tau) - H^0(\tau)H_2(\tau) &\geq 0; \\
 H^1(\tau)H^2(\tau) - H^0(\tau)H_3(\tau) &\geq 0.
 \end{aligned}
 \tag{30}$$

In the *near-horizon limit* $\tau \rightarrow -\infty$, Eq. (29) yields the *purely charge-dependent, critical* expressions of the scalars at the BH event horizon, given by Eq. (4.9) of [50]. In the same limit, the constraints (30) consistently yield the constraints (27). The *integrability conditions* (13) clearly hold also in this case.

Consistently with the analysis of [47], the general non-BPS $Z = 0$ attractor flow solution (29) of the *stu* model can be *axion-free* only for the configurations $D0 - D6$, $D0 - D4$ (*magnetic*) and $D2 - D6$ (*electric*).

A consequence of Eq. (29) is that Γ_∞ satisfies the *non-BPS $Z = 0$ Attractor Eqs.* (see e.g. [23] and [32]). Analogously to what happens for the $\frac{1}{2}$ -BPS attractor flow, this determines a sort of “*Attractor Mechanism at spatial infinity*”.

Analogously to what happens in the $\frac{1}{2}$ -BPS case, the absence of *flat* directions in the non-BPS $Z = 0$ attractor flow (which is *not* a general feature of $\mathcal{N} = 2$, $d = 4$ *ungauged* supergravity coupled to Abelian vector multiplets, but however holds for the *stu* model [40, 43]) is crucial for the validity of the expression (29).

By exploiting the strict relation with the $\frac{1}{2}$ -BPS attractor flow, one can also determine the explicit expression of the *fake superpotential* $\mathcal{W}_{non-BPS,Z=0}$ for the non-BPS $Z = 0$ attractor flow. Considering the absolute value of the $\mathcal{N} = 2$, $d = 4$ *central charge function* Z given by Eq. (5) and flipping two dilatons out of three in the “*polarization*” given by Eq. (26), one obtains the following non-BPS $Z = 0$ *fake superpotential* (notice that K , as given by the first Eq. of (4), is invariant under such a flipping):

$$\begin{aligned}
 \mathcal{W}_{non-BPS,Z=0,s} &= e^{K/2} |q_0 + q_1 s + q_2 \bar{t} + q_3 \bar{u} + p^0 s \bar{t} \bar{u} - p^1 \bar{t} \bar{u} - p^2 s \bar{u} - p^3 s \bar{t}| = \\
 &= |Z(s, \bar{t}, \bar{u})| = \mathcal{W}_{\frac{1}{2}-BPS}(s, \bar{t}, \bar{u}),
 \end{aligned}
 \tag{31}$$

where the subscript “*s*” denotes the modulus untouched by the considered flipping of dilatons; in the last step we used that aforementioned fact that for all $\mathcal{N} = 2$, $d = 4$ *ungauged* supergravities it holds that $\mathcal{W}_{\frac{1}{2}-BPS} = |Z|$.

Clearly, the flipping (26) is not the only possible one. By triality symmetry, two other equivalent flippings exist, namely

$$y^1 \rightarrow -y^1, y^2 \rightarrow y^2, y^3 \rightarrow -y^3;
 \tag{32}$$

$$y^1 \rightarrow -y^1, y^2 \rightarrow -y^2, y^3 \rightarrow y^3,
 \tag{33}$$

obtained by cyclic permutations from (26). Such equivalent flippings respectively determine the following (respectively “*t-polarized*” and “*u-polarized*”) non-BPS $Z = 0$ *fake superpotentials*, completely

equivalent to the “*s-polarized*” one given by Eq. (31):

$$\begin{aligned} \mathcal{W}_{non-BPS,Z=0,t} &= e^{K/2} |q_0 + q_1\bar{s} + q_2t + q_3\bar{u} + p^0\bar{s}t\bar{u} - p^1t\bar{u} - p^2\bar{s}\bar{u} - p^3\bar{s}t| = \\ &= |Z(\bar{s}, t, \bar{u})| = \mathcal{W}_{\frac{1}{2}-BPS}(\bar{s}, t, \bar{u}); \end{aligned} \tag{34}$$

$$\begin{aligned} \mathcal{W}_{non-BPS,Z=0,u} &= e^{K/2} |q_0 + q_1\bar{s} + q_2\bar{t} + q_3u + p^0\bar{s}\bar{t}u - p^1\bar{t}u - p^2\bar{s}u - p^3\bar{s}\bar{t}| = \\ &= |Z(\bar{s}, \bar{t}, u)| = \mathcal{W}_{\frac{1}{2}-BPS}(\bar{s}, \bar{t}, u). \end{aligned} \tag{35}$$

It can be shown by straightforward computations that the *real*, moduli- and charge- dependent functions given by Eqs. (31), (34) and (35) do satisfy all the conditions defining a *first order fake superpotential* (see the treatment in [37, 42], recently reviewed in [70, 74]), and thus they respectively are an “*s-polarized*”, “*t-polarized*” and “*u-polarized*” non-BPS $Z = 0$ *fake superpotential*.

Eqs. (31), (34) and (35) can also be rewritten respectively as follows:

$$\mathcal{W}_{non-BPS,Z=0,s} = \sqrt{g^{1\bar{1}} (D_s Z) \overline{D_{\bar{s}} Z}}; \tag{36}$$

$$\mathcal{W}_{non-BPS,Z=0,t} = \sqrt{g^{2\bar{2}} (D_t Z) \overline{D_{\bar{t}} Z}}; \tag{37}$$

$$\mathcal{W}_{non-BPS,Z=0,u} = \sqrt{g^{3\bar{3}} (D_u Z) \overline{D_{\bar{u}} Z}}, \tag{38}$$

where $D_s Z \equiv [\partial_s + \frac{1}{2}(\partial_s K)] Z$ is the covariant derivative of Z along the direction s (and analogously for the moduli t and u).

Due to the complete factorization of the manifold $\left(\frac{SU(1,1)}{U(1)}\right)^3$ (determining the diagonality of the metric $g^{i\bar{j}}$, given by the second Eq. of (4)), Eqs. (36), (38) and (37) are *manifestly H-invariant*. This result is consistent with the *H-invariance* imposed by the relation (given by Eq. (2.21) of [37], as well as by Eq. (13) of [42], and reported here for completeness’ sake)

$$\frac{dU(\tau, \Gamma)}{d\tau} = e^{U(\tau)} W(z(\tau), \bar{z}(\tau), \Gamma) \tag{39}$$

between \mathcal{W} and the *warp factor* $U(\tau)$ appearing in the *Ansatz* for the static, spherically symmetric, asymptotically flat, *extremal dyonic BH metric*:

$$ds^2 = -e^{2U(\tau)} dt^2 + e^{-2U(\tau)} d\vec{x}^2. \tag{40}$$

Actually, in the treatment given in the present paper, Eq. (39) has been crucial in order to guess (and thus check) the analytical form of the *fake superpotential* $\mathcal{W}(z(\tau), \bar{z}(\tau), \Gamma)$, by knowing the analytical solution for the *warp factor* $U(\tau, \Gamma)$ relevant to the considered charge configuration.

Under the *integrability conditions* (13), the flow (29) is the most general solution (within the “*s-polarization*” defined by Eq. (26)) of the “non-BPS $Z = 0$ analogue” of the $\frac{1}{2}$ -BPS *stabilization Eqs.* (14), namely of:

$$\mathcal{H}^T(\tau) = 2e^{K(z(\tau), \bar{z}(\tau))} Im \left[g^{i\bar{j}}(z(\tau), \bar{z}(\tau)) \begin{pmatrix} (D_i X^\Lambda)(z(\tau), \bar{z}(\tau)) \\ (D_i F_\Lambda)(z(\tau), \bar{z}(\tau)) \end{pmatrix} (\bar{\partial}_{\bar{j}} \overline{W})(\bar{z}(\tau), \mathcal{H}(\tau)) \right], \tag{41}$$

which we can refer to as the *non-BPS $Z = 0$ stabilization Equations*. It is easy realized that they can be obtained from the *non-BPS $Z = 0$ Attractor Eqs.* (see the treatment in [23])

$$\Gamma^T = 2e^{K(z,\bar{z})} Im \left[g^{i\bar{j}}(z, \bar{z}) \begin{pmatrix} (D_i X^\Lambda)(z, \bar{z}) \\ (D_i F_\Lambda)(z, \bar{z}) \end{pmatrix} (\bar{\partial}_{\bar{j}} \bar{W})(\bar{z}, \Gamma) \right] \tag{42}$$

by simply replacing Γ with $\mathcal{H}(\tau)$. Consistently, Eq. (42) is the *near-horizon* ($\tau \rightarrow -\infty$) *limit* of Eq. (41).

A remarkable consequence of the *first order formalism* for the non-BPS $Z = 0$ attractor flow is that Eqs. (41) and (42) can actually be recast in the following ($\frac{1}{2}$ -)BPS-like forms in terms of the non-BPS $Z = 0$ *fake superpotential(s)*, respectively:

$$\mathcal{H}^T(\tau) = 2e^{K(z(\tau), \bar{z}(\tau))} Im \left[\mathfrak{W}_{non-BPS, Z=0}(z(\tau), \bar{z}(\tau), \mathcal{H}(\tau)) \begin{pmatrix} \bar{X}_f^\Lambda(z(\tau), \bar{z}(\tau)) \\ \bar{F}_{\Lambda, f}(z(\tau), \bar{z}(\tau)) \end{pmatrix} \right]; \tag{43}$$

$$\Gamma^T = 2e^{K(z, \bar{z})} Im \left[\mathfrak{W}_{non-BPS, Z=0}(z, \bar{z}, \Gamma) \begin{pmatrix} \bar{X}_f^\Lambda(z, \bar{z}) \\ \bar{F}_{\Lambda, f}(z, \bar{z}) \end{pmatrix} \right], \tag{44}$$

where

$$\mathfrak{W}_{non-BPS, Z=0}(z, \bar{z}, \Gamma) = Z(s, \bar{t}, \bar{u}, \Gamma) \text{ in the “}s\text{-polarization” (Eq. 26);} \tag{45}$$

$$\mathfrak{W}_{non-BPS, Z=0}(z, \bar{z}, \Gamma) = Z(\bar{s}, t, \bar{u}, \Gamma) \text{ in the “}t\text{-polarization” (Eq. 32);} \tag{46}$$

$$\mathfrak{W}_{non-BPS, Z=0}(z, \bar{z}, \Gamma) = Z(\bar{s}, \bar{t}, u, \Gamma) \text{ in the “}u\text{-polarization” (Eq. 33),} \tag{47}$$

such that in general

$$\mathcal{W}_{non-BPS, Z=0} = |\mathfrak{W}_{non-BPS, Z=0}|. \tag{48}$$

The subscript “ f ” in Eqs. (43) and (44) indicates that a flipping of the dilatons has been performed ((Eqs. (26), (32) and (33), respectively for the choices (45), (46) and (47)). Notice that such a flipping destroys the holomorphicity of X^Λ and F_Λ in the moduli.

Eq. (29) is the most general solution of Eq. (43) with the choice (45), and equivalent expressions for the most general non-BPS $Z = 0$ attractor flow can be obtained by solving Eq. (43) with the choice (46) or (47).

Such a ($\frac{1}{2}$ -)BPS-like reformulation of the *non-BPS $Z = 0$ stabilization Equations* (41) and of their *near-horizon* ($\tau \rightarrow -\infty$) *limit* given by the *non-BPS $Z = 0$ Attractor Eqs.* (42) is possible due to the strict similarity between the most general $\frac{1}{2}$ -BPS and non-BPS $Z = 0$ attractor flows in the considered *stu* model, which actually are related through a flipping of two dilatons out of three. This in turn is related once again to the absence of *flat* directions along such flows, such that *all* moduli are explicitly determined as functions of dyonic BH charges (and as functions of τ) *all along the attractor flow*.

Now, by exploiting the *first order formalism* [118] for $d = 4$ extremal BHs [37, 42] (see also [70] and [74]), one can compute the relevant BH parameters of the non-BPS $Z = 0$ attractor flow of the

stu model starting from the expression of the non-BPS $Z = 0$ fake superpotential $\mathcal{W}_{non-BPS,Z=0}$ given by Eqs. (31), (34) or (35). The choice of “*s*-polarization”, “*t*-polarization” or “*u*-polarization” is immaterial, due to the underlying *triality symmetry* of the moduli *s*, *t* and *u*. Thus, without loss of generality, we choose to perform computations in the “*s*-polarization” (equivalent results in the other two “*polarizations*” can be obtained by cyclic permutations of the moduli).

Eqs. (31) and (19) yield the following expressions of the ADM mass of the non-BPS $Z = 0$ attractor flow of the *stu* model:

$$\begin{aligned}
 M_{ADM,non-BPS,Z=0}(z_\infty, \bar{z}_\infty, \Gamma) &\equiv \lim_{\tau \rightarrow 0^-} \mathcal{W}_{non-BPS,Z=0,s}(z(\tau), \bar{z}(\tau), \Gamma) = \\
 &= \lim_{\tau \rightarrow 0^-} |Z(s(\tau), \bar{t}(\tau), \bar{u}(\tau))| = \\
 &= \frac{|q_0 + q_1 s_\infty + q_2 \bar{t}_\infty + q_3 \bar{u}_\infty + p^0 s_\infty \bar{t}_\infty \bar{u}_\infty - p^1 \bar{t}_\infty \bar{u}_\infty - p^2 s_\infty \bar{u}_\infty - p^3 s_\infty \bar{t}_\infty|}{\sqrt{-i(s_\infty - \bar{s}_\infty)(t_\infty - \bar{t}_\infty)(u_\infty - \bar{u}_\infty)}}.
 \end{aligned}
 \tag{49}$$

Eq. (49) yields that the *marginal bound* [111] is not *saturated* by non-BPS $Z = 0$ states, because $M_{ADM,non-BPS,Z=0}$ is *not* equal to the sum of the ADM masses of four *D6*-branes with appropriate fluxes (for further detail, see the discussion in [59]). This is actually expected, due to the strict similarity, discussed above, between $\frac{1}{2}$ -BPS and non-BPS $Z = 0$ attractor flows in the considered *stu* model; such a similarity can be explained by noticing that both such flows can be uplifted to the *same* $\frac{1}{8}$ -BPS *non-degenerate* attractor flow of $\mathcal{N} = 8, d = 4$ supergravity (see *e.g.* the discussion in [50]).

Concerning the (*covariant*) *scalar charges* of the non-BPS $Z = 0$ attractor flow of the *stu* model, they can be straightforwardly computed (in the “*s*-polarization”, without loss of generality) by using Eqs. (31) and (20):

$$\begin{aligned}
 \Sigma_{s,non-BPS,Z=0}(z_\infty, \bar{z}_\infty, \Gamma) &\equiv \lim_{\tau \rightarrow 0^-} (\partial_s \mathcal{W}_{non-BPS,Z=0,s})(z(\tau), \bar{z}(\tau), \Gamma) = \\
 &= \lim_{\tau \rightarrow 0^-} \partial_s |Z(s(\tau), \bar{t}(\tau), \bar{u}(\tau))| = \\
 &= \lim_{\tau \rightarrow 0^-} \frac{e^{K/2}}{2} \left[(\partial_s K) |W(s, \bar{t}, \bar{u})| + (\partial_s W(s, \bar{t}, \bar{u})) \sqrt{\frac{\overline{W}(\bar{s}, t, u)}{W(s, \bar{t}, \bar{u})}} \right] = \\
 &= \frac{1}{2\sqrt{-i(s_\infty - \bar{s}_\infty)(t_\infty - \bar{t}_\infty)(u_\infty - \bar{u}_\infty)}} \cdot \\
 &\cdot \left[\frac{|q_0 + q_1 s_\infty + q_2 \bar{t}_\infty + q_3 \bar{u}_\infty + p^0 s_\infty \bar{t}_\infty \bar{u}_\infty - p^1 \bar{t}_\infty \bar{u}_\infty - p^2 s_\infty \bar{u}_\infty - p^3 s_\infty \bar{t}_\infty|}{(s_\infty - \bar{s}_\infty)} + \right. \\
 &\quad \left. + (q_1 + p^0 \bar{t}_\infty \bar{u}_\infty - p^2 \bar{u}_\infty - p^3 \bar{t}_\infty) \cdot \right. \\
 &\quad \left. \sqrt{\frac{q_0 + q_1 \bar{s}_\infty + q_2 t_\infty + q_3 u_\infty + p^0 \bar{s}_\infty t_\infty u_\infty - p^1 t_\infty u_\infty - p^2 \bar{s}_\infty u_\infty - p^3 \bar{s}_\infty t_\infty}{q_0 + q_1 s_\infty + q_2 \bar{t}_\infty + q_3 \bar{u}_\infty + p^0 s_\infty \bar{t}_\infty \bar{u}_\infty - p^1 \bar{t}_\infty \bar{u}_\infty - p^2 s_\infty \bar{u}_\infty - p^3 s_\infty \bar{t}_\infty}} \right] = \\
 &= \Sigma_{s,\frac{1}{2}-BPS} \Big|_{t_\infty \rightarrow \bar{t}_\infty, u_\infty \rightarrow \bar{u}_\infty}.
 \end{aligned}
 \tag{50}$$

$$\begin{aligned}
 \Sigma_{t,non-BPS,Z=0}(z_\infty, \bar{z}_\infty, \Gamma) &\equiv \lim_{\tau \rightarrow 0^-} (\partial_t \mathcal{W}_{non-BPS,Z=0,s})(z(\tau), \bar{z}(\tau), \Gamma) = \\
 &= \lim_{\tau \rightarrow 0^-} \partial_t |Z(s(\tau), \bar{t}(\tau), \bar{u}(\tau))| = \\
 &= \lim_{\tau \rightarrow 0^-} \frac{e^{K/2}}{2} \left[(\partial_t K) |W(s, \bar{t}, \bar{u})| + (\partial_t \bar{W}(\bar{s}, t, u)) \sqrt{\frac{W(s, \bar{t}, \bar{u})}{\bar{W}(\bar{s}, t, u)}} \right] = \\
 &= \frac{1}{2\sqrt{-i(s_\infty - \bar{s}_\infty)(t_\infty - \bar{t}_\infty)(u_\infty - \bar{u}_\infty)}} \cdot \\
 &\cdot \left[\frac{|q_0 + q_1 s_\infty + q_2 \bar{t}_\infty + q_3 \bar{u}_\infty + p^0 s_\infty \bar{t}_\infty \bar{u}_\infty - p^1 \bar{t}_\infty \bar{u}_\infty - p^2 s_\infty \bar{u}_\infty - p^3 s_\infty \bar{t}_\infty|}{(t_\infty - \bar{t}_\infty)} + \right. \\
 &\quad \left. + (q_2 + p^0 \bar{s}_\infty u_\infty - p^1 u_\infty - p^3 \bar{s}_\infty) \cdot \right. \\
 &\quad \left. \sqrt{\frac{q_0 + q_1 s_\infty + q_2 \bar{t}_\infty + q_3 \bar{u}_\infty + p^0 s_\infty \bar{t}_\infty \bar{u}_\infty - p^1 \bar{t}_\infty \bar{u}_\infty - p^2 s_\infty \bar{u}_\infty - p^3 s_\infty \bar{t}_\infty}{q_0 + q_1 \bar{s}_\infty + q_2 t_\infty + q_3 u_\infty + p^0 \bar{s}_\infty t_\infty u_\infty - p^1 t_\infty u_\infty - p^2 \bar{s}_\infty u_\infty - p^3 \bar{s}_\infty t_\infty}} \right].
 \end{aligned}
 \tag{51}$$

$$\begin{aligned}
 \Sigma_{u,non-BPS,Z=0}(z_\infty, \bar{z}_\infty, \Gamma) &\equiv \lim_{\tau \rightarrow 0^-} (\partial_u \mathcal{W}_{non-BPS,Z=0,s})(z(\tau), \bar{z}(\tau), \Gamma) = \\
 &= \lim_{\tau \rightarrow 0^-} \partial_u |Z(s(\tau), \bar{t}(\tau), \bar{u}(\tau))| = \\
 &= \lim_{\tau \rightarrow 0^-} \frac{e^{K/2}}{2} \left[(\partial_u K) |W(s, \bar{t}, \bar{u})| + (\partial_u \bar{W}(\bar{s}, t, u)) \sqrt{\frac{W(s, \bar{t}, \bar{u})}{\bar{W}(\bar{s}, t, u)}} \right] = \\
 &= \frac{1}{2\sqrt{-i(s_\infty - \bar{s}_\infty)(t_\infty - \bar{t}_\infty)(u_\infty - \bar{u}_\infty)}} \cdot \\
 &\cdot \left[\frac{|q_0 + q_1 s_\infty + q_2 \bar{t}_\infty + q_3 \bar{u}_\infty + p^0 s_\infty \bar{t}_\infty \bar{u}_\infty - p^1 \bar{t}_\infty \bar{u}_\infty - p^2 s_\infty \bar{u}_\infty - p^3 s_\infty \bar{t}_\infty|}{(u_\infty - \bar{u}_\infty)} + \right. \\
 &\quad \left. + (q_3 + p^0 \bar{s}_\infty t_\infty - p^1 t_\infty - p^2 \bar{s}_\infty) \cdot \right. \\
 &\quad \left. \sqrt{\frac{q_0 + q_1 s_\infty + q_2 \bar{t}_\infty + q_3 \bar{u}_\infty + p^0 s_\infty \bar{t}_\infty \bar{u}_\infty - p^1 \bar{t}_\infty \bar{u}_\infty - p^2 s_\infty \bar{u}_\infty - p^3 s_\infty \bar{t}_\infty}{q_0 + q_1 \bar{s}_\infty + q_2 t_\infty + q_3 u_\infty + p^0 \bar{s}_\infty t_\infty u_\infty - p^1 t_\infty u_\infty - p^2 \bar{s}_\infty u_\infty - p^3 \bar{s}_\infty t_\infty}} \right].
 \end{aligned}
 \tag{52}$$

Also, it is here worth computing the difference between the squared non-BPS $Z = 0$ fake superpotential and the squared absolute value of the $\mathcal{N} = 2, d = 4$ central charge along the considered non-BPS $Z = 0$ attractor flow. This amounts to computing the difference generalizing the BPS bound [112] to the

whole attractor flow (without loss of generality, we perform calculations in the “*s-polarization*”):

$$\begin{aligned} \Theta(\mathcal{X}, \mathcal{Y}, \Gamma) &\equiv \mathcal{W}_{s, non-BPS, Z=0}^2 - |Z|^2 = \\ &= -\frac{1}{8(p^1 p^3 - p^0 q_2) \mathcal{Y}^2} \left[(p^\Lambda q_\Lambda - 2p^2 q_2 - 2(p^1 p^3 - p^0 q_2) \mathcal{X}^2)^2 + 4(p^1 p^3 - p^0 q_2)^2 (\mathcal{Y}^2)^2 + \mathcal{I}_4 \right] - \\ &\quad -\frac{1}{8(p^1 p^2 - p^0 q_3) \mathcal{Y}^2} \left[(p^\Lambda q_\Lambda - 2p^3 q_3 - 2(p^1 p^2 - p^0 q_3) \mathcal{X}^3)^2 + 4(p^1 p^2 - p^0 q_3)^2 (\mathcal{Y}^2)^2 + \mathcal{I}_4 \right] > 0. \end{aligned} \tag{53}$$

Such an expression for the scalar-dependent, strictly positive Θ was obtained by using the following relations:

$$\begin{aligned} (p^\Lambda q_\Lambda - 2p^2 q_2)^2 + \mathcal{I}_4 &= 4(p^1 p^3 - p^0 q_2)(q_1 q_3 + p^2 q_0); \\ (p^\Lambda q_\Lambda - 2p^3 q_3)^2 + \mathcal{I}_4 &= 4(p^1 p^2 - p^0 q_3)(q_1 q_2 + p^3 q_0). \end{aligned} \tag{54}$$

Thus, the *BPS bound* [112] holds not only at the BH event horizon ($r = r_H$), but actually (in a scalar-dependent way) all along the non-BPS $Z = 0$ attractor flow (*i.e.* $\forall r \geq r_H$).

5. The most General Non-BPS $Z \neq 0$ Attractor Flow

All the features holding for $\frac{1}{2}$ -BPS and non-BPS $Z = 0$ attractor flows (respectively treated in Sects. 3. and 4.) do not directly hold for the non-BPS $Z \neq 0$ attractor flow, which actually turns out to be rather different from (and structurally much more intricate than) such two attractor flows.

As mentioned in the Introduction, the non-BPS $Z \neq 0$ attractor flow of the *stu* model has been already considered in literature in particular cases, namely for the $D0 - D4$ (*magnetic*) [51, 59], $D0 - D6$ [59], $D2 - D6$ (*electric*) [28, 62] $D0 - D2 - D4$ (*magnetic with D2*) [62], $D0 - D2 - D4 - D6$ (without *B-fields*) [28] supporting BH charge configurations.

In the present Section we determine the explicit expression of the non-BPS $Z \neq 0$ attractor flow for *the most general* supporting BH charge configuration, with *all* electric and magnetic charges switched on, namely for the non-BPS $Z \neq 0$ -supporting branch of the $D0 - D2 - D4 - D6$ configuration. Thence, as already done for $\frac{1}{2}$ -BPS and non-BPS $Z = 0$ attractor flows, by exploiting the *first order (fake supergravity) formalism* [37, 42, 118], we compute the *ADM masses* as well as the *covariant scalar charges*, and study the issue of *marginal stability* [111], completing and refining the treatment given in [51, 59, 62].

5.1. *U-Duality Transformations along the Orbit $\mathcal{O}_{non-BPS, Z \neq 0}$*

In order to derive the explicit expression of the non-BPS $Z \neq 0$ attractor flow when all BH charges are non-vanishing, we exploit a method already used in [28], [59] and [62], based on performing suitable symplectic transformations along the relevant (*i.e.* supporting) charge orbit of the *U-duality* group. In Eqs. (16) and (28) we recalled the form of the $\frac{1}{2}$ -BPS- and non-BPS $Z = 0$ - supporting BH charge orbits of the *tri-fundamental* representation $(\mathbf{2}, \mathbf{2}, \mathbf{2})$ of the *U-duality* group G (given by Eq.(1)) of the *stu* model, also commenting on their separation [26]. The corresponding non-BPS $Z \neq 0$ -supporting BH charge orbit reads [26]

$$\mathcal{O}_{non-BPS, Z \neq 0} = \frac{(SU(1, 1))^3}{(SO(1, 1))^2}, \tag{55}$$

defined by the constraint

$$\mathcal{I}_4(\Gamma) < 0. \tag{56}$$

As done in [59] and [62], in order to perform a symplectic transformation along the charge orbit $\mathcal{O}_{non-BPS,Z \neq 0}$ of the $(2, 2, 2)$ representation of the U -duality, we take advantage of the complete factorization of the special Kähler manifold $\left(\frac{SU(1,1)}{U(1)}\right)^3$ (ultimately determining the *triality symmetry*), which allows one to deal with the product of three distinct 2×2 matrices of $SL(2, \mathbb{R})$, rather than with a unique 8×8 matrix of the U -duality group embedded in the relevant symplectic group $Sp(8, \mathbb{R})$.

The first step is to perform an $Sp(8, \mathbb{R})$ -transformation from the basis (p^Λ, q_Λ) to a basis \mathcal{A}_{abc} ($a, b, c = 0, 1$ throughout) of BH charges explicitly transforming under the $(2, 2, 2)$ of the U -duality. Such a transformation is given by Eq. (5.1) of [59] (equivalent to Eq. (3.5) of the second Ref. of [3]; see also Section 5 of [50]). The explicit action of a generic symplectic transformation of the U -duality on the BH charges \mathcal{A}_{abc} is given, up to some change of notation, by Eqs. (5.2) and (5.3) of [59], which we report below for simplicity's sake:

$$\mathcal{A}'_{a'b'c'} = (M_1)_{a'}^a (M_2)_{b'}^b (M_3)_{c'}^c a_{abc}; \tag{57}$$

$$M_i \equiv \begin{pmatrix} \mathbf{a}_i & \mathbf{b}_i \\ \mathbf{c}_i & \mathbf{d}_i \end{pmatrix} \in SL(2, \mathbb{R}), \det(M_i) = 1, \forall i = 1, 2, 3, \tag{58}$$

where each matrix pertains to the degrees of freedom of only one modulus (e.g. M_1 to s , M_2 to t , M_3 to u). The transformation (57)-(58) of $(SL(2, \mathbb{R}))^3 \subset Sp(8, \mathbb{R})$ induces also a *linear fractional (Möbius)* transformation on the moduli z^i as follows (no summation on repeated indices; once again, we report Eq. (5.3) of [59] for simplicity's sake; also recall Eq. (2)):

$$z'^i = \frac{\mathbf{a}_i z^i + \mathbf{b}_i}{\mathbf{c}_i z^i + \mathbf{d}_i}. \tag{59}$$

As done in [59] and [62], we use the configuration $D0 - D6$ as “*pivot*” in order to perform the transformation (57)-(59). Indeed, such a BH charge configuration supports only non-BPS $Z \neq 0$ attractors, as it can be easily realized by computing the corresponding *quartic* U -invariant, given by Eq. (8): $\mathcal{I}_4(\Gamma_{D0-D6}) < 0$ (see also the treatment of [47]). Thus, we want to transform from the configuration $D0 - D6$ (corresponding to charges (q_0, p^0) , which we denote here (q, p)) to the most general configuration $D0 - D2 - D4 - D6$, corresponding to all BH charges switched on: (q_0, q_i, p^i, p^0) . By exploiting the transformation (57)-(59), the parameters $\mathbf{a}_i, \mathbf{b}_i, \mathbf{c}_i, \mathbf{d}_i$ of the M_i s dualizing from $D0 - D6$ to $D0 - D2 - D4 - D6$ must satisfy the following set of constraints:

$$\begin{aligned} -q_0 &= -\mathbf{a}_1 \mathbf{a}_2 \mathbf{a}_3 q + \mathbf{b}_1 \mathbf{b}_2 \mathbf{b}_3 p; \\ q_i &= -\frac{1}{2} s_{ijk} \mathbf{c}_i \mathbf{a}_j \mathbf{a}_k q + \frac{1}{2} s_{ijk} \mathbf{d}_i \mathbf{b}_j \mathbf{b}_k p; \\ p^i &= -\frac{1}{2} s_{ijk} \mathbf{a}_i \mathbf{c}_j \mathbf{c}_k q + \frac{1}{2} s_{ijk} \mathbf{b}_i \mathbf{d}_j \mathbf{d}_k p; \\ p^0 &= -\mathbf{c}_1 \mathbf{c}_2 \mathbf{c}_3 q + \mathbf{d}_1 \mathbf{d}_2 \mathbf{d}_3 p, \end{aligned} \tag{60}$$

where $s_{ijk} \equiv |\epsilon_{ijk}|$. Notice that the system (60) admits solutions iff the condition (56) is met; this implies the transformation (57)-(59) to belong to the U -duality orbit $\mathcal{O}_{non-BPS,Z \neq 0}$ given by Eq. (55). The sign of the BH charges q and p is actually irrelevant for the condition (56) to be satisfied; thus, without loss of any generality, one can choose e.g. $q > 0, p > 0$. Within such a choice, the explicit form of the matrices M_i s under consideration (and of their inverse) reads as follows:

$$M_i = -\frac{sgn(\xi)}{\sqrt{(\varsigma_i + \varrho_i)} \xi} \begin{pmatrix} \varsigma_i \xi & -\varrho_i \\ \xi & 1 \end{pmatrix} \Leftrightarrow M_i^{-1} = -\frac{sgn(\xi)}{\sqrt{(\varsigma_i + \varrho_i)} \xi} \begin{pmatrix} 1 & \varrho_i \\ -\xi & \varsigma_i \xi \end{pmatrix}; \tag{61}$$

$$\xi \equiv \left(\frac{p}{q}\right)^{1/3} \left[\frac{2p^1 p^2 p^3 + p^0 (\sqrt{-\mathcal{I}_4} - p^\Lambda q_\Lambda)}{2p^1 p^2 p^3 - p^0 (\sqrt{-\mathcal{I}_4} - p^\Lambda q_\Lambda)} \right]^{1/3} \in \mathbb{R}; \tag{62}$$

$$\varsigma_i \equiv \frac{\sqrt{-\mathcal{I}_4} + p^\Lambda q_\Lambda - 2p^i q_i}{s_{ijk} p^j p^k - 2p^0 q_i} \in \mathbb{R} \text{ (no sum on } i); \tag{63}$$

$$\varrho_i \equiv \frac{\sqrt{-\mathcal{I}_4} - p^\Lambda q_\Lambda + 2p^i q_i}{s_{ijk} p^j p^k - 2p^0 q_i} \in \mathbb{R} \text{ (no sum on } i). \tag{64}$$

The definitions (63) and (64), together with Eq. (8), imply that (no sum on i)

$$\varsigma_i \varrho_i = -\frac{s_{ijk} q_j q_k + 2q_0 p^i}{s_{ijk} p^j p^k - 2p^0 q_i}. \tag{65}$$

As expected since the transformation (57)-(59) belongs to the orbit $\mathcal{O}_{non-BPS,Z \neq 0}$ of the U -duality, it leaves \mathcal{I}_4 unchanged:

$$\mathcal{I}_4(\Gamma_{D0-D2-D4-D6}) = - (p^\Lambda q_\Lambda)^2 + 4 \sum_{i < j} p^i q_i p^j q_j - 4p^0 q_1 q_2 q_3 + 4q_0 p^1 p^2 p^3 = - (pq)^2 = \mathcal{I}_4(\Gamma_{D0-D6}). \tag{66}$$

It should be stressed that the transformation (57)-(59) (along with Eqs. (61)-(64)) is *not* the most general transformation of $\mathcal{O}_{non-BPS,Z \neq 0}$ mapping the $D0-D6$ into the $D0-D2-D4-D6$ configuration (and *vice versa*). Indeed, it may be further generalized by replacing ξ with a triplet ξ_i , constrained by $\xi_1 \xi_2 \xi_3 = \xi^3$. Such a *two-parameter generalization* of the above transformation indicates, as mentioned above, the presence of a real, 2-dim. *moduli space* (namely $(SO(1,1))^2$ [41, 43]) *all along the non-BPS $Z \neq 0$ attractor flow*; this will become evident when looking at the explicit form of such a flow, presented further below.

5.2. $D0 - D6$: the Most General Flow and Fake Superpotential

The most general non-BPS $Z \neq 0$ attractor flow in the $D0 - D6$ configuration reads as follows [59]:

$$\begin{aligned} &exp[-4U_{non-BPS,Z \neq 0}(\tau)] = \\ &= \left[a - (-\mathcal{I}_4)^{1/4} \tau \right] \left[k^1 - (-\mathcal{I}_4)^{1/4} \tau \right] \left[k^2 - (-\mathcal{I}_4)^{1/4} \tau \right] \left[k^3 - (-\mathcal{I}_4)^{1/4} \tau \right] - b^2; \end{aligned} \tag{67}$$

$$x_{non-BPS,Z \neq 0}^i(\tau) = \xi_0^{-1} e^{\alpha_i} \cdot \frac{\left[k^j - (-\mathcal{I}_4)^{1/4} \tau \right] \left[k^l - (-\mathcal{I}_4)^{1/4} \tau \right] - \left[a - (-\mathcal{I}_4)^{1/4} \tau \right] \left[k^i - (-\mathcal{I}_4)^{1/4} \tau \right]}{\left[k^j - (-\mathcal{I}_4)^{1/4} \tau \right] \left[k^l - (-\mathcal{I}_4)^{1/4} \tau \right] + \left[a - (-\mathcal{I}_4)^{1/4} \tau \right] \left[k^i - (-\mathcal{I}_4)^{1/4} \tau \right] - 2b}; \tag{68}$$

$$y_{non-BPS,Z \neq 0}^i(\tau) = 2\xi_0^{-1} e^{\alpha_i} \cdot \frac{\exp[-2U_{non-BPS,Z \neq 0}(\tau)]}{\left[k^j - (-\mathcal{I}_4)^{1/4} \tau \right] \left[k^l - (-\mathcal{I}_4)^{1/4} \tau \right] + \left[a - (-\mathcal{I}_4)^{1/4} \tau \right] \left[k^i - (-\mathcal{I}_4)^{1/4} \tau \right] - 2b}, \tag{69}$$

where

$$\xi_0 \equiv (p/q)^{1/3}, \tag{70}$$

$a \in \mathbb{R}_0, b, k^i \in \mathbb{R}$ (k^i 's cannot all vanish), and the triplet of real constants α_i satisfies the constraint

$$\sum_i \alpha_i = 0. \tag{71}$$

It is worth pointing out that the $D0 - D6$ configuration supports *axion-free* non-BPS $Z \neq 0$ attractor flow(s); when considering the *near-horizon limit*, and thus the critical, charge-dependent values of the moduli, this is consistent with the analysis performed in [15, 41, 47]. An *axion-free* attractor flow solution of Eqs. (67)-(69) can be obtained *e.g.* by putting

$$k^i = a \forall i = 1, 2, 3, \tag{72}$$

and it reads as follows:

$$\begin{aligned} \exp[-4U_{non-BPS,Z \neq 0,axion-free}(\tau)] &= \left[a - (-\mathcal{I}_4)^{1/4} \tau \right]^4 - b^2; \\ x_{non-BPS,Z \neq 0,axion-free}^i(\tau) &= 0; \\ y_{non-BPS,Z \neq 0,axion-free}^i(\tau) &= \xi_0^{-1} e^{\alpha_i} \sqrt{\frac{\left[a - (-\mathcal{I}_4)^{1/4} \tau \right]^2 + b}{\left[a - (-\mathcal{I}_4)^{1/4} \tau \right]^2 - b}}. \end{aligned} \tag{73}$$

The non-BPS $Z \neq 0$ *fake superpotential* of the *first order formalism* can be computed to have the following form in the $D0 - D6$ configuration:

$$\begin{aligned} \mathcal{W}_{non-BPS,Z \neq 0}(z, \bar{z}, q, p) &= \frac{1}{4} e^{K/2} \left[\prod_i |q^{1/3} + p^{1/3} e^{-\alpha_i} z^i| \right] \cdot \\ &\cdot \left[1 + \sum_{i < j} \frac{\left(q^{2/3} - p^{2/3} e^{-2\alpha_i} |z^i|^2 \right) \left(q^{2/3} - p^{2/3} e^{-2\alpha_j} |z^j|^2 \right) - e^{-\alpha_i - \alpha_j} q^{2/3} p^{2/3} (z^i - \bar{z}^i)(z^j - \bar{z}^j)}{|q^{1/3} + p^{1/3} e^{-\alpha_i} z^i|^2 |q^{1/3} + p^{1/3} e^{-\alpha_j} z^j|^2} \right]. \end{aligned} \tag{74}$$

The *axion-free* version of such a *fake superpotential* (e.g. pertaining to the solution (73)-(73)) reads as follows:

$$\mathcal{W}_{non-BPS,Z\neq 0,axion-free}(y, q, p) = \frac{1}{2^3\sqrt{2}} \frac{1}{\sqrt{y^1 y^2 y^3}} \left[\prod_l |q^{1/3} - ip^{1/3} e^{-\alpha_l} y^l| \right] \cdot \left[1 + \sum_{i<j} \frac{[q^{2/3} - p^{2/3} e^{-2\alpha_i} (y^i)^2] [q^{2/3} - p^{2/3} e^{-2\alpha_j} (y^j)^2] + 4e^{-\alpha_i - \alpha_j} q^{2/3} p^{2/3} y^i y^j}{|q^{1/3} - ip^{1/3} e^{-\alpha_i} y^i|^2 |q^{1/3} - ip^{1/3} e^{-\alpha_j} y^j|^2} \right]. \tag{75}$$

Now, by exploiting the *first order formalism* [118] for $d = 4$ extremal BHs [37, 42] (see also [70] and [74]), one can compute the relevant BH parameters of the non-BPS $Z \neq 0$ attractor flow of $d = 4$ *stu* model in the $D0 - D6$ configuration, starting from the expression of the non-BPS $Z \neq 0$ *fake superpotential* $\mathcal{W}_{non-BPS,Z\neq 0}$ given by Eq. (74).

Eqs. (74) and (19) yield, after some algebra, the following expression for the *ADM mass*:

$$M_{ADM,non-BPS,Z\neq 0}(z_\infty, \bar{z}_\infty, \Gamma_{D0-D6}) = \frac{P}{2^{7/2}} \left[\prod_i \sqrt{[(\Lambda^i)^{-1} + B^i]^2 + 1} \right] \cdot \left\{ 1 + \sum_{i<j} \frac{[(\Lambda^i)^{-2} - (B^i)^2 - 1] [(\Lambda^j)^{-2} - (B^j)^2 - 1] + 4(\Lambda^i)^{-1} (\Lambda^j)^{-1}}{[(\Lambda^i)^{-1} + B^i]^2 + 1} \frac{1}{[(\Lambda^j)^{-1} + B^j]^2 + 1} \right\}, \tag{76}$$

where the quantities

$$\Lambda^i \equiv \xi_0 y_\infty^i; \quad B^i \equiv \frac{x_\infty^i}{y_\infty^i}, \quad P \equiv p \sqrt{y_\infty^1 y_\infty^2 y_\infty^3}, \quad Q \equiv \frac{q}{\sqrt{y_\infty^1 y_\infty^2 y_\infty^3}} \tag{77}$$

were introduced, and, for simplicity's sake, the α_i s were chosen *all* to vanish (i.e. $\alpha_i = 0 \forall i = 1, 2, 3$). P and Q are the *dressed charges*, i.e. a sort of *asymptotical redefinition* of the charges pertaining to $D6$ and $D0$ branes, respectively. On the other hand, Λ^i and B^i are usually named (*asymptotical brane*) *fluxes* and *B-fields*, respectively.

Eq. (5.2.) (along with the definitions (77)) coincides with Eq. (5.56) of [59], provided that the following condition is met (see Eq. (5.44) of [59]):

$$\Lambda^1 [1 + (B^1)^2] - (\Lambda^1)^{-1} = \Lambda^2 [1 + (B^2)^2] - (\Lambda^2)^{-1} = \Lambda^3 [1 + (B^3)^2] - (\Lambda^3)^{-1}. \tag{78}$$

As observed in [59], Eq. (5.2.) (along with the definitions (77)) yields the *marginal bound* [111] to be *saturated*, because $M_{ADM,non-BPS,Z\neq 0}$ is equal to the sum of the *ADM masses* of four $D6$ -branes with appropriate fluxes (for further detail, see the discussion in [59]).

Concerning the (*covariant*) *scalar charges*, they can be straightforwardly computed by recalling Eqs. (74) and (20), but their expressions are rather cumbersome. For simplicity's sake, here we limit ourselves to give the *scalar charges* for the "*t³-degeneration*" of the *stu* model (in which all charges and moduli are equal, insensitive to i -index; see e.g. Eq. (116) below, as well as the treatment in Sect. 5 of [50]). By denoting $t \equiv x - iy$, the *covariant scalar charges* of axion and dilaton in the $D0 - D6$ configuration

respectively read

$$\Sigma_{x,non-BPS,Z \neq 0}(x_\infty, y_\infty, \Gamma_{D0-D6}) = \sqrt{2} P x_\infty \frac{(\Lambda^{-1}B + B^2 + 1)}{\sqrt{(\Lambda^{-1} + B)^2 + 1}}; \tag{79}$$

$$\Sigma_{y,non-BPS,Z \neq 0}(x_\infty, y_\infty, \Gamma_{D0-D6}) = -\frac{P y_\infty}{\sqrt{2}} \frac{[\Lambda^{-4} + \Lambda^{-3}B + \Lambda^{-1}B(B^2 - 1) + B^4 - 1]}{\sqrt{(\Lambda^{-1} + B)^2 + 1}}. \tag{80}$$

5.3. D0 – D2 – D4 – D6: the Most General Flow and Fake Superpotential

Now, by performing the *U*-duality transformation (57)-(59) (along with Eqs. (61)-(65)) and using the most general non-BPS $Z \neq 0$ attractor flow in the *D0 – D6* configuration given by Eqs. (67)-(69), it is a matter of long but straightforward computations to determine the most general non-BPS $Z \neq 0$ attractor flow in the most general configuration, namely in the *D0 – D2 – D4 – D6* one, in which all BH charges are switched on. It reads as follows (the moduli are here denoted as $\mathcal{Z}^i \equiv \mathcal{X}^i - i\mathcal{Y}^i$; $i \neq j \neq l$ and no sum on repeated *i*-indices throughout):

$$\exp[-4U_{non-BPS,Z \neq 0}(\tau)] = h_0(\tau) h_1(\tau) h_2(\tau) h_3(\tau) - b^2; \tag{81}$$

$$\mathcal{X}_{non-BPS,Z \neq 0}^i(\tau) = \frac{\left\{ \begin{aligned} &\varsigma_i e^{2\alpha_i \nu^2} [h_j(\tau) h_l(\tau) + h_0(\tau) h_i(\tau) + 2b] + \\ &+ e^{\alpha_i \nu} (\varsigma_i - \varrho_i) [h_j(\tau) h_l(\tau) - h_0(\tau) h_i(\tau)] + \\ &- \varrho_i [h_j(\tau) h_l(\tau) + h_0(\tau) h_i(\tau) - 2b] \end{aligned} \right\}}{\left\{ \begin{aligned} &e^{2\alpha_i \nu^2} [h_j(\tau) h_l(\tau) + h_0(\tau) h_i(\tau) + 2b] + \\ &+ 2e^{\alpha_i \nu} [h_j(\tau) h_l(\tau) - h_0(\tau) h_i(\tau)] + \\ &+ h_j(\tau) h_l(\tau) + h_0(\tau) h_i(\tau) - 2b \end{aligned} \right\}}; \tag{82}$$

$$\mathcal{Y}_{non-BPS,Z \neq 0}^i(\tau) = \frac{2\nu e^{\alpha_i} (\varsigma_i + \varrho_i) \exp[-2U_{non-BPS,Z \neq 0}(\tau)]}{\left\{ \begin{aligned} &e^{2\alpha_i \nu^2} [h_j(\tau) h_l(\tau) + h_0(\tau) h_i(\tau) + 2b] + \\ &+ 2e^{\alpha_i \nu} [h_j(\tau) h_l(\tau) - h_0(\tau) h_i(\tau)] + \\ &+ h_j(\tau) h_l(\tau) + h_0(\tau) h_i(\tau) - 2b \end{aligned} \right\}}, \tag{83}$$

where ς_i and ϱ_i have been defined in Eqs. (63) and (64), respectively. The constants α_i and b have been introduced in Eqs. (67)-(69). Furthermore, the new quantities (see Eqs. (62) and (70), as well)

$$\nu \equiv \frac{\xi}{\xi_0} = \left[\frac{2p^1 p^2 p^3 + p^0 (\sqrt{-\mathcal{I}_4} - p^\Lambda q_\Lambda)}{2p^1 p^2 p^3 - p^0 (\sqrt{-\mathcal{I}_4} - p^\Lambda q_\Lambda)} \right]^{1/3} \in \mathbb{R}; \tag{84}$$

$$h_\Lambda(\tau) \equiv b_\Lambda - (-\mathcal{I}_4)^{1/4} \tau, \tag{85}$$

where b_Σ are real constants, have been defined.

In the particular case in which $b = 0$, the expression of $\exp[-4U_{non-BPS,Z \neq 0}(\tau)]$ can be recast in the form

$$\exp[-4U_{non-BPS,Z \neq 0}(\tau)] = -\mathcal{I}_4(\mathcal{H}(\tau)),$$

consistently with the results of [28] and [59], and on the same ground of (the first of) Eqs. (9) and (29), respectively holding for the $\frac{1}{2}$ -BPS and non-BPS $Z = 0$ attractor flows.

By performing the *near-horizon* (i.e. $\tau \rightarrow -\infty$) *limit*, Eqs. (82) and (83) respectively yield the following *critical* values of the moduli (the subscript “H” stands for “horizon”):

$$\mathcal{X}_{non-BPS,Z \neq 0,H}^i \equiv \lim_{\tau \rightarrow -\infty} \mathcal{X}_{non-BPS,Z \neq 0}^i(\tau) = \frac{\varsigma_i e^{2\alpha_i \nu^2} - \varrho_i}{e^{2\alpha_i \nu^2} + 1}; \tag{86}$$

$$\mathcal{Y}_{non-BPS,Z \neq 0,H}^i \equiv \lim_{\tau \rightarrow -\infty} \mathcal{Y}_{non-BPS,Z \neq 0}^i(\tau) = \frac{1}{2} \frac{e^{\alpha_i} (\varsigma_i + \varrho_i) \nu}{e^{2\alpha_i \nu^2} + 1} = \frac{\sqrt{-\mathcal{I}_4} e^{\alpha_i \nu}}{(s_{ijk} p^j p^k - 2p^0 q_i) (e^{2\alpha_i \nu^2} + 1)}. \tag{87}$$

It is worth pointing out that the $D0 - D2 - D4 - D6$ configuration does *not* support *axion-free* non-BPS $Z \neq 0$ attractor flow(s); when considering the *near-horizon limit*, and thus the critical, charge-dependent values of the moduli, this is consistent with the analysis performed in [15, 41, 47].

The solution (81)-(83) (along with the definitions (84) and (85)) generalizes the result of [28]. As mentioned in the Introduction, in [28] it was shown that, within the (non-BPS $Z \neq 0$ -supporting branches of the) $D2 - D6$ (*electric*) and $D0 - D2 - D4 - D6$ configurations, in absence of (some of the) *B-fields* the attractor flow solution can be obtained by replacing the $Sp(8, \mathbb{R})$ -covariant vector Γ of charges (defined by Eq. (7)) with the $Sp(8, \mathbb{R})$ -covariant vector $\mathcal{H}(\tau)$ of harmonic functions (defined by Eqs. (10)-(13)) in the corresponding critical, horizon solution.

For the $\frac{1}{2}$ -BPS and non-BPS $Z = 0$ attractor flows, respectively treated in Sects. 3. and 4., such a procedure allows one to determine the most general attractor flow solution starting from the corresponding critical, horizon solution.

On the other hand, for the non-BPS $Z \neq 0$ attractor flow such a procedure fails in presence of non-vanishing *B-fields*. In other words, it can be shown that the completely general non-BPS $Z \neq 0$ attractor flow solution (81)-(83) is *not* a solution of the *would-be non-BPS $Z \neq 0$ stabilization Eqs.* (see e.g. the

treatments of [23], [19] and [32])

$$\begin{aligned} \mathcal{H}^T(\tau) = & 2e^{K(z(\tau), \bar{z}(\tau))} \text{Im} \left[W(z(\tau)) \begin{pmatrix} \bar{X}^\Lambda(\bar{z}(\tau)) \\ \bar{F}_\Lambda(\bar{z}(\tau)) \end{pmatrix} + \right. \\ & + \frac{i}{2} \frac{\bar{W}(\bar{z}(\tau))}{|W|^2(z(\tau), \bar{z}(\tau))} \bar{C}_{ijk}(z(\tau), \bar{z}(\tau)) g^{i\bar{i}}(z(\tau), \bar{z}(\tau)) g^{j\bar{j}}(z(\tau), \bar{z}(\tau)) g^{k\bar{k}}(z(\tau), \bar{z}(\tau)) \cdot \\ & \left. \cdot (D_j W)(z(\tau), \bar{z}(\tau), \mathcal{H}(\tau)) (D_k W)(z(\tau), \bar{z}(\tau), \mathcal{H}(\tau)) \begin{pmatrix} (D_i X^\Lambda)(z(\tau), \bar{z}(\tau)) \\ (D_i F_\Lambda)(z(\tau), \bar{z}(\tau)) \end{pmatrix} \right], \end{aligned} \tag{88}$$

which can be obtained from the *non-BPS $Z \neq 0$ Attractor Eqs.* (see e.g. the treatments in [23] and [32])

$$\begin{aligned} \Gamma^T = & 2e^{K(z, \bar{z})} \text{Im} \left[W(z) \begin{pmatrix} \bar{X}^\Lambda(\bar{z}) \\ \bar{F}_\Lambda(\bar{z}) \end{pmatrix} + \right. \\ & + \frac{i}{2} \frac{\bar{W}(\bar{z})}{|W|^2(z, \bar{z})} \bar{C}_{ijk}(z, \bar{z}) g^{i\bar{i}}(z, \bar{z}) g^{j\bar{j}}(z, \bar{z}) g^{k\bar{k}}(z, \bar{z}) \cdot \\ & \left. \cdot (D_j W)(z, \bar{z}, \Gamma) (D_k W)(z, \bar{z}, \Gamma) \begin{pmatrix} (D_i X^\Lambda)(z, \bar{z}) \\ (D_i F_\Lambda)(z, \bar{z}) \end{pmatrix} \right] \end{aligned} \tag{89}$$

by simply replacing Γ with $\mathcal{H}(\tau)$. On the other hand, the non-BPS $Z \neq 0$ attractor flow solutions obtained in [28] for the $D2 - D6$ and $D0 - D2 - D4 - D6$ configurations (both without *B-fields*) consistently do satisfy Eq. (88).

Furthermore, Eq. (89) is the *near-horizon* ($\tau \rightarrow -\infty$) *limit* of Eq. (88), as it has to be.

The issue concerning whether (in all non-BPS $Z \neq 0$ -supporting configurations) the *actual non-BPS $Z \neq 0$ stabilization Eqs.* (if any) admit a ($\frac{1}{2}$ -)BPS-like reformulation in terms of a non-BPS $Z \neq 0$ *fake superpotential* (whose general form is given by Eq. (90) below) is open, and its investigation is left for future work.

Next, we can compute the non-BPS $Z \neq 0$ *fake superpotential* of the *first order formalism* in the $D0 - D2 - D4 - D6$ configuration. To do this, we apply the *U-duality transformation* (57-(59) (along with Eqs. (61)-(65)) to the expression of the non-BPS $Z \neq 0$ *fake superpotential* in the $D0 - D6$ configuration given by Eq. (74), and, by noticing that \mathcal{W} does *not* have any further covariance property

under such a transformation, after some algebra one achieves the following result:

$$\begin{aligned}
 & \mathcal{W}_{non-BPS,Z \neq 0}(\mathcal{Z}, \bar{\mathcal{Z}}, p^0, p^1, p^2, p^3, q_0, q_1, q_2, q_3) = \\
 &= \frac{1}{4} \frac{\nu^{3/2} (-\mathcal{I}_4)^{1/4}}{\sqrt{\prod_i (\varsigma_i + \varrho_i)}} e^{K/2} \left[\prod_i |\varsigma_i - \mathcal{Z}^i + (\varrho_i + \mathcal{Z}^i) e^{-\alpha_i \nu^{-1}}| \right] \cdot \\
 & \cdot \left(1 + \sum_{i < j} \frac{[|\varsigma_i - \mathcal{Z}^i|^2 - |\varrho_i + \mathcal{Z}^i|^2 e^{-2\alpha_i \nu^{-2}}] [|\varsigma_j - \mathcal{Z}^j|^2 - |\varrho_j + \mathcal{Z}^j|^2 e^{-2\alpha_j \nu^{-2}}]}{|\varsigma_i - \mathcal{Z}^i + (\varrho_i + \mathcal{Z}^i) e^{-\alpha_i \nu^{-1}}|^2 |\varsigma_j - \mathcal{Z}^j + (\varrho_j + \mathcal{Z}^j) e^{-\alpha_j \nu^{-1}}|^2} + \right. \\
 & \left. - \sum_{i < j} \frac{e^{-\alpha_i - \alpha_j \nu^{-2}} (\varsigma_i + \varrho_i) (\varsigma_j + \varrho_j) (\mathcal{Z}^i - \bar{\mathcal{Z}}^i) (\mathcal{Z}^j - \bar{\mathcal{Z}}^j)}{|\varsigma_i - \mathcal{Z}^i + (\varrho_i + \mathcal{Z}^i) e^{-\alpha_i \nu^{-1}}|^2 |\varsigma_j - \mathcal{Z}^j + (\varrho_j + \mathcal{Z}^j) e^{-\alpha_j \nu^{-1}}|^2} \right). \tag{90}
 \end{aligned}$$

Consistently with the *first order formalism* [118] for $d = 4$ extremal BHs [37, 42] (see also [70] and [74]), it is easy to check that the *near-horizon limit* of $\mathcal{W}_{non-BPS,Z \neq 0}^2$ yields the square root of $-\mathcal{I}_4$ (given by Eq. (8)), or equivalently the square root of the *Cayley’s hyperdeterminant* $Det(\Psi)$:

$$\begin{aligned}
 & \mathcal{W}_{non-BPS,Z \neq 0,H}^2(\Gamma_{D0-D2-D4-D6}) \equiv \\
 & \equiv \lim_{\tau \rightarrow -\infty} \mathcal{W}_{non-BPS,Z \neq 0}^2(\mathcal{Z}(\tau), \bar{\mathcal{Z}}(\tau), \Gamma_{D0-D2-D4-D6}) = \\
 & = \sqrt{-\mathcal{I}_4} = \sqrt{Det(\Psi)} = \frac{S_{BH,non-BPZ,Z \neq 0}(\Gamma_{D0-D2-D4-D6})}{\pi}, \tag{91}
 \end{aligned}$$

where in the last step the Bekenstein-Hawking entropy-area formula [95] was used.

Now, as done for the $D0 - D6$ configuration in the previous Subsection, by exploiting the *first order formalism* [118] for $d = 4$ extremal BHs [37, 42] (see also [70] and [74]), one can compute the relevant BH parameters, such as the *ADM mass* (Eq. (19)) and the *covariant scalar charges* (Eq. (20)), starting from the *fake superpotential* $\mathcal{W}_{non-BPS,Z \neq 0}$ given by Eq. (90). The computations are long but straightforward, and they yield cumbersome results (also *e.g.* in the limit of “ t^3 -degeneration”, see Eq. (116) below), which we thus decide to omit here. We will explicitly analyze some particular configurations in Sect. 6..

However, it is easy to realize that Eq. (90) implies the *marginal bound* [111] to be *saturated*, because (see Eq. (19))

$$\begin{aligned}
 & M_{ADM,non-BPS,Z \neq 0}(\mathcal{Z}_\infty, \bar{\mathcal{Z}}_\infty, \Gamma_{D0-D2-D4-D6}) = \\
 & = \mathcal{W}_{non-BPS,Z \neq 0}(\mathcal{Z}_\infty, \bar{\mathcal{Z}}_\infty, \Gamma_{D0-D2-D4-D6}) \equiv \\
 & \equiv \lim_{\tau \rightarrow 0^-} \mathcal{W}_{non-BPS,Z \neq 0}(\mathcal{Z}(\tau), \bar{\mathcal{Z}}(\tau), \Gamma_{D0-D2-D4-D6}) \tag{92}
 \end{aligned}$$

is equal to the sum of the *ADM masses* of four $D6$ -branes with appropriate fluxes (for further detail on definition of such brane fluxes, see the related discussion in [59]). Thus, generalizing the related results of [59] and [62], it can be stated that the *marginal stability* holds for the most general non-BPS $Z \neq 0$ attractor flow of the $\mathcal{N} = 2, d = 4$ *stu* model.

6. Analysis of Particular Configurations

In this Section we analyze in depth some particularly simple configurations, generalizing some results in literature [28, 51, 59, 62].

6.1. Magnetic (D0 – D4)

The configuration D0 – D4 (also named *magnetic*) of the *stu* model has been previously treated in [51] and [59]. In this case, the quantities of the *U*-duality transformation (57)-(59) along $\mathcal{O}_{non-BPS,Z\neq 0}$ defined by Eqs. (62)-(64) undergo a major simplification:

$$\xi = \xi_0; \quad \varsigma_i = \varrho_i = \sqrt{\frac{-q_0 p^i}{\frac{1}{2} s_{ijk} p^j p^k}}. \tag{93}$$

Correspondingly, the non-BPS $Z \neq 0$ attractor flow (81)-(83) acquires the following form (as above, the moduli are here denoted as $\mathcal{Z}^i \equiv \mathcal{X}^i - i\mathcal{Y}^i; i \neq j \neq l$ and no sum on repeated *i*-indices throughout):

$$\exp[-4U_{non-BPS,Z\neq 0}(\tau)] = h_0(\tau) h_1(\tau) h_2(\tau) h_3(\tau) - b^2; \tag{94}$$

$$\mathcal{X}_{non-BPS,Z\neq 0}^i(\tau) = \varsigma_i \cdot$$

$$\cdot \frac{e^{2\alpha_i} [h_j(\tau) h_l(\tau) + h_0(\tau) h_i(\tau) + 2b] - [h_j(\tau) h_l(\tau) + h_0(\tau) h_i(\tau) - 2b]}{\left\{ \begin{array}{l} e^{2\alpha_i} [h_j(\tau) h_l(\tau) + h_0(\tau) h_i(\tau) + 2b] + 2e^{\alpha_i} [h_j(\tau) h_l(\tau) - h_0(\tau) h_i(\tau)] + \\ + h_j(\tau) h_l(\tau) + h_0(\tau) h_i(\tau) - 2b \end{array} \right\}}; \tag{95}$$

$$\mathcal{Y}_{non-BPS,Z\neq 0}^i(\tau) = 4\varsigma_i \cdot$$

$$\cdot \frac{e^{\alpha_i} \exp[-2U_{non-BPS,Z\neq 0}(\tau)]}{\left\{ \begin{array}{l} e^{2\alpha_i} [h_j(\tau) h_l(\tau) + h_0(\tau) h_i(\tau) + 2b] + 2e^{\alpha_i} [h_j(\tau) h_l(\tau) - h_0(\tau) h_i(\tau)] + \\ + h_j(\tau) h_l(\tau) + h_0(\tau) h_i(\tau) - 2b \end{array} \right\}}. \tag{96}$$

It is worth pointing out that the D0 – D4 configuration supports *axion-free* non-BPS $Z \neq 0$ attractor flow(s); when considering the *near-horizon limit*, and thus the critical, charge-dependent values of the moduli, this is consistent with the analysis performed in [15, 41, 47]. An *axion-free* attractor flow solution of Eqs. (94)-(96) can be obtained *e.g.* by putting

$$\alpha_i = 0 \quad \forall i = 1, 2, 3; \tag{97}$$

$$b = 0, \tag{98}$$

and it reads as follows:

$$\begin{aligned} \exp[-4U_{non-BPS,Z\neq 0,axion-free}(\tau)] &= h_0(\tau) h_1(\tau) h_2(\tau) h_3(\tau); \\ \mathcal{X}_{non-BPS,Z\neq 0,axion-free}^i(\tau) &= 0; \\ \mathcal{Y}_{non-BPS,Z\neq 0,axion-free}^i(\tau) &= 4\varsigma_i \frac{\sqrt{h_0(\tau) h_1(\tau) h_2(\tau) h_3(\tau)}}{4h_j(\tau) h_l(\tau)}. \end{aligned} \tag{99}$$

Furthermore, within the additional assumption (97), Eqs. (94)-(96) yield the solution obtained in [59].

Always considering a framework in which the assumption (97) holds true, Eqs. (90) yields that the non-BPS $Z \neq 0$ fake superpotential in the $D0 - D4$ configuration has the following expression:

$$\mathcal{W}_{non-BPS,Z \neq 0}|_{\alpha_i=0 \forall i}(\mathcal{Z}, \bar{\mathcal{Z}}, \Gamma_{D0-D4}) = e^{K/2} \cdot \left[-q_0 + \frac{p^1}{2} (\mathcal{Z}^2 \bar{\mathcal{Z}}^3 + \mathcal{Z}^3 \bar{\mathcal{Z}}^2) + \frac{p^2}{2} (\mathcal{Z}^1 \bar{\mathcal{Z}}^3 + \mathcal{Z}^3 \bar{\mathcal{Z}}^1) + \frac{p^3}{2} (\mathcal{Z}^2 \bar{\mathcal{Z}}^1 + \mathcal{Z}^1 \bar{\mathcal{Z}}^2) \right]. \tag{100}$$

The axion-free version of such a fake superpotential (e.g. pertaining to the solution (99)-(99)) reads as follows:

$$\mathcal{W}_{non-BPS,Z \neq 0, axion-free}(\mathcal{Y}, \Gamma_{D0-D4}) = \frac{1}{2\sqrt{2}} \frac{[-q_0 + p^1 \mathcal{Y}^2 \mathcal{Y}^3 + p^2 \mathcal{Y}^1 \mathcal{Y}^3 + p^3 \mathcal{Y}^1 \mathcal{Y}^2]}{\sqrt{\mathcal{Y}^1 \mathcal{Y}^2 \mathcal{Y}^3}}. \tag{101}$$

The existence of a first order formalism in the non-BPS $Z \neq 0$ -supporting (branch of the) $D0 - D4$ configuration of the *stu* model, based on the fake superpotential given by Eq. (100), gives a simple explanation of the integrability of the equations of motion of scalars, answering to the question raised in Appendix A of [59].

Now, as done above for the $D0 - D6$ and $D0 - D2 - D4 - D6$ configurations, by exploiting the first order formalism for $d = 4$ extremal BHs, one can compute the relevant BH parameters, such as the ADM mass and the covariant scalar charges, starting from the fake superpotential $\mathcal{W}_{non-BPS,Z \neq 0}|_{\alpha_i=0 \forall i}$ given by Eq. (100).

Concerning the ADM mass, by recalling Eq. (19) and using Eq. (100) one obtains an explicit expression which, after introducing suitable dressed charges (see Eq. (104)) and putting (see Eq. (77))

$$B^1 = B^2 = B^3 = B, \tag{102}$$

is given by Eq. (4.6) of [59], which we report here for completeness' sake:

$$\begin{aligned} M_{ADM, non-BPS, Z \neq 0}|_{\alpha_i=0 \forall i}(\mathcal{Z}_\infty, \bar{\mathcal{Z}}_\infty, \Gamma_{D0-D4}) &= \\ = \lim_{\tau \rightarrow 0^-} \mathcal{W}_{non-BPS,Z \neq 0}|_{\alpha_i=0 \forall i}(\mathcal{Z}(\tau), \bar{\mathcal{Z}}(\tau), \Gamma_{D0-D4}) &= \\ = \frac{1}{2\sqrt{2}} \left[|Q_0| + (1 + B^2) \sum_i P^i \right], & \end{aligned} \tag{103}$$

where the dressed charges are defined as follows (no summation on repeated indices; notice the different definitions with respect to the $D0 - D6$ configuration, whose dressed charges are given by Eq. (77)):

$$Q_0 \equiv \frac{q_0}{\sqrt{\mathcal{Y}_\infty^1 \mathcal{Y}_\infty^2 \mathcal{Y}_\infty^3}}, \quad P^i \equiv \frac{\sqrt{\mathcal{Y}_\infty^1 \mathcal{Y}_\infty^2 \mathcal{Y}_\infty^3}}{\mathcal{Y}_\infty^i} p^i. \tag{104}$$

By recalling Eq. (20) and using Eq. (100), one can compute the covariant scalar charges of the non-BPS $Z \neq 0$ attractor flow in the $D0 - D4$ configuration. Within the simplifying assumptions (97) and (102), one obtains the following explicit expressions ($i \neq j \neq l$, no sum on repeated indices):

$$\begin{aligned} \Sigma_{\mathcal{X}, i, non-BPS, Z \neq 0}(\mathcal{X}_\infty, \mathcal{Y}_\infty, \Gamma_{D0-D4}) &\equiv \\ \equiv \lim_{\tau \rightarrow 0^-} \left(\frac{\partial \mathcal{W}_{non-BPS,Z \neq 0}|_{\alpha_m=0 \forall m}}{\partial \mathcal{X}^i} \right) (\mathcal{Z}(\tau), \bar{\mathcal{Z}}(\tau), \Gamma_{D0-D4}) &= \\ = \sqrt{2} \mathcal{X}_\infty^i (P^j + P^l); & \end{aligned} \tag{105}$$

$$\begin{aligned} & \Sigma_{\mathcal{Y},i,non-BPS,Z\neq 0}(\mathcal{X}_\infty, \mathcal{Y}_\infty, \Gamma_{D0-D4}) \equiv \\ \equiv & \lim_{\tau \rightarrow 0^-} \left(\frac{\partial \mathcal{W}_{non-BPS,Z\neq 0}|_{\alpha_m=0 \forall m}}{\partial \mathcal{Y}^i} \right) (\mathcal{Z}(\tau), \bar{\mathcal{Z}}(\tau), \Gamma_{D0-D4}) = \\ = & \frac{\mathcal{Y}_\infty^i}{\sqrt{2}} \left(-|Q_0| - 2P^i + (1 - B^2) \sum_k P^k \right), \end{aligned} \tag{106}$$

where the split in *axionic scalar charges* $\Sigma_{\mathcal{X},i}$ and *dilatonic scalar charges* $\Sigma_{\mathcal{Y},i}$ was performed.

It is here worth computing the difference between the squared non-BPS $Z \neq 0$ *fake superpotential* and the squared absolute value of the $\mathcal{N} = 2, d = 4$ central charge along the non-BPS $Z \neq 0$ attractor flow. This amounts to computing the difference generalizing the *BPS bound* [112] to the whole attractor flow (in the non-BPS $Z \neq 0$ -supporting branch of the *magnetic* charge configuration):

$$\Delta(\mathcal{Y}, \Gamma) \equiv \mathcal{W}_{non-BPS,Z\neq 0}^2 - |Z|^2 = \frac{|q_0|}{2} \left(\frac{p^1}{\mathcal{Y}^1} + \frac{p^2}{\mathcal{Y}^2} + \frac{p^3}{\mathcal{Y}^3} \right) > 0. \tag{107}$$

Δ is dilaton-dependent and strictly positive all along the non-BPS $Z \neq 0$ attractor flow. At the infinity, by using the *dressed charges* defined by Eq. (104), the result given by Eq. (4.8) of [59] is recovered. Thus, the *BPS bound* [112] holds not only at the BH event horizon ($r = r_H$), but actually (in a dilaton-dependent way) all along the non-BPS $Z \neq 0$ attractor flow (*i.e.* $\forall r \geq r_H$).

Of course, by relaxing the simplifying conditions (97) and/or (102), *i.e.* by considering non-vanishing α_i s (constrained by Eq. (71)) and/or different, *i*-indexed *B-fields*, a much richer situation arises, but the main features of the framework, outlined above, are left unchanged.

6.2. Electric (D2 – D6)

The configuration *D2 – D6* (also named *electric*) of the *stu* model has been previously treated in [28] and [62]. Analogously to what happens in the *D0 – D4* (*magnetic*) configuration, in this case the quantities of the *U*-duality transformation (57)-(59) along $\mathcal{O}_{non-BPS,Z\neq 0}$ defined by Eqs. (62)-(64) undergo a major simplification (the prime denotes the charges in the considered configuration):

$$\xi = -\xi_0; \quad \varsigma_i = \varrho_i = -\sqrt{\frac{\frac{1}{2} s_{ijk} q'_j q'_k}{p'^0 q'_i}}. \tag{108}$$

Correspondingly, the non-BPS $Z \neq 0$ attractor flow (81)-(83) acquires the following form (as above,

the moduli are here denoted as $\mathcal{Z}^i \equiv \mathcal{X}^i - i\mathcal{Y}^i; i \neq j \neq l$ and no sum on repeated i -indices throughout):

$$\exp[-4U_{non-BPS,Z \neq 0}(\tau)] = h_0(\tau) h_1(\tau) h_2(\tau) h_3(\tau) - b^2; \tag{109}$$

$$\mathcal{X}_{non-BPS,Z \neq 0}^i(\tau) = \varsigma_i \cdot \left. \frac{e^{2\alpha_i} [h_j(\tau) h_l(\tau) + h_0(\tau) h_i(\tau) + 2b] - [h_j(\tau) h_l(\tau) + h_0(\tau) h_i(\tau) - 2b]}{\left\{ \begin{array}{l} e^{2\alpha_i} [h_j(\tau) h_l(\tau) + h_0(\tau) h_i(\tau) + 2b] - 2e^{\alpha_i} [h_j(\tau) h_l(\tau) - h_0(\tau) h_i(\tau)] + \\ + h_j(\tau) h_l(\tau) + h_0(\tau) h_i(\tau) - 2b \end{array} \right\}} \right\}; \tag{110}$$

$$\mathcal{Y}_{non-BPS,Z \neq 0}^i(\tau) = -4\varsigma_i \cdot \left. \frac{e^{\alpha_i} \exp[-2U_{non-BPS,Z \neq 0}(\tau)]}{\left\{ \begin{array}{l} e^{2\alpha_i} [h_j(\tau) h_l(\tau) + h_0(\tau) h_i(\tau) + 2b] - 2e^{\alpha_i} [h_j(\tau) h_l(\tau) - h_0(\tau) h_i(\tau)] + \\ + h_j(\tau) h_l(\tau) + h_0(\tau) h_i(\tau) - 2b \end{array} \right\}} \right\}. \tag{111}$$

It is worth pointing out that the $D2 - D6$ configuration supports *axion-free* non-BPS $Z \neq 0$ attractor flow(s); when considering the *near-horizon limit*, and thus the critical, charge-dependent values of the moduli, this is consistent with the analysis performed in [15, 41, 47]. An *axion-free* attractor flow solution of Eqs. (109)-(111) can be obtained *e.g.* by assuming the conditions given by Eqs. (97) and (98), and it reads as follows:

$$\exp[-4U_{non-BPS,Z \neq 0,axion-free}(\tau)] = h_0(\tau) h_1(\tau) h_2(\tau) h_3(\tau); \tag{112}$$

$$\mathcal{X}_{non-BPS,Z \neq 0,axion-free}^i(\tau) = 0;$$

$$\mathcal{Y}_{non-BPS,Z \neq 0,axion-free}^i(\tau) = -\frac{\varsigma_i}{h_i(\tau)} \sqrt{\frac{h_1(\tau) h_2(\tau) h_3(\tau)}{h_0(\tau)}}. \tag{113}$$

As done for the *magnetic* configuration, in order to further simplify Eqs. (109)-(111) and (112)-(113), one can consider the particular case constrained by Eq. (97). Within such an additional assumption, the solution obtained in [62], generalizing the one of [28], is recovered.

Furthermore, within the simplifying assumption (97), Eq. (90) yields that the non-BPS $Z \neq 0$ *fake superpotential* in the $D2 - D6$ configuration has the following expression:

$$\mathcal{W}_{non-BPS,Z \neq 0}|_{\alpha_i=0 \forall i}(\mathcal{Z}, \bar{\mathcal{Z}}, \Gamma_{D2-D6}) = e^{K/2} |\mathcal{Z}^1| |\mathcal{Z}^2| |\mathcal{Z}^3| \cdot \left[p' + \frac{q'_1}{2} \frac{(\mathcal{Z}^2 \bar{\mathcal{Z}}^3 + \mathcal{Z}^3 \bar{\mathcal{Z}}^2)}{|\mathcal{Z}^2|^2 |\mathcal{Z}^3|^2} + \frac{q'_2}{2} \frac{(\mathcal{Z}^1 \bar{\mathcal{Z}}^3 + \mathcal{Z}^3 \bar{\mathcal{Z}}^1)}{|\mathcal{Z}^1|^2 |\mathcal{Z}^3|^2} + \frac{q'_3}{2} \frac{(\mathcal{Z}^2 \bar{\mathcal{Z}}^1 + \mathcal{Z}^1 \bar{\mathcal{Z}}^2)}{|\mathcal{Z}^1|^2 |\mathcal{Z}^2|^2} \right]. \tag{114}$$

The *axion-free* version of such a *fake superpotential* (e.g. pertaining to the solution (112)-(113)) reads as follows:

$$\mathcal{W}_{non-BPS,Z\neq 0}|_{\alpha_i=0 \forall i, axion-free}(\mathcal{Z}, \bar{\mathcal{Z}}, \Gamma_{D2-D6}) = \frac{1}{2\sqrt{2}} \sqrt{\mathcal{Y}^1 \mathcal{Y}^2 \mathcal{Y}^3} \left[p'^0 + \frac{q'_1}{\mathcal{Y}^2 \mathcal{Y}^3} + \frac{q'_2}{\mathcal{Y}^1 \mathcal{Y}^3} + \frac{q'_3}{\mathcal{Y}^1 \mathcal{Y}^2} \right], \tag{115}$$

coinciding with the *fake superpotential* given by Eq. (4-20) of [37].

The existence of a *first order formalism* in the non-BPS $Z \neq 0$ -supporting (branch of the) $D2 - D6$ configuration of the *stu* model, based on the *fake superpotential* given by Eq. (114), gives a explanation of the *integrability* of the equations of motion of scalars supported by the *electric* configuration (see the treatment of [62]).

Let us now consider the “ t^3 -degeneration” of the *stu* model, in which all charges and moduli are equal, insensitive to i -index; in the considered configuration this amounts to putting

$$\mathcal{Z}^1 = \mathcal{Z}^2 = \mathcal{Z}^3 = \mathcal{Z}, \quad q'_1 = q'_2 = q'_3 = q'/3 \tag{116}$$

(see the treatment in Sect. 5 of [50]). By doing so, Eq. (114) yields the non-BPS $Z \neq 0$ *fake superpotential* given by Eq. (5.5) of [37], which we report here for completeness' sake:

$$\mathcal{W}_{non-BPS,Z\neq 0}|_{\alpha_i=0 \forall i, t^3-deg.} = \left| \frac{p'^0 (\mathcal{Z})^2 \bar{\mathcal{Z}} + q' \mathcal{Z}}{\sqrt{-i(\mathcal{Z} - \bar{\mathcal{Z}})^3}} \right|. \tag{117}$$

The *axion-free* version of such a *fake superpotential* (e.g. pertaining to “ t^3 -degeneration” of the solution (112)-(113)) reads as follows:

$$\mathcal{W}_{non-BPS,Z\neq 0, axion-free, t^3-deg.} = \frac{1}{2\sqrt{2}} \left| \frac{p'^0 \mathcal{Y}^3 + q' \mathcal{Y}}{\sqrt{\mathcal{Y}^3}} \right|. \tag{118}$$

Now, as done above for the $D0 - D6$, $D0 - D2 - D4 - D6$ and $D0 - D4$ configurations, by exploiting the *first order formalism* for $d = 4$ extremal BHs, one can compute the relevant BH parameters, such as the *ADM mass* and the *covariant scalar charges*, starting from the *fake superpotential* $\mathcal{W}_{non-BPS,Z\neq 0}|_{\alpha_i=0 \forall i}$ given by Eq. (114).

Concerning the *ADM mass*, by recalling Eq. (19) and using Eqs. (114), (77) and (102), one obtains an explicit expression which, after introducing suitable *dressed charges* (see Eq. (120)), reads as follows:

$$\begin{aligned} M_{ADM, non-BPS, Z\neq 0}|_{\alpha_i=0 \forall i}(\mathcal{Z}_\infty, \bar{\mathcal{Z}}_\infty, \Gamma_{D2-D6}) &= \\ &= \lim_{\tau \rightarrow 0^-} \mathcal{W}_{non-BPS, Z\neq 0}|_{\alpha_i=0 \forall i}(\mathcal{Z}(\tau), \bar{\mathcal{Z}}(\tau), \Gamma_{D2-D6}) = \\ &= \frac{\sqrt{1+B^2}}{2\sqrt{2}} \left[(1+B^2)P'^0 + \sum_i Q'_i \right], \end{aligned} \tag{119}$$

where the *dressed charges* are defined as follows (no summation on repeated indices; notice the different definitions with respect to the $D0 - D6$ and $D0 - D4$ configurations, whose *dressed charges* are given by Eqs. (77) and (104), respectively):

$$P'^0 \equiv p'^0 \sqrt{\mathcal{Y}_\infty^1 \mathcal{Y}_\infty^2 \mathcal{Y}_\infty^3}, \quad Q'_i \equiv \frac{\mathcal{Y}_\infty^i}{\sqrt{\mathcal{Y}_\infty^1 \mathcal{Y}_\infty^2 \mathcal{Y}_\infty^3}} q'_i. \tag{120}$$

Up to redefinition of the *dressed charges*, Eq. (119) is nothing but Eq. (5.2) of [62].

By recalling Eq. (20) and using Eq. (114), one can compute the *covariant scalar charges* of the non-BPS $Z \neq 0$ attractor flow in the $D2 - D6$ configuration. Within the simplifying assumptions (97) and (102), one obtains the following explicit expressions (no sum on repeated indices):

$$\begin{aligned} & \Sigma_{\mathcal{X},i,non-BPS,Z\neq 0}(\mathcal{X}_\infty, \mathcal{Y}_\infty, \Gamma_{D2-D6}) \equiv \\ & \equiv \lim_{\tau \rightarrow 0^-} \left(\frac{\partial \mathcal{W}_{non-BPS,Z\neq 0}|_{\alpha_m=0 \forall m}}{\partial \mathcal{X}^i} \right) (\mathcal{Z}(\tau), \bar{\mathcal{Z}}(\tau), \Gamma_{D2-D6}) = \\ & = \sqrt{2} \frac{\mathcal{X}_\infty^i}{\sqrt{1+B^2}} [(1+B^2)P'^0 + Q'_i]; \end{aligned} \tag{121}$$

$$\begin{aligned} & \Sigma_{\mathcal{Y},i,non-BPS,Z\neq 0}(\mathcal{X}_\infty, \mathcal{Y}_\infty, \Gamma_{D2-D6}) \equiv \\ & \equiv \lim_{\tau \rightarrow 0^-} \left(\frac{\partial \mathcal{W}_{non-BPS,Z\neq 0}|_{\alpha_m=0 \forall m}}{\partial \mathcal{Y}^i} \right) (\mathcal{Z}(\tau), \bar{\mathcal{Z}}(\tau), \Gamma_{D2-D6}) = \\ & = \frac{\mathcal{Y}_\infty^i}{\sqrt{2} \sqrt{1+B^2}} \left[(B^4 - 1) P'^0 - 2Q'_i + (1+B^2)^2 \sum_j Q'_j \right], \end{aligned} \tag{122}$$

where, as for the $D0 - D4$ configuration, the split in *axionic scalar charges* $\Sigma_{\mathcal{X},i}$ and *dilatonic scalar charges* $\Sigma_{\mathcal{Y},i}$ was performed.

As done for the *magnetic* configuration in Subsect. 6.1., also for *electric* configuration it is worth computing the difference between the squared non-BPS $Z \neq 0$ *fake superpotential* and the squared absolute value of the $\mathcal{N} = 2, d = 4$ central charge along the non-BPS $Z \neq 0$ attractor flow:

$$\Delta(\mathcal{X}, \mathcal{Y}, \Gamma) \equiv \mathcal{W}_{non-BPS,Z\neq 0}^2 - |Z|^2 = \frac{p'^0}{2} \left(q'_1 \frac{(\mathcal{X}^1)^2 + (\mathcal{Y}^1)^2}{\mathcal{Y}^1} + q'_2 \frac{(\mathcal{X}^2)^2 + (\mathcal{Y}^2)^2}{\mathcal{Y}^2} + q'_3 \frac{(\mathcal{X}^3)^2 + (\mathcal{Y}^3)^2}{\mathcal{Y}^3} \right) > 0. \tag{123}$$

Differently from what happens for the *magnetic* configuration, for *electric* configuration Δ does depend also on axions, but nevertheless it is still strictly positive all along the non-BPS $Z \neq 0$ attractor flow. At infinity, by using the *dressed charges* defined by Eq. (120), the following result is achieved:

$$\Delta(\mathcal{X}_\infty, \mathcal{Y}_\infty, \Gamma) = \frac{P'^0}{2} (1+B^2) \sum_i Q'_i. \tag{124}$$

Thus, the *BPS bound* [112] holds not only at the BH event horizon ($r = r_H$), but actually (in a scalar-dependent way) all along the non-BPS $Z \neq 0$ attractor flow (*i.e.* $\forall r \geq r_H$).

Of course, by relaxing the simplifying conditions (97) and/or (102), *i.e.* by considering non-vanishing α_i s (constrained by Eq. (71)) and/or different, i -indexed B -fields, a much richer situation arises, but the main features of the framework, outlined above, are left unchanged.

By noticing that the $D0 - D4$ (*magnetic*) and $D2 - D6$ (*electric*) configurations are reciprocally *dual* in $d = 4$ and recalling the treatment of Subsect. 5.1., it is worth computing the matrices $M_{i,D0-D4 \rightarrow D2-D6}$ representing the U -duality transformation along the charge orbit $\mathcal{O}_{non-BPS,Z\neq 0}$ which connects (the non-BPS $Z \neq 0$ -supporting *branches* of) such two charge configurations. In order to do this, we exploit the treatment given in Subsect. 5.1., by performing the following steps:

$$D0 - D4 \xrightarrow{(q_0, p^i)} D0 - D6 \xrightarrow{(q, p)} D2 - D6. \tag{125}$$

For the step $D0 - D4 \rightarrow D0 - D6$, we consider M_i^{-1} given by Eq. (61), along with the definitions (62)-(64) specified for the configuration $D0 - D4$, obtaining $M_{i,D0-D4 \rightarrow D0-D6}^{-1}$. Thence, for the the step $D0 - D6 \rightarrow D2 - D6$, we take M_i given by Eq. (61), along with the definitions (62)-(64) specified for the configuration $D2 - D6$, obtaining $M_{i,D0-D6 \rightarrow D2-D6}$. Thus (no sum on repeated index $i = 1, 2, 3$ throughout; also recall Eq. (57)):

$$\begin{aligned} (M_{i,D0-D4 \rightarrow D2-D6})_{a'}^{b'} &= (M_{i,D0-D6 \rightarrow D2-D6})_{a'}^a (M_{i,D0-D4 \rightarrow D0-D6})_a^{b'} = \\ &= \begin{pmatrix} 0 & -\sqrt[4]{-\frac{q_0 p^i s_{ijk} q'_j q'_k}{p'^0 q'_i s_{ijk} p^j p^k}} \\ \sqrt[4]{-\frac{p'^0 q'_i s_{ijk} p^j p^k}{q_0 p^i s_{ijk} q'_j q'_k}} & 0 \end{pmatrix}. \end{aligned} \tag{126}$$

Consequently, by recalling Eqs. (59) and (65) one can directly relate the non-BPS $Z \neq 0$ attractor flows $\mathcal{Z}_{non-BPS,Z \neq 0,D2-D6}^i(\tau)$ and $\mathcal{Z}_{non-BPS,Z \neq 0,D0-D4}^i(\tau)$ (respectively given by Eqs. (109)-(111) and (94)-(96); recall that $\mathcal{Z}^i(\tau) = \mathcal{X}^i(\tau) - i\mathcal{Y}^i(\tau)$) by the following expression, explicitly showing the duality between the $D0 - D4$ (magnetic) and $D2 - D6$ (electric) configurations in $d = 4$:

$$\mathcal{Z}_{non-BPS,Z \neq 0,D2-D6}^i(\tau) = -\sqrt{-\frac{q_0 p^i s_{ijk} q'_j q'_k}{p'^0 q'_i s_{ijk} p^j p^k}} \frac{1}{\mathcal{Z}_{non-BPS,Z \neq 0,D0-D4}^i(\tau)}. \tag{127}$$

6.3. $D0 - D2 - D4$

The configuration $D0 - D2 - D4$ of the *stu* model has been previously treated in [62]. In this case, the quantities of the U -duality transformation (57)-(59) along $\mathcal{O}_{non-BPS,Z \neq 0}$ defined by Eqs. (62)-(64) have the following form (no summation on repeated index $i = 1, 2, 3$ throughout):

$$\xi = \xi_0; \quad \varsigma_i = \frac{\sqrt{-\mathcal{I}_4} + p^l q_l - 2p^i q_i}{s_{ijk} p^j p^k}; \quad \varrho_i = \frac{\sqrt{-\mathcal{I}_4} - p^l q_l + 2p^i q_i}{s_{ijk} p^j p^k}. \tag{128}$$

Within the additional assumption (97) (considered for simplicity' sake), the non-BPS $Z \neq 0$ attractor flow (81)-(83) correspondingly acquires the following form (as above, the moduli are here denoted as $\mathcal{Z}^i \equiv \mathcal{X}^i - i\mathcal{Y}^i; i \neq j \neq l$, and no sum on repeated i -indices throughout):

$$\exp[-4U_{non-BPS,Z \neq 0}(\tau)] = h_0(\tau) h_1(\tau) h_2(\tau) h_3(\tau) - b^2; \tag{129}$$

$$\mathcal{X}_{non-BPS,Z \neq 0}^i(\tau) = \frac{\sqrt{-\mathcal{I}_4}}{s_{ikm} p^k p^m} \frac{b}{h_j(\tau) h_l(\tau)} + \frac{p^n q_n - 2p^i q_i}{s_{ikm} p^k p^m}; \tag{130}$$

$$\mathcal{Y}_{non-BPS,Z \neq 0}^i(\tau) = \frac{\sqrt{-\mathcal{I}_4}}{s_{ikm} p^k p^m} \frac{\exp[-2U_{non-BPS,Z \neq 0}(\tau)]}{h_j(\tau) h_l(\tau)}. \tag{131}$$

This is nothing but the solution obtained in [62].

It is worth pointing out that, as the general case $D0 - D2 - D4 - D6$ (see Subsect. 5.3.), the $D0 - D2 - D4$ configuration does not support axion-free non-BPS $Z \neq 0$ attractor flow(s); when considering the near-horizon limit, and thus the critical, charge-dependent values of the moduli, this is consistent with the analysis performed in [15, 41, 47].

Furthermore, always within the simplifying assumption (97), Eq. (90) yields that the non-BPS $Z \neq 0$ fake superpotential in the $D0 - D2 - D4$ configuration has the following expression:

$$\begin{aligned} & \mathcal{W}_{non-BPS, Z \neq 0} |_{\alpha_i=0 \forall i} (\mathcal{Z}, \bar{\mathcal{Z}}, \Gamma_{D0-D2-D4}) = \\ & = e^{K/2} \left[-q_0 - \frac{q_1}{2} (\mathcal{Z}^1 + \bar{\mathcal{Z}}^1) - \frac{q_2}{2} (\mathcal{Z}^2 + \bar{\mathcal{Z}}^2) - \frac{q_3}{2} (\mathcal{Z}^3 + \bar{\mathcal{Z}}^3) + \right. \\ & \left. + \frac{p^1}{2} (\mathcal{Z}^2 \bar{\mathcal{Z}}^3 + \mathcal{Z}^3 \bar{\mathcal{Z}}^2) + \frac{p^2}{2} (\mathcal{Z}^1 \bar{\mathcal{Z}}^3 + \mathcal{Z}^3 \bar{\mathcal{Z}}^1) + \frac{p^3}{2} (\mathcal{Z}^1 \bar{\mathcal{Z}}^2 + \mathcal{Z}^2 \bar{\mathcal{Z}}^1) \right]. \end{aligned} \tag{132}$$

The existence of a *first order formalism* in the non-BPS $Z \neq 0$ -supporting (branch of the) $D0 - D2 - D4$ configuration of the *stu* model, based on the *fake superpotential* given by Eq. (132), gives a explanation of the *integrability* of the equations of motion of scalars supported by such a configuration (see the treatment of [62]).

Now, by exploiting the *first order formalism* for $d = 4$ extremal BHs, one can compute the relevant BH parameters, such as the *ADM mass* and the *covariant scalar charges*, starting from the *fake superpotential* $\mathcal{W}_{non-BPS, Z \neq 0} |_{\alpha_i=0 \forall i}$ given by Eq. (132).

Concerning the *ADM mass*, by recalling Eq. (19) and using Eq. (132) one obtains the following result:

$$\begin{aligned} & M_{ADM, non-BPS, Z \neq 0} |_{\alpha_i=0 \forall i} (\mathcal{Z}_\infty, \bar{\mathcal{Z}}_\infty, \Gamma_{D0-D2-D4}) = \\ & = \lim_{\tau \rightarrow 0^-} \mathcal{W}_{non-BPS, Z \neq 0} |_{\alpha_i=0 \forall i} (\mathcal{Z}(\tau), \bar{\mathcal{Z}}(\tau), \Gamma_{D0-D2-D4}) = \\ & = \frac{1}{2\sqrt{2}} \left[|Q_0| - \sum_i Q_i B_i + \sum_i P^i + \sum_{i \neq j \neq k} P^i B_j B_k \right], \end{aligned} \tag{133}$$

where the *dressed charges* are defined by Eqs. (104) and (120).

By recalling Eq. (20) and using Eq. (132), one can compute the *covariant scalar charges* of the non-BPS $Z \neq 0$ attractor flow in the $D0 - D2 - D4$ configuration. Always within the assumption (97), one obtains the following explicit expressions (no sum on repeated index $i = 1, 2, 3$):

$$\begin{aligned} & \Sigma_{\mathcal{X}, i, non-BPS, Z \neq 0} (\mathcal{X}_\infty, \mathcal{Y}_\infty, \Gamma_{D0-D2-D4}) \equiv \\ & \equiv \lim_{\tau \rightarrow 0^-} \left(\frac{\partial \mathcal{W}_{non-BPS, Z \neq 0} |_{\alpha_m=0 \forall m}}{\partial \mathcal{X}^i} \right) (\mathcal{Z}(\tau), \bar{\mathcal{Z}}(\tau), \Gamma_{D0-D2-D4}) = \\ & = \sqrt{2} \mathcal{Y}_\infty^i (s_{ijk} P^j B_k - Q_i); \end{aligned} \tag{134}$$

$$\begin{aligned} & \Sigma_{\mathcal{Y}, i, non-BPS, Z \neq 0} (\mathcal{X}_\infty, \mathcal{Y}_\infty, \Gamma_{D0-D2-D4}) \equiv \\ & \equiv \lim_{\tau \rightarrow 0^-} \left(\frac{\partial \mathcal{W}_{non-BPS, Z \neq 0} |_{\alpha_m=0 \forall m}}{\partial \mathcal{Y}^i} \right) (\mathcal{Z}(\tau), \bar{\mathcal{Z}}(\tau), \Gamma_{D0-D2-D4}) = \\ & = \frac{\mathcal{Y}_\infty^i}{\sqrt{2}} \left[-|Q_0| - 2P^i + \sum_j Q_j B_j + \sum_j P^j - \sum_{i \neq j \neq k} P^i B_j B_k \right], \end{aligned} \tag{135}$$

where, as above, the split in *axionic scalar charges* $\Sigma_{\mathcal{X}, i}$ and *dilatonic scalar charges* $\Sigma_{\mathcal{Y}, i}$ was performed, and the definition (77) of *B-fields* was used.

As done for the *magnetic* and *electric* configurations (respectively in Subjects. 6.1. and 6.2.), also for $D0 - D2 - D4$ configuration it is worth computing the difference between the squared non-BPS

$Z \neq 0$ fake superpotential and the squared absolute value of the $\mathcal{N} = 2, d = 4$ central charge along the non-BPS $Z \neq 0$ attractor flow. For simplicity's sake, we decide to perform computations in the “ t^3 -degeneration” of the stu model, in which all charges and moduli are equal, insensitive to i -index; in the considered configuration this amounts to putting (see the treatment in Sect. 5 of [50])

$$\mathcal{Z}^1 = \mathcal{Z}^2 = \mathcal{Z}^3 = \mathcal{Z} \equiv \mathcal{X} - i\mathcal{Y}, \quad q_1 = q_2 = q_3 = q/3, \quad p^1 = p^2 = p^3 = p, \quad (136)$$

which generalizes Eq. (116), also implying the assumption (102). Then, one obtains the following result:

$$\Delta(\mathcal{Y}, \Gamma) \equiv \mathcal{W}_{non-BPS, Z \neq 0}^2 - |Z|^2 = \frac{1}{8\mathcal{Y}} (12|q_0|p - q^2) > 0. \quad (137)$$

In this case, Δ is strictly positive all along the non-BPS $Z \neq 0$ attractor flow, due to the fact that \mathcal{I}_4 is strictly negative. At infinity, by using the dressed charges defined by Eqs. (104) and (120), the following result is achieved:

$$\Delta(\mathcal{Y}_\infty, \Gamma) = \frac{1}{8} (12|Q_0|P - Q^2). \quad (138)$$

Thus, the BPS bound [112] is found to hold not only at the BH event horizon ($r = r_H$), but actually (in a scalar-dependent way) all along the non-BPS $Z \neq 0$ attractor flow (i.e. $\forall r \geq r_H$).

It is here worth pointing out that, by exploiting the procedure outlined in Subsect. 5.1., the results (107), (123) and (137) can be related one to the others by performing suitable U -duality transformations. In such a way (or equivalently by recalling the results of Sect. 2. and Subsect. 5.3.), one can also compute Δ for the non-BPS $Z \neq 0$ -supporting branch of the most general (i.e. $D0 - D2 - D4 - D6$) BH charge configuration.

Of course, by relaxing the simplifying condition (97) and/or the “ t^3 -degeneration” described by Eq. (136) (in turn implying the condition(102)), a much richer situation arises, but the main features of the framework, outlined above, are left unchanged.

7. Conclusion

In the present paper the analysis and solution of the equations of motion of the scalar fields of the so-called stu model [15, 39, 41, 50, 97–100], consisting of $\mathcal{N} = 2, d = 4$ ungauged supergravity coupled to 3 Abelian vector multiplets whose complex scalars span the special Kähler manifold $\frac{G}{H} = \left(\frac{SU(1,1)}{U(1)}\right)^3$, has been performed in full detail. The obtained results complete and generalize the ones already present in literature [28, 51, 59, 62].

The 3 classes of non-degenerate attractor flows of the stu model have been presented and/or determined in full generality, and their features have been studied and compared. We sketchily list the essential facts below:

- The $\frac{1}{2}$ -BPS attractor flow, known since [105]- [109] (as well as the third of Refs. [111]), has been reviewed. It corresponds to $\mathcal{I}_4 > 0$, and it does not yield any associated moduli space, at the BH event horizon nor along the flow. The most general flow solution (9) can be obtained starting from the known most general horizon, critical solution [15, 28, 39, 41, 50, 99, 100], and replacing the BH charges with suitable harmonic functions. Correspondingly, the $\frac{1}{2}$ -BPS Attractor Eqs. (15), determining the attractor solution at BH event horizon, can be extended to the whole

flow into the so-called $\frac{1}{2}$ -BPS stabilization Eqs. (14) by simply substituting the BH charges with the corresponding harmonic functions in the radial parameter τ . The (first order) $\frac{1}{2}$ -BPS fake superpotential (which is nothing but the absolute value of the $\mathcal{N} = 2$, $d = 4$ central charge function Z given by Eq. (5), and thus it is manifestly H -invariant) has been explicitly determined, and the relevant BH parameters, namely the gravitational ADM mass (22) and (covariant) scalar charges, have been computed, as functions of BH charges and (spatially) asymptotical scalars. The marginal bound has been shown to be never saturated, and thus marginal stability [111] does not hold for $\frac{1}{2}$ -BPS states in the considered framework.

- The non-BPS $Z = 0$ attractor flow, hitherto unknown (up to a short comment in [59]), has been studied and derived in full generality. As the $\frac{1}{2}$ -BPS attractor flow, it corresponds to $\mathcal{I}_4 > 0$, and it does not yield any associated moduli space, at the BH event horizon nor along the flow. Due to the underlying triality symmetry of the stu model, 3 different “polarizations” of the results are possible, and choosing one of them does not imply any loss of generality. The most general flow solution (29) can be obtained starting from the known most general horizon, critical solution [50], and replacing the BH charges with suitable harmonic functions. Correspondingly, the non-BPS $Z = 0$ Attractor Eqs. (42), determining the attractor solution at BH event horizon, can be extended to the whole flow into the so-called non-BPS $Z = 0$ stabilization Eqs. (41) by simply substituting the BH charges with the corresponding harmonic functions in the radial parameter τ . The (first order) non-BPS $Z = 0$ fake superpotential has been explicitly determined, and it is given by the manifestly H -invariant Eqs. (31) and (36) (or equivalently, by Eqs. (34) and (38), or (35) and (37)). The relevant BH parameters, namely the gravitational ADM mass (49) and (covariant) scalar charges, have been computed, as functions of BH charges and (spatially) asymptotical scalars. Furthermore, the BPS bound [112] has been shown to hold all along the non-BPS $Z = 0$ attractor flow (i.e. $\forall r \geq r_H$), and not only at the BH event horizon ($r = r_H$). The marginal bound has been shown to be never saturated, and thus marginal stability [111] does not hold for non-BPS $Z = 0$ states in the considered framework. The strict similarity between $\frac{1}{2}$ -BPS and non-BPS $Z = 0$ attractor flows can be explained by noticing that both such flows can be uplifted to the same $\frac{1}{8}$ -BPS non-degenerate attractor flow of $\mathcal{N} = 8$, $d = 4$ supergravity (see e.g. the discussion in [50]).
- The non-BPS $Z \neq 0$ attractor flow, studied in [28, 51, 59, 62] in various configurations, has been here studied and analyzed in full generality. It corresponds to $\mathcal{I}_4 < 0$, and it yields an associated moduli space $(SO(1,1))^2$ (which is nothing but the scalar manifold of the $d = 5$ uplift of the stu model), both at the BH event horizon [41, 43] and along the flow [59]. Consistently with the analysis performed in [51] for the $D0 - D4$ (magnetic) configuration, the most general flow solution (81)-(83), supported by the (relevant branch of the) $D0 - D2 - D4 - D6$ configuration (with all charges switched on), cannot be obtained starting from the known most general horizon, critical solution, and replacing the BH charges with suitable harmonic functions. The opposite claim of [28] for the $D2 - D6$ (electric) and $D0 - D2 - D4 - D6$ configurations is actually due to the fact that (some) B -fields were chosen to vanish therein. Correspondingly, the non-BPS $Z \neq 0$ Attractor Eqs. (89), determining the attractor solution at BH event horizon, cannot be extended to the

whole flow into the so-called *non-BPS $Z \neq 0$ stabilization Eqs.* (whose *would-be* version is given by Eq. (88)) by simply substituting the BH charges with the corresponding harmonic functions in the radial parameter τ . The (*first order*) *non-BPS $Z \neq 0$ fake superpotential* has been explicitly determined, and it is given by the manifestly non- H -invariant Eq. (90), which reproduces the few known results [37] in the corresponding particular cases. The relevant BH parameters, namely the *gravitational ADM mass* and (*covariant*) *scalar charges*, have been computed in Subsect. 5.2. and Sect. 6. for various configurations, as functions of BH charges and (spatially) asymptotical scalars. Furthermore, the *BPS bound* [112] has been shown to hold all along the non-BPS $Z \neq 0$ attractor flow (*i.e.* $\forall r \geq r_H$), and not only at the BH event horizon ($r = r_H$). This has been explicitly computed for non-BPS $Z \neq 0$ -supporting branches of *magnetic* (Subsect. 6.1.), *electric* (Subsect. 6.2.) and $D0 - D2 - D4$ (Subsect. 6.3.) configurations. On the other hand, by exploiting the procedure outlined in Subsect. 5.1., or equivalently by recalling the results of Sect. 2. and Subsect. 5.3., one can also prove the *BPS bound* to hold for the non-BPS $Z \neq 0$ -supporting branch of the most general (*i.e.* $D0 - D2 - D4 - D6$) BH charge configuration. From the very Eq. (90), the *marginal bound* turns out to be saturated, and thus *marginal stability* [111] holds for non-BPS $Z \neq 0$ states in the considered framework. The manifest non- H -invariance of the general (*first order*) *non-BPS $Z \neq 0$ fake superpotential* (90) seems clash with the H -invariance imposed by Eq. (2.21) of [37] (or equivalently by Eq. (13) of [42]), reported as eq. (39) above, relating the *fake superpotential* and the *warp factor* $U(\tau)$ appearing in the *Ansatz* (40) for the static, spherically symmetric, asymptotically flat, *extremal* dyonic BH metric. A way out to such an apparent contradiction consists in recalling the treatment of Subsect. 2.2. of [37], and thus observing that the fake superpotential is *not* unique within the same attractor flow, the various equivalent superpotentials being related through a matrix R satisfying the conditions (2.28) and (2.29) of [37] (see in general Subsect. 2.2 of [37]). If R is scalar-dependent, it may relate (manifestly) H -invariant *fake superpotentials* to (manifestly) non- H -invariant ones, and *vice versa*. Thus, one may state that a suitable scalar-dependent matrix R exists, satisfying Eqs. (2.28) and (2.29) of [37], and mapping the *non-BPS $Z \neq 0$ fake superpotential* (90) into an equivalent, but (manifestly) H -invariant one. It would be interesting to determine explicitly such a matrix; we leave such an issue for future work.

Various comments, remarks, ideas for further developments along the lines of research considered in the present paper are listed below.

- By exploiting the approach considered in Sect. 5 of [50], the *stu* can be consistently related to the so-called st^2 and t^3 models, respectively with 2 and 1 *complex* scalars. Through such a procedure, all the results obtained for the *stu* model can be considered to hold for such models. Furthermore, by performing the *near-horizon* (*i.e.* $\tau \rightarrow -\infty$) *limit* on the attractor flow solutions, one obtains the corresponding attractor solution at the event horizon of the extremal BH. This is particularly relevant for the non-BPS $Z \neq 0$ horizon attractor solutions, hitherto analytically known (in a rather intricate form) only for the t^3 model, so far the only $\mathcal{N} = 2$, $d = 4$ supergravity model based on *cubic* special Kähler geometry whose Attractor Eqs. had been completely solved. In the *near-horizon limit*, the results of the present paper yield the non-BPS $Z \neq 0$ horizon attractor solutions

for both st^2 and stu models.

- It should be recalled once again that the stu model is a sector of *all* $\mathcal{N} > 2$, $d = 4$ supergravities, as well as of *all* $\mathcal{N} = 2$, $d = 4$ supergravities based on homogeneous (both *symmetric* [97, 113–115] and *non-symmetric* - see *e.g.* [116, 117] -) scalar manifolds based on cubic geometries. Thus, stu model captures the essential features of extremal BHs in all such theories (see *e.g.* the stu interpretation of $\mathcal{N} = 8$, $d = 4$ attractors, and the observations in [59]). Consequently, (the core of) the results holding for stu model can be thought to hold *at least* for all such theories. For instance, it would be interesting to try to extend them to some of the theories considered in [74], also in relation to the issue of the *effective horizon radius* treated therein.
- The stu model has been recently shown to be relevant for the analogy between pure states of multipartite entanglement of *qubits* in *quantum information theory* and extremal stringy BHs [3] (see also [120] for further recent developments). In the seventh of Refs. [3] the relation between quantum information theory and the theory of extremal stringy BHs was studied within the stu model, showing that the *three-qubit interpretation* of supersymmetric, $\frac{1}{2}$ -BPS attractors can be extended also to include non-supersymmetric, non-BPS (both $Z \neq 0$ and $Z = 0$) ones, performing a classification of the attractor solutions based on the charge codes of *quantum error correction*. However, only *double-extremal* solutions, with *constant, non-dynamical* scalars *all along* the attractor flow, were discussed therein. Thus, as also observed in [62], it would be interesting to extend the analysis of the seventh of Refs. [3] using the full general non-BPS (both $Z \neq 0$ and $Z = 0$) attractor flow solutions obtained in the present paper.
- The existence of a *first order formalism* for the equations of motion of the scalar fields (also named *attractor flow Eqs.*) in the background of an extremal BH [37, 42] *in principle* implies the *integrability* of such equations, regardless their eventual intricate form. This answers to the question raised in Appendix A of [59], and it is particularly relevant for the non-BPS $Z \neq 0$ attractor flow, as pointed out in Subsects. 6.1.-6.3.. It would be interesting to study the integrability of the equations of motion of the scalars in presence of *quantum* (*perturbative* and/or *non-perturbative*) corrections to the considered stu model. For instance, it would be interesting to study the attractor flow Eqs. for a quantum corrected prepotential $f = stu + i\lambda$, with $\lambda \in \mathbb{R}$, which is the only correction which preserves the *axion shift symmetry* and modifies the geometry of the scalar manifold (see [58] and Refs. therein). A tempting ideas, inspired by the intriguing connection between quantum information theory and extremal BHs mentioned at the previous point, is to consider the quantum, axion-shift-consistent parameter λ as related to the *quantum noise* of the system (see *e.g.* [123] and Refs. therein).
- As found in [121], observed also in [59] and noticed in Sect. 3., an immediate consequence of the general form of $\frac{1}{2}$ -BPS attractor flow given by Eq. (9) is that Γ_∞ satisfies the $\frac{1}{2}$ -BPS Attractor Eqs. [106]. This determines a sort of “*Attractor Mechanism at spatial infinity*”, mapping the 6 *real* moduli $(x^1, x^2, x^3, y^1, y^2, y^3)$ into the 8 *real* constants $(p_\infty^1, p_\infty^2, p_\infty^3, q_{1,\infty}, q_{2,\infty}, q_{3,\infty})$, arranged as Γ_∞ and constrained by the 2 *real* conditions (13). As noticed in [59], the absence of *flat* directions in the $\frac{1}{2}$ -BPS attractor flow (which is a general feature of $\mathcal{N} = 2$, $d = 4$ *ungauged* supergravity

coupled to Abelian vector multiplets, *at least* as far as the metric of the scalar manifold is strictly positive definite $\forall \tau \in \mathbb{R}^-$ [10]) is crucial for the validity of the expression (9). As pointed out in Sect. 4., the same holds for the non-BPS $Z = 0$ case. Indeed, a consequence of the general form of non-BPS $Z = 0$ attractor flow given by Eq. (29) is that Γ_∞ satisfies the *non-BPS $Z = 0$ Attractor Eqs.* (see e.g. [23] and [32]), determining a sort of “*Attractor Mechanism at spatial infinity*”. Analogously to what happens in the $\frac{1}{2}$ -BPS case, the absence of *flat* directions in the non-BPS $Z = 0$ attractor flow (which is *not* a general feature of $\mathcal{N} = 2, d = 4$ *ungauged* supergravity coupled to Abelian vector multiplets, but however holds for the *stu* model [40, 43]) is crucial for the validity of the expression (29). In view of the crucial differences among the non-BPS $Z \neq 0$ attractor flow and the $\frac{1}{2}$ -BPS and non-BPS $Z = 0$ attractor flows (such as the presence of a 2-dim. real *moduli space* $(SO(1,1))^2$ *all along* the non-BPS $Z \neq 0$ attractor flow), it would be interesting to investigate the non-BPS $Z \neq 0$ “*Attractor Mechanism at spatial infinity*”, if any.

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