STUDENTIZATION IN EDGEWORTH EXPANSIONS FOR ESTIMATES OF SEMIPARAMETRIC INDEX MODELS *

by

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Abstract

We establish valid theoretical and empirical Edgeworth expansions for density-weighted averaged derivative estimates of semiparametric index models.

Keywords: Edgeworth expansions; semiparametric estimates; averaged derivatives.

JEL Nos.: C21, C24

1. INTRODUCTION

During the 1970's, Takeshi Amemiya considerably advanced the asymptotic theory of estimation of parametric econometric models for cross-sectional data. Previously, most work had concerned closed form estimates, such as generalized least squares and instrumental variable estimates of linear regressions, or two- or three-stage least squares estimates of linear-in-equations-and parameters simultaneous equations systems. Prompted by Jennrich's (1969) work on strong consistency and asymptotic normality of nonlinear least squares, Amemiya developed asymptotic theory for implicitly-defined extremum estimates of a variety of econometric models.

Let Y_i, X_i, $i = 1, 2, \ldots,$ be sequences of, respectively, scalar and $d \times 1$ vector observables, and define

$$Y_i = (\beta^\tau X_i + \varepsilon_i) 1(\beta^\tau X_i + \varepsilon_i > 0), \quad i = 1, 2, \ldots, \tag{1.1}$$

where ε_i, $i = 1, 2, \ldots,$ is a sequence of unobservable zero-mean random variables, β is a $d \times 1$ unknown vector, τ denotes transposition, and $1(\cdot)$ is the indicator function. (1.1) is called a Tobit model. Least squares regression of Y_i on X_i, using either all observations or all observations such that $Y_i > 0$, inconsistently estimates β. Assuming the ε_i are independent and identically distributed (iid) normal variates, maximum likelihood (ML) estimates based on (1.1) can be consistent. These, however, are only implicitly defined. Amemiya (1973) established their strong consistency and asymptotic normality, later extending these results (Amemiya (1974a)) to a multivariate version of (1.1).

Another model of econometric interest is

$$\left(\frac{Y_i^\lambda - 1}{\lambda}\right) 1(\lambda > 0) + (\log Y_i) 1(\lambda = 0) = \beta^\tau X_i + \varepsilon_i, \tag{1.2}$$

where the scalar λ is unknown. This is called a Box-Cox transformation model. If λ is specified incorrectly, least squares regression inconsistently estimates β. Thus, methods have been proposed for estimating β and λ simultaneously. One such purports to be ML, based on normal ε_i, but unless $\lambda = 0$ or $1/\lambda$ is odd the left hand side of (1.2) cannot possibly be conditionally normal. Alternative, logically consistent, distributions have been proposed, e.g., Amemiya and Powell (1981), but if the distribution is misspecified inconsistent estimates again result. Amemiya and Powell (1981) also applied non-linear two-stage least squares estimation, which applies to a general class of models including (1.2), and whose asymptotic theory was earlier developed by Amemiya (1974b). This estimate, which again is only implicitly defined, is consistent over a wide class of ε_i. Amemiya (1977) also developed asymptotic theory for non-linear three-stage least squares and ML estimates of non-linear simultaneous equations, to provide an extension to vector dependent variables.

Both models (1.1) and (1.2) are of the single linear index type

$$E(Y_i|X_i) = G(\beta^\tau X_i), \; i=1,2,..., \tag{1.3}$$

almost surely (a.s.), for a function $G: R \rightarrow R$. Let *F* be the distribution function of ε_i. In (1.1),

$$G(u) = u\int_{-u}^{\infty} dF(v) - \int_{-\infty}^{-u} v dF(v).$$

If *F* is an unknown, nonparametric function, then so is *G*. Then β can be identified only up to scale. But if we can estimate β up to scale in (1.3), with unknown *G*, we have a form of robustness with respect to *F*. In (1.2),

$$G(u) = \int \{1+\lambda(u+v)\}^{1/\lambda} dF(v) 1(\lambda>0) + e^{u}\int e^{v} dF(v) 1(\lambda=0),$$

so the same considerations arise. As already noted, we can robustly estimate β (and also λ) in (1.2) using nonlinear two-stage least squares. However, the general index form (1.3) indicates that we may be able to estimate β up to scale whether or not the transformation of Y_i is of Box-Cox type. Note that β can be identifiable on the basis of objective functions used in other semiparametric methods such as LAD (Powell, 1984), symmetrically trimmed least squares (Powell, 1986), semiparametric M-estimation (Horowitz, 1988), and semiparametric least squares (Horowitz, 1986; Lee, 1992).

We can estimate β up to scale by the density-weighted averaged derivative statistic

$$U = \binom{n}{2}^{-1} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} U_{ij},$$

where

$$U_{ij} = h^{-d-1}K'(\frac{X_i - X_j}{h})(Y_i - Y_j) \ ,$$

such that $K'(u) = (\partial/\partial u)K(u),$ where $K : R^d \to R$ is a differentiable (kernel) function such that $\int_{R^d} K(u)du = 1,$ and $h = h_n$ is a positive (bandwidth or smoothing) sequence which tends to zero slowly as $n \to \infty$. For an unknown scalar *c*, $n^{1/2}(U - c\beta)$ was shown to be asymptotically normal when the Y_i, X_i are iid (Powell, Stock and Stoker (1989)) and when they are weakly dependent (Robinson (1989)), and to be possibly asymptotically non-normal in case of an element of long range dependence (Cheng and Robinson (1994)).

Thus, in case of the Tobit model (1.1), for example, *U* achieves the same rate of convergence as that of the ML estimate established by Amemiya (1973) where the ε_i are normal, and by Robinson (1982) where the ε_i are normal but actually weakly dependent. (Robinson (1982) also established consistency when the ε_i are long range dependent normal). On the other hand, the smoothing entailed in *U* might be expected to produce inferior higher-order asymptotic properties, since these more closely approximate the finite sample situation. We know of no explicit treatment of higher-order properties of the Tobit MLE (or of the Box-Cox estimates we have mentioned), but general results of Pfanzagl (1971, 1973), Bhattacharya and Ghosh (1978), Linton (1996) suggest that, under suitable conditions, they are likely to have an $O(n^{-1/2})$ Berry-Esseen bound (uniform rate of convergence to normality) and valid Edgeworth expansion in powers of $n^{-1/2}$, while Robinson (1991) established a Berry-Esseen bound for an optimal version of Amemiya's (1977) nonlinear three-stage least squares estimate. Robinson (1995) showed that while in general *U* has a Berry-Esseen bound of order greater than $n^{-1/2}$, it can be implemented (using suitable *h* and *K*) to have an $O(n^{-1/2})$ bound. Correspondingly, Nishiyama and Robinson (1998) (hereafter NR) established that the leading Edgeworth expansion term is $O(n^{-1/2})$ or larger.

Theorems 1 and 2 of NR established valid theoretical and empirical Edgeworth expansions of $Z = n^{1/2}\sigma_\nu^{-1}\nu^\tau(U - c\beta)$ for any $d \times 1$ vector ν, where $\sigma_\nu^2 = \nu^\tau \Sigma \nu$ and Σ

is the asymptotic variance matrix of $n^{1/2}(U - c\beta)$. Of course Σ is unknown so that these Edgeworth expansions fall short of being operational. For a consistent estimate, $\hat{\Sigma}$, of Σ, we are led to consideration of $\hat{Z} = n^{1/2}\hat{\sigma}_{\nu}^{-1}\nu^{\tau}(U - c\beta)$, where $\hat{\sigma}_{\nu}^{2} = \nu^{\tau}\hat{\Sigma}\nu$. NR in fact proposed such a (jackknife) estimate $\hat{\Sigma}$, and reported valid theoretical and empirical Edgeworth expansions for $\hat{Z}$ in their Theorems 3 and 4. NR also derived a choice of *h* that is optimal in the sense of minimizing the maximal deviation of Edgeworth correction terms from the normal approximation, and proposed also a consistent estimate of the scale factor of this, leading to a feasible approximately optimal *h*. NR also reported a Monte Carlo examination of their Edgeworth expansions, and of their bandwidth choice proposal. However, NR did not include the proofs of their Theorems 3 and 4, which entail additional regularity conditions and a considerable and lengthy development beyond that of their Theorems 1 and 2. By marked contrast with the routine application of Slutsky's lemma which is all that is needed to deduce asymptotic normality of $\hat{Z}$ from that of *Z*, the Edgeworth expansions for $\hat{Z}$ involve considerable extra work and actually differ from those for *Z*. The present paper fills this gap, by providing the proofs of NR's Theorems 3 and 4, while taking for granted the proofs of their Theorems 1 and 2. Callaert and Veraverbeke (1981), Helmers (1985, 1991) have established higher-order asymptotics for studentized versions of standard U-statistics. Though we follow their broad approach, our *U* is a U-statistic with an *n*-dependent "kernel" (through *h*) which significantly complicates matters, whereby we must also make substantial use of lemmas established by Robinson (1995) and NR.

The following section presents regularity conditions and theorem statements. Section 3 contains the main details of the proofs, with some detailed technical material left to appendices.

2. THEORETICAL AND EMPIRICAL EDGEWORTH EXPANSIONS

Our conditions below imply that X has a probability density, $f(x)$, and existence of the conditional moments $g = E(Y|X)$, $q = E(Y^2|X)$, $r = E(Y^3|X)$, where, for a function $h: R^d \to R$, we write $h = h(X)$. For such a function, suitably smooth, we define $h' = (\partial/\partial X)h(X)$, $h'' = (\partial/\partial X^\tau)h'(X)$ and $h''' = (\partial/\partial X^\tau)vec(h'')$. Write $e = fg$, $\mu = \mu(X,Y) = Yf' - e'$, $a = g'f - E(g'f)$, $\mu. = E(\mu) = -E(g'f)$ and $\Sigma = 4Var(\mu)$. We introduce the following assumptions.

(i) $E(Y^6) < \infty$.

(ii) Σ is finite and positive definite.

(iii) The underlying measure of (X^τ, Y) can be written as $\mu_X \times \mu_Y$, where μ_X and μ_Y are Lebesgue measure on R^d and R respectively, and (X_i^τ, Y_i) are iid observations on (X^τ, Y).

(iv) f is $(L+1)$ times differentiable, and f and its first $(L+1)$ derivatives are bounded, for $2L>d+2$.

(v) g is $(L+1)$ times differentiable, and e and its first $(L+1)$ derivatives are bounded for $L \ge 1$.

(vi) q is twice differentiable and $q', q'', g', g'', g''', E(|Y|^3|X)f$ and qf' are bounded.

(vii) $f, gf, g'f$ and qf vanish on the boundaries of their convex (possibly infinite) supports.

(viii) $K(u)$ is even and differentiable,

$$\int_{R^d}\{(1+\|u\|^L)|K(u)| + \|K'(u)\|\}du + \sup_{u \varepsilon R^d}\|K'(u)\| < \infty,$$

and for the same L as in (iv),

$$\int_{R^d} u_1^{l_1} \cdots u_d^{l_d} K(u)du \begin{cases} = 1, & if \quad l_1 + \cdots + l_d = 0 \\ = 0, & if \ 0 < l_1 + \cdots + l_d < L \\ \neq 0, & if \quad l_1 + \cdots + l_d = L \end{cases}.$$

(ix) $\frac{(\log n)^9}{nh^{d+2}} + nh^{2L} \to 0$ as $n \to \infty$.

(x) $\sup_{\nu^\tau \nu = 1} \limsup_{|t| \to \infty} |E\exp\{it\sigma_\nu^{-1}\nu^\tau(\mu - \mu.)\}| < 1$.

These assumptions are the same as those of Theorem 1 of NR except (i) strengthens their third moment assumption to sixth moments, our treatment of studentization requiring finite third moments of certain squared terms.

In our studentized statistic $\hat{Z}$ we take

$$\hat{\Sigma} = \frac{4}{(n-1)(n-2)^2}\sum_{i=1}^{n}\left\{\sum_{j\neq i}^{n}(U_{ij}-U)\right\}\left\{\sum_{k\neq i}^{n}(U_{ik}-U)^{\tau}\right\}, \tag{2.1}$$

a jackknife estimate of Σ. We are concerned with approximating

$$\hat{F}(z) = P(\hat{Z} \leq z)$$

by the Edgeworth expansion

$$F^{+}(z) = \Phi(z) - \varphi(z)\left[n^{1/2}h^{L}\kappa_1 - \frac{\kappa_2}{nh^{d+2}}z - \frac{4}{3n^{1/2}}\left\{(2z^2+1)\kappa_3 + 3(z^2+1)\kappa_4\right\}\right],$$

where $\Phi(z)$ and $\varphi(z)$ are respectively the distribution function and density function of the standard normal, and, with

$$\Delta^{(l_1,\cdots,l_d)} = \frac{\partial^{(l_1+\cdots+l_d)}}{\partial x_1^{l_1}\cdots\partial x_d^{l_d}},$$

$$\kappa_1 = \frac{2(-1)^{L}\sigma_{\nu}^{-1}}{L!}\sum_{\substack{0\leq l_1,\cdots,l_d\leq L\\ l_1+\cdots+l_d=L}}\cdots\sum\left\{\int\prod_{i=1}^{d}u_i^{l_i}K(u)du\right\}E\left[\left(\Delta^{(l_1,\cdots,l_d)}\nu^{\tau}f'\right)g\right],$$

$$\kappa_2 = 2\sigma_{\nu}^{-2}\int\left\{\nu^{\tau}K'(u)\right\}^2du\, E\{(q-g^2)f\},$$

$$\kappa_3 = \sigma_{\nu}^{-3}E\left[\{r-3(q-g^2)g-g^3\}(\nu^{\tau}f')^3-3(q-g^2)(\nu^{\tau}f')^2(\nu^{\tau}a)-(\nu^{\tau}a)^3\right]$$

$$\kappa_4 = -\sigma_{\nu}^{-3}E\Big[f(q-g^2)(\nu^{\tau}f')(\nu^{\tau}a'\nu)-f(\nu^{\tau}f')\{\nu^{\tau}(q'-2gg')\}(\nu^{\tau}a)$$
$$-f(q-g^2)(\nu^{\tau}a)(\nu^{\tau}f''\nu)+f(\nu^{\tau}g')(\nu^{\tau}a)^2\Big],$$

<u>THEOREM A</u> : Under assumptions (i)-(x), as $n\rightarrow\infty$

$$\sup_{\nu:\nu^{\tau}\nu=1}\sup_{z}|\hat{F}(z) - F^{+}(z)| = o(n^{-1/2}+n^{-1}h^{-d-2}+n^{1/2}h^{L}).$$

The correction terms in $F^{+}(z)$ are of the same orders as those in the unstudentized case (see Theorem 1 of NR), though their coefficients are different.

The κ_i are unknown, but a feasible, empirical Edgeworth expansion is

$$\hat{F}^{+}(z) = \Phi(z) - \varphi(z)\left[n^{1/2}h^{L}\tilde{\kappa}_1 - \frac{\tilde{\kappa}_2}{nh^{d+2}}z - \frac{4}{3n^{1/2}}\left\{(2z^2+1)\tilde{\kappa}_3 + 3(z^2+1)\tilde{\kappa}_4\right\}\right] ,$$

where

$$\tilde{\kappa}_1 = \frac{2(-1)^L\hat{\sigma}_\nu^{-1}}{L!}\sum_{\substack{0\le l_1,\cdots,l_d\le L\\ l_1+\cdots+l_d=L}}\cdots\sum\left\{\int\prod_{i=1}^{d}u_i^{l_i}K(u)du\right\}\frac{1}{n}\sum_{i=1}^{n}\left\{\Delta^{(l_1,\cdots,l_d)}\nu^{\tau}\tilde{f}^{/}(X_i)\right\}Y_i ,$$

$$\tilde{\kappa}_2 = \hat{\sigma}_\nu^{-2}\binom{n}{2}^{-1}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}h^{d+2}\overline{W}_{ij}^{\,2} , \qquad \tilde{\kappa}_3 = \frac{\hat{\sigma}_\nu^{-3}}{n}\sum_{i=1}^{n}\overline{V}_i^{\,3} ,$$

$$\tilde{\kappa}_4 = \frac{\hat{\sigma}_\nu^{-3}}{n(n-1)}\sum_{i=1}^{n}\sum_{j\ne i}^{n}\nu^{\tau}U_{ij}\overline{V}_i\overline{V}_j ,$$

where for positive b and a function $H : R^d \to R$

$$\tilde{f}(X_i) = \frac{1}{(n-1)b^d}\sum_{j\ne i}^{n}H\left(\frac{X_i-X_j}{b}\right) ,$$

and

$$\overline{U}_i = \frac{1}{n-1}\sum_{j\ne i}^{n}U_{ij} , \ \overline{V}_i = \nu^{\tau}(\overline{U}_i - U) , \ \overline{W}_{ij} = \nu^{\tau}(U_{ij} - \overline{U}_i - \overline{U}_j + U) . \quad (2.2)$$

We impose the following additional assumptions, which are identical to those of Theorem 2 of NR.

(iv)’ f is $(L+2)$ times differentiable, and f and its first $(L+2)$ derivatives are bounded, where $2L>d+2$.

(v)’ g is $(L+2)$ times differentiable, and e and its first $(L+2)$ derivatives are bounded.

(ix)’ $\frac{(\log n)^9}{nh^{d+3}} + nh^{2L} \to 0$ as $n\to\infty$.

(xi) $H(u)$ is even and $(L+1)$ times differentiable,

$$\int_{R^d}H(u)du = 1 ,$$

$$\int_{R^d}\|\Delta^{(l_1,\cdots,l_d)}H^{/}(u)\|du + \sup_{u\varepsilon R^d}\|\Delta^{(l_1,\cdots,l_d)}H^{/}(u)\| < \infty$$

for any integers $l_1,...,l_d$ satisfying $0\le l_1+\cdots+l_d\le L$ and $0\le l_i\le L$, $i=1,...,l$.

(xii) $b\to 0$ and $\frac{(\log n)^2}{nb^{d+2+2L}} = O(1)$ as $n\to\infty$.

THEOREM B : *Under assumptions* (i)-(iii), (iv)', (v)', (vi) - (viii), (ix)' *and* (x)-(xii),

$$\sup_{\nu:\nu^\tau\nu=1}\sup_z |\hat{F}(z) - \hat{F}^+(z)| = o(n^{-1/2} + n^{-1}h^{-d-2} + n^{1/2}h^L) \quad a.s.$$

3. PROOF OF THEOREMS A AND B

Proof of Theorem A.

In the sequel, C denotes a generic, finite, positive constant and the qualification "for sufficiently large n" may be omitted.

As is standard in U-statistic theory, we write

$$n^{1/2}\sigma_\nu^{-1}\nu^\tau(U-\mu.) = \frac{2}{\sqrt{n}}\sum_{i=1}^n V_i + n^{1/2}\binom{n}{2}^{-1}\sum_{i=1}^{n-1}\sum_{j=i+1}^n W_{ij} + n^{1/2}\sigma_\nu^{-1}\nu^\tau(EU-\mu.)$$

$$= \bar{V} + \bar{W} + \Delta . \tag{3.1}$$

where $U_i = E(U_{ij}|i)$, $V_i = \sigma_\nu^{-1}\nu^\tau(U_i - EU)$, $W_{ij} = \sigma_\nu^{-1}\nu^\tau(U_{ij}-EU) - V_i - V_j$, such that $E(\cdot|i_1,\cdots,i_r) = E(\cdot|(X_{i_j},Y_{i_j}), j=1,\cdots,r)$. Writing $S = 4Var(U_i)$, $s^2 = \sigma_\nu^{-2}\nu^\tau S\nu$, Taylor's theorem gives

$$\sigma_\nu\hat{\sigma}_\nu^{-1} = s^{-1} - \frac{s^{-3}}{2}(\hat{\sigma}_\nu^2 - s^2) + \frac{3}{4}\{s^2+\theta(\hat{\sigma}_\nu^2-s^2)\}^{-5/2}(\hat{\sigma}_\nu^2-s^2)^2$$

$$= s^{-1} + \tilde{R} + \tilde{\tilde{R}} \tag{3.2}$$

for some $\theta \varepsilon [0,1]$. Similarly to Callaert and Veraverbeke (1981), we expand $\tilde{R}$ as follows. With $\tilde{V}_i = E(V_j W_{ij}|i)$, $\tilde{W}_{jk} = E(W_{ij}W_{ik}|j,k)$, we have

$$\tilde{R} = T + Q + R \; , \quad T = T_1 + T_2 + T_3 \; , \quad Q = Q_1 + Q_2 \; , \quad R = R_1+R_2+R_3+R_4+R_5$$

where

$$T_1 = \frac{4\delta n}{(n-2)^2}E(W_{12}^2) \; , \; T_2 = \frac{\delta}{n}\sum_{i=1}^n\left\{(4V_i^2-s^2)+8\tilde{V}_i\right\} \; , \; T_3 = 4\delta\binom{n-1}{2}^{-1}\sum_{i<j}^n \tilde{W}_{ij} \; ,$$

$$Q_1 = 4\delta\binom{n}{2}^{-1}\sum_{i<j}\left\{(V_i+V_j)W_{ij} - \tilde{V}_i - \tilde{V}_j\right\} \; , \; Q_2 = -\frac{8\delta}{n}\binom{n-1}{2}^{-1}\sum_{i=1}^n\sum_{k<m}^{n} {}^{(i)} V_i W_{km} \; ,$$

$$R_1 = -4\delta\binom{n}{2}^{-1}\sum_{i<j} V_i V_j \ , \quad R_2 = \frac{4\delta}{n-2}\binom{n-1}{2}^{-1}\sum_{i=1}^{n}\sum_{k<m}^{n\ (i)} (W_{ik}W_{im} - \tilde{W}_{km}) \ ,$$

$$R_3 = \frac{4\delta n}{(n-2)^2}\binom{n}{2}^{-1}\sum_{i<j} \{W_{ij}^2 - \tilde{W}_{ii} - \tilde{W}_{jj} + E(W_{12}^2)\} \ ,$$

$$R_4 = \frac{8\delta}{(n-2)^2}\sum_{i=1}^{n} \{\tilde{W}_{ii} - E(W_{12}^2)\} \ , \quad R_5 = -\frac{4\delta n(n-1)}{(n-2)^2}\left\{\binom{n-1}{2}^{-1}\sum_{i<j} W_{ij}\right\}^2 \ ,$$

where $\delta = -s^{-3}/2$ and $\sum_{k<m}^{n\ (i)}$ denotes summation with respect to k and m for $1 \le k < m \le n$ excluding $k=i$ and $m=i$. Because

$$\hat{Z} = (s^{-1} + \tilde{R} + \tilde{\tilde{R}})(\bar{V} + \bar{W} + \Delta) \ ,$$

by a standard inequality

$$\sup_z |\hat{F}(z) - F^+(z)| \le \sup_z \left|P\left((s^{-1} + T + Q)(\bar{V} + \bar{W}) + s^{-1}\Delta \le z\right) - F^+(z)\right|$$
$$+ P\left(|(R + \tilde{\tilde{R}})(\bar{V} + \bar{W} + \Delta) + (T + Q)\Delta| \ge a_n \right) + O(a_n) \qquad (3.3)$$

for $a_n > 0$, where here and subsequently we drop reference to $\sup_{\nu:\nu^\tau\nu=1}$. Taking $a_n = \frac{1}{\log n}\max(n^{-1/2}, n^{-1}h^{-d-2}, n^{1/2}h^L)$, we bound the second term on the right of (3.3) by

$$P\left(|(R + \tilde{\tilde{R}})(\bar{V} + \bar{W} + \Delta)| \ge \frac{a_n}{2} \right) + P\left(|(T + Q)\Delta| \ge \frac{n^{1/2}h^L}{2\log n} \right)$$
$$\le P\left(|R + \tilde{\tilde{R}}| \ge \frac{a_n}{2\log n} \right) + P\left(|\bar{V} + \bar{W} + \Delta| \ge \log n\right) + P\left(|(T + Q)\Delta| \ge \frac{n^{1/2}h^L}{2\log n} \right) \qquad (3.4)$$

The first term in (3.4) is, by an elementary inequality, bounded by

$$P(|R| \ge \frac{a_n}{4\log n}) + P\left(\frac{|\tilde{\tilde{R}}|}{\tilde{R}^2} \ge C_0 \right) + P\left(\tilde{R}^2 \ge \frac{a_n}{4C_0 \log n} \right) \qquad (3.5)$$

for a constant C_0 determined later. The third term of (3.5) is bounded by

$$P\left(T_2^2 \ge \frac{a_n}{12C_0 \log n} \right) + P\left(|T_1 + T_3|^2 \ge \frac{a_n}{12C_0 \log n} \right) + P\left(|Q + R|^2 \ge \frac{a_n}{12C_0 \log n} \right)$$

$$= \quad (a) \quad + \quad (b) \quad + \quad (c) \ .$$

Lemmas 10-19 and Markov's inequality give, for $\zeta > 0$,

$$(a) \le \frac{E|T_2|^{2(1+\zeta)}}{\left(\frac{a_n}{12C_0 \log n}\right)^{1+\zeta}} \le \frac{Cn^{-(1+\zeta)}(\log n)^{2(1+\zeta)}}{n^{-\frac{1}{2}(1+\zeta)}} = o(n^{-\frac{1}{2}}) ,$$

$$(b) \le \frac{E|T_1 + T_3|^{2(1+\zeta)}}{\left(\frac{a_n}{12C_0 \log n}\right)^{1+\zeta}} \le \frac{C(n^{-1}h^{-d-2})^{2(1+\zeta)}(\log n)^{2(1+\zeta)}}{(n^{-1}h^{-d-2})^{1+\zeta}} = o(n^{-1}h^{-d-2}) ,$$

$$(c) \le \frac{E|R+Q|^2}{\frac{a_n}{12C_0 \log n}} \le \frac{Cn^{-2}h^{-d-2}(\log n)^2}{n^{-1/2}} = o(n^{-1}h^{-d-2}) ,$$

where $\zeta = \frac{2}{7}$ suffices in (b), and ζ arbitrarily small suffices in (a).

The first term of (3.5) is, using Markov's inequality, (ix) and Lemmas 15-19, bounded by

$$\frac{16\,E(R^2)(\log n)^2}{a_n^2} \le C\,(n^{-1} + n^{-2}h^{-2d-4})(\log n)^4 = o\,(n^{-1/2} + n^{-1}h^{-d-2}) .$$

Now, in view of (3.2), $\tilde{\tilde{R}} = 3s(1-2\theta s\tilde{R})^{-5/2}\tilde{R}^2$ so that because $\tilde{\tilde{R}} \ge 0$ and $0 \le \theta \le 1$,

$$P\left(\frac{\tilde{\tilde{R}}}{\tilde{R}^2} \ge C_0\right) = P\left(3s(1-2\theta s\tilde{R})^{-5/2} \ge C_0\right)$$

$$\le P\left(|\tilde{R}| \ge \frac{1}{2s}\{1-(\frac{3s}{C_0})^{2/5}\}\right) . \tag{3.6}$$

Taylor's expansion of s^r around $s^2 = 1$ and Lemma 2 of Robinson(1995) give for integer r,

$$s^r = 1 + O\,(\sigma_\nu^{-2}\nu^\tau(S-\Sigma)\nu) = 1 + O\,(h^L) \tag{3.7}$$

so that we can choose C_0 such that $C_0 > 3s$ for sufficiently large *n* by (ii). Then by (3.7) and Markov's inequality, (3.6) is bounded by a constant times $E|T+Q+R|^3$ $= O\,(n^{-3/2} + n^{-3}h^{-3d-6})$ from Lemmas 10-19, so that the second term of (3.5) is $O\,(n^{-3/2} + n^{-3}h^{-3d-6})$. Therefore,

$$P\left(|R+\tilde{\tilde{R}}| \ge \frac{a_n}{\log n}\right) = o\,(n^{-\frac{1}{2}} + n^{-1}h^{-d-2}) . \tag{3.8}$$

Put $F(z) = P[n^{1/2}\sigma_\nu^{-1}\nu^\tau(U-\mu.) \le z]$. Then

$$P(|\bar{V}+\bar{W}+\Delta| \ge \log n) = 1-F(\log n)+F(-\log n) . \tag{3.9}$$

NR proved in Theorem 1 that

$$\sup_z |F(z) - \tilde{F}(z)| = o(n^{-1/2} + n^{-1}h^{-d-2} + n^{1/2}h^{L})$$

where

$$\tilde{F}(z) = \Phi(z) - \varphi(z)\left\{n^{1/2}h^{L}\kappa_1 + \frac{\kappa_2}{nh^{d+2}}z + \frac{4(\kappa_3 + 3\kappa_4)}{3n^{1/2}}(z^2 - 1)\right\} \tag{3.10}$$

which implies that for any z

$$1 - F(z) + F(-z) = 1 - \tilde{F}(z) + \tilde{F}(-z) + o(n^{-1/2} + n^{-1}h^{-d-2} + n^{1/2}h^{L}) \ . \tag{3.11}$$

Now by (3.10),

$$1 - \tilde{F}(z) + \tilde{F}(-z) = 1 - \Phi(z) + \Phi(-z) + \varphi(z)\frac{2\kappa_2}{nh^{d+2}}z$$
$$= 2 - 2\Phi(z) + \varphi(z)\frac{2\kappa_2}{nh^{d+2}}z \ . \tag{3.12}$$

Substituting (3.12) into (3.11) and putting $z = \log n$, because $1 - \Phi(\log n) = o(n^{-1/2})$ and $\varphi(\log n)\log n = o(n^{-1/2})$, we have

$$1 - F(\log n) + F(-\log n) = o(n^{-1/2} + n^{-1}h^{-d-2} + n^{1/2}h^{L}) \ . \tag{3.13}$$

By (3.9) and (3.13),

$$P\left(|\bar{V} + \bar{W} + \Delta| \geq \log n\right) = o(n^{-1/2} + n^{-1}h^{-d-2} + n^{1/2}h^{L}) \ . \tag{3.14}$$

Finally, Markov's inequality, (ix), Lemma 1 of Robinson (1995), and Lemmas 10-14 bound the last term of (3.4) by

$$\frac{\Delta^2 E|T + Q|^2 (2\log n)^2}{nh^{2L}} \leq C(n^{-1} + n^{-2}h^{-2d-4})(\log n)^2 = o(n^{-1/2} + n^{-1}h^{-d-2}). \tag{3.15}$$

Substituting (3.8), (3.14) and (3.15) into (3.4),

$$P\left(|(R + \approx{R})(\bar{V} + \bar{W} + \Delta) + (T + Q)\Delta| \geq a_n \right) = o(n^{-1/2} + n^{-1}h^{-d-2} + n^{1/2}h^{L}) \ . \tag{3.16}$$

To deal with the first term on the right of (3.3), write $b_2 = s^{-1}\bar{V}$, $b_3 = s^{-1}\bar{W}$, $\tilde{b}_2 = (T + Q)\bar{V}$, $\tilde{b}_3 = (T + Q)\bar{W}$, $b_1 = b_2 + b_3$, $\tilde{b}_1 = \tilde{b}_2 + \tilde{b}_3$, $B = b_1 + \tilde{b}_1$, and define

$$\chi^{+}(t) = \int e^{itz}dF^{+}(z)$$

$$= e^{-\frac{t^2}{2}}\left[1 + \left\{n^{1/2}h^{L}\kappa_1 - \frac{4(\kappa_3 + 2\kappa_4)}{n^{1/2}}\right\}(it)\right.$$

$$- \frac{\kappa_2}{nh^{d+2}}(it)^2 - \frac{4(2\kappa_3+3\kappa_4)}{3n^{1/2}}(it)^3 \Bigg| \ .$$

Esseen's smoothing lemma gives for $N_0 = \log n \min(\eta n^{1/2}, nh^{d+2}), \eta = (E|2s^{-1}V_1|^3)^{-1}$

$$\sup_z \left|P\left((s^{-1}+T+Q)(\bar{V}+\bar{W}) + s^{-1}\Delta \leq z\right) - F^+(z)\right|$$

$$\leq \int_{-N_0}^{N_0} \left| \frac{Ee^{it(B+s^{-1}\Delta)} - \chi^+(t)}{t} \right| dt + O(N_0^{-1}) \ ,$$

which, for $p = \min(\log n, \varepsilon n^{1/2}, nh^{d+2})$, is bounded by

$$\int_{-p}^{p} \left| \frac{Ee^{it(B+s^{-1}\Delta)} - \chi^+(t)}{t} \right| dt + \int_{p \leq |t| \leq N_0} \left| \frac{Ee^{it(B+s^{-1}\Delta)}}{t} \right| dt$$

$$+ \int_{|t| \geq p} \left| \frac{\chi^+(t)}{t} \right| dt + o(n^{-1/2} + n^{-1}h^{-d-2})$$

$$= (I) + (II) + (III) + o(n^{-1/2} + n^{-1}h^{-d-2}).$$

Here we can set $\varepsilon \varepsilon (0, \eta]$ for sufficiently large n as discussed in the proof of the Theorem of Robinson (1995) using (3.7) also so that $p \leq N_0$. We first mention an inequality frequently used hereafter:

$$|e^{ix} - 1 - ix - \cdots - \frac{(ix)^{k-1}}{(k-1)!}| \leq \frac{|x|^k}{k!} \tag{3.17}$$

for integer k and real x .

<u>Estimation of (I)</u>

Since $s^{-1}\Delta$ is nonstochastic,

$$E\left\{e^{it(B+s^{-1}\Delta)}\right\} = e^{its^{-1}\Delta} E\left(e^{itB}\right) , \tag{3.18}$$

where (3.7) and (3.18) yield

$$e^{its^{-1}\Delta} = 1 + it\Delta + O(t^2\Delta^2 + |t|h^L\Delta) \ . \tag{3.19}$$

Writing $\tilde{b}_2 = \tilde{b}_2^{/} + \tilde{b}_2^{//}$ where $\tilde{b}_2^{/} = T\bar{V}$ and $\tilde{b}_2^{//} = Q\bar{V}$, and applying (3.17) repeatedly, we have

$$E(e^{itB}) = E(e^{itb_1}) + E(e^{itB} - e^{itb_1})$$

$$= E(e^{itb_1}) + \left\{Ee^{it(b_1+\tilde{b}_2+\tilde{b}_3)} - Ee^{it(b_1+\tilde{b}_2^{/})}\right\} + \left\{Ee^{it(b_1+\tilde{b}_2^{/})} - E(e^{itb_1})\right\}$$

$$= E(e^{itb_1}) + O(|t|E|\tilde{b}_2^{//}+\tilde{b}_3|) + \left\{Ee^{it(b_1+\tilde{b}_2^{/})} - E(e^{itb_1}) - itE(\tilde{b}_2^{/}e^{itb_1})\right\}$$

$$+ itE(\tilde{b}_2^{/}e^{itb_1} - \tilde{b}_2^{/}e^{itb_2}) + itE(\tilde{b}_2^{/}e^{itb_2})$$

$$= E(e^{itb_1}) + itE(\tilde{b}_2^{/}e^{itb_2}) + O\left(|t|E|\tilde{b}_2^{//}+\tilde{b}_3| + t^2(E|\tilde{b}_2^{/}|^2 + E|\tilde{b}_2^{/}b_3|)\right) . \qquad (3.20)$$

Write

$$E(e^{itb_1}) = E\left[e^{itb_2}\left\{1 + itb_3 + \frac{(it)^2}{2}b_3^2\right\}\right] + O(|t|^3E|b_3|^3) , \qquad (3.21)$$

and put $\gamma(t) = E(e^{it\frac{2}{\sqrt{n}s}V_1})$. As in Appendix A of NR,

$$E(e^{itb_2}) = \{\gamma(t)\}^n , \qquad (3.22)$$

$$E(b_3e^{itb_2}) = \{\gamma(t)\}^{n-2}\left[\frac{4(it)^2}{n^{1/2}}E(W_{12}V_1V_2) + O\left(\frac{t^2h^L}{n^{1/2}} + (\frac{t^4}{n^{3/2}} + \frac{|t|^3}{n})h^{-\frac{2}{3}d-1}\right)\right] , \qquad (3.23)$$

$$E(b_3^2e^{itb_2}) = \frac{2}{n-1}\{\gamma(t)\}^{n-2}\left[E(W_{12}^2) + O(\frac{h^L}{nh^{d+2}} + |t|n^{-1/2}h^{-\frac{4}{3}d-2})\right]$$

$$+ \{\gamma(t)\}^{n-3}O(|t|n^{-3/2}h^{-\frac{4}{3}d-2})$$

$$+ \{\gamma(t)\}^{n-4}O(t^4n^{-1} + t^8n^{-3}h^{-\frac{4}{3}d-2} + t^6n^{-2}h^{-\frac{4}{3}d-2}) . \qquad (3.24)$$

Since, for m=0,1,2,3,

$$\{\gamma(t)\}^{n-m} = e^{-\frac{t^2}{2}}\left\{1 + \frac{E(2V_1)^3}{6n^{1/2}s^3}(it)^3\right\} + o\left(n^{-1/2}(|t|^3+t^6)e^{-\frac{t^2}{4}}\right) , \qquad (3.25)$$

by Lemma 1 of Robinson (1995), Appendix B-(a) and (3.18)-(3.25),

$$E\{e^{it(B+s^{-1}\Delta)}\} = \left\{1 + it\Delta + O\left(t^2nh^{2L} + |t|n^{1/2}h^{2L}\right)\right\}$$

$$\times\left\{\left[e^{-\frac{t^2}{2}}\left\{1 + \frac{4E(V_1^3)}{3n^{1/2}s^3}(it)^3\right\} + o\left(n^{-1/2}(|t|^3+t^6)e^{-\frac{t^2}{4}}\right)\right]\right.$$

$$\times\left[1 + \frac{4(it)^3}{n^{1/2}}E(W_{12}V_1V_2) + \frac{(it)^2}{n}E(W_{12}^2) - \frac{2(it)^2}{n}E(W_{12}^2)\right.$$

$$- \frac{(it)^3+(it)}{n^{1/2}}\left\{4E(V_1^3)+8E(W_{12}V_1V_2)\right\} + O(A_n)\Bigg]$$

$$+ \left(|t|E|\tilde{b}_2^{\prime\prime}+\tilde{b}_3| + t^2(E|\tilde{b}_2^{\prime}|^2+E|\tilde{b}_2^{\prime}b_3|)\right)\Bigg\} \qquad (3.26)$$

where

$$A_n = \frac{|t|^3h^L}{n^{1/2}} + \left(\frac{|t|^5}{n^{3/2}}+\frac{t^4}{n}\right)h^{-\frac{2}{3}d-1} + \frac{t^2}{n^2h^{d+2}} + \frac{t^2h^L}{n^2h^{d+2}} + \frac{|t|^3}{n^{3/2}h^{\frac{4}{3}d+2}} + \frac{t^6}{n}$$

$$+ \frac{t^{10}}{n^3h^{\frac{4}{3}d+2}} + \frac{t^8}{n^2h^{\frac{4}{3}d+2}} + \frac{|t|^3}{(nh^{d+2})^{3/2}}$$

$$+ \frac{t^2}{n^2h^{d+2}} + \frac{|t|^3}{n^{3/2}h^{d+2}} + \frac{t^2h^L}{nh^{d+2}} + \frac{t^2}{n}$$

$$+ \frac{|t|^3+t^4}{n} + \frac{|t|^3}{n^{3/2}} + \frac{|t|^7}{n^3h^{d+2}} + \frac{t^6}{n^{5/2}h^{d+2}} + \frac{|t|^5+t^4}{n^2h^{d+2}}$$

$$= o\left(\frac{t^2+t^{10}}{nh^{d+2}} + \frac{t^2+t^6}{n^{1/2}} \right) .$$

Expanding (3.26), we have

$$E\{e^{it(B+s^{-1}\Delta)}\} = e^{-\frac{t^2}{2}}\Bigg[1+\left\{\Delta-\frac{4E(V_1^3)+8E(W_{12}V_1V_2)}{3n^{1/2}}\right\}(it) - \frac{E(W_{12}^2)}{n}(it)^2$$

$$-\frac{4\{2E(V_1^3)+3E(W_{12}V_1V_2)\}}{3n^{1/2}}(it)^3\Bigg] + D_n \ , \qquad (3.27)$$

where

$$D_n = O\Bigg(\left\{e^{-\frac{t^2}{2}}\frac{|t|^3}{n^{1/2}} + o(n^{-1/2}(t^6+|t|^3)e^{-\frac{t^2}{4}})\right\}\left\{\frac{t^2}{n^2h^{d+2}} + \frac{|t|^3+|t|}{n^{1/2}} + A_n\right\}$$

$$+ e^{-\frac{t^2}{2}}(|t|n^{1/2}h^L + t^2h^{2L} + |t|n^{1/2}h^{2L})\left(\frac{t^2}{n^2h^{d+2}} + \frac{|t|^3+|t|}{n^{1/2}} + A_n\right)$$

$$+ (|t|n^{1/2}h^L + t^2nh^{2L})\left\{e^{-\frac{t^2}{2}}\frac{|t|^3}{n^{1/2}} + o(n^{-1/2}(t^6+|t|^3)e^{-\frac{t^2}{4}})\right\}$$

$$+ (|t|n^{1/2}h^L + t^2nh^{2L})\left\{e^{-\frac{t^2}{2}}\frac{|t|^3}{n^{1/2}} + o(n^{-1/2}(t^6+|t|^3)e^{-\frac{t^2}{4}})\right\}$$

$$\times\left(\frac{t^2}{n^2h^{d+2}} + \frac{|t|^3+|t|}{n^{1/2}} + A_n\right)$$

$$+ \ (|t| + t^2 n^{1/2} h^L + |t|^3 n h^{2L}) E|\tilde{b}_2'' + \tilde{b}_3|$$

$$+ \ (t^2 + |t|^3 n^{1/2} h^L + t^4 n h^{2L})(E|\tilde{b}_2'|^2 + E|\tilde{b}_2' b_3|)\Big) \ . \tag{3.28}$$

By Hölder's inequality, equation (14) of Robinson (1995) and Lemmas 9-14,

$$E|\tilde{b}_2''| = E|Q\,\bar{V}| \le (E|Q|^2 E|\bar{V}|^2)^{1/2} = O(n^{-1} h^{-\frac{d+2}{2}}) \tag{3.29}$$

$$E|\tilde{b}_3| = E|(T+Q)\bar{W}| \le (E|T+Q|^2 E|\bar{W}|^2)^{1/2}$$

$$= O\big((n^{-1/2} + n^{-1} h^{-d-2})(n^{-1} h^{-d-2})^{1/2}\big) \ . \tag{3.30}$$

Writing $E|\tilde{b}_2'|^2 \le C(|T_1|^2 E|\bar{V}|^2 + E|T_2\bar{V}|^2 + E|T_3\bar{V}|^2)$, Lemmas 9, 10, 12 and Hölder's inequality give

$$|T_1|^2 E|\bar{V}|^2 + E|T_3\bar{V}|^2 \le |T_1|^2 E|\bar{V}|^2 + (E|T_3|^4 E|\bar{V}|^4)^{1/2} = O(n^{-2} h^{-2d-4}) \ ,$$

and (3.7), (i), (iii), Lemma 1-(d) of NR and (A.1) give

$$E|T_2\bar{V}|^2 \le \frac{C}{n^3} E\Big|\sum_{i=1}^{n} (4V_i^2 - s^2 + 8\tilde{V}_i)\sum_{j=1}^{n} V_j\Big|^2$$

$$= \frac{C}{n^3}\Big\{nE|(4V_1^2 - s^2 + 8\tilde{V}_1)V_1|^2 + n(n-1)E|(4V_1^2 - s^2 + 8\tilde{V}_1)V_2|^2\Big\}$$

$$= O(n^{-1}) \ .$$

Thus

$$E|\tilde{b}_2'|^2 = E|T\,\bar{V}|^2 = O(n^{-1} + n^{-2} h^{-2d-4}) \ . \tag{3.31}$$

Hölder's inequality, (3.31) and equation (14) of Robinson (1995) yield

$$E|\tilde{b}_2' b_3| = (E|\tilde{b}_2'|^2 E|b_3|^2)^{1/2} = O\big((n^{-1/2} + n^{-1} h^{-d-2})(n^{-1} h^{-d-2})^{1/2}\big) \ . \tag{3.32}$$

It is straightforward due to (C.1) of NR that $E(V_1^3) = E(v_1^3) + o(1)$ and $E(W_{12}V_1V_2) = E(W_{12}v_1v_2) + o(1)$ where $v_i = \sigma_\nu^{-1}\nu^\tau\{\mu(X_i, Y_i) - \mu.\}$. Therefore, using (3.27)-(3.32) and Lemmas 11-13 of NR,

$$(I) \le \int_{-logn}^{logn} \Big|\frac{E\,e^{it(B+s^{-1}\Delta)} - \chi^+(t)}{t}\Big| dt = o(n^{-1/2} + n^{-1} h^{-d-2} + n^{1/2} h^L) \ .$$

<u>Estimation of (II)</u>

Put $\tilde{b}_3' = T\bar{W}$, $\tilde{b}_3'' = Q\bar{W}$, then noting that $\tilde{b}_3 = \tilde{b}_3' + \tilde{b}_3''$ and $B = b_1 + \tilde{b}_2 +$

$(\tilde{b}_3^{/} + \tilde{b}_3^{//})$, we have, using (3.17),

$$|Ee^{itB}| \leq |Ee^{itB} - Ee^{it(b_1+\tilde{b}_2+\tilde{b}_3^{/})} - itE\tilde{b}_3^{//}e^{it(b_1+\tilde{b}_2+\tilde{b}_3)}| + |Ee^{it(b_1+\tilde{b}_2+\tilde{b}_3^{/})}|$$

$$+ |t||E\tilde{b}_3^{//}e^{it(b_1+\tilde{b}_2+\tilde{b}_3^{/})}|$$

$$\leq |t|^2 E|\tilde{b}_3^{//}|^2 + |Ee^{it(b_1+\tilde{b}_2+\tilde{b}_3^{/})}| + |t||E\tilde{b}_3^{//}e^{it(b_1+\tilde{b}_2+\tilde{b}_3^{/})}| \ . \qquad (3.33)$$

Writing $E|b_3^{//}|^2 \leq C(E|Q_1\overline{W}|^2 + E|Q_2\overline{W}|^2)$, Hölder's inequality, equation (14) of Robinson (1995) and Lemma 14 give

$$E|Q_2\overline{W}|^2 \leq (E|Q_2|^4)^{1/2}(E|\overline{W}|^4)^{1/2} = O\big((n^{-4}h^{-3d-4})^{1/2}n^{-1}h^{-d-2}\big) \qquad (3.34)$$

and

$$E|Q_1\overline{W}|^2 \leq \frac{C}{n^7}E|\sum_{i<j}\sum\{(V_i+V_j)W_{ij} - \tilde{V}_i - \tilde{V}_j\}\sum_{k<l}\sum W_{kl}|^2$$

$$\leq \frac{C}{n^7}E|\sum_{i<j<k<l}\sum\sum\sum\{(V_i+V_j)W_{ij} - \tilde{V}_i - \tilde{V}_j\}W_{kl}|^2$$

$$+ \frac{C}{n^7}E|\sum_{i<j<l}\sum\sum\{(V_i+V_j)W_{ij} - \tilde{V}_i - \tilde{V}_j\}W_{il}|^2$$

$$+ \frac{C}{n^7}E|\sum_{i<j}\sum\{(V_i+V_j)W_{ij} - \tilde{V}_i - \tilde{V}_j\}W_{ij}|^2 \ ,$$

$$\leq \frac{C}{n^7}\sum_{i<j<k<l}\sum\sum\sum E|\{(V_i+V_j)W_{ij} - \tilde{V}_i - \tilde{V}_j\}W_{kl}|^2$$

$$+ \frac{C}{n^7}\sum_{j>}\sum_{l\geq 2} n^2 E|\{(V_1+V_j)W_{1j} - \tilde{V}_1 - \tilde{V}_j\}W_{1l}|^2$$

$$+ \frac{C}{n^7}n^4 E|\{(V_1+V_2)W_{12} - \tilde{V}_1 - \tilde{V}_2\}W_{12}|^2$$

$$= O(n^{-3}h^{-3d-4}) \ , \qquad (3.35)$$

where the third inequality uses the Theorem of Dharmardhikari et.al.(1968; abbreviated to DFJ hereafter), and the equality uses nested conditional expectation, Lemmas 1-(d), 4 of NR, Lemma 4 of Robinson (1995) and Lemma 2. Therefore by (3.34) and (3.35),

$$E|\tilde{b}_3^{//}|^2 = E|Q\,\overline{W}|^2 = O(n^{-3}h^{-3d-4}) \ . \qquad (3.36)$$

To investigate the second term of (3.33), let

$$d_i = (4V_i^2 - s^2) + 8\tilde{V}_i \ , \ e_{ij} = 4\left\{(V_i+V_j)W_{ij} - \tilde{V}_i - \tilde{V}_j + \frac{n}{(n-2)}\tilde{W}_{ij}\right\} \ , \qquad (3.37)$$

then

$$\tilde{b}_2 = -\frac{s^{-3}}{2}\left\{\frac{4n}{(n-2)^2}E(W_{12}^2) + \frac{1}{n}\sum_{i=1}^{n} d_i + \binom{n}{2}^{-1}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} e_{ij}\right.$$

$$\left. - \frac{4}{n}\binom{n-1}{2}^{-1}\sum_{i=1}^{n}\ \sum_{k=1}^{n-1(i)}\ \sum_{l=k+1}^{n(i)} V_i W_{kl}\right\}\bar{V} \ ,$$

$$\tilde{b}_3^{/} = -\frac{s^{-3}}{2}\left\{\frac{4n}{(n-2)^2}E(W_{12}^2) + \frac{1}{n}\sum_{i=1}^{n} d_i + \binom{n}{2}^{-1}\sum_{j=1}^{n-1}\sum_{k=j+1}^{n} 4\tilde{W}_{jk}\right\}\bar{W} \ .$$

Define

$$b_{3m} = s^{-1} n^{\frac{1}{2}}\binom{n}{2}^{-1}\sum_{i=1}^{m}\sum_{j=i+1}^{n} W_{ij} \qquad ,$$

$$\tilde{b}_{2m} = -\frac{s^{-3}}{2}\left[\frac{8n^{1/2}}{(n-2)^2}E(W_{12}^2)\sum_{i=1}^{m} V_i + \frac{2}{n^{3/2}}\left(\sum_{i=1}^{n}\sum_{s=1}^{m} d_i V_s + \sum_{i=1}^{m}\sum_{s=m+1}^{n} d_i V_s\right)\right.$$

$$+ \frac{2}{n^{1/2}}\binom{n-1}{2}^{-1}\left(\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\sum_{s=1}^{m} e_{ij}V_s + \sum_{i=1}^{m}\sum_{j=i+1}^{n}\sum_{s=m+1}^{n} e_{ij}V_s\right)$$

$$- \frac{8}{n^{3/2}}\binom{n-1}{2}^{-1}\left(\sum_{i=1}^{n}\sum_{k<l}{}^{(i)}\sum_{s=1}^{m} V_i W_{kl} V_s + \sum_{i=1}^{n}\sum_{k=1}^{m}{}^{(i)}\sum_{l=k+1}^{n}{}^{(i)}\sum_{s=m+1}^{n} V_i W_{kl} V_s\right.$$

$$\left.\left. + \sum_{i=1}^{m}\ \sum_{k=m+1}^{n-1\,(i)}\ \sum_{l=k+1}^{n\,(i)}\ \sum_{s=m+1}^{n} V_i W_{kl} V_s\right)\right] \ ,$$

$$\tilde{b}_{3m}^{/} = -\frac{s^{-3}}{2}\left[\frac{4n^{3/2}}{(n-2)^2}E(W_{12}^2)\binom{n}{2}^{-1}\sum_{l=1}^{m}\sum_{s=l+1}^{n} W_{ls}\right.$$

$$+ \frac{1}{\sqrt{n}}\binom{n}{2}^{-1}\left(\sum_{i=1}^{m}\sum_{l=1}^{n-1}\sum_{s=l+1}^{n} d_i W_{ls} + \sum_{i=m+1}^{n}\sum_{l=1}^{m}\sum_{s=l+1}^{n} d_i W_{ls}\right)$$

$$+ n^{1/2}\binom{n}{2}^{-2}\left(\sum_{j=1}^{m}\sum_{k=j+1}^{n}\sum_{l=1}^{n-1}\sum_{s=l+1}^{n} 4\tilde{W}_{jk}W_{ls} + \sum_{j=m+1}^{n-1}\sum_{k=j+1}^{n}\sum_{l=1}^{m}\sum_{s=l+1}^{n} 4\tilde{W}_j\right.$$

for $m=1,\ldots,n-1$. Note that b_1-b_{3m}, $\tilde{b}_2-\tilde{b}_{2m}$ and $\tilde{b}_3^{/}-\tilde{b}_{3m}^{/}$ are independent of $(X_1^{\tau},Y_1),\cdots,(X_m^{\tau},Y_m)$. Putting $\bar{B}_m=(b_1-b_{3m})+(\tilde{b}_2-\tilde{b}_{2m})+(\tilde{b}_3^{/}-\tilde{b}_{3m}^{/})$, and using (3.17) repeatedly, we have

$$|Ee^{it(b_1+\tilde{b}_2+\tilde{b}_3^{/})}| \le \frac{t^2}{2}E|\tilde{b}_{2m}+\tilde{b}_{3m}^{/}|^2 + |Ee^{it(\bar{B}_m+b_{3m})}|$$

$$+|t||Ee^{it(\bar{B}_m+b_{3m})}(\tilde{b}_{2m}+\tilde{b}_{3m}^{/})|$$

$$\le \frac{t^2}{2}E|\tilde{b}_{2m}+\tilde{b}_{3m}^{/}|^2 + \left[\frac{|t|^3}{6}E|b_{3m}|^3 + |Ee^{it\bar{B}_m}\{1+itb_{3m}+\frac{(it)^2}{2}b_{3m}^2\}|\right]$$

$$+\left[t^2E|b_{3m}||\tilde{b}_{2m}+\tilde{b}_{3m}^{/}| + |t||Ee^{it\bar{B}_m}(\tilde{b}_{2m}+\tilde{b}_{3m}^{/})|\right]$$

$$= \left[\frac{t^2}{2}E|\tilde{b}_{2m}+\tilde{b}_{3m}^{/}|^2 + t^2E|b_{3m}||\tilde{b}_{2m}+\tilde{b}_{3m}^{/}|\right]$$

$$+\left[\frac{|t|^3}{6}E|b_{3m}|^3 + |Ee^{it\bar{B}_m}\{1+itb_{3m}+\frac{(it)^2}{2}b_{3m}^2\}|\right]$$

$$+\left[|t||Ee^{it\bar{B}_m}(\tilde{b}_{2m}+\tilde{b}_{3m}^{/})|\right] . \tag{3.38}$$

By elementary inequalities, (ix), Appendix B-(d), (e) and equation (14) of Robinson (1995), the first bracketed term is bounded by

$$Ct^2\left\{E|\tilde{b}_{2m}|^2+E|\tilde{b}_{3m}^{/}|^2+\left(E|b_{3m}|^2\right)^{1/2}\left(E|\tilde{b}_{2m}|^2+E|\tilde{b}_{3m}^{/}|^2\right)^{1/2}\right\}$$

$$\le Cmt^2\left\{\frac{1}{n^3h^{2d+4}}+\frac{1}{n^2}+\frac{1}{n^4h^{3d+6}}\right.$$

$$\left.+\frac{1}{(n^2h^{d+2})^{1/2}}\left(\frac{1}{(n^3h^{2d+4})^{1/2}}+\frac{1}{n}+\frac{1}{(n^4h^{3d+6})^{1/2}}\right)\right\}$$

$$\le Cmt^2\left(\frac{1}{n^2h^{\frac{d+2}{2}}}+\frac{1}{n^{\frac{5}{2}}h^{\frac{3d+6}{2}}}\right) . \tag{3.39}$$

The second bracketed term on the right of (3.38) is bounded by

$$C\left[|t|^3(\frac{m}{n^2h^{d+2}})^{3/2}+\left\{1+\frac{m|t|}{n^{1/2}h}+\frac{m|t|}{n^2h^{d+2}}+\frac{t^2m^2}{nh^2}\right\}|\gamma(t)|^{m-4}\right] , \tag{3.40}$$

which is verified as in equations (13)-(19) of Robinson (1995), because s^{-1} is bounded due to (3.7) and $\bar{B}_m$ is the sum of $\frac{2}{\sqrt{n}s}\sum_{i=1}^{n}V_i$ and $(b_3-b_{3m})+(\tilde{b}_2-\tilde{b}_{2m})+(\tilde{b}_3^{/}-\tilde{b}_{3m}^{/})$, the latter being independent of (X_1^{τ}, Y_1) , ... , (X_m^{τ}, Y_m) . Appendix B-(b), (c) bound the last term in (3.38) by

$$\frac{C m|t|}{n^{1/2}h^3}|\gamma(t)|^{m-4} \ . \tag{3.41}$$

Now we investigate the third term on the right of (3.33). Using elementary inequalities, (3.17), (3.36), equation (14) of Robinson (1995), Appendix B-(d), (e), (f),

$$\begin{aligned}
&|E\tilde{b}_3^{\prime\prime}e^{it(b_1+\tilde{b}_2+\tilde{b}_3^{\prime})}| \\
&\le |E\tilde{b}_3^{\prime\prime}e^{it(b_1+\tilde{b}_2+\tilde{b}_3^{\prime})} - E\tilde{b}_3^{\prime\prime}e^{it(b_1-b_{3m}+\tilde{b}_2-\tilde{b}_{2m}+\tilde{b}_3^{\prime}-\tilde{b}_{3m}^{\prime})}| + |E\tilde{b}_3^{\prime\prime}e^{it\bar{B}_m}| \\
&\le |t|E|\tilde{b}_3^{\prime\prime}||b_{3m}+\tilde{b}_{2m}+\tilde{b}_{3m}^{\prime}| + |E\tilde{b}_3^{\prime\prime}e^{it\bar{B}_m}| \\
&\le C|t|(E|\tilde{b}_3^{\prime\prime}|^2)^{1/2}\left[(E|b_{3m}|^2)^{1/2}+(E|\tilde{b}_{2m}|^2)^{1/2}+(E|\tilde{b}_{3m}^{\prime}|^2)^{1/2}\right] + |E\tilde{b}_3^{\prime\prime}e^{it\bar{B}_m}| \\
&\le \frac{C|t|h}{(nh^{d+2})^{3/2}}\left[(\frac{m}{n^2h^{d+2}})^{1/2}+(\frac{m}{n^3h^{2d+4}})^{1/2}+(\frac{m}{n^2})^{1/2}+(\frac{m}{n^4h^{3d+6}})^{1/2}\right] \\
&\qquad + \frac{Cn^{1/2}}{h^2}|\gamma(t)|^{m-5} \\
&\le \frac{C|t|hm^{1/2}}{n^{1/2}(nh^{d+2})^2} + \frac{Cn^{1/2}}{h^2}|\gamma(t)|^{m-5} \ .
\end{aligned} \tag{3.42}$$

Therefore, by (3.33), (3.36), (3.38)-(3.42),

$$\begin{aligned}
|Ee^{itB}| &\le \frac{C\,t^2}{n^3h^{3d+4}} + Cmt^2\left(\frac{1}{n^2h^{\frac{d+2}{2}}} + \frac{1}{n^{\frac{5}{2}}h^{\frac{3d+6}{2}}}\right) \\
&\quad + C\left[|t|^3(\frac{m}{n^2h^{d+2}})^{3/2} + \left\{1+\frac{m|t|}{n^{1/2}h}+\frac{m|t|}{n^2h^{d+2}}+\frac{t^2m^2}{nh^2}\right\}|\gamma(t)|^{m-4}\right] \\
&\quad + \frac{Cm|t|}{n^{1/2}h^3}|\gamma(t)|^{m-4} \\
&\quad + \frac{Chm^{1/2}t^2}{n^{1/2}(nh^{d+2})^2} + \frac{Cn^{1/2}|t|}{h^2}|\gamma(t)|^{m-5} \ .
\end{aligned} \tag{3.43}$$

Now divide (3.43) by $|t|$ and integrate over $p \le |t| \le N_0$, where we partition the range of integration into two parts, $p \le |t| \le N_1$ and $N_1 \le |t| \le N_0$, for $N_1 = \min(\eta n^{1/2}, nh^{d+2})$.

(i) $p \le |t| \le N_1$

We can choose $m = [9n\,logn/t^2]$ to satisfy $1 \le m \le n-1$ for large *n*. For this m , since $E(2V_1/s)=0$ and $Var(2V_1/s)=1$,

$$|\gamma(t)|^{m-4} \le \exp(-\frac{m-4}{3n}t^2) \le C\exp(-3\,logn) = \frac{C}{n^3} \ . \tag{3.44}$$

By (3.43), (3.44) and (ix), we obtain

$$\int_{p\le|t|\le N_1}\left|\frac{Ee^{itB}}{t}\right|dt$$
$$\le \frac{C}{n^3h^{3d+4}}\int_{p\le|t|\le nh^{d+2}}|t|dt$$
$$+ C\left(\frac{n\log n}{n^2h^{\frac{d+2}{2}}}+\frac{n\log n}{n^{\frac{5}{2}}h^{\frac{3d+6}{2}}}\right)\int_{p\le|t|\le\eta n^{1/2}}\frac{dt}{|t|}$$
$$+ C\left(\frac{n\log n}{n^2h^{d+2}}\right)^{3/2}\int_{p\le|t|\le\eta n^{1/2}}\frac{dt}{|t|}$$
$$+\frac{C}{n^3}\int_{p\le|t|\le\eta n^{1/2}}\left\{\frac{1}{|t|}+\frac{n\log n}{n^{1/2}ht^2}+\frac{(n\log n)^2}{nh^2|t|^3}\right\}dt$$
$$+ \frac{C\,n\log n}{n^{7/2}h^3}\int_{p\le|t|\le\eta n^{1/2}}\frac{dt}{t^2}$$
$$+ \frac{C\,h(n\log n)^{1/2}}{n^{1/2}(nh^{d+2})^2}\int_{p\le|t|\le nh^{d+2}}dt + \frac{C}{n^{5/2}h^2}\int_{p\le|t|\le\eta n^{1/2}}dt$$
$$= o\left(n^{-1/2}+n^{-1}h^{-d-2}\right)\ . \qquad (3.45)$$

(ii) $N_1\le|t|\le N_0$

For sufficiently large n, there exists $\xi>0$ such that $|\gamma(t)|<1-\xi$ by assumption (x). We may take $m=\left\lfloor-\frac{3\log n}{\log(1-\xi)}\right\rfloor$ to satisfy $1\le m\le n-1$ for sufficiently large n. Since $|\gamma(t)|^{m-4}\le Cn^{-3}$,

$$\int_{N_1\le|t|\le N_0}\left|\frac{Ee^{itB}}{t}\right|dt$$
$$\le \frac{C}{n^3h^{3d+4}}\int_{N_1\le|t|\le nh^{d+2}\log n}|t|dt$$
$$+ C\left(\frac{\log n}{n^2h^{\frac{d+2}{2}}}+\frac{\log n}{n^{\frac{5}{2}}h^{\frac{3d+6}{2}}}\right)\int_{N_1\le|t|\le n^{1/2}\log n}|t|dt$$
$$+ C\left(\frac{\log n}{n^2h^{d+2}}\right)^{3/2}\int_{N_1\le|t|\le n^{1/2}\log n}|t|^2dt$$
$$+ \frac{C}{n^3}\int_{N_1\le|t|\le n^{1/2}\log n}\left\{\frac{1}{|t|}+\frac{\log n}{n^{1/2}h}+\frac{(\log n)^2}{nh^2}|t|\right\}dt$$
$$+ \frac{C\log n}{n^{7/2}h^3}\int_{N_1\le|t|\le n^{1/2}\log n}dt$$

$$+ \frac{Ch(\log n)^{1/2}}{n^{1/2}(nh^{d+2})^2}\int_{N_1 \le |t| \le nh^{d+2}\log n} |t| dt$$

$$+ \frac{C}{n^{5/2}h^2}\int_{N_1 \le |t| \le n^{1/2}\log n} dt$$

$$= o\,(n^{-1/2} + n^{-1}h^{-d-2}) \tag{3.46}$$

by (ix). Therefore, by (3.45) and (3.46),

$$(II) = o\,(n^{-1/2} + n^{-1}h^{-d-2}) \ .$$

Estimation of (III).

$$(III) \le C\left[\int_p^\infty \frac{1}{t} e^{-\frac{t^2}{2}} dt + n^{1/2}h^L \int_p^\infty e^{-\frac{t^2}{2}} dt + \frac{1}{nh^{d+2}}\int_p^\infty t e^{-\frac{t^2}{2}} dt \right.$$

$$\left. + \frac{1}{n^{1/2}}\int_p^\infty (t + t^2) e^{-\frac{t^2}{2}} dt \right] . \tag{3.47}$$

The first integral in (3.47) is bounded by

$$\frac{1}{p^2}\int_p^\infty t e^{-\frac{t^2}{2}} dt = \frac{1}{p^2} e^{-\frac{p^2}{2}} = o(n^{-1}) \ ,$$

because $p = \min(\log n, \varepsilon n^{1/2})$. The remaining integrals are clearly $o(1)$ as $p \to \infty$.

Therefore,

$$(III) = o(n^{-1/2} + n^{-1}h^{-d-2} + n^{1/2}h^L) \ ,$$

to complete the proof.

Proof of Theorem B.

In view of the proof of Theorem 2 of NR, $\tilde{\kappa}_i \to \kappa_i$, $i = 1,2,3,4,$ a.s. Combine this with Theorem A. □

APPENDIX A

Lemma 1. Under assumptions (i), (iii), (iv), (v), (vi), (vii) and (viii),

$$E|V_1W_{12}|^r = E|V_2W_{12}|^r = O(h^{-(r-1)d-r}) \text{ for } 1 \le r \le 3 .$$

Proof. Using Lemmas 1-(d) and 4 of NR,

$$E|V_1W_{12}|^r \le E\{|V_1|^r E(|W_{12}|^r|1)\}$$

$$\le CE\{(|Y_1|^r+1)^2\} h^{-(r-1)d-r}$$

$$\le Ch^{-(r-1)d-r} \text{ for } 1 \le r \le 3 \text{ by (i).}$$

$E|V_1W_{12}|^r = E|V_2W_{12}|^r$ is obvious by the symmetry of W_{12} and (iii). □

Lemma 2. Under assumptions (i), (iii), (iv), (v), (vi), (vii) and (viii),

$$E|\tilde{V}_1|^r = O(1) \text{ for } 1 \le r \le 6 .$$

Proof. As in the proof of Lemma 3 of Robinson (1995),

$$|\tilde{V}_1|^r = |E(V_2W_{12}|1)|^r \le C(|Y_1|^r+1) \text{ a.s.} \tag{A.1}$$

so (i) immediately produces the conclusion. □

Lemma 3. Under assumptions (i), (iii), (iv), (v), (vi) and (viii),

(a) $E|\tilde{W}_{11}|^r = O(h^{-r(d+2)})$ for $1 \le r \le 3$,

(b) $E|\tilde{W}_{12}|^r = O(h^{-(r-1)d-2r})$ for $1 \le r \le 6$.

Proof.

(a). $\tilde{W}_{11} = E\left(W_{12}^2|1\right) \le C(|Y_1|^2+1)h^{-d-2}$ a.s. by Lemma 4 of NR so again application of (i) completes the proof.

(b). Apply Lemma 6 of NR. □

Lemma 4. Under assumptions (i), (iii), (iv), (v), (vi), (vii) and (viii), for d_1 given in (3.37),

(a) $E|d_1 V_2|^r = O(1)$ for $1 \le r \le 3$,

(b) $E|d_1 V_1|^r = O(1)$ for $1 \le r \le 2$.

Proof.

(a) By (iii), $E|d_1 V_2|^r = E|d_1|^r E|V_2|^r$, where the second factor is bounded due to Lemma 1-(d) of NR. From Lemma 1-(d) of NR and (A.1),

$$|d_1|^r \le C\left(|V_1^2|^r + |\tilde{V}_1|^r + 1 \right) \le C(|Y_1|^{2r} + 1) , \quad \text{(A.2)}$$

then apply (i). □

(b) By an elementary inequality and (3.7), $E|d_1 V_1|^r \le C(E|V_1^3|^r + E|\tilde{V}_1 V_1|^r + E|V_1|^r)$. By Lemma 1-(d) of NR and (A.1), $E|V_1^3|^r + E|V_1|^r = O(1)$ for $1 \le r \le 2$, and

$$E|\tilde{V}_1 V_1|^r \le CE(|Y_1|^r + 1)^2 = O(1) \quad \text{(A.3)}$$

for $1 \le r \le 3$ by (i). □

Lemma 5. Under assumptions (i), (iii), (iv), (v), (vi), (vii) and (viii),

(a) $E|W_{12} V_1 V_3|^r = E|W_{12} V_2 V_3|^r = O(h^{-(r-1)d-r})$ for $1 \le r \le 3$,

(b) $E|W_{12} V_1^2|^r = E|W_{12} V_2^2|^r = O(h^{-(r-1)d-r})$ for $1 \le r \le 2$.

Proof.

(a). Using (iii), Lemma 1-(d) of NR and Lemma 1, for $1 \le r \le 3$,

$$E|W_{12} V_1 V_3|^r = E|W_{12} V_1|^r E|V_3|^r = O(h^{-(r-1)d-r}) .$$

$E|W_{12} V_1 V_3|^r = E|W_{12} V_2 V_3|^r$ is straightforward by (iii) and symmetry of W_{12} .

(b). By Lemmas 1-(d) and 4 of NR, the left side is

$$E\left\{|V_1|^{2r}E(|W_{12}|^r|1)\right\} \le E\left\{C(|Y_1|^{3r}+1)h^{-(r-1)d-r}\right\} = O(h^{-(r-1)d-r}) \ .$$

$E|W_{12}V_1^2|^r = E|W_{12}V_2^2|^r$ is straightforward by (iii) and symmetry of W_{12} . □

Lemma 6. Under assumptions (i), (iii), (iv), (v), (vi), (vii) and (viii), with e_{12} given in (3.37),

(a) $E|e_{12}V_3|^r = O(h^{-(r-1)d-2r})$ for $1 \le r \le 3$,

(b) $E|e_{12}V_1|^r = E|e_{12}V_2|^r = O(h^{-(r-1)d-2r})$ for $1 \le r \le 2$.

Proof.

(a). By (iii) and Lemma 1-(d) of NR, write

$$E|e_{12}V_3|^r = E|e_{12}|^r E|V_3|^r \le C(E|V_1W_{12}|^r + E|\tilde{V}_1|^r + E|\tilde{W}_{12}|^r) \ .$$

Then apply Lemmas 1, 2 and 3-(b).

(b). An elementary inequality gives

$$E|e_{12}V_1|^r \le C\left(E|V_1V_2W_{12}|^r + E|V_1^2W_{12}|^r + E|\tilde{V}_1V_1|^r + E|\tilde{V}_1V_2|^r + E|\tilde{W}_{12}V_1|^r\right) \ . \quad \text{(A.4)}$$

Writing $E|V_1V_2W_{12}|^r = E\left\{|V_1|^r E(|V_2W_{12}|^r|1)\right\}$, the proof of Lemma 4 of NR applies to yield $E(|V_2W_{12}|^r|1) \le C(|Y_1|^r+1)h^{-(r-1)d-r}$ a.s. Thus, for $1 \le r \le 3$,

$$E|V_1V_2W_{12}|^r = O(h^{-(r-1)d-r}) \ . \quad \text{(A.5)}$$

The second term in (A.4) has the same order bound as (A.5) by Lemma 5-(b) for $1 \le r \le 2$. The third term in (A.4) is bounded due to (A.3), while the fourth term is bounded due to Lemma 1-(d) of NR and Lemma 2. We handle the last term in (A.4) similarly to Lemma 6 of NR :

$$E|\tilde{W}_{12}V_1|^r = E|E(W_{13}W_{23}|1,2)V_1|^r = O(h^{-(r-1)d-2r}) \ . \quad \text{(A.6)}$$

$E|e_{12}V_1|^r = E|e_{12}V_2|^r$ is straightforward by (iii) and symmetry of e_{12} . □

Lemma 7. Under assumptions (i), (iii), (iv), (v), (vi), (vii) and (viii),

(a) $E|d_1W_{23}|^r = O(h^{-(r-1)d-r})$ for $1 \le r \le 3$,

(b) $E|d_1W_{12}|^r = E|d_2W_{12}|^r = O(h^{-(r-1)d-r})$ for $1 \le r \le 2$.

Proof.

(a). Using (A.2) and Lemma 4 of Robinson (1995),

$$E|d_1W_{23}|^r = E|d_1|^r E|W_{23}|^r = O(h^{-(r-1)d-r})$$

for $1 \le r \le 3$.

(b). Using (A.2) and Lemma 4 of NR the left side is

$$E\left\{|d_1|^r E(|W_{12}|^r|1)\right\} \le E\left\{(|Y_1|^{2r}+1)\,C(|Y_1|^r+1)h^{-(r-1)d-r}\right\}$$
$$\le C E(|Y_1|^{3r}+1)h^{-(r-1)d-r} = O(h^{-(r-1)d-r})$$

for $1 \le r \le 2$ under (i). $E|d_1W_{12}|^r = E|d_2W_{12}|^r$ is straightforward by (iii) and symmetry of W_{12} .

□

Lemma 8. Under assumptions (i), (iii), (iv), (v), (vi), (vii) and (viii),

(a) $E|\tilde{W}_{12}W_{12}|^r = O(h^{-(2r-1)d-3r})$ for $1 \le r \le 3$,

(b) $E|\tilde{W}_{12}W_{13}|^r = O(h^{-2(r-1)d-3r})$ for $1 \le r \le 3$,

(c) $E|\tilde{W}_{12}W_{23}|^r = O(h^{-2(r-1)d-3r})$ for $1 \le r \le 3$,

(d) $E|\tilde{W}_{12}W_{34}|^r = O(h^{-2(r-1)d-3r})$ for $1 \le r \le 6$.

Proof.

(a). In view of the proof of Lemma 6 of NR,

$$E|\tilde{W}_{12}W_{12}|^r = E|\tilde{W}_{12}|^r|W_{12}|^r$$
$$\le h^{-r(d+2)} C E(1+|Y_1|^r+|Y_2|^r+|Y_1|^r|Y_2|^r)|W_{12}|^r$$
$$\le C h^{-r(d+2)}\left(E|W_{12}|^r + E|Y_1W_{12}|^r + E|Y_2W_{12}|^r + E|Y_1Y_2W_{12}|^r\right) .$$

The first term in parentheses is $O(h^{-(r-1)d-r})$ by Lemma 4 of Robinson (1995). From inspecting their proofs, Lemma 1 and (A.5) still hold with V_1 and V_2 replaced by Y_1 and Y_2 so that the other terms are $O(h^{-(r-1)d-r})$ for $1 \le r \le 3$.

(b). Using Lemma 4 of NR, for $1 \leq r \leq 6$,

$$E|\tilde{W}_{12}W_{13}|^r = E\left\{|\tilde{W}_{12}|^r E(|W_{13}|^r|1,2)\right\}$$
$$\leq E\left\{|\tilde{W}_{12}|^r C(|Y_1|^r + 1)h^{-(r-1)d-r}\right\}$$
$$= Ch^{-(r-1)d-r}\left(E|\tilde{W}_{12}Y_1|^r + E|\tilde{W}_{12}|^r\right) .$$

We may replace V_1 by Y_1 in (A.6), so that using also Lemma 3-(b),

$$E|\tilde{W}_{12}W_{13}|^r = O(h^{-2(r-1)d-3r}) \text{ for } 1 \leq r \leq 3 .$$

(c). The proof is as in (b).

(d). Writing $E|\tilde{W}_{12}W_{34}|^r = E|\tilde{W}_{12}|^r E|W_{12}|^r$ by (iii), the proof is straightforward by Lemma 4 of Robinson (1995) and Lemma 3-(b). □

Lemma 9. Under assumptions (i), (iii), (iv), (v), (vi), (vii) and (viii),

$$E|\bar{V}|^r = O(1) \text{ for } 2 \leq r \leq 6 .$$

Proof. Since V_i , i=1,...,n is an iid sequence, the result follows straightforwardly by DFJ and Lemma 1-(d) of NR. □

Lemma 10. Under assumptions (i), (v), (vi), (vii) and (viii)

$$|T_1|^r = O(n^{-r}h^{-r(d+2)}) \quad \text{for } r > 0 .$$

Proof. Using Lemma 4 of Robinson (1995) and $|\delta| < C$ due to (3.7),

$$|T_1|^r \leq \frac{C}{n^r}|E(W_{12}^2)|^r = O(n^{-r}h^{-r(d+2)}) .$$ □

Lemma 11. Under assumptions (i), (iii), (iv), (v), (vi), (vii) and (viii),

$$E|T_2|^r = O(n^{-\frac{r}{2}}) \text{ for } 2 \le r \le 3 \ .$$

Proof. Using (3.7), write

$$E|T_2|^r \le \frac{C}{n^r}E|\sum_{i=1}^{n}(4V_i^2 - s^2)|^r + \frac{C}{n^r}E|\sum_{i=1}^{n}8\tilde{V}_i|^r \ .$$

Since $E(4V_i^2) = s^2$ and $E(\tilde{V}_i) = 0$, by (iii) both the $4V_i^2 - s^2$ and $\tilde{V}_i$ are martingale differences and thus the theorem of DFJ applies to yield

$$E|\sum_{i=1}^{n}(4V_i^2 - s^2)|^r \le C n^{\frac{r}{2}} E|4V_1^2 - s^2|^r = O(n^{\frac{r}{2}})$$

for $2 \le r \le 3$ by (3.7) and Lemma 1-(d) of NR and

$$E|\sum_{i=1}^{n}8\tilde{V}_i|^r \le C n^{\frac{r}{2}} E|\tilde{V}_1|^r = O(n^{\frac{r}{2}})$$

by Lemma 2. □

Lemma 12. Under assumptions (i), (iii), (iv), (v), (vi) and (viii)

$$E|T_3|^r = O(n^{-r}h^{-(r-1)d-2r}) \text{ for } 2 \le r \le 6 \ .$$

Proof. Using (3.7), write

$$E|T_3|^r \le C n^{-2r} E\,|\sum_{k=1}^{n-1} Z_k\,|^r \tag{A.7}$$

where $Z_k = \sum_{m=k+1}^{n} \tilde{W}_{km}$, for k=1, ... , n-1. Since

$$E(\tilde{W}_{12}|2) = E(\tilde{W}_{12}|1) = E\left\{E(W_{13}W_{23}|1,2)|1\right\} = E(W_{13}W_{23}|1) = 0 \text{ a.s.},$$

Z_k , k = n-1, ... ,1 is a martingale difference sequence. Thus we apply DFJ to bound (A.7) by $C n^{-2r}(n-1)^{\frac{r}{2}-1}\sum_{k=1}^{n-1} E|Z_k|^r$. Since $E(\tilde{W}_{km}|m) = 0$ a.s. for m=k+1,...,n, the $\tilde{W}_{km}$ are martingale differences. We use DFJ again and get, by Lemma 3-(b),

$$E|\,Z_k|^r \le C(n-k)^{\frac{r}{2}-1}\sum_{m=k+1}^{n} E|\tilde{W}_{km}|^r \le C(n-k)^{\frac{r}{2}-1}(n-k)\,h^{-(r-1)d-2r} \ .$$ □

Lemma 13. Under assumptions (i), (iii), (iv), (v), (vi), (vii) and (viii),

$$E|Q_1|^r = O(n^{-r}h^{-(r-1)d-r}) \quad \text{for } 2 \le r \le 3 \ .$$

Proof. Write $P_{ij} = (V_i + V_j)W_{ij} - \tilde{V}_i - \tilde{V}_j$. Then $\sum_{j=i+1}^{n} P_{ij}$ is a martingale difference sequence for i=n-1,...,1. We can proceed by replacing $\tilde{W}_{km}$ in Lemma 12 by P_{ij} due to

the property $E(P_{ij}|j) = 0$ a.s. for $i \neq j$. Applying DFJ and (3.7),

$$E|Q_1|^r \leq C\binom{n}{2}^{-r} E\left|\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} P_{ij}\right|^r \leq C\binom{n}{2}^{-r} n^{\frac{r}{2}-1}\sum_{i=1}^{n-1} E|\sum_{j=i+1}^{n} P_{ij}|^r .$$

Since P_{ij} , $j = n, \ldots, i+1$ is a martingale difference for fixed i, we can apply the theorem of DJF again and obtain $E|\sum_{j=i+1}^{n} P_{ij}|^r \leq C(n-i)^{\frac{r}{2}-1}\sum_{j=i+1}^{n} E|P_{ij}|^r$. By Lemmas 1 and 2, $E|P_{ij}|^r \leq C[E|\tilde{V}_i|^r + E|V_i W_{ij}|^r] = O(h^{-(r-1)d-r})$ for $1 \leq r \leq 3$. □

Lemma 14. Under assumptions (i), (iii), (iv), (v), (vi), (vii) and (viii),

$$E|Q_2|^r = O(n^{-r}h^{-(r-1)d-r}) \quad \text{for} \quad 2 \leq r \leq 6 .$$

Proof. By an elementary inequality and (3.7),

$$E|Q_2|^r \leq \frac{C}{n^r}\binom{n-1}{2}^{-r} E|\sum_{i=1}^{n}\sum_{k=1}^{n-1\,(i)}\sum_{m=k+1}^{n\,(i)} V_i W_{km}|^r$$

$$= \frac{C}{n^r}\binom{n-1}{2}^{-r} n^{r-1}\sum_{i=1}^{n} E|\sum_{k=1}^{n-1\,(i)}\sum_{m=k+1}^{n\,(i)} V_i W_{km}|^r .$$

$V_i W_{km}$, m=k+1,...,n is a martingale difference for fixed i, k, $k \neq i$ and $m \neq i$, and $\sum_{m=k+1}^{n\,(i)} V_i W_{km}$, k=n-1,...,1 is also a martingale difference for fixed i and $k \neq i$ so that we apply DFJ repeatedly as in the proof of the previous Lemma and get

$$\sum_{i=1}^{n} E|\sum_{k=1}^{n-1\,(i)}\sum_{m=k+1}^{n\,(i)} V_i W_{km}|^r \leq C\sum_{i=1}^{n}(n-2)^{\frac{r}{2}-1}\sum_{k=1}^{n-1\,(i)} E|\sum_{m=k+1}^{n\,(i)} V_i W_{ij}|^r$$

$$\leq C(n-1)^{\frac{r}{2}-1}\sum_{i=1}^{n}\sum_{k=1}^{n-1\,(i)} E(n-k)^{\frac{r}{2}-1}\sum_{m=k+1}^{n\,(i)} E|V_i W_{km}|^r$$

$$\leq C n^{r+1} h^{-(r-1)d-r}$$

for $2 \leq r \leq 6$ by (iii), Lemma 1-(d) of NR and Lemma 4 of Robinson (1995). □

Lemma 15. Under assumptions (i), (iii), (iv), (v), (vi), (vii) and (viii),

$$E|R_1|^r = O(n^{-r}) \text{ for } 2 \le r \le 6 .$$

Proof. Writing $E|R_1|^r \le C\binom{n}{2}^{-r} E|\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} V_i V_j|^r$ due to (3.7), as in Lemma 12 or 13, $V_i V_j$, i=1,..., j-1 is a martingale difference sequence for fixed j as well as $\sum_{j=i+1}^{n} V_i V_j$, i= n-1,...,1. We use DFJ repeatedly again and (i), (iii) and Lemma 1-(d) of NR to obtain

$$E|\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} V_i V_j|^r \le C(n-1)^{\frac{r}{2}-1}\sum_{i=1}^{n-1} E|\sum_{j=i+1}^{n} V_i V_j|^r$$

$$\le C(n-1)^{\frac{r}{2}-1}\sum_{i=1}^{n-1}(n-i)^{\frac{r}{2}-1}\sum_{j=i+1}^{n} E|V_i V_j|^r = O(n^r) . \qquad \square$$

Lemma 16. Under assumptions (i), (iii), (iv), (v), (vi) and (viii),

$$E|R_2|^r = O(n^{-3(r-1)}h^{-2(r-1)d-2r}) \text{ for } 2 \le r \le 3 .$$

Proof. Using (3.7), write

$$E|R_2|^r = \frac{C}{n^r}\binom{n-1}{2}^{-r} E|\sum_{i=1}^{n}\sum_{k=1}^{n-1}{}^{(i)}\sum_{m=k+1}^{n}{}^{(i)}(W_{ik}W_{im} - \tilde{W}_{km})|^r .$$

Since R_2 has the same martingale structure as Q_2 , the same method of proof as in Lemma 14 applies. The difference is in the moment bounds of the two summands, i.e.

$$E|V_i W_{km}|^r = O(h^{-(r-1)d-r}) \text{ for } 1 \le r \le 6$$

and

$$E|W_{ik}W_{im} - \tilde{W}_{km}|^r = O(h^{-2(r-1)d-2r}) , \quad i \ne k \ne m$$

by Lemmas 1-(d), 4 of NR and Lemma 3-(b). $\square$

Lemma 17. Under assumptions (i), (iii), (iv), (v), (vi) and (viii)

$$E|R_3|^r = O(n^{-2r}h^{-(2r-1)d-2r}) \text{ for } 2 \le r \le 3 .$$

Proof. Write $E|R_3|^r \le \frac{C}{n^r}\binom{n}{2}^{-r} E|\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\{W_{ij}^2 - \tilde{W}_{ii} - \tilde{W}_{jj} + E(W_{12}^2)\}|^r$ using (3.7). Since

$$E\{W_{ij}^2-\tilde{W}_{ii}-\tilde{W}_{jj}+E(W_{12}^2)|\,j\}=E\{W_{ij}^2-\tilde{W}_{ii}-\tilde{W}_{jj}+E(W_{12}^2)|\,i\}=0$$

for $j>i$, R_3 has the same martingale structure as T_3 . Therefore, we apply DFJ to obtain

$$E|\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\{W_{ij}^2-\tilde{W}_{ii}-\tilde{W}_{jj}+E(W_{12}^2)\}|^r$$

$$\leq C(n-1)^{\frac{r}{2}-1}\sum_{i=1}^{n-1}E|\sum_{j=i+1}^{n}\{W_{ij}^2-\tilde{W}_{ii}-\tilde{W}_{ii}E(W_{12}^2)\}|^r$$

$$\leq C(n-1)^{\frac{r}{2}-1}\sum_{i=1}^{n-1}(n-i)^{\frac{r}{2}-1}\sum_{j=i+1}^{n}E|\{W_{ij}^2-\tilde{W}_{ii}-\tilde{W}_{jj}+E(W_{12}^2)\}|^r$$

$$=O(n^r h^{-(2r-1)d-2r})$$

by Lemma 4 of Robinson (1995) and Lemma 3-(a). □

Lemma 18. Under assumptions (i), (iv), (v), (vi) and (viii)

$$E|R_4|^r=O(n^{-\frac{3}{2}r}h^{-r(d+2)})\quad\text{for } 1\leq r\leq 3 .$$

Proof. Write $E|R_4|^r\leq\frac{C}{n^{2r}}E|\sum_{i=1}^{n}\{\tilde{W}_{ii}-E(W_{12}^2)\}|^r$ using (3.7). Since $\tilde{W}_{ii}-E(W_{12}^2)$ is a martingale difference, by (iii), DFJ and Lemma 3-(a),

$$E|\sum_{i=1}^{n}\{\tilde{W}_{ii}-E(W_{12}^2)\}|^r\leq Cn^{\frac{r}{2}-1}\sum_{i=1}^{n}E|\tilde{W}_{ii}-E(W_{12}^2)|^r=O(n^{\frac{r}{2}}h^{-r(d+2)}) .\quad □$$

Lemma 19. Under assumptions (i), (iii), (iv), (v), (vi), (vii) and (viii),

$$E|R_5|^r=O(n^{-2r}h^{-r(d+2)})\quad\text{for } 1\leq r\leq 3 .$$

Proof. Using (3.7), DFJ and Lemma 6 of Robinson(1995),

$$E|R_5|^r\leq\frac{C}{n^{4r}}E\left|\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}W_{ij}\right|^{2r}\leq\frac{C}{n^{4r}}(n-1)^{r-1}\sum_{i=1}^{n-1}E|\sum_{j=i+1}^{n}W_{ij}|^{2r}$$

$$=O(n^{-2r}h^{-r(d+2)}) .\quad □$$

APPENDIX B

Here, we present some of the derivations used in the proof of Theorem A, namely:

(a) $E(\tilde{b}_2^{/} e^{itb_2})$

$$= -\{\Upsilon(t)\}^{n-1}\left\{ it\frac{2}{n}E(W_{12}^2) + O\left(\frac{|t|}{n^2h^{d+2}} + \frac{t^2}{n^{3/2}h^{d+2}} + \frac{|t|h^L}{nh^{d+2}}\right)\right\}$$

$$- \{\Upsilon(t)\}^{n-1}\left\{\frac{4E(V_1^3) + 8E(W_{12}V_1V_2)}{n^{1/2}} + O\left(\frac{|t|}{n}\right)\right\}$$

$$- \{\Upsilon(t)\}^{n-2}\left[\frac{(it)^2}{n^{1/2}}\{4E(V_1^3) + 8E(W_{12}V_1V_2)\} + O\left(\frac{t^2+|t|^3}{n} + \frac{t^4}{n^{3/2}}\right)\right]$$

$$+ \{\Upsilon(t)\}^{n-3}\left[O\left(\frac{|t|^3+|t|}{n} + \frac{t^2}{n^{3/2}h^{d+2}} + \frac{t^6}{n^3h^{d+2}} + \frac{|t|^5}{n^{5/2}h^{d+2}} + \frac{t^4+|t|^3}{n^2h^{d+2}}\right)\right] ,$$

(b) $|E(\tilde{b}_{2m} e^{it\bar{B}_m})| \le \dfrac{Cm}{n^{1/2}h^2}|\Upsilon(t)|^{m-4}$,

(c) $|E(\tilde{b}_{3m}^{/} e^{it\bar{B}_m})| \le \dfrac{Cm}{n^{1/2}h^3}|\Upsilon(t)|^{m-4}$,

(d) $E|\tilde{b}_{2m}|^2 \le C\,m\left(\dfrac{1}{n^3h^{2d+4}} + \dfrac{1}{n^2}\right)$,

(e) $E|\tilde{b}_{3m}^{/}|^2 \le \dfrac{C\,m}{n^4h^{3d+6}}$,

(f) $|E\tilde{b}_3^{//} e^{it\bar{B}_m}| \le \dfrac{C\,n^{1/2}}{h^2}|\Upsilon(t)|^{m-5}$.

for $1 \le m \le n-1$.

<u>Proof.</u>

(a) Write

$$E(\tilde{b}_2^{/} e^{itb_2}) = E(T\bar{V}e^{itb_2})$$

$$= E\left(T_1\frac{2}{\sqrt{n}}\sum_{j=1}^{n}V_je^{itb_2}\right) + E\left(T_2\frac{2}{\sqrt{n}}\sum_{j=1}^{n}V_je^{itb_2}\right) + E\left(T_3\frac{2}{\sqrt{n}}\sum_{j=1}^{n}V_je^{itb_2}\right)$$

$$= (A) + (B) + (C) \qquad . \tag{B.1}$$

Thus

$$(A) = -\frac{4n^{1/2}}{(n-2)^2s^3}E(W_{12}^2)\sum_{j=1}^{n}E(V_je^{itb_2}) . \tag{B.2}$$

Due to (iii), (3.17), $\gamma(t) = E(e^{it\frac{2}{\sqrt{n}s}V_1})$, $4E(V_1^2) = s^2$, (3.7) and Lemma 1-(d) of NR,

$$E(V_j e^{itb_2}) = E(V_j e^{it\frac{2}{\sqrt{n}s}V_j})\,E(e^{it\frac{2}{\sqrt{n}s}\sum_{k\neq j}^{n} V_k})$$

$$= \left[E\left\{V_j(e^{it\frac{2}{\sqrt{n}s}V_j} - 1 - it\frac{2V_j}{n^{1/2}s})\right\} + (it)E(\frac{2V_j^2}{n^{1/2}s})\right]\{\gamma(t)\}^{n-1}$$

$$= \{\gamma(t)\}^{n-1}\left\{\frac{its}{2n^{1/2}} + O(\frac{t^2}{n})\right\}$$

$$= \{\gamma(t)\}^{n-1}\left\{\frac{it}{2n^{1/2}} + O(\frac{t^2}{n} + \frac{|t|h^L}{n^{1/2}})\right\} \tag{B.3}$$

Substituting (B.3) into (B.2),

$$(A) = -\{\gamma(t)\}^{n-1}\frac{4n^{3/2}}{(n-2)^2 s^3}E(W_{12}^2)\left\{\frac{it}{2n^{1/2}} + O(\frac{t^2}{n} + \frac{|t|h^L}{n^{1/2}})\right\}$$

$$= -\frac{\{\gamma(t)\}^{n-1}}{s^3}\left\{\frac{2it}{n}E(W_{12}^2) + O(\frac{|t|}{n^2h^{d+2}} + \frac{t^2}{n^{3/2}h^{d+2}} + \frac{|t|h^L}{nh^{d+2}})\right\} . \tag{B.4}$$

Now write $(B) = (B)^{/} + (B)^{//}$, where

$$(B)^{/} = -\frac{1}{n^{3/2}s^3}\sum_{j=1}^{n}E(4V_j^2 - s^2 + 8\tilde{V}_j)V_j e^{itb_2} , \tag{B.5}$$

$$(B)^{//} = -\frac{1}{n^{3/2}s^3}\sum_{j=1}^{n}\sum_{k\neq j}^{n}E(4V_j - s^2 + 8\tilde{V}_j)V_k e^{itb_2} . \tag{B.6}$$

The summand of $(B)^{/}$ is, using (3.17) and Lemma 1-(d) of NR,

$$E\{(4V_1^2 - s^2 + 8\tilde{V}_1)V_1 e^{it\frac{2V_1}{\sqrt{n}s}}\}E(e^{it\frac{2}{\sqrt{n}s}\sum_{l\neq 1}V_l})$$

$$= \{\gamma(t)\}^{n-1}E\{(4V_1^2 - s^2 + 8\tilde{V}_1)V_1 e^{it\frac{2V_1}{\sqrt{n}s}}\}$$

$$= \{\gamma(t)\}^{n-1}\left\{4E(V_1^3) + 8E(W_{12}V_1V_2) + O(\frac{|t|}{n^{1/2}})\right\} . \tag{B.7}$$

Substituting (B.7) into (B.5),

$$(B)^{/} = -\frac{1}{n^{1/2}s^3}\{\gamma(t)\}^{n-1}\left\{4E(V_1^3) + 8E(W_{12}V_1V_2) + O(\frac{|t|}{n^{1/2}})\right\} . \tag{B.8}$$

For $j\neq k$, the summand of $(B)^{//}$ is, due to (iii),

$$E\{(4V_1^2 - s^2 + 8\tilde{V}_1)V_2 e^{it\frac{2(V_1+V_2)}{\sqrt{n}s}}\}E(e^{it\frac{2}{\sqrt{n}s}\sum_{l\neq 1,2}V_l})$$

$$= \{\gamma(t)\}^{n-2}E\{(4V_1^2 - s^2) + 8\tilde{V}_1)V_2 e^{it\frac{2(V_1+V_2)}{\sqrt{n}s}}\}$$

$$= \{\gamma(t)\}^{n-2}E\{(4V_1^2 - s^2) + 8\tilde{V}_1)e^{it\frac{2V_1}{\sqrt{n}s}}\}E(V_2 e^{it\frac{2V_2}{\sqrt{n}s}})$$

$$= \{\gamma(t)\}^{n-2}\left[E\left\{(4V_1^2-s^2+8\tilde{V}_1)\left(e^{it\frac{2V_1}{\sqrt{n}s}}-1-it\frac{V_1}{\sqrt{n}s}\right)\right\}+itE\frac{V_1}{\sqrt{n}s}(4V_1^2-s^2+8\tilde{V}_1)\right]$$

$$\times\left[E\left\{V_2\left(e^{it\frac{2V_2}{\sqrt{n}s}}-1-it\frac{V_2}{\sqrt{n}s}\right)\right\}+itE\frac{V_2^2}{\sqrt{n}s}\right]$$

$$= \{\gamma(t)\}^{n-2}\left[itE\frac{V_1}{\sqrt{n}s}(4V_1^2-s^2+8\tilde{V}_1)+O\left(\frac{|t|^2}{n}\right)\right]\left[it\frac{E(V_2^2)}{\sqrt{n}s}+O\left(\frac{|t|^2}{n}\right)\right]$$

$$= \{\gamma(t)\}^{n-2}\left[\frac{it}{\sqrt{n}s}\{E(4V_1^3)+8E(W_{12}V_1V_2)\}+O\left(\frac{|t|^2}{n}\right)\right]\left[\frac{its}{\sqrt{n}}+O\left(\frac{|t|^2}{n}\right)\right]$$

$$= \{\gamma(t)\}^{n-2}\left[\frac{(it)^2}{n}\{E(4V_1^3)+8E(W_{12}V_1V_2)\}+O\left(\frac{|t|^3}{n^{3/2}}+\frac{|t|^4}{n^2}\right)\right] \qquad \text{(B.9)}$$

by (3.17). Therefore, substituting (B.9) into (B.6) yields

$$(B)'' = -\frac{n(n-1)}{n^{3/2}s^3}\left[\frac{(it)^2}{n}\{4E(V_1^3)+8E(W_{12}V_1V_2)\}+O\left(\frac{|t|^3}{n^{3/2}}+\frac{|t|^4}{n^2}\right)\right]\{\gamma(t)\}^{n-2}. \qquad \text{(B.10)}$$

By (B.8) and (B.10),

$$(B) = -\frac{\{\gamma(t)\}^{n-1}}{s^3}\left\{\frac{4E(V_1^3)+8E(W_{12}V_1V_2)}{n^{1/2}}+O\left(\frac{|t|}{n}\right)\right\}$$

$$-\frac{\{\gamma(t)\}^{n-2}}{s^3}\left[\frac{(it)^2}{n^{1/2}}\{4E(V_1^3)+8E(W_{12}V_1V_2)\}\right.$$

$$\left.+O\left(\frac{t^2+|t|^3}{n}+\frac{t^4}{n^{3/2}}\right)\right] \qquad \text{(B.11)}$$

Now, write

$$(C) = -\frac{4}{n^{1/2}s^3}E\left\{\binom{n-1}{2}^{-1}\sum_{1\le j<k\le n}\tilde{W}_{jk}\right\}\frac{1}{\sqrt{n}}\sum_{l=1}^{n}V_l e^{itb_2}$$

$$= -\frac{4}{n^{1/2}s^3}\binom{n-1}{2}^{-1}\sum_{l=1}^{n}\sum_{j<k}{}^{(l)}E(\tilde{W}_{jk}V_l e^{itb_2})$$

$$-\frac{4}{n^{1/2}s^3}\binom{n-1}{2}^{-1}\sum_{j=1}^{n-1}\sum_{k=j+1}^{n}E(\tilde{W}_{jk}V_j e^{itb_2})$$

$$-\frac{4}{n^{1/2}s^3}\binom{n-1}{2}^{-1}\sum_{j=1}^{n-1}\sum_{k=j+1}^{n}E(\tilde{W}_{jk}V_k e^{itb_2})$$

$$= (C)' + (C)'' + (C)'''.$$

Using (iii), Lemma 1-(d) of NR, $E(V_j)=0$ and $E(\tilde{W}_{km})=E(\tilde{W}_{km}|k)=E(\tilde{W}_{km}|m)=0$, the summand of $(C)'$ is

$$E\left\{\tilde{W}_{12}V_3 e^{it\frac{2}{\sqrt{n}s}(V_1+V_2+V_3)}\right\}E(e^{it\frac{2}{\sqrt{n}s}\sum_{l\neq 1,2,3}^{n}V_l}) = E\left\{\tilde{W}_{12}V_3 e^{it\frac{2}{\sqrt{n}s}(V_1+V_2+V_3)}\right\}\{\gamma(t)\}^{n-3}$$

$$= \left\{E\,\tilde{W}_{12}(e^{it\frac{2V_1}{\sqrt{n}s}}-1-it\frac{2V_1}{\sqrt{n}s})(e^{it\frac{2V_2}{\sqrt{n}s}}-1-it\frac{2V_2}{\sqrt{n}s})\right.$$

$$+(it)^2\frac{4}{ns^2}E(\tilde{W}_{12}V_1V_2)+(it)\frac{2}{\sqrt{n}s}E\{\tilde{W}_{12}V_1(e^{it\frac{2V_2}{\sqrt{n}s}}-1-it\frac{2V_2}{\sqrt{n}s})\}$$

$$\left.+(it)\frac{2}{\sqrt{n}s}E\{\tilde{W}_{12}V_2(e^{it\frac{2V_1}{\sqrt{n}s}}-1-it\frac{2V_1}{\sqrt{n}s})\}\right\}$$

$$\times\left\{E\{V_3(e^{it\frac{2V_3}{\sqrt{n}s}}-1-it\frac{2V_3}{\sqrt{n}s})\}+(it)\frac{2}{\sqrt{n}s}E(V_3^2)\right\}\{\gamma(t)\}^{n-3}$$

$$=\left\{(it)^2\frac{4}{ns^2}E(\tilde{W}_{12}V_1V_2)+O(\frac{t^4}{n^2h^{d+2}}+\frac{|t|^3}{n^{3/2}h^{d+2}})\right\}$$

$$\times\left\{(it)\frac{2}{\sqrt{n}s}E(V_3^2)+O(\frac{t^2}{n})\right\}\{\gamma(t)\}^{n-3}\ .$$

The last equality uses (3.17) and

$$E|\tilde{W}_{12}V_1^2V_2^2| \le \{E|\tilde{W}_{12}|^3\}^{1/2}(E|V_1V_2|^3)^{2/3} \le C\,h^{-\frac{2}{3}d-2} = O(h^{-d-2})$$

due to Hölder's inequality, Lemma 3-(b), (i), (iii) and Lemma 1-(d) of NR.

$$(C)' = -\frac{\{\gamma(t)\}^{n-3}}{s^3}\frac{8n(n-1)(n-2)}{n^{5/2}}$$

$$\times\left\{\frac{8(it)^3}{n^{3/2}s^3}E(\tilde{W}_{12}V_1V_2)s^2+O(\frac{t^6}{n^3h^{d+2}}+\frac{|t|^5}{n^{5/2}h^{d+2}}+\frac{t^4}{n^2h^{d+2}})\right\}$$

$$=\frac{\{\gamma(t)\}^{n-3}}{s^3}\left[O\left(\frac{|t|^3}{n}+\frac{t^6}{n^3h^{d+2}}+\frac{|t|^5}{n^{5/2}h^{d+2}}+\frac{t^4+|t|^3}{n^2h^{d+2}}\right)\right]. \tag{B.12}$$

Here we use, due to (iii) and Lemma 2,

$$E(\tilde{W}_{12}V_1V_2)=E(W_{13}W_{23}V_1V_2)=E[E(W_{13}V_1|3)E(W_{23}V_2|3)]=E(\tilde{V}_3^{\,2})=O(1)\,. \tag{B.13}$$

The summand of (C)'' can be expressed as follows using (iii), $E(\tilde{W}_{12}V_1e^{it\frac{2}{\sqrt{n}s}V_1})=0$, Lemma 3-(b) and (3.17).

$$E(\tilde{W}_{jk}V_j e^{itb_2}) = \{\gamma(t)\}^{n-2}E(\tilde{W}_{12}V_1 e^{it\frac{2}{\sqrt{n}s}(V_1+V_2)})$$

$$= \{\gamma(t)\}^{n-2}E\left\{\tilde{W}_{12}V_1 e^{it\frac{2V_1}{\sqrt{n}s}}(e^{it\frac{2V_2}{\sqrt{n}s}}-1-it\frac{2V_2}{\sqrt{n}s}) + \frac{2it}{\sqrt{n}s}\tilde{W}_{12}V_1V_2\right\}$$

$$= \{\gamma(t)\}^{n-2}\left\{\frac{2it}{\sqrt{n}s}E(\tilde{W}_{12}V_1V_2) + O(\frac{|t|^2}{nh^{d+2}})\right\} .$$

Thus, using (3.7) and (B.13),

$$(C)'' = \frac{\{\gamma(t)\}^{n-2}}{s^3}\left| -\frac{8n(n-1)}{n^{5/2}}\left\{\frac{2it}{\sqrt{n}s}E(\tilde{W}_{12}V_1V_2) + O(\frac{t^2}{nh^{d+2}})\right\}\right|$$

$$= \frac{\{\gamma(t)\}^{n-2}}{s^3}\,O(\frac{|t|}{n} + \frac{t^2}{n^{3/2}h^{d+2}}) . \tag{B.14}$$

Similarly,

$$(C)''' = \frac{\{\gamma(t)\}^{n-2}}{s^3}\,O(\frac{|t|}{n} + \frac{t^2}{n^{3/2}h^{d+2}}) . \tag{B.15}$$

By (B.12), (B.14) and (B.15),

$$(C) = (C)' + (C)'' + (C)'''$$

$$= \frac{\{\gamma(t)\}^{n-3}}{s^3}\left[O\left(\frac{|t|^3+|t|}{n} + \frac{t^2}{n^{3/2}h^{d+2}} + \frac{t^6}{n^3h^{d+2}} + \frac{|t|^5}{n^{5/2}h^{d+2}} + \frac{t^4+|t|^3}{n^2h^{d+2}}\right)\right] . \tag{B.16}$$

Therefore, by (3.7), (B.1), (B.4), (B.11) and (B.16),

$$E(\tilde{b}_2' e^{itb_2}) = (A)+(B)+(C)$$

$$= -\{\gamma(t)\}^{n-1}\left\{it\frac{2}{n}E(W_{12}^2) + O(\frac{|t|}{n^2h^{d+2}} + \frac{t^2}{n^{3/2}h^{d+2}} + \frac{|t|h^L}{nh^{d+2}})\right\}$$

$$- \{\gamma(t)\}^{n-1}\left\{\frac{4E(V_1^3)+8E(W_{12}V_1V_2)}{n^{1/2}} + O(\frac{|t|}{n})\right\}$$

$$- \{\gamma(t)\}^{n-2}\left[\frac{(it)^2}{n^{1/2}}\{4E(V_1^3)+8E(W_{12}V_1V_2)\} + O(\frac{t^2+|t|^3}{n} + \frac{t^4}{n^{3/2}})\right]$$

$$+ \{\gamma(t)\}^{n-3}\left[O\left(\frac{|t|^3+|t|}{n} + \frac{t^2}{n^{3/2}h^{d+2}} + \frac{t^6}{n^3h^{d+2}} + \frac{|t|^5}{n^{5/2}h^{d+2}} + \frac{t^4+|t|^3}{n^2h^{d+2}}\right)\right] .$$

(b) Writing, using (3.7) and Lemma 4 of Robinson (1995),

$$|E(\tilde{b}_{2m}e^{it\bar{B}_m})| \le \frac{C}{n^{3/2}h^{d+2}}\sum_{j=1}^{m}|E(V_j e^{it\bar{B}_m})|$$

$$+ \frac{C}{n^{3/2}}\left\{\sum_{j=1}^{n}\sum_{k=1}^{m}|E(d_jV_ke^{it\bar{B}_m})| + \sum_{j=1}^{m}\sum_{k=m+1}^{n}|E(d_jV_ke^{it\bar{B}_m})|\right\}$$

$$+ \frac{C}{n^{5/2}}\left\{\sum_{j=1}^{n-1}\sum_{k=j+1}^{n}\sum_{s=1}^{m}|E(e_{jk}V_se^{it\bar{B}_m})| + \sum_{j=1}^{m}\sum_{k=j+1}^{n}\sum_{s=m+1}^{n}|E(e_{jk}V_se^{it\bar{B}_m})|\right\}$$

$$+ \frac{C}{n^{7/2}}\left[\sum_{j=1}^{n}\sum_{k<l}^{n\,(j)}\sum_{s=1}^{m}|E(V_jW_{kl}V_se^{it\bar{B}_m})|\right.$$

$$+ \sum_{j=1}^{n}\sum_{k=1}^{m\,(j)}\sum_{l=k+1}^{n\,(j)}\sum_{s=m+1}^{n}|E(V_jW_{kl}V_se^{it\bar{B}_m})|$$

$$\left.+ \sum_{j=1}^{m}\sum_{k=m+1}^{n-1\,(j)}\sum_{l=k+1}^{n\,(j)}\sum_{s=m+1}^{n}|E(V_jW_{kl}V_se^{it\bar{B}_m})|\right], \tag{B.17}$$

$$|E(V_je^{it\bar{B}_m})| = |E(V_je^{it\frac{2}{\sqrt{n}s}V_j})E\{e^{it(\frac{2}{\sqrt{n}s}\sum_{k\neq j}^{n}V_k + b_3 - b_{3m} + \tilde{b}_2 - \tilde{b}_{2m} + \tilde{b}_3' - \tilde{b}_{3m}')}\}|$$

$$\leq E|V_j||\gamma(t)|^{m-1} \tag{B.18}$$

for $j=1,\cdots,m$, since $b_3 - b_{3m} + \tilde{b}_2' - b_{2m}' + \tilde{b}_3' - \tilde{b}_{3m}'$ is independent of $V_1,\cdots,V_m$.

For $j\leq m$, $k\leq m$ and $j\neq k$,

$$|E(d_jV_ke^{it\bar{B}_m})|$$

$$= |E[d_jV_ke^{it\{\frac{2}{\sqrt{n}s}(V_j+V_k+\sum_{l=m+1}^{n}V_l)+b_3-b_{3m}+\tilde{b}_2-\tilde{b}_{2m}+\tilde{b}_3'-\tilde{b}_{3m}'\}}]||E\{e^{it\frac{2}{\sqrt{n}s}\sum_{l\neq j,k}^{m}V_l}\}|$$

$$\leq E|d_jV_k||\gamma(t)|^{m-2}. \tag{B.19}$$

For $j=k\leq m$,

$$|E(d_jV_je^{it\bar{B}_m})|$$

$$= |E[d_jV_je^{it\{\frac{2}{\sqrt{n}s}(V_j+\sum_{k=m+1}^{n}V_k)+b_3-b_{3m}+\tilde{b}_2-\tilde{b}_{2m}+\tilde{b}_3'-\tilde{b}_{3m}'\}}]||E\{e^{it\frac{2}{\sqrt{n}s}\sum_{k\neq j}^{m}V_k}\}|$$

$$\leq E|d_jV_j||\gamma(t)|^{m-1} \leq E|d_jV_j||\gamma(t)|^{m-2}. \tag{B.20}$$

For $j\leq m$ and $k\geq m+1$,

$$|E(d_jV_ke^{it\bar{B}_m})|$$

$$= |E[d_jV_ke^{it\{\frac{2}{\sqrt{n}s}(V_j+\sum_{l=m+1}^{n}V_l)+b_3-b_{3m}+\tilde{b}_2-\tilde{b}_{2m}+\tilde{b}_3'-\tilde{b}_{3m}'\}}]||E\{e^{it\frac{2}{\sqrt{n}s}\sum_{l\neq j}^{m}V_l}\}|$$

$$\leq E|d_jV_k||\gamma(t)|^{m-1} \leq E|d_jV_k||\gamma(t)|^{m-2}. \tag{B.21}$$

For $j\geq m+1$ and $k\leq m$, similarly to (B.21),

$$|E(d_jV_ke^{it\bar{B}_m})| \leq E|d_jV_k||\gamma(t)|^{m-2} \tag{B.22}$$

Therefore, by (B.19)-(B.22) and Lemma 4, for all j, k,

$$|E(d_j V_k e^{it\bar{B}_m})| \le E|d_j V_k| \, |\gamma(t)|^{m-2} \tag{B.23}$$

Similarly to the derivation of (B.23), for any j, k, l, s,

$$|E(e_{jk} V_s e^{it\bar{B}_m})| \le E|e_{jk} V_s| \, |\gamma(t)|^{m-3} \ , \tag{B.24}$$

$$|E(V_j W_{kl} V_s e^{it\bar{B}_m})| \le E|V_j W_{kl} V_s| \, |\gamma(t)|^{m-4} \ . \tag{B.25}$$

Substituting (B.18), (B.23)-(B.25) into (B.17), using $|\gamma(t)| \le 1$,

$$\begin{aligned} |E(\tilde{b}_{2m} e^{it\bar{B}_m})| \le C \, |\gamma(t)|^{m-4} \times \\ \Bigg[\frac{1}{n^{3/2} h^{d+2}} \sum_{j=1}^{m} E|V_j| + \frac{1}{n^{3/2}} \left(\sum_{j=1}^{n} \sum_{k=1}^{m} E|d_j V_k| + \sum_{j=1}^{m} \sum_{k=m+1}^{n} E|d_j V_k| \right) \\ + \frac{1}{n^{5/2}} \left(\sum_{j=1}^{n-1} \sum_{k=j+1}^{n} \sum_{s=1}^{m} E|e_{jk} V_s| + \sum_{j=1}^{m} \sum_{k=j+1}^{n} \sum_{s=m+1}^{n} E|e_{jk} V_s| \right) \\ + \frac{1}{n^{7/2}} \left(\sum_{j=1}^{n} \sum_{k<l}^{n \ (j)} \sum_{s=1}^{m} E|V_j W_{kl} V_s| + \sum_{j=1}^{n} \sum_{k=1}^{m \ (j)} \sum_{l=k+1}^{n \ (j)} \sum_{s=m+1}^{n} E|V_j W_{kl} V_s| \right. \\ \left. + \sum_{j=1}^{m} \sum_{k=m+1}^{n-1 \ (j)} \sum_{l=k+1}^{n \ (j)} \sum_{s=m+1}^{n} E|V_j W_{kl} V_s| \right) \Bigg] \ . \end{aligned} \tag{B.26}$$

The summations in the square brackets have the following bounds.

$$\sum_{j=1}^{m} E|V_j| \le C\, m \quad \text{by Lemma 1-(d) of NR.} \tag{B.27}$$

$$\begin{aligned} \sum_{j=1}^{n} \sum_{k=1}^{m} E|d_j V_k| &= \sum_{j=1}^{m} E|d_j V_j| + \sum_{j=1}^{n} \sum_{k=1}^{m \ (j)} E|d_j V_k| \\ &\le C\,(m + mn) \quad \text{by Lemma 4.} \end{aligned} \tag{B.28}$$

$$\sum_{j=1}^{m} \sum_{s=m+1}^{n} E|d_j V_s| \le C\, mn \quad \text{by Lemma 4-(a).} \tag{B.29}$$

$$\begin{aligned} \sum_{j=1}^{n-1} \sum_{k=j+1}^{n} \sum_{s=1}^{m} E|e_{jk} V_s| &= \sum_{j=1}^{n-1} \sum_{k=j+1}^{n} \sum_{s=1}^{m \ (j,k)} E|e_{jk} V_s| + \sum_{j=1}^{m} \sum_{k=j+1}^{n} E|e_{jk} V_j| \\ &\quad + \sum_{j=1}^{m-1} \sum_{k=j+1}^{m} E|e_{jk} V_k| \\ &\le C\,(mn^2 + mn + m^2) h^{-2} \end{aligned} \tag{B.30}$$

by Lemma 6, $\sum_{s}^{(i_1,i_2,\cdots,i_r)}$ denoting summations excluding $s=i_1,i_2,\cdots,i_r$.

$$\sum_{j=1}^{m}\sum_{k=j+1}^{n}\sum_{s=m+1}^{n} E|e_{jk}V_s| = \sum_{j=1}^{m}\sum_{k=j+1}^{n}\sum_{s=m+1}^{n(j)} E|e_{jk}V_s| + \sum_{j=1}^{m}\sum_{k=j+1}^{n} E|e_{jk}V_k|$$

$$\le C(mn^2+mn)h^{-2} \quad \text{by Lemma 6.} \tag{B.31}$$

$$\sum_{j=1}^{n}\sum_{k<l}^{n(j)}\sum_{s=1}^{m} E|V_jW_{kl}V_s|$$

$$= \sum_{j=1}^{n}\sum_{k<l}^{n(j)}\sum_{s=1}^{m\,(j,k,l)} E|V_j|E|W_{kl}|E|V_s| + \sum_{j=1}^{m}\sum_{k<l}^{n(j)} E|V_j^2W_{kl}|$$

$$+ \sum_{j=1}^{n}\sum_{k<l}^{m(j)} E|V_jW_{kl}V_k| + \sum_{j=1}^{n}\sum_{k<l}^{m(j)} E|V_jW_{kl}V_l|$$

$$\le C(mn^3+mn^2+m^2n)h^{-1} \tag{B.32}$$

by (iii), Lemma 1-(d) of NR, Lemma 4 of Robinson(1995) and Lemma 5.

$$\sum_{j=1}^{n}\sum_{k=1}^{m(j)}\sum_{l=k+1}^{n(j)}\sum_{s=m+1}^{n} E|V_jW_{kl}V_s|$$

$$= \sum_{j=1}^{n}\sum_{k=1}^{m(j)}\sum_{l=k+1}^{n(j)}\sum_{s=m+1}^{n\,(j,l)} E|V_j|E|W_{kl}|E|V_s|$$

$$+ \sum_{j=m+1}^{n}\sum_{k=1}^{m(j)}\sum_{l=k+1}^{n(j)} E|V_j^2W_{kl}| + \sum_{j=1}^{n}\sum_{k=1}^{m(j)}\sum_{l=k+1}^{n(j)} E|V_jW_{kl}V_l|$$

$$\le C(mn^3+mn^2)h^{-1} \tag{B.33}$$

by (iii), Lemma 1-(d) of NR, Lemma 4 of Robinson (1995) and Lemma 5.

$$\sum_{j=1}^{m}\sum_{m<k<l}^{n(j)}\sum_{s=m+1}^{n} E|V_jW_{kl}V_s|$$

$$= \sum_{j=1}^{m}\sum_{m<k<l}^{n(j)}\sum_{s=m+1}^{n\,(k,l)} E|V_j|E|W_{kl}|E|V_s| + \sum_{j=1}^{m}\sum_{m<k<l}^{n(j)} (E|V_jW_{kl}V_k| + E|V_jW_{kl}V_l|)$$

$$\le C(mn^3+mn^2)h^{-1} \tag{B.34}$$

by (iii), Lemma 1-(d) of NR, Lemma 4 of Robinson (1995) and Lemma 5. Therefore, substituting (B.27)-(B.34) into (B.26), using $1\le m\le n-1$,

$$|E(\tilde{b}'_{2m}e^{it\bar{B}_m})| \le C\,|\gamma(t)|^{m-4}\left(\frac{m}{n^{3/2}h^{d+2}} + \frac{mn}{n^{3/2}} + \frac{mn^2}{n^{5/2}h^2} + \frac{mn^3}{n^{7/2}h}\right)$$
$$\le \frac{C_1 m}{n^{1/2}h^2}|\gamma(t)|^{m-4}\ , \qquad \text{(B.35)}$$

the third term in parentheses dominating for sufficiently large n by assumption (ix).

(c) Using (3.7) and Lemma 4 of Robinson(1995), we write

$$|E(\tilde{b}'_{3m}e^{it\bar{B}_m})|$$
$$\le C\Bigg[\frac{1}{n^{5/2}h^{d+2}}\sum_{l=1}^{m}\sum_{s=l+1}^{n}|E(W_{ls}e^{it\bar{B}_m})|$$
$$+\frac{1}{n^{5/2}}\left(\sum_{j=1}^{m}\sum_{l<s}^{n}|E(d_jW_{ls}e^{it\bar{B}_m})| + \sum_{j=m+1}^{n}\sum_{l=1}^{m}\sum_{s=l+1}^{n}|E(d_jW_{ls}e^{it\bar{B}_m})|\right)$$
$$+\frac{1}{n^{7/2}}\left(\sum_{j=1}^{m}\sum_{k=j+1}^{n}\sum_{l<s}^{n}|E(\tilde{W}_{jk}W_{ls}e^{it\bar{B}_m})| + \sum_{m<j<k}^{n}\sum_{l=1}^{m}\sum_{s=l+1}^{n}|E(\tilde{W}_{jk}W_{ls}e^{it\bar{B}_m})|\right)\Bigg]\ . \qquad \text{(B.36)}$$

Similarly to (B.23)-(B.25), for all j, k, l, s,

$$|E(W_{ls}e^{it\bar{B}_m})| \le E|W_{ls}||\gamma(t)|^{m-2}\ , \qquad \text{(B.37)}$$

$$|E(d_jW_{ls}e^{it\bar{B}_m})| \le E|d_jW_{ls}||\gamma(t)|^{m-3}\ , \qquad \text{(B.38)}$$

$$|E(\tilde{W}_{jk}W_{ls}e^{it\bar{B}_m})| \le E|\tilde{W}_{jk}W_{ls}||\gamma(t)|^{m-4}\ . \qquad \text{(B.39)}$$

Substituting (B.37)-(B.39) into (B.36), we have, due to $|\gamma(t)| \le 1$,

$$|E(\tilde{b}'_{3m}e^{it\bar{B}_m})|$$
$$\le C\,|\gamma(t)|^{m-4}\Bigg[\frac{1}{n^{5/2}h^{d+2}}\sum_{l=1}^{m}\sum_{s=l+1}^{n}E|W_{ls}|$$
$$+\frac{1}{n^{5/2}}\left(\sum_{j=1}^{m}\sum_{l<s}^{n}E|d_jW_{ls}| + \sum_{j=m+1}^{n}\sum_{l=1}^{m}\sum_{s=l+1}^{n}E|d_jW_{ls}|\right)$$
$$+\frac{1}{n^{7/2}}\left(\sum_{j=1}^{m}\sum_{k=j+1}^{n}\sum_{l<s}^{n}E|\tilde{W}_{jk}W_{ls}| + \sum_{j=m+1}^{n-1}\sum_{k=j+1}^{n}\sum_{l=1}^{m}\sum_{s=l+1}^{n}E|\tilde{W}_{jk}W_{ls}|\right)\Bigg]\ .$$

Applying Lemma 4 of Robinson (1995), Lemmas 7 and 8, and (ix),

$$|E(\tilde{b}_{3m}^{/} e^{it\bar{B}_m})| \le C\,|\gamma(t)|^{m-4}\left\{\frac{mn}{n^{5/2}h^{d+3}} + \frac{mn^2}{n^{5/2}h} + \frac{1}{n^{7/2}}\left(\frac{mn}{h^{d+3}} + \frac{mn^3}{h^3}\right)\right\}$$

$$= C\,m\,|\gamma(t)|^{m-4}\left(\frac{1}{n^{3/2}h^{d+3}} + \frac{1}{n^{1/2}h} + \frac{1}{n^{5/2}h^{d+3}} + \frac{1}{n^{1/2}h^3}\right)$$

$$\le \frac{C\,m}{n^{1/2}h^3}|\gamma(t)|^{m-4} \quad .$$

(d) Write, using (3.7) and Lemma 4 of Robinson(1995),

$$E|\tilde{b}_{2m}|^2 \le C\Bigg[\frac{1}{n^3h^{2d+4}}E|\sum_{i=1}^{m}V_i|^2 + \frac{1}{n^3}\left(E|\sum_{i=1}^{n}\sum_{s=1}^{m}d_iV_s|^2 + E|\sum_{i=1}^{m}\sum_{s=m+1}^{n}d_iV_s|^2\right)$$
$$+\frac{1}{n^5}\left(E|\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\sum_{s=1}^{m}e_{ij}V_s|^2 + E|\sum_{i=1}^{m}\sum_{j=i+1}^{n}\sum_{s=m+1}^{n}e_{ij}V_s|^2\right)$$
$$+\frac{1}{n^7}\left(E|\sum_{i=1}^{n}\sum_{k<l}{}^{(i)}\sum_{s=1}^{m}V_iW_{kl}V_s|^2 + E|\sum_{i=1}^{n}\sum_{k=1}^{m}{}^{(i)}\sum_{l=k+1}^{n}{}^{(i)}\sum_{s=m+1}^{n}V_iW_{kl}V_s|^2\right.$$
$$\left.+E|\sum_{i=1}^{m}\sum_{k=m+1}^{n-1}{}^{(i)}\sum_{l=k+1}^{n}{}^{(i)}\sum_{s=m+1}^{n}V_iW_{kl}V_s|^2\right)\Bigg] \quad . \qquad \text{(B.40)}$$

We show bounds only of some typical terms. Since V_i is an iid sequence with zero mean, due to Lemma 1-(d) of NR, $E|\sum_{i=1}^{m}V_i|^2 = m\,E|V_1|^2 \le C\,m$. Writing

$$E|\sum_{i=1}^{n}\sum_{s=1}^{m}d_iV_s|^2 \le C\left(E|\sum_{i=1}^{m}d_iV_i|^2 + E|\sum_{i=1}^{m-1}\sum_{s=i+1}^{m}d_iV_s|^2 + E|\sum_{s=1}^{m}\sum_{i=s+1}^{n}d_iV_s|^2\right) \quad , \qquad \text{(B.41)}$$

the first term in parentheses is bounded by

$$mE|d_1V_1|^2 + m(m-1)E|d_1V_1|E|d_2V_2| \le C\,m^2 \qquad \text{(B.42)}$$

due to (iii) and Lemma 4-(b). Since d_i and V_s are iid with zero mean,

$$E|\sum_{i=1}^{m-1}\sum_{s=i+1}^{m}d_iV_s|^2 = \sum_{i=1}^{m-1}E(d_i^2)\sum_{s=i+1}^{m}E(V_s^2) \le C\,m^2 \qquad \text{(B.43)}$$

by Lemma 1-(d) of NR and (A.2) under (i). Similarly, using Lemma 4-(a),

$$E|\sum_{s=1}^{m}\sum_{i=s+1}^{n}d_iV_s|^2 \le \sum_{s=1}^{m}\sum_{i=s+1}^{n}E(d_i^2)E(V_s^2) \le C\,m\,n \quad . \qquad \text{(B.44)}$$

From (B.41)-(B.44),

$$E|\sum_{i=1}^{n}\sum_{s=1}^{m}d_iV_s|^2 \le C\,(m^2 + mn) \quad .$$

Similarly,

$$E|\sum_{i=1}^{m}\sum_{j=m+1}^{n} d_i V_s|^2 = \sum_{i=1}^{m} E(d_i^2) \sum_{s=m+1}^{n} E(V_s^2) \leq C\, mn\,.$$

We next consider

$$E|\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\sum_{s=1}^{m} e_{ij}V_s|^2 \leq C\Big\{E|\sum_{s=1}^{m-1}\sum_{i=s+1}^{n-1}\sum_{j=i+1}^{n} e_{ij}V_s|^2 + E|\sum_{i=1}^{m}\sum_{j=i+1}^{n} e_{ij}V_i|^2$$

$$+ E|\sum_{i=1}^{m-1}\sum_{s=i+1}^{m}\sum_{j=s+1}^{n} e_{ij}V_s|^2 + E|\sum_{i=1}^{m-1}\sum_{j=i+1}^{m} e_{ij}V_j|^2 + E|\sum_{i=1}^{m-2}\sum_{j=i+1}^{m-1}\sum_{s=j+1}^{m} e_{ij}V_s|^2\Big\}\,. \qquad \text{(B.45)}$$

Due to (iii), $E(e_{ij}|i) = E(e_{ij}|j) = 0$, $E(V_s) = 0$ and Lemma 6, the triple summation terms on the right of (B.45) is $O\big((m^3 + m^2n + mn^2)h^{-d-4}\big)$. Using Lemma 6 and Hölder's inequality, the second term in (B.45) is

$$\sum_{i=1}^{m}\sum_{j=i+1}^{n} E(e_{ij}V_i)^2 + 2\sum_{i=1}^{m-1}\sum_{k=i+1}^{m}\sum_{j=k+1}^{n} E(e_{ij}V_i e_{kj}V_k)$$

$$\leq C\,[mnE(e_{12}V_1)^2 + m^2n\{E(e_{13}V_1)^2E(e_{23}V_2)^2\}^{1/2}] \leq C\,m^2nh^{-d-4}\,. \qquad \text{(B.46)}$$

Similarly, the fourth term of (B.45) is $O\,(m^3h^{-d-4})$. Using Lemma 5, as above, the terms involving $V_iW_{kl}V_s$ in (B.40) are $O\big((m^4 + m^3n + m^2n^2 + mn^3)h^{-d-2}\big)$, so by (ix)

$$E|\tilde{b}_{2m}|^2 \leq \frac{C}{nh^{d+2}}\left(\frac{m}{n^2h^{d+2}}\right) + \frac{C}{n^3}(m^2 + mn)$$

$$+ \frac{C}{n^5}(m^3 + m^2n + mn^2)h^{-d-4}$$

$$+ \frac{C}{n^7}(m^4 + m^3n + m^2n^2 + mn^3)h^{-d-2}$$

$$\leq C\, m\,\Big(\frac{1}{n^3h^{2d+4}} + \frac{1}{n^2}\Big)\,.$$

(e) The derivation is similar using Lemma 4 of Robinson (1995), Lemmas 7 and 8. As in (d), we can show

$$E|\tilde{b}_{3m}'|^2 \leq \frac{C}{n^5h^{2d+4}}mnh^{-d-2} + \frac{C}{n^5}(m^3 + m^2n + mn^2)h^{-d-2}$$

$$+ \frac{C}{n^7}(m^4 + m^3n + m^2n^2 + mn^3)h^{-3d-6}$$

$$\leq \frac{C\,m}{n^4h^{3d+6}}\,.$$

(f) Write

$$|E\tilde{b}_3^{//} e^{it\bar{B}_m}| = |EQ\,\bar{W}\,e^{it\bar{B}_m}| \le |EQ_1\bar{W}\,e^{it\bar{B}_m}| + |EQ_2\bar{W}\,e^{it\bar{B}_m}| \ . \tag{B.47}$$

By (3.7),

$$|E(Q_1\bar{W}\,e^{it\bar{B}_m})|$$

$$\le \frac{C}{n^{7/2}}\Big|\sum_{j=1}^{n-1}\sum_{k=j+1}^{n}\sum_{l=1}^{n-1}\sum_{s=l+1}^{n} E\{(V_j+V_k)W_{jk}-\tilde{V}_j-\tilde{V}_k\}W_{ls}e^{it\bar{B}_m}\Big|$$

$$\le \frac{6C}{n^{7/2}}\sum_{j=1}^{n-3}\sum_{k=j+1}^{n-2}\sum_{l=k+1}^{n-1}\sum_{s=l+1}^{n} E|\{(V_j+V_k)W_{jk}-\tilde{V}_j-\tilde{V}_k\}W_{ls}||\gamma(t)|^{m-4}$$

$$+ \frac{6C}{n^{7/2}}\sum_{j=1}^{n-2}\sum_{k=j+1}^{n-1}\sum_{s=k+1}^{n} E|\{(V_j+V_k)W_{jk}-\tilde{V}_j-\tilde{V}_k\}W_{ks}||\gamma(t)|^{m-3}$$

$$+ \frac{C}{n^{7/2}}\sum_{j=1}^{n-1}\sum_{k=j+1}^{n} E\,|\{(V_j+V_k)W_{jk}-\tilde{V}_j-\tilde{V}_k\}W_{jk}||\gamma(t)|^{m-2}$$

$$\le C\,n^{1/2}E|\{(V_1+V_2)W_{12}-\tilde{V}_1-\tilde{V}_2\}W_{34}||\gamma(t)|^{m-4}$$

$$+ \frac{C}{n^{1/2}}E|\{(V_1+V_2)W_{12}-\tilde{V}_1-\tilde{V}_2\}W_{13}||\gamma(t)|^{m-3}$$

$$+ \frac{C}{n^{3/2}}E|\{(V_1+V_2)W_{12}-\tilde{V}_1-\tilde{V}_2\}W_{12}||\gamma(t)|^{m-3} \ . \tag{B.48}$$

Using (i), (iii), Lemmas 1-(d), 4 of NR, (A.1) and Lemma 4 of Robinson (1995), the first expectation of (B.48) is bounded by

$$C\,E\{(|Y_1|+|Y_2|+1)|W_{12}|\}E|W_{34}| \le C\,h^{-2} \ .$$

Using (i), Lemmas 1-(d), 4 of NR, (A.1) and Lemma 4 of Robinson (1995), the second expectation of (B.48) is bounded by

$$C\,E\{(|Y_1|+|Y_2|+1)|W_{12}||W_{13}|\}$$

$$\le C\,E\{(|Y_1|+|Y_2|+1)|W_{12}|\,E(|W_{13}|\,|1)\}$$

$$\le C\,h^{-1}E\{(|Y_1|+|Y_2|+1)(|Y_1|+1)|W_{12}|\} \ .$$

Similarly to Lemma 1 and (A.5), $E|Y_1W_{12}|+E|Y_1^2W_{12}|+E|Y_1Y_2W_{12}| = O(h^{-1})$ so that the above quantity is $O(h^{-2})$. The third expectation of (B.48) is bounded by

$$C\,E|V_1 W_{12}^2| + E|\tilde{V}_1 W_{12}| \le C\,(h^{-d-2} + h^{-1}) = O(h^{-d-2})$$

due to Lemmas 1-(d), 4 of NR, Lemma 4 of Robinson (1995) and Lemma 2. Therefore,

$$|E(Q_1 \bar{W} e^{itB_m})| \quad \le C\left(\frac{n^{1/2}}{h^2} + \frac{1}{n^{3/2}h^{d+2}}\right)|\gamma(t)|^{m-4} \le \frac{C n^{1/2}}{h^2}|\gamma(t)|^{m-4}.$$

The second term of (B.47) is bounded by, using (3.7),

$$\frac{C}{n^{9/2}}\left|\sum_{r=1}^{n} \sum_{j=1}^{n-1\,(r)} \sum_{k=j+1}^{n\,(r)} \sum_{l=1}^{n-1} \sum_{s=l+1}^{n} E(V_r W_{jk} W_{ls} e^{it\bar{B}_m})\right|$$

$$\le \frac{C}{n^{9/2}} \sum_{r=1}^{n-4} \sum_{j=r+1}^{n-3} \sum_{k=j+1}^{n-2} \sum_{l=k+1}^{n-1} \sum_{s=l+1}^{n} E|V_r W_{jk} W_{ls}||\gamma(t)|^{m-5}$$

$$+ \frac{C}{n^{9/2}} \sum_{r=1}^{n-3} \sum_{j=r+1}^{n-2} \sum_{k=j+1}^{n-1} \sum_{s=k+1}^{n} E|V_r W_{jk} W_{ks}||\gamma(t)|^{m-4}$$

$$+ \frac{C}{n^{9/2}} \sum_{r=1}^{n-3} \sum_{j=r+1}^{n-2} \sum_{k=j+1}^{n-1} \sum_{s=k+1}^{n} E|V_r W_{jk} W_{rs}||\gamma(t)|^{m-4}$$

$$+ \frac{C}{n^{9/2}} \sum_{r=1}^{n-2} \sum_{j=r+1}^{n-1} \sum_{k=j+1}^{n} E|V_r W_{jk} W_{jk}||\gamma(t)|^{m-3}$$

$$+ \frac{C}{n^{9/2}} \sum_{r=1}^{n-2} \sum_{j=r+1}^{n-1} \sum_{k=j+1}^{n} E|V_r W_{jk} W_{rk}||\gamma(t)|^{m-3}$$

$$\le C\,n^{1/2} E|V_1| E|W_{23}| E|W_{45}||\gamma(t)|^{m-5}$$

$$+ \frac{C}{n^{1/2}}(E|V_1| E|W_{23} W_{24}| + E|V_1 W_{14}| E|W_{23}|)|\gamma(t)|^{m-4}$$

$$+ \frac{C}{n^{3/2}}(E|V_1| E|W_{23}|^2 + E|V_1 W_{23} W_{13}|)|\gamma(t)|^{m-3}$$

$$\le C\left(\frac{n^{1/2}}{h^2} + \frac{1}{n^{1/2}h^2} + \frac{1}{n^{3/2}h^{d+2}}\right)|\gamma(t)|^{m-5}$$

by (i), (iii), Lemmas 1-(d), 4 of NR, Lemma 4 of Robinson (1995) and Lemma 1. Then apply (ix).□

References

Amemiya,T.(1973) "Regression analysis when the dependent variable is truncated normal", Econometrica, 41, 997-934.

Amemiya,T.(1974a) "Multivariate regression and simultaneous equation models when the dependent variables are truncated normal", Econometrica, 42, 999-1012.

Amemiya,T.(1974b) "The non-linear two stage least squares estimator", Journal of Econometrics, 2, 105-110.

Amemiya,T.(1977) "The maximum likelihood and the nonlinear three-stage least squares estimator in the general nonlinear simultaneous equation model", Econometrica, 45, 955-968.

Amemiya,T., and J.L.Powell(1973) "A comparison of the Box-Cox maximum likelihood estimator and the non-linear two-stage least squares estimator", Journal of Econometrics, 17, 351-381.

Bhattacharya,R.N. and J.K.Ghosh(1978) "On the validity of the formal Edgeworth expansion", Annals of Statistics, 6, 434-451.

Bhattacharya,R.N. and R.R.Rao(1976) Normal Approximation and Asymptotic Expansions. Wiley, New York.

Callaert,H and N.Veraverbeke(1981) "The order of normal approximation to a studentized U-statistic", Annals of Statistics, 9, 194-200.

Cheng,B. and P.M.Robinson(1994) "Semiparametric estimation for time series with long range dependence", Journal of Econometrics, 64, 335-353.

Dharmardhikari,S.W., V.Fabian and K.Jogdeo(1968) "Bounds on the moments of martingales", Annals of Mathematical Statistics, 39, 1719-1723.

Helmers,R.(1985) "The Berry-Esseen bound for studentized U-statistics", Canadian Journal of Statistics, 13, 1, 79-82.

Helmers,R.(1991) "On the Edgeworth expansion and the bootstrap approximation for a studentized U-statistic", Annals of Statistics, 19, 1, 470-484.

Horowitz, J.L.(1986) "A Distribution-Free Least Squares Estimator for Censored Linear

Regression Models," Journal of Econometrics, 32, 59-84.

Horowitz, J.L.(1988) "Semiparametric M-Estimation of Censored Linear Regression Models," in Advances in Econometrics, Vol. 7, 45-83.

Jennrich, R.I.(1969) "Asymptotic properties of non-linear least squares estimators", Annals of Mathematical Statistics, 40, 633-643.

Lee L.-F.(1992) "Semiparametric Nonlinear Least Squares Estimation of Truncated Regression Models," Econometric Theory, 8, 52-94.

Linton, O.B.(1996) "Edgeworth approximation for MINPIN estimators in semiparametric regression models", Econometric Theory, 12, 30-60.

Nishiyama, Y. and Robinson P.M.(1999) "Edgeworth expansions for semiparametric averaged derivatives", Econometrica, forthcoming.

Pfanzagl,J.(1971) "The Berry-Esseen bound for minimum contrast estimates", Metrika, 17, 82-91.

Pfanzagl,J.(1973) "Asymptotic expansions related to minimum contrast estimators", Annals of Statistics, 1, 993-1026.

Powell,J.L.(1984) "Least Absolute Deviations estimation for the Censored Regression Model," Journal of Econometrics, 25, 303-325.

Powell,J.L.(1986) "Symmetrically Trimmed Least Squared Estimation for Tobit Models," Econometrica, 54, 1435-1460.

Powell,J.L., J.H.Stock and T.M.Stoker(1989) "Semiparametric estimation of index coefficients", Econometrica, 57, No.6, 1403-1430.

Robinson,P.M.(1982) "On the asymptotic properties of estimators of models containing limited dependent variables", Econometrica, 50, 27-41.

Robinson,P.M.(1989) "Hypothesis testing in semiparametric and non-parametric models for econometric time series", Review of Economic Studies, 56, 511-534.

Robinson,P.M.(1991) "Best nonlinear three-stage least squares estimation of certain econometric models", Econometrica, 59, 755-786.

Robinson,P.M.(1995) "The normal approximation for semiparametric averaged derivatives", Econometrica, 63, No.3, 667-680.

Serfling,R.J.(1980) Approximation Theorems of Mathematical Statistics. Wiley, New York.