Educ Stud Math (2012) 81:365–384
DOI 10.1007/s10649-012-9408-8

Students' understanding of the general notion of a function of two variables

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Published online: 20 April 2012

Abstract In this study we analyze students' understanding of two-variable function; in particular we consider their understanding of domain, possible arbitrary nature of function assignment, uniqueness of function image, and range. We use APOS theory and semiotic representation theory as a theoretical framework to analyze data obtained from interviews with thirteen students who had taken a multivariable calculus course. Results show that few students were able to construct an object conception of function of two variables. Most students showed difficulties finding domains of functions, in particular, when they were restricted to a specific region in the *xy* plane. They also showed that they had not fully coordinated their R^3, set, and function of one variable schemata. We conclude from the analysis that many of the interviewed students' notion of function can be considered as pre-Bourbaki.

Keywords APOS · Schema · Two-variable function · Representations · Semiotic representation theory

1 Introduction

The notion of a multivariable function is of fundamental importance in advanced mathematics and its applications. Our review of the literature found several published articles presenting classroom material for multivariable calculus topics, but few papers investigate student understanding of this type of function. In one of the few articles we found that explicitly treats function of two variables, Yerushalmy (1997) considered linear function of two variables in a modeling context with precollege students. Referring to the generalization

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involved in the transition from function of one variable to function of two variables, she insisted on the importance of the interplay between different representations to generalize key aspects of these functions and to identify changes in what seemed to be fixed properties of each type of function or representation. Representations in the case of functions have interested mathematics education research for a long time (for example, Artigue, 1992; Dufour-Janvier, Bednarz & Belanger, 1987; Hitt, 1998). More recently, and relating to multivariable functions, Kabael (2009) studied the effect that using the "function machine" might have on student understanding of function of two variables, and concluded that it had a positive impact in their learning. In other related work, Montiel, Wilhelmi, Vidakovic and Elstak (2009) considered student understanding of the relationship between rectangular, cylindrical, and spherical coordinates in a multivariable calculus course. They found that focusing on conversion among representation registers and on individual processes of objectification, conceptualization and meaning contributes to a coherent view of mathematical knowledge.

This study is a continuation of that by Trigueros and Martínez-Planell (2010) in which student understanding of the graphical representation of two-variable functions was analyzed. In that study we concluded that the generalization of the concept of one-variable function to two-variable function, in particular in the case of graphical representation, is not direct. We also found that students who had constructed a schema for R^3, which included strong relationships between different subsets of points, had a better understanding of the graphs of two-variable functions.

The present study focuses on student understanding of formal aspects of the function concept. In particular, we are interested in studying students' understanding of the notions of domain, range, uniqueness of function value, and possible arbitrary nature of a functional relation.

There is a vast literature discussing the parallels and limitations in comparing historical development of a concept, and development of the concept in an individual (Furinghetti & Radford, 2008; Radford, 1997). One point of view is summarized by Sfard (1995, pp. 15–16) when she stated "difficulties experienced by an individual learner at different stages of knowledge formation may be quite close to those that once challenged generations of mathematicians." Somewhat more cautiously, Piaget and García (1983) stated "We mustn't exaggerate the parallel between historical and individual development, but in broad outline, there certainly are stages that are the same" (as cited in Furinghetti & Radford, 2008, p. 630). Hence, in order to gain insights on the possible constructions needed to understand these functions we reviewed the historical development of the concept as summarized by Kleiner (1989) and Sfard (1992): The development of the concept can be divided in stages: The first formal definition of function was given by Johann Bernoulli in 1718: "one calls here function of a variable a quantity composed in any manner whatever of this variable and constants." Spurred by the debate on the vibrating string problem, Euler in 1748 admitted functions defined by several analytic expressions on different intervals and even curves drawn free-hand. This was a definite step towards admitting the possible arbitrary nature of a functional relation. Developments in the early 1800's, motivated by the need to make Fourier's results on heat conduction mathematically acceptable, required a redefinition of the function concept: Dirichlet (see Kleiner, 1989, p. 291) formulated his definition in 1829 as "y is a function of the variable x defined on the interval $a<x<b$, if to every value of the variable x in this interval there corresponds a definite value of the variable y. Also, it is

irrelevant in what way this correspondence is established." This definition stresses the arbitrary nature of the functional correspondence, and extends the notion by allowing functions that are not given by one, or even several, analytic expressions, or by a curve drawn free-hand. This was also one of the first definitions to explicitly restrict the domain of a function to an interval. Further developments resulting from the great growth experienced in all fields of mathematics in the late 19th and early 20th centuries, led further abstraction in the function concept as new types of objects were admitted on the domain of functions: vectors in the linear transformations of vector spaces, functions in linear functionals, group homeomorphisms in abstract algebra, and so on. All this required extending the function concept to its prevalent view as a mapping between arbitrary sets and Bourbaki's definition of 1939 (Kleiner, 1989, p. 299): Let *E* and *F* be two sets, which may or may not be distinct. A relation between a variable element x of *E* and a variable element y of *F* is called a *functional relation* in y if, for all *x* in *E,* there exists a unique y in *F* which is in the given relation with x. Bourbaki went on to give the definition of function as a certain subset of $E \times F$, which is the modern definition of a function as a set of ordered pairs (Kleiner, 1989).

The modern concept of function plays a very important role in today's mathematics. There is plenty of literature dealing with the many complexities involved in its teaching and learning (see for example: Harel & Dubinsky, 1992; Leinhardt, Zaslovsky & Stein, 1990). In this study we focus on some aspects related to students' understanding of functions when they need to extend the definition for one-variable function to include that of two-variable function. Our research questions are: What are the constructions that students need in order to understand two-variable function? Which of those constructions can be associated with the understanding of the notions of domain, range, uniqueness of functional value, and the possible arbitrary nature of a functional relation?

2 Theoretical framework

This study is based in two complementary theoretical frameworks. APOS theory (Asiala et al., 1996; Dubinsky, 1991, 1994) is used to describe constructions needed and observed in the development of the concept of two-variable function. Semiotic representation theory (Duval, 1999, 2006) is used in the analysis of the use of different representational registers and how flexibility in this use impacts the evolution of the mathematical ideas under consideration. Since this theoretical framework, including the relationship between APOS and semiotic representation theory is discussed in the paper by Trigueros and Martínez-Planell (2010) we will only complement the presentation given there.

To apply APOS theory to do research on students' construction of specific mathematical knowledge and in the design of teaching materials, a genetic decomposition needs to be developed. A genetic decomposition is a model that describes specific mental constructions students may make to construct specific mathematical knowledge. This is used in the methodology of the theory to design activities that may enable researchers to "observe" indirectly, through the work of students, the constructions they have made. Research instruments are intended to probe the model. If constructions described are observed, the model is considered as valid. If it is found that most students construct the concept in a different way than the one described, the model is discarded and a new one is designed. Results of research can also be used to refine a genetic decomposition so that it better reflects students' constructions. Once a genetic decomposition is validated in terms of observation of the predicted constructions, it can be used to design activities to teach the concept. Students' work on those activities and reflection on their work is supposed to contribute to construction of the concept. As with other models, a genetic decomposition is not unique, different researchers can propose different genetic decompositions for the same concept; what is important is that the constructions modeled by the genetic decomposition can be found in research with students.

We used a genetic decomposition for the two-variable function concept that refines that presented in the paper by Trigueros and Martínez-Planell (2010): The schemata that we consider students should have developed previously to be able to understand the concept of two-variable function are as follows: A schema of intuitive three-dimensional space, which consists of a construction of the external material world; a schema of Cartesian plane which includes the concept of points as objects and relations between variables, such as curves, functions, and regions as processes resulting from the generalization of the action of representing their component points; a schema for real numbers which includes the concept of number as an object, and arithmetic and algebraic transformations as processes, a schema for sets; a schema for real function of one real variable including function as process, operations with functions, and coordination of their different representations.

The Cartesian plane, real numbers, and the intuitive notion of space schemata must be coordinated in order to construct the Cartesian space of dimension three, R^3, through the action of assigning real numbers to points in R^2, and the actions of representing the results of those actions as 3-tuples, or as points in space. These actions are interiorized into processes that make it possible to consider different subsets, in particular fundamental planes (planes of the form $x = c$, $y = c$, or $z = c$, where c is constant), in each representation register. These processes can be encapsulated into objects on which further treatment actions or processes can be performed. These treatment actions or processes include intersecting fundamental planes with other surfaces to form transversal sections, contour curves and projections, and processes of conversion (Duval, 2006) of those sets and subsets among representations in a schema which evolves and that can be thematized as a schema for three-dimensional space, R^3.

The R^3 schema is coordinated with the schemata for one-variable function and set through actions of assigning one and only one specific height to each point in a given subset of R^2, given in a particular representation register. These actions are interiorized into the process of assigning a height to each point on a subset of R^2 to construct a two-variable function, and are also interiorized in the processes of conversion needed to relate its different representations. These processes are generalized to consider any possible function of two variables, as a specific relation between subsets of R^2 and R and can be encapsulated into an object.

Students' conceptions can be described in terms of the type of constructions they consistently use when working in related mathematical tasks, and it has been observed that these conceptions often appear as a dialectical progression where there can be partial developments, passages and returns from one conception to the other (Czarnocha, Dubinsky, Prabhu & Vidakovic, 1999). What the theory states is that the way an individual works with diverse mathematical tasks related to the concept is different depending on his or her conception. The notion of conception is used to differentiate among students' constructions and identify the mechanisms they need to develop in order construct a deeper understanding of a concept. When a student's responses involve mainly actions, it is said that he or she shows an action conception of the concept. This is intended to stress that he or she may need more opportunities to interiorize those actions into processes. Students may show different difficulties, all related to an action conception; the same can be said of difficulties related to processes. Students in these two groups also struggle with the encapsulation of different processes. Students show an object conception of the concept when they demonstrate, through their responses, that they can work with the concept as a whole, and that they can de-encapsulate the object into the processes needed in its construction. APOS theory helps researchers identify students' differences in terms of their conceptions which are related, in turn, with needs in the evolution of those constructions.

APOS theory is a cognitive theory. As any other theory, it is limited to descriptions and predictions of constructions given in terms of the elements that constitute the theory. On some occasions these elements may appear not to give a full account of the phenomenon under study; it is then possible to further refine the genetic decomposition by introducing some elements from other complementary theories, taking care on maintaining the coherence of the approach. It is clear that mathematical learning involves many aspects other than the cognitive one, but approaching this aspect sheds fundamental light on the understanding of mathematical learning. APOS theory has been used to study different complex concepts and has proven to give important insights on students learning of mathematics. Also, and very importantly, it has been tested in the classroom and has proven effective in promoting students' learning of different concepts (Dubinsky & McDonald, 2001, has an annotated bibliography).

3 Method

The first step in a research study using APOS theory involves probing the genetic decomposition with students who have taken a course on the concept under study, in this case, multivariable functions. For this study thirteen students were chosen from two different groups of undergraduate students: one group at a private university in Mexico and another at a public university in Puerto Rico. They all had taken a multivariable calculus course the previous semester. The material covered in their respective courses was equivalent to that of a standard undergraduate course, as found, for example, in Stewart's calculus text (Stewart, 2006). Both teachers introduced multivariable functions in a similar way. They introduced the functions as a generalization of one-variable functions, then introduced graphs of two-variable functions and explained they are surfaces, introduced planes in R^3, and used fundamental planes to show students how to graph two-variable functions. Students were given tasks to practice, where the graphs of quadric surfaces and finding domain and range of functions played an important role. Teachers also discussed the fact that substituting a number for a variable in an equation with three variables corresponds to intersecting a fundamental plane with the graph of the equation, and explained contours and projections stressing their usefulness in understanding the behavior of functions.

An instrument to test students' understanding of the different components of the proposed genetic decomposition was designed to conduct semi-structured interviews with students. Some questions were taken from a previous questionnaire (Trigueros & Martínez-Planell, 2010), and others were designed specifically for this study. The instructors chose what they judged to be three good, seven average, and three weak students for the interview, and reviewed both the genetic decomposition and the instrument questions before they were used with the students.

The response of each question, even if it seems to be simple, is not straightforward. It requires students to do several of the constructions described in the genetic decomposition. We now present each of the questions of the interview instrument together with the constructions, including treatments and conversions (Duval, 2006), that we conjecture students would need to do in answering the questions:

1) The following table defines a function *f* whose domain is represented with the variables *x*, *y* and whose range is represented with the variable *z*: *z=f(x,y)*. The values for *x* are in the first column of the table, the ones corresponding to *y* are in the first row of the table:

$x \setminus y$	2	3	4	5
0	3	1	2	4
1	4	3	2	3
2	6	5	2	2
3	6	7	2	1

a. Find the domain of function *f*.
b. Represent geometrically a point on the graph of *f*.
c. Find the range of *f*.

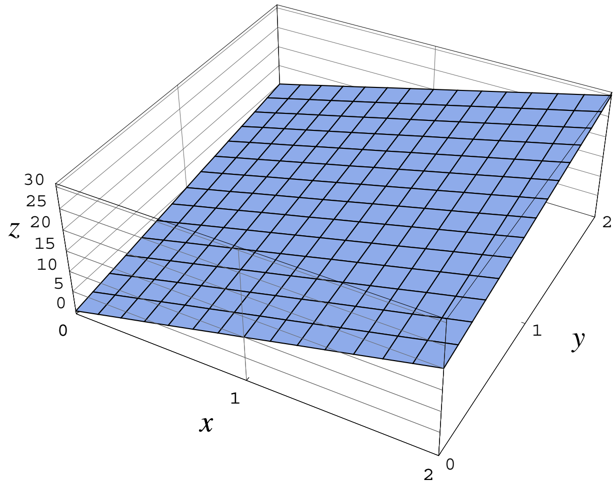
This problem probes students' constructions regarding domain, range, and graph of a function given by a table. In part (a), students need to do actions on the table and a conversion to identify the domain elements *(x,y)*. In part (b), students need to do another conversion in order to identify *(x,y)* with the point *(x,y,0)* on the *xy* plane of a three-dimensional geometric representation, then perform the action of assigning a point in space to the given point in the plane. In part (c), students need to do a treatment to list the range elements from the given table.

2) A function *f* is defined using the formula $f(x,y) = x^2 + y^2 + 1$ where the domain is restricted to the pairs *(x,y)* that satisfy: $-1 \leq x \leq 1$ and $-1 \leq y \leq 1$.

a. Represent the domain of *f* as a subset of the Cartesian plane.
b. Find the range of *f*.

This problem explores students' constructions when two-variable functions are given by a formula. Part (a) requires converting from the given algebraic representation of the domain to a geometrical representation on the Cartesian plane. It probes if students' constructions allow them to work with a function of two variables with a domain restriction, and if they can coordinate the set, one-variable function, and R^3 schemata to identify the set given by the restriction as the domain of the two-variable function. Part (b) explores interiorization of the action of assigning a value to a point in the domain into a process for a function given by a formula and the coordination of this action with the set schema.

3) The following is the complete graph of a function *f*:



a. Find the domain of *f*.
b. Evaluate *f(0,0)*, *f(2,0)*, *f(2,2)*, *f(0,2)*.
c. Find the range of *f*.

Here we consider student constructions when the function is given by a graph. Part (a) requires a conversion from the graphical to the symbolic register. Part (b) probes if students can do the action of assigning a height to a given point in R^2. It requires a conversion to locate the given ordered pair as a domain element in the given graph and a treatment to graphically evaluate the function. Part (c) requires the interiorization of the actions of assigning heights to points in R^2 into a process when the given function is represented graphically.

4) In each of the following cases state if the given rule defines a function. Justify your answer. If it is a function, what can you say about its domain and range?

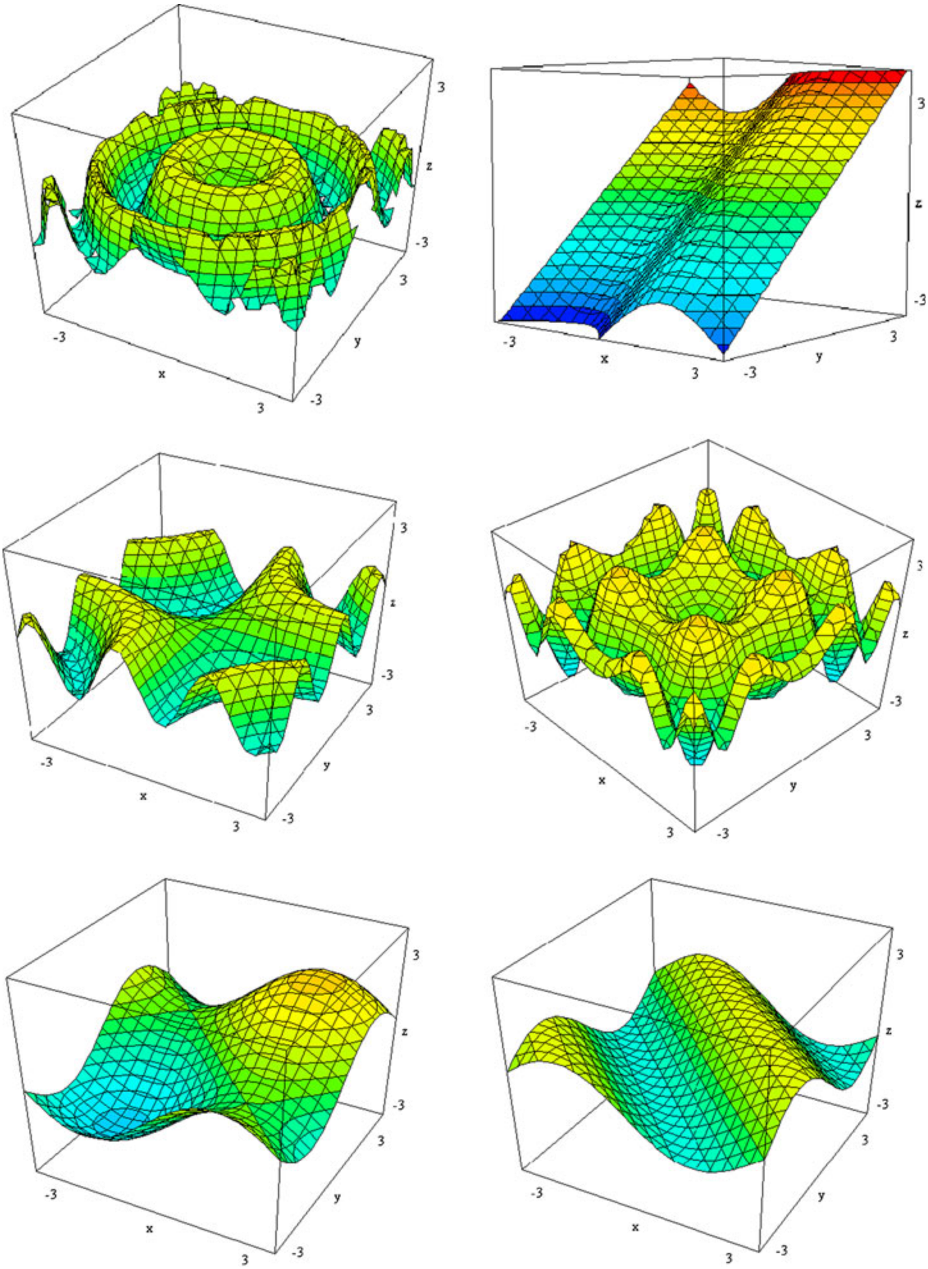
a. Input: weight in kilograms and height in centimeters
Output: name of person with that weight and height
b. In the set of RUM students, as of today;
Input: student number and complete name of the student
Output: student's weight in kilograms

5) Define as carefully as you can the notion of a function of two variables.

The above two questions probe student formal understanding of the function concept. One, (Question 4) examines students' understanding of uniqueness of functional image and the possible arbitrary nature of a functional relation. By not referring to a function of two variables it gives information of the students' general notion of function. The other (Question 5) directly examines students' definition of a function of two variables.

6) State which figure corresponds to which formula. Completely justify your answer.

$g(x, y) = \sin(x) + y$ $h(x, y) = \sin(xy)$



This question probes students' use of sections in identifying graphs. Its answer requires coordination of the schema for function of one variable, and the schemas for R^2 and R^3. Consider, for example, the mental transformations that might take place when identifying the graph of one of these functions, say $g(x,y)=sin(x)+y$. There are different strategies a student may use in this problem. One possible way of doing the problem is to start with the action of substituting a number, for example 0, for x to obtain $z=y$. Do the action or process of conversion from the algebraic to the graphical register to obtain the corresponding graph in R^2. Coordinate the R^2 and R^3 schemas to position the graph in the corresponding fundamental plane in R^3. Do the treatment of comparing the resulting curve with the graphical options given in the problem.

All interviews lasted 45–60 min, were audio-recorded, transcribed, and all students' work on paper was kept as part of the data. All the data were independently analyzed by two researchers, and the conclusions negotiated.

4 Results

Analysis of the data is divided in two parts. First, we use questions 1, 2, 3, and 6 to describe differences in responses given by students who showed different conceptions of two-variable function. We grouped their responses in order to describe typical arguments given by students who demonstrated, throughout the interview, what we considered to be action, process, or object conceptions. Then, we analyzed differences in students' general conception of function, as gathered from their response to questions 4 and 5 of the interview, to find out possible relationships between action, process, or object conceptions that they showed for the two-variable case and their general notion of function. Student schema development is not analyzed in this paper.

4.1 Action, process, and object conceptions of function of two variables

4.1.1 Students showing an action conception of function of two variables:

In APOS, an action is a transformation of a mathematical object which is perceived as external; it may be a manipulation of objects or using memorized facts. Four students showed an action conception of function of two variables. They demonstrated that they had not constructed a schema for R^3 because they had not coordinated the schemata for R^2, for R and for an intuitive idea of space. They all were able to carry out actions on points in R^2 as objects, including treatments and conversions, but had difficulties generalizing them to sets of points in the plane. They thus had difficulties considering sets of points, or regions of the plane as domains of functions of two variables, describing the range of these functions; they also had difficulties with the processes involved in treatments and conversions among different representations (Duval, 2006). These students demonstrated difficulties associated with different actions (Table 1), as well as a tendency to use memorized facts in their responses.

For example, when working with the domain of the function in question 1, Isaac stated:

I: 4 of x and 4 of y, are in the domain.

Int.: what would be an example of an element of the domain?

I: if the domain is defined with x comma y, then 1…

Int.: the number 1?…the domain consists of numbers?

I: yes.

Table 1 Students' difficulties and their relation to students' conception of function

Observed difficulty	Number of students according to conception	
	Action	Process
Incomplete Cartesian space R^3 construction.	4	
Difficulty with the action of assigning a number to a point in R^2.	1	
Recognizing domain set of points in the plane.	4	
Domain as *x* axis and range as *y* axis.	2	
Difficulties with restricted domains.	4	4
Lack of coordination between the process for intersecting the graph of the function with the fundamental plane $z=0$, and the process of projecting the graph onto the *xy* plane.	4	3
Difficulties in describing and finding the range of functions.	4	2
Difficulties with conversions among different representations.	4	2
Difficulty to use sections in graphical analysis of functions.	4	5

Throughout the interview Isaac generalized his notion of domain of a one-variable function, to construct a notion of the domain of a function of two variables consisting of two sets of real numbers, numbers for the x and y coordinates as separate sets.

Patricia, showed similar constructions. She directly imported from one-variable function an action conception of "domain" and "range". Referring to question 1:

P: part c…, so 1, 2, 3, 4, it goes from 0 to 3, the domain always came from the x

Int.: …can you give me an example of an element of the domain?

P: number 1

Int.: the y doesn't play any role?

P: …no, because y is in the range, it is the image.

All four students demonstrated an action conception in the sense that they seem to be constrained to the memorized application of a series of facts to functions which are represented analytically. Rodrigo, for example, when discussing the range of $f(x,y) = x^2 + y^2 + 1$ with domain restricted, as in problem 2, responded:

R: …an element in the range of f?, would be…1 (writing f(0,0)=1).

Int.: could you tell me which are going to be all the elements of the range?

R: …they'd be the results of the equation but with the values that satisfy the condition of the domain.

Even after further insistence by the interviewer, Rodrigo could not give a specific interval for the range. He seemed to be unable to imagine the result of taking all pairs *(x,y)* that satisfy the domain restriction and plugging them into $x^2 + y^2 + 1$, which is evidence of not having interiorized the action of function evaluation into a process. Rodrigo also exemplified difficulties of how students who showed an action conception of two-variable function worked with problems asking to obtain domain and range information from a graph. Even though he was able to represent points in three-dimensional space by doing a conversion from a tabular representation to a physical representation of a function, he could not carry out conversions from a graphical representation to an algebraic one. This difficulty with conversion between registers can be explained from the point of view of APOS theory, which asserts that students would need a process conception in order to be able to reverse a process or to coordinate a process with other processes. While working in problem 3:

Int.: Can you identify a point on the graph of f?

R: it would be 1,0,5

Int.: …Can you obtain an element of the domain given that this point is on the graph?

R: …I don't know.

Indeed, he required an equation to be able to respond when asked to evaluate *f(2,0)*:

R: well, first they'd have to give me the equation, isn't it? To be able to evaluate.

Int.: Can't you get it from the graph?

R: Yes, one can, they'd be…0, 5…I'm trying to get the equation of the plane but…gee whiz…I couldn't find it without the equation of the graph.

These results demonstrate that the schema for R^3 constructed by all the students who showed an action conception of two-variable function consists of isolated actions which are also constrained to specific representation registers and to very simple tasks. The use of this schema constrains them to repeat memorized facts or to perform some simple algorithms relaying only on the analytic representation of two-variable functions.

4.1.2 Students showing a process conception of function of two variables:

A process in APOS theory is the internal transformation of an action. Students with a process conception are able to describe or reflect upon a series of function evaluations without actually performing them. Five students were considered as using a process conception of two-variable function throughout the interview. Their work showed they had interiorized the actions described in the genetic decomposition. Again, there were differences among these students depending on the coordination of processes they seemed to have constructed (Table 1).

For example, María showed confusion between having real numbers or pair of numbers in the domain of a function, but she was able to demonstrate she had interiorized the actions of finding elements in the domain into the process of finding all the elements in the domain. In question 1 she reconsidered:

M: An element in the domain is (0,2).

Int.: And, in total, how many will there be?

M: 4…a list? ok (0,2), (0,3),…ahh no,…there's more…(1,2), (1,3)…because x and y are the domain…

Int.: Do you have an idea of how many will there be?

M: …do I count them as pairs or separate?…they are pairs…16.

She also showed that she had constructed the processes involved in conversions (Duval, 2006), and in the consideration of limited domains for functions as that required in problem 3:

M: …then the domain…goes from…x from 0 to 2 and y from 0 a 2…Find the range of f…it is of z, from 0 to 30.

Students who showed a process conception of function of two variables were able to do, in general, conversions between different representations, but they struggled with some coordination of processes which seem to be important in the construction of an object conception. For example, 4 of the 5 students considered to have a process conception showed they could not coordinate the process of considering a given set for the domain of a function with that of finding its range when the domain was restricted and functions were defined algebraically. Particularly, their difficulties involved conversions of restrictions given in an analytic representation register to the geometric one, or coordination of the schemata for set, function, and R^3. Most of these difficulties could be traced back to an underdeveloped R^3 schema. However, in spite of the difficulties encountered, all these students attempted strategies to solve the problems. One of the observed strategies, used by 2 of the 5 students, was trying to obtain the domain of the function by drawing its graph (a conversion) and projecting onto the *xy* plane (a treatment), without taking the restriction into account, as shown in Gracielle's explanation when drawing the domain of the function in problem 2:

G: …x goes from −1 to 1, y goes from −1 to 1, a circle.

Int.: why is it a circle?

G: Because the graph is a paraboloid…

Int.: is the point (1,1) in the domain of the function?

G: Yes, it satisfies that they are less than or equal to1 and less than or equal to 1 in both cases.

Int.: then, is it in the domain?

G: …yes, because a paraboloid would go up one, one up, that is, that it would be in (1,1) and the bowl would open like this…from here to here and this would be thus in (1,1) [pointing to her drawing].

Gracielle knows the point (1,1) satisfies the conditions defining the domain. She is able to convert the function from the analytical to the graphical representation, but still needs to decide this issue by making reference to the graph. She shows some coordination of processes for function of one variable, where the strategy of projecting a graph onto an axis to find a domain (a reversal process) is likely to have been used in other courses. However, she does not seem to be able to coordinate these processes with the process of considering the elements in the set that defines the restriction, as a separate entity, the defined domain of the function. Nor is she able to coordinate this process with that involved in finding the range of the function.

As concluded by Trigueros and Martínez-Planell (2010), most students only achieve an intra-R^3, or at most inter-R^3, stage of development[1] for the schema of subsets of R^3, and this makes the graphical analysis of functions of two variables very difficult. This is evident in students who, not recognizing the domain restriction, tried to obtain the domain by attempting to graph the section $f(x,y) = 0$ (a conversion). These students did not show coordination between the process for intersecting the graph of the function with the fundamental plane z=0 (a treatment), and the process of projecting the graph onto the *xy* plane (a treatment), as can be seen in Paola's strategy for problem 2:

[1] In APOS theory, the evolution of a schema may be described by the intra, inter, trans level of the "triad" of Piaget and García (1983). These stages are defined by the relations, groupings, interactions and structures utilized by the learner at particular points in time.

Int.: … you drew a circle of radius 1, why did you draw it?

Pa: …I don't know…so thinking that…this is the formula…and because if z is 0 then it is a circle of radius1.

Int.: …but if you make z equal to 0…

Pa: ah, then no…

Then the interviewer drew her attention to the restriction:

Int.: …what are they telling you with that?

Pa: that that is the domain…well, this region here,…a square.

Then, when asked about the range of the function she said:

Pa: …I'm not sure what it means that it is restricted to the pairs…in terms of the range…

To define the domain of a function through a restriction requires further coordination with the schema for sets. A multivariable calculus student might have a notion of function that allows each input to be assigned a unique output, but she might not be able to coordinate it with the set schema to recognize the given restriction as defining a domain object which is independent of the function formula. When guided, Paola was able to draw the unit square and even said that it was the domain; however she could not coordinate the domain with her notion of range of a two-variable function.

"The mathematical abstraction is drawn not from the object which is acted upon, but from the action itself" Piaget said (as cited by Sfard, 1991, p.17). Hence, the difficulty exhibited by students in dealing with domain restrictions and ranges is a reminder that they need further experience applying the action of restricting domains and that of finding ranges, and considering different types of functions before the act of reflective abstraction can result in an encapsulated notion of function which is in accordance with modern practice. As stated by Kleiner (1989, p. 283), "Why define an abstract notion of function unless one had many examples from which to abstract?"

Similarly, when some students were asked to represent the domain of the function as a subset of the Cartesian plane (a conversion), they considered bringing together the plane $z=0$ and the formula for the function. They coordinate these treatment processes through the action of computing a section of the graph of the function. They didn't seem to recognize that substituting a number for a variable in $z=f(x,y)$ corresponds to intersecting a fundamental plane with the graph of the function.

Most of these students tried to identify the graph of a function using other properties and characteristics of functions, such as examining the possible range of the functions through evaluation at some points, rather than using sections. They needed some help to be able to use sections to identify graphs, and, even when they succeeded to identify one function using sections when guided, they quickly reverted to other strategies for other functions. Paola's work exemplifies this difficulty:

Int.: (for the function $g(x,y)=\sin(x)+y$ …if I tell you to give a value to x, say x=3, how can you use that?

Pa: in no way because I don't know what is sin(3).

Int.: but…it's a number.

Pa: …this graph [points to the correct graph].

Int.: …and how did you decide that is the correct one?

Pa: because if the sine of x were 0, z is equal to y and then one is left with, one gets a line [pointing to a section of the graph].

We see that when assisted, she could coordinate the process of conversion needed to draw the graph of a function of one variable with the treatment process of positioning the resulting curve in its appropriate fundamental plane in R^3, a coordination which is beyond the scope of students with an action conception. However, when asked to identify the graph of function *h*, she was unable to use a similar strategy on her own. Hence we conclude that she showed difficulties coordinating the one-variable function schema with that for subsets of R^3.

These results show that processes of conversion involved in the use of sections for the graphical analysis of functions include recognition that substituting a number for a variable in $z = f(x, y)$ corresponds to intersecting a fundamental plane with the graph of the function. It also includes recognizing transformations on one-variable function or families of plane curves that may be obtained by substituting values for one variable in $z = f(x, y)$.

As we can see, coordination of a schema for R^3 as described in the genetic decomposition, seems to play an important role in enabling students to encapsulate the process conception of two-variable function. All students who showed a process conception of this concept found it difficult to use sections when doing graphical analysis of functions.

4.1.3 Students showing an object conception of two-variable function

In APOS theory an object is constructed when the student becomes aware of a process as a totality and realizes that transformations can act on it. Only four students demonstrated an object conception of two-variable function. Daniel is an example. He was able to consider functions with restricted domains independently of the representation register (Duval, 2006) in which they were given:

D: Problem 2…(function given by formula) we have here that x goes from −1 to 1 and y…here [he draws the domain correctly].

Int.: Does the domain include the points inside the square?

D: Yes…and the range goes from 1 to 3…

D: Number 3…(function given graphically) the domain here is x, it goes from 0 to 2, and y from 0 to 2 as well.

Daniel correctly identified graphs corresponding to different analytic representation of functions by performing treatment and conversion actions or processes of intersection of graphs with selected fundamental planes, as in problem 6:

Int.: You can start with $g(x, y) = \sin(x) + y$.

D: ok, it is this one.

Int.: you are pointing to the graph in the third row and second column, why that one?

D: let’s see,…because if we take the xz plane it looks like the graph of sine…but this one also…

Int.: but why you did not choose the graph in the second row and first column?

D: Oh! Yes…these graphs are confusing, there are several with that form, several that look like a sine graph, then what you have are transformations…they are transformations, for example, this one is the graph of sine in the xz plane, then it…, when it… grows it is supposed to get up…to get up in z…

Int.: and if you use one value for x, say, x=0, does that tell you something?

D: it tells me that z=y, so it would be on the zy plane and it would be a line z=y, and would be this one [choosing the correct one].

Daniel seemed to look at the function given by $\sin(x) + y$ and concentrate on the term sin*(x)*. His first choice, like several of the possible options, shows a sine-like oscillation on the plane $y=0$. Daniel went on to describe the process he used to find the section corresponding to $x=0$. Given a surface, he starts looking for $x=0$ as a point on the line segment labeled from $x=-3$ to $x=3$. He then moves under the surface in the y direction (actually on the plane $z=-3$) and tries to figure out the changes in z values. This procedure helps him identify the graph of the function. Daniel repeated this strategy when presented with other functions such as $h(x, y) = \sin(xy)$. It can be observed that Daniel uses fundamental planes in a dynamic way, using treatments and conversions (Duval, 2006) to describe a family of curves resulting from those intersections and, at the same time, to reconstruct the surface. He clearly demonstrates he has constructed a well-structured R^3 schema.

In general, all students who demonstrated an object conception of two-variable function showed a well-structured R^3 schema. They were able to carry out treatments and conversions on sets or functions presented in different representation registers; in particular they could consider functions and fundamental planes as objects and were able to visualize the result of the process of intersecting them without having to do the actual intersection. They were also able to consider functions with different domains and how restrictions in the domain of the function are reflected in its graph and range.

4.2 General notion of a function: uniqueness and arbitrary nature of functional relation

Consistent with Bernoulli's 1718 definition of function, eight of the thirteen interviewed students have a conception of function as a relationship between variables; they seem to need a formula to consider it as such. For example, Patricia explains:

P: (problem 4)…if one gives the student number…and supposing that the name is here…student number and name should give you the weight, the weight in kilograms of the student,… it is not very logical…

Int.: …why do you find it illogical?

P: …because, no! [Laughs] how are the name of a student and the student number going to give you the weight, the weight…no!, it is not valid!

Int.: what would you need?

P: …a formula?…yes.

These students show a conception of function related to algebraic processes with numbers, or function as a formula. In history, mathematicians started from a notion of function which grew from algebraic processes with numbers, and had to work for a long time with the idea of a function as a formula before developing a notion of function which does not depend on the idea of variable. Likewise, an individual may start from an idea of function as a process of algebraic evaluation, before the need to make transformations on this idea requires encapsulating it into an object, thus emancipating his/her notion of function from the idea of numerical evaluation and algebraic manipulation. Problems are what provide the stimulus for reflective abstraction, as was seen historically with the vibrating string problem, and Fourier's work on heat conduction. Most likely these students have not had the need to consider functions other than those given by formulas. They even consider functions given by a table or graph as coming from a formula.

Student's difficulties with domain restrictions (eight of thirteen students) are also suggested by history. Domain restrictions were explicitly considered around 1829 when Dirichlet defined function. Student difficulties persist even though they have met instances where domain restrictions have been used, as when defining inverse trigonometric functions, or considering extrema of two-variable functions in closed and bounded subsets of R^2. It seems that problem situations met by interviewed students did not require most of them to change their working definition.

We found that three of thirteen students did not recognize uniqueness of function image as a requirement for a functional relation. For example, Gaddis, who was considered as having an object conception of two-variable function, responded in problem 4a to a question asking if it was a function:

Gd: …yes, I think so.

Int.: So if I give you the weight, say 60 kg and height 2 m, what could be the output?

Gd: It would be the name of a person with that data.

Int.: and, if there were more than one person with that data?

Gd: …the output would be the name of the person.

Gaddis seems undisturbed by the lack of uniqueness in the value being assigned by the correspondence. This fact is further corroborated when the interviewer presented the equation $x^2 + y^2 = 1$ to him, and asked again if it was a function of x. He responded:

Gd: …it could be, if we solve for y.

Int.: …I solve for y and get $\pm\sqrt{1-x^2}$, then y, is it a function of x?…

Gd: I think it is.

When discussing part b of the same problem:

Gd: …here I am doubting, one can put the name of a student as if it were a number, since with a name the function does not tell you the weight in kilograms of a student

Int.: are you saying that one would have to put the name as if it were a number?

Gd: exactly.

Int.: that is, that it couldn't be a function if the inputs are not numbers?

Gd: well, in order to be able to evaluate a function there must always be numbers.

Indeed, he goes on to define function in terms of algebraic expressions and variable dependence.

What is interesting about Gaddis is that he did everything else in the interview correctly and without hesitation, even the use of sections in graphical analysis. As mathematicians in the 18th and 19th centuries, who were able to perform a wide variety of transformations on functions manipulating a mathematical "object" which, from the modern perspective, was an incomplete construction, Gaddis constructed an object which can be flexibly used in the activities of an undergraduate calculus course, but which does not reflect all the constructions required in a genetic decomposition of the modern function concept.

Students completing a multivariable course have studied real valued functions of one and two variables. They also have studied other types of functions, such as curves in space, that is, functions from R or an interval in R to R^3. It could be expected that through these experiences they would be able to develop a schema which includes relationships among different processes or objects that can be considered as functions.

However, it seems that in spite of all this, few students can be considered to be on their way to constructing a modern notion of function.

5 Discussion

According to results of this study, the refined genetic decomposition proposed can be a good model of students' constructions. Comparison among students who demonstrate differences in their understanding of the notion of two-variable function enables us to recognize which of those constructions need to be taken into account to help students overcome the obstacles encountered even by those who successfully finished a multivariable calculus course, and which are needed for developing a deeper understanding of this concept.

The data also show that the coordination of one-variable function, set, and R^3 schemata may be at the heart of students' potential to develop a deeper understanding of the notions of domain and range of two-variable function. Evidence of this is found in the tendency of some students to generalize their knowledge of functions of one variable to find domains of two-variable functions, as when they identify subsets of R as domain of functions of two variables. Evidence is also found in some students' need to draw the graph in order to find its domain as the projection in the *xy* plane, and their attempt to find domains by using the section $f(x, y) = 0$. We hypothesize that the encapsulation of functions of two variables as objects is needed in order to recognize the set describing a domain restriction as the domain of a two-variable function.

From our point of view, a well developed schema of subsets of R^3 can also explain the success of students in the graphical analysis of functions of two variables, and in particular in the use of sections to draw and identify graphs.

Analysis of the data shows that students need more opportunities to reflect on actions related to the construction of the domain, range and graphs of a rich diversity of functions and to the transformations among representation registers to foster the development of a process construction of two-variable function. According to our results, the construction of an object conception of two-variable function depends on the coordination of specific processes such as those that allow to predict the effect that a restriction on the definition of the domain of a function has on its range and on its graph, and those involved when intersecting fundamental planes and surfaces to find projections or sections needed in the treatments and conversions involved in the graphical analysis of these functions.

Markovitz, Eylon, and Bruckheimer (1986) found that junior high school students exhibited a general neglect of domain and range, and had difficulty with domain restrictions; we found the same difficulties with students who have taken a multivariable calculus course, even though they have had a broader exposition to different types of functions. Like Even (1993) in the case of functions of one variable, we found that students, including those who were quite successful in their multivariable calculus course, had difficulty with uniqueness of functional values and arbitrariness of a functional relation. We also put forward evidence that many students still exhibit difficulties with the concepts of domain and range as was found in Tall and Bakar (1992), and Schwingendorf, Hawks and Beineke (1992). Our study, however, goes beyond the observation of these difficulties; our results suggest specific actions and processes that would need to be part of instructional treatments that may help students overcome them.

Our results indicate in particular the need to develop tools to help students build a schema for subsets of R^3, specifically the potential to use fundamental planes in obtaining sections, contours and projections in different representation registers. Dedicating explicit attention to having students achieve flexibility in treatments and conversions between different representational registers throughout the multivariable calculus course seems to be needed. Also, more work with functions defined on restricted domains and comparison of different kinds of functions, parameterized curves for example, could help students build a general function schema that it is closer to the modern formal definition of this concept.

It is interesting to note that students' conceptions of two-variable function mirror those encountered in the evolution of the function concept: dependence on symbolic representations, expectation of a single analytic expression in the definition of a function, introduction of domain restrictions, introduction of functions defined on sets other than numbers, and a clear notion of domain and range as separate entities.

The use of APOS theory and semiotic representation provided us with two different but coherent lenses to analyze students' work. The description of specific actions or processes involved in treatments and conversions among representation registers put forward aspects of transformations which are important in the construction of knowledge. Semiotic representation theory provided elements to clearly identify the necessity of using or describing transformations in each particular register and among registers to find evidence of the cognitive importance of those processes in the construction of an object conception of two-variable function.

6 Conclusions

This study gives evidence of the complexities regarding the generalization of the concepts involved in the transition between understanding function of one and of two variables. One important contribution of this research is the presentation of a refined genetic decomposition which accurately reflects the constructions needed to learn the concept of two-variable function.

In the previous paper (Trigueros and Martínez-Planell, 2010) we described students conceptions and difficulties regarding the graph of two-variable functions. In this paper we complement these results with new findings related to conceptions about these functions in general, which have not been reported previously. When looking at differences in students' conceptions of two-variable functions, we found specific constructions which can be associated with the understanding of the notions of their domain and range. In particular our results show that even though students had already taken a course in multivariable calculus, there were still some students who showed difficulties considering domains of two variable functions as sets of points and who needed a formula to be able to work with these functions. According to our results, the development of a process conception requires specifically the interiorization of actions related to finding domains of functions which are restricted to specific regions of the *xy* plane presented in different representation registers. We also found that in order to encapsulate this notion, students need more opportunities to work with examples where they can carry out treatment processes in each representation register to be able to distinguish how restrictions in the domain of the function are reflected in their range, and examples where conversion processes can be encapsulated in a function object which relates their tabular, algebraic and graphic representations.

Regarding the development of a general concept of function we found that most of these students showed a concept of function strongly related with a formula and with non restricted domains. They had not constructed a modern notion of function. These results had been found by other researchers with students who had only been exposed to one variable functions. This study shows that exposure to other types of functions does not guarantee that students will develop a modern conception of function and that explicit actions are needed to help them reflect on the specific particularities of the accepted definition of function.

As found in this study, the specific constructions that seem to be necessary for a thorough understanding of two-variable function involve the coordination of the schemata of R^3, function of one variable, and sets. This is shown to play an important role in students' potential to: (1) identify domain and range of functions given in different representational registers; (2) carry out the necessary transformations to be able to relate information across different representational registers; (3) use sections to analyze graphs of functions, and (4) develop a modern notion of function as a set of ordered pairs.

While our previous study (Trigueros and Martínez-Planell, 2010) was focused on students' geometrical understanding, the present study contributes new information on students' formal understanding of functions of two variables. Existing mathematics education literature deals mainly with functions of one variable; we know of no other study exploring students' notion of domain, range, and uniqueness of functional values as it pertains to functions of two variables and, as we have seen, the particularities of this type of function require their being studied on their own. Our study adds valuable new knowledge in this regard. Finally, we can assert that results of this study, together with those of the previous one, show a more complete and detailed picture of students' understanding of two-variable function which can encourage teachers to design activities that include the exploration of a diversity of examples of two-variable functions in a variety of representational registers, as well as explicit comparison with one-variable functions, in order to help students construct a deeper notion of this type of function and a more coherent schema for function in general.

Acknowledgments This project was partially supported by Asociación Mexicana de Cultura AC.

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