Mon. Not. R. astr. Soc. (1967) **135**, 345–360.

STUDIES IN RADIO SOURCE STRUCTURE

I. A Relativistically Expanding Model for Variable Quasi-Stellar Radio Sources

M. J. Rees

(Received 1966 September 28)

Summary

The rapid variations in the flux from some quasars at centimetre wavelengths are known to be incompatible with most models of these objects which assume that they emit synchrotron radiation and are at cosmological distances. We show that this difficulty can be resolved if the sources expand relativistically. The flux variations from a source of synchrotron radiation expanding with a velocity $\sim c$ can, in principle, be arbitrarily rapid. It is argued that relativistic explosive velocities could in fact arise in compact radio sources, and that models of this kind, with reasonable parameters, could account for the observed radio fluctuations of quasars.

1. *Introduction.* Variations have been observed in the flux at centimetre wavelengths from several radio sources (Dent 1965, Maltby & Moffett 1965, Pauliny-Toth & Kellermann 1966). At least one variable source is associated with a Seyfert galaxy (NGC 1275), but the limited information available suggests that the sources most likely to exhibit this behaviour are those associated with quasi-stellar objects (quasars). Conventional theories of quasars assume that they are at cosmological distances. If their radio emission is synchrotron radiation, the observed variations present serious theoretical difficulties. The surface brightness temperature of a source emitting synchrotron radiation is limited to a value $\propto H^{1/2}$ (when H is the magnetic field strength) by the onset of self-absorption (LeRoux 1961, Slish 1963, Williams 1963). If the distance of quasars, and hence their power output, can be deduced from the Hubble law, we can calculate a lower limit ($\propto H^{1/4}$) to their size, and even if we assume that the magnetic field strength has its smallest plausible value it proves to be difficult to construct models which can vary fast enough to explain the observations.

If it turned out to be impossible to devise a reasonable synchrotron model for a variable quasar at a cosmological distance, the following alternatives suggest themselves:

(i) The radio variations may not be intrinsic to the source but could be due to irregularities in the intervening medium.

This is unlikely to be the correct explanation, because variations have only been observed at short wavelength (< 30 cm), whereas one would expect the interstellar and intergalactic medium to have a greater influence on longer wavelength radiation.

(ii) The sources may not be at the great distances inferred by interpreting their red-shifts as cosmological. Their linear dimensions might then be small enough to be consistent with rapid variability.

This view has been advocated by some authors (Terrell 1964, Hoyle & Burbidge 1966), who have suggested that quasars are 'local' objects moving with relativistic velocities. There are many difficulties associated with this hypothesis, such as the absence of blue-shifts, the isotropic distribution of quasars, and the peculiar properties of the hypothetical explosion which accelerated the objects to velocities $\sim c$.

(iii) The variable flux might arise from some mechanism other than the ordinary synchrotron process which permits a higher brightness temperature without the occurrence of self-absorption. The source could then be smaller than a synchrotron source emitting the same flux.

Some such emission mechanism is almost certainly needed to explain other phenomena, such as the radio bursts from flare stars and the small low frequency component of the Crab Nebula (Hewish & Okoye 1965), and it is not unlikely that the same process may operate in quasars. Ginzburg & Ozernoy (1966) have suggested that some of the radio emission from quasars might arise from coherent plasma oscillations. Synchrotron radiation from protons has also been invoked as a possible mechanism. This latter process permits a surface brightness $(m_p/m_e)^{3/2}$ times greater than ordinary (electron) synchrotron radiation.

Future observations of quasars may show that their radio emission is not ordinary synchrotron radiation, but is best explained by a theory of class (iii). However we shall demonstrate in this paper that the occurrence of rapid variations is not in itself strong evidence for the operation of exotic radiation mechanisms. We shall discuss a model of a spherical synchrotron source expanding with a relativistic velocity, and show that relativistic effects permit surprisingly rapid changes in the flux received from such a source. It will be argued that the relativistic bulk velocities required by the model might in fact arise in intense compact radio sources.

In Section 2 we describe the basic geometry relevant to the model. We calculate in Section 3 the variations in the flux density from the model, and apply the results to the particular case of 3C 273 in Section 4. The possibility of cyclic variations is briefly discussed in Section 5.

2. *Geometry of relativistic explosion.* Suppose that at a time $t=0$ an observer sees an explosion at a distant point S. We suppose that the debris from the explosion is ejected from S in all directions with velocities which may be relativistic, and that these velocities subsequently remain constant. In this section we shall consider what the observer O sees at times $t>0$. We assume for simplicity that O is at rest relative to S—if O recedes from S with velocity corresponding to a red shift z, t must be replaced by $t/1+z$ in what follows. In Section 3 these results will be applied to models of spherically symmetric expanding radio sources. (It is however worth pointing out that they are also relevant to other astronomical problems, such as, for example, a calculation of the distribution over the sky of objects ejected in random directions from a distant galaxy (Faulkner, Gunn & Peterson 1966)).

At a time $t>0$ all the material ejected from S with a particular velocity v will be observed to lie on a surface enclosing S. If $v \ll c$ this surface is obviously a sphere centred on S with radius vt. However if $v \sim c$ the distance which a particle with given v appears to have moved in a time t depends on its direction of motion, being in fact proportional to the Doppler shift. The material with velocity v

appears to lie at time t on the surface given by

$$r = \frac{vt}{1 - \frac{v}{c}\cos\theta} \qquad (1)$$

when r is measured from S and θ is the angle made with SO. This is not a sphere, but a spheroid with axis along SO, eccentricity v/c, and semi-latus rectum vt. The particles which appear to have the largest transverse velocity are those moving in the direction making an angle $\cos^{-1}(v/c)$ with SO (*not* those moving perpendicular to SO). They have an apparent transverse velocity γv (where $\gamma = (1 - v^2/c^2)^{-1/2}$), which can be $\gg c$. Therefore the angular size of an exploding object can increase surprisingly rapidly, and it is basically this fact which enables the intensity of the radio source models described in Section 3 to fluctuate rapidly.

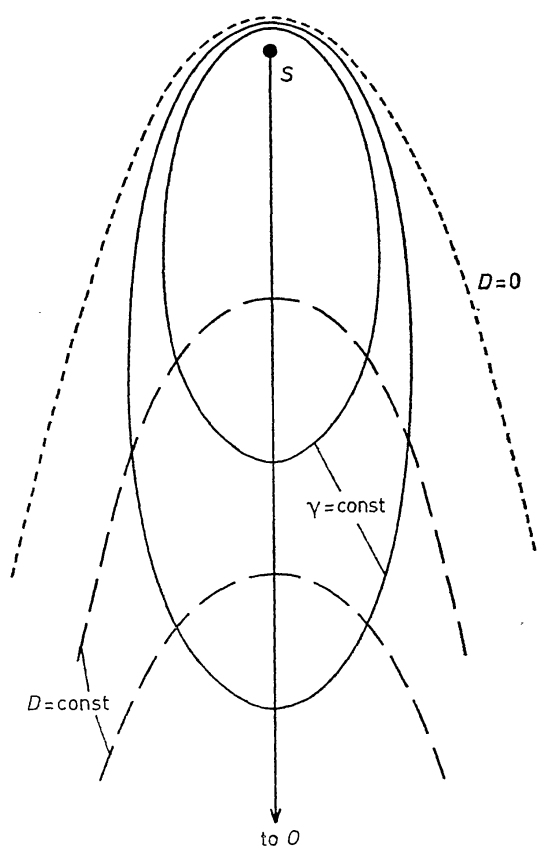

FIG. 1. *An explosion at S is assumed to have been observed by O when $t=0$. At a time $t>0$, particles with the same velocity v lie on spheroids with eccentricity v/c and semi-minor axis γvt (when $\gamma=(1-v^2/c^2)^{-1/2}$). Particles with the same Doppler shift D lie on congruent paraboloids.*

The two spheroids drawn in Fig. 1 correspond to $\gamma \sim 3$ and $\gamma \sim 4$. Particles moving in the direction SA have a Doppler shift γ, and those coming directly towards the observer have a Doppler shift $\gamma(1 + v/c)$.

If the explosion ejects material with different velocities we can also calculate the locus, at time t, of the particles with a particular Doppler shift D.

$$D = \frac{1}{\gamma(1 - v/c\cos\theta)} = \frac{r}{\gamma vt} \qquad (2)$$

and from (1) and (2) the required locus is

$$\left(1+\frac{r}{ct}\cos\theta\right)^2 = D^2 + \left(\frac{r}{ct}\right)^2. \tag{3}$$

If D is taken to be a constant, this gives the equation of a paraboloid with focus at a distance $\frac{1}{2}ctD^2$ from S. The paraboloid corresponding to $D=0$ is the locus of points from which 'light echoes' of the explosion would be observed at time t, and the surfaces corresponding to other values of D are congruent to it but displaced towards O by amounts depending on D.

The foregoing is valid provided that the distance SO ($=d$, say) is much greater than the other dimensions. This requires that the condition $\gamma^2 ct \ll d$ must be satisfied.

The two families of surfaces shown in Fig. 1—those of constant γ (or v) and those of constant D—will be used in the model described in the next section.

3. *Expanding radio source models.* The mechanism which accelerates the relativistic electrons responsible for the synchrotron radiation from radio sources is not understood. However it is reasonable to assume that the fluctuations in the radio output of quasars are related to variations in the associated massive object, and that an increase in the flux density observed from such sources is due to the injection of a burst of electrons into the magnetic field surrounding the object. We can distinguish two extreme types of model, according to whether

A. the magnetic field is so strong as to be unaffected by the injection of the relativistic particles, or

B. the magnetic field is weak, in the sense that $H^2/8\pi \lesssim$ the energy density of the relativistic particles.

In case A the motion of the relativistic electrons would be governed by the pre-existing field, though they would tend to diffuse outwards, and could move freely along the field lines. In case B the magnetic field would be dragged outwards with the particles, and an electric field would be induced in a frame fixed relative to the massive object.

In models of class A the magnetic field is in general so strong that a typical relativistic electron cannot cross the source in its synchrotron lifetime. The radiation at a frequency ν from such a source would come from a region of radius $\sim R(\nu)$ within which self-absorption occurred at frequencies $\lesssim \nu$, and the flux $S(\nu)$ would depend only on $R(\nu)$. $R(\nu)$ (and consequently the flux) would alter in response to changes in the rate of acceleration of electrons (Rees & Sciama 1965), and the results of the last section would be relevant if $R(\nu) \sim c$. Although this is possible, we shall not discuss a detailed model of this kind here.

The expansion velocity of class B models will be comparable with the Alfvén or sound speed (whichever is larger) within the region containing the magnetic field and relativistic particles. If there were no other material within the region, both these speeds would be $\sim c$, and the source would expand with relativistic velocity. (The expansion would not be significantly retarded by the gravitational attraction of the massive object.) If gas were present in the same region, it would be coupled to the relativistic particles by the magnetic field, and would reduce the expansion velocity to a non-relativistic value if its rest mass energy were greater than the total energy of the magnetic field and relativistic particles. Furthermore the expanding source would sweep up external gas, and would be slowed down significantly when the rest mass energy of the gas swept up was comparable with the energy within the source. Without discussing the details of the explosion which led to the formation of the expanding radio source, we can only make rough estimates of the expected gas density and decide whether relativistic velocities are plausible.

In Section 4 we shall show that an appropriate model for the variable component of 3C 273 would be one in which the magnetic field and relativistic particles have a total energy $\gtrsim 10^{56}$ erg, and the source expands with a relativistic velocity at least until it attains a radius ~ 30 pc. We would require for this model that the mass of gas within a sphere of radius ~ 30 pc surrounding the site of the explosion should also be below this limit. (The latter requirement would be satisfied if the gas density did not exceed the mean density of the interstellar medium in our galaxy.) This is not an unreasonably low limit on the gas density. The gas which emits the optical spectral lines from which the red-shift is determined would have a much greater mass, but would presumably not be involved in the relativistic expansion. Furthermore, if the energy of the exploding source were $\gg 10^{56}$ erg (and such energies are known to exist in some radio sources), a higher density of gas would be permitted. There is therefore no physical reason to rule out the relativistic expansion velocity which is an essential feature of the source model we shall describe.

A model in which a cloud containing relativistic particles and a magnetic field expands adiabatically has been discussed by Shklovsky (1960), who applied it first to supernova remnants. Shklovský (1965) has subsequently applied the same idea to extragalactic sources, and other authors (Pauliny-Toth & Kellermann 1966, Van der Laan 1966) have considered similar models. If a homogeneous source expands uniformly, the magnetic field $H \propto R^{-2}$, when R is a length scale. If the magnetic field is random, so that the velocity distribution of the electrons within the source remaining isotropic during the expansion, the energy of the individual electrons $\propto R^{-1}$ (neglecting losses other than adiabatic losses). We note that the ratio of the particle and magnetic energy densities is a constant. If the electrons have a power law energy spectrum, and synchrotron energy losses and self-absorption can be neglected, it is easy to show that the flux density S from the source $\propto R^{-2(2\alpha+1)}$, where α is the spectral index of the radiation. However at low frequencies where self-absorption is important, $S \propto R^2 H^{-1/2} \propto R^3$. At any instant the flux density has a maximum at some frequency ν_m (provided that $\alpha > 0$); the slope of the spectrum is $2\cdot 5$ for $\nu \ll \nu_m$ and $-\alpha$ for $\nu \gg \nu_m$ (assuming that the source is fairly homogeneous). As the expansion proceeds the source becomes transparent at progressively lower frequencies, and $\nu_m \propto R^{-[(4\alpha+5)/(\alpha+2\cdot 5)]}$. The flux density $S(\nu)$ from such a source would initially increase $\propto R^3$, but would decrease when $\nu_m < \nu$. S would be observed to decrease at high frequencies while it was still increasing at lower frequencies.

Models of this kind can satisfactorily reproduce the observed relations between variations at different frequencies, but they cannot vary quickly enough to explain all the observations unless an unreasonably weak magnetic field is assumed, even if we put $\dot{R} \sim c$. However if parts of the source expand at speeds $\sim c$ the above simple theory is invalid, and the discussion must be modified to include relativistic effects. We now consider the simplest generalization of Shklovsky's original model to the case when the boundary of the source expands relativistically.

We suppose that the source expands uniformly, in the sense that *any* observer situated inside the source and sharing the mean motion of his immediate surroundings would see the other parts of the source receding from him isotropically. The required velocity law is

$$v = c \tanh \frac{r}{ct}$$

(this is a generalization of the non-relativistic law $v = r/t$). We suppose that the source has a boundary corresponding to $v = v_0$, $\gamma = \gamma_0$. Such a model is obviously, like Shklovsky's original model, an idealization. In particular, no account is taken of the dynamics of the explosion. However it represents a reasonable approximation to the behaviour of models in which the magnetic and relativistic gas pressure is high enough to impart relativistic expansion velocities to the source. It is permissible, at least in the early stages of the expansion, to neglect the braking due to external gas, and to assume that the source boundary expands outwards with a constant velocity $v_0 \sim c$.

In calculating the time-dependence of $S(\nu)$ during the expansion, we must allow for the fact that the Doppler shift D is not the same for all parts of the source. The radiation reaching the observer at time t from different elements of the source would have been emitted at different frequencies ν/D, when different proper times $t_s = Dt$ had elapsed in the source.

We can distinguish three phases in the source's evolution:

1. $t < t_1(\nu)$. While the source is small and the field strong all the flux received at frequency ν is emitted under conditions when self-absorption is important, and $S(\nu) \propto t^3$.
2. $t_1(\nu) \lesssim t \lesssim t_2(\nu)$. During this phase parts of the source will be transparent, whereas self-absorption will still be important in other parts (as we shall see, self absorption is more important for the small values of D).
3. $t > t_2(\nu)$. Eventually self-absorption becomes negligible throughout the source, and $S(\nu) \propto t^{-2(2\alpha+1)}$.

The intermediate phase 2, during which we shall find that $S(\nu)$ decreases, involves the most complicated calculations. We shall derive expressions for $S(\nu)$ as a function of t which are appropriate to each of the phases. During phases 1 and 3 the flux density has the same time dependence as in Shklovsky's model, although the constants of proportionality are greatly affected by the relativistic corrections.

Phase 1

When synchrotron self-absorption is important, the observed flux effectively comes from a thin surface layer moving with radial velocity v_0 relative to S. The radiation received at a given instant t was therefore emitted from the surface of a spheroid with eccentricity v_0/c, as described in Section 2. Only those parts of the surface whose velocity makes an angle $\leq \cos^{-1} v_0/c$ $(= \sin^{-1} 1/\gamma_0)$ will contribute to the observed flux, the rest of the surface being invisible owing to aberration. The flux reaching O can be calculated by integrating the contributions from the visible part of the surface (equivalent to integrating over half the spheroid).

The flux received at frequency ν from an element of a self-absorbing source with Doppler shift D, subtending a solid angle $\Delta\Omega$, is

$$K\nu^{2.5} D^{1/2} H^{-1/2} \Delta\Omega \text{ flux units.} \quad (4)$$

(1 flux unit is 10^{-26} w.m^{-2} (c/s)$^{-1}$). If ν is measured in Mc/s, H in gauss, and $\Delta\Omega$ in steradians, K is approximately $3 \cdot 10^7$, but its exact value depends on the energy spectrum of the relativistic electrons. The solid angle subtended by the part of the spheroid whose velocity makes an angle between θ and $\theta + d\theta$ with SO is

$$\frac{2\pi v_0^2 t^2}{d^2} \frac{(\cos\theta - v_0/c)\sin\theta\, d\theta}{(1 - v_0/c\cos\theta)^3}$$

where d is the distance SO. The flux density due to this ring-shaped element is therefore

$$2\pi K \nu^{2\cdot 5} D^{1/2} H^{-1/2} \frac{v_0^2 t^2}{d^2} \frac{(\cos\theta - v_0/c)\sin\theta}{(1 - v_0/c\cos\theta)^3}.$$

But $H \propto D^{-2} t^{-2}$ ($= H_0 D^{-2} t^{-2}$ if H_0 is the field measured at a point in the source when $t_s = 1$) and D is given in terms of θ by (2).

So

$$S(\nu) = 2\pi K \nu^{2\cdot 5} \frac{v_0^2 t^3 H_0^{-1/2}}{d^2 \gamma_0^{3/2}} \int_0^{\cos^{-1} v_0/c} \frac{(\cos\theta - v_0/c)\sin\theta\, d\theta}{(1 - v_0/c\cos\theta)^{3/2}} \tag{5}$$

$$= \frac{4\pi K c^2 t^2 H_0^{-1/2} \gamma_0^{7/2} \nu^{2\cdot 5}}{35 d^2} \left[2 + \left(1 + \frac{v_0}{c}\right)^{5/2} \left(5\left(1 + \frac{v}{c}\right) - 7\right)\right]. \tag{6}$$

When $v_0/c \to 0$ this expression reduces to

$$\frac{\pi K v_0^2 t^3 H_0^{-1/2} \nu^{2\cdot 5}}{d^2}. \tag{7}$$

However when $v_0 \to c$ and γ_0 becomes large it tends to the value

$$\frac{8}{35}(1 + 6\sqrt{2}) \frac{\pi K c^2 t^3 H_0^{-1/2} \gamma_0^{7/2} \nu^{2\cdot 5}}{d^2}. \tag{8}$$

Previous authors have estimated the maximum rate of increase of $S(\nu)$ by putting $v \sim c$ in (7). However this is a gross error, since the correct formula (8) contains an extra factor $\sim \gamma_0^{7/2}$, which can be very large. Even if we take $\nu_0 \sim 6$, by no means the largest value which is physically reasonable, this factor ~ 500. All the flux increases so far observed in quasars can easily be accounted for by this type of model (see Section 4).

The factor $\sim \gamma_0^{7/2}$ is made up as follows. The apparent radius of the source at time t is not $v_0 t$ but $\gamma_0 v_0 t$, and this contributes an extra factor γ_0^2 to the solid angle subtended at time t; the flux all comes from parts of the source with $D \geqslant \gamma_0$, so (from (4)) $S(\nu)$ is increased by a factor $\sim \gamma_0^{1/2}$; furthermore the proper time which has elapsed in the visible part of the source is $\geqslant Dt$, and since $H \propto D^{-2} t^{-2}$ we gain, from (4) again, a further factor $\sim \gamma_0$.

Phase 2

Only the radiation emitted from positions near the source boundary escapes freely during the initial stages of the expansion. However self-absorption eventually becomes negligible, and we can define a time $\tau_s(\nu_s)$ measured in a frame sharing the local motion of the source, after which radiation emitted at frequency ν_s from a typical location within the source is unlikely to be reabsorbed. In fact

$\tau_s(\nu_s) \propto \nu_s^{-[(\alpha+2\cdot5)/(4\alpha+5)]}$. The radiation reaching O at time t will have been emitted when $t_s = Dt$ at a frequency $\nu_s = \nu/D$. It is easily deduced that the radiation received from a part of the source with Doppler shift D is only affected by self-absorption before a time

$$t \propto D^{-[(3\alpha+2\cdot5)/(4\alpha+5)]}. \tag{9}$$

Therefore the first parts of the source to appear to become transparent are those for which D is largest. $t_1(\nu)$ (corresponding to the onset of phase 2) is the time when the part with $D = \gamma_0(1 + v_0/c)$ becomes transparent, and

$$t_2(\nu) = t_1(\nu)\,[\gamma_0(1 + v_0/c)]^{[(6\alpha+5)/(4\alpha+5)]}.$$

At times between $t_1(\nu)$ and $t_2(\nu)$ the situation is represented schematically by Fig. 2. Self-absorption is important for the radiation coming from parts of the source with $D \lesssim D'$ (say) which occupy the shaded region II of the diagram. The parts with $D \gtrsim D'$ (labelled I) are transparent. As t increases and the source expands, D' decreases according to (9), and the paraboloid which is the locus of points with $D = D'$ moves back towards S.

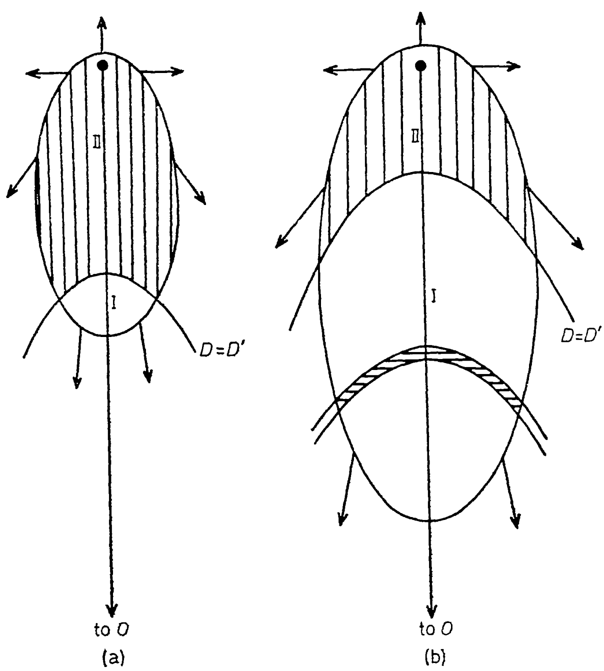

FIG. 2. *During phase 2 of the expansion of the source model, radiation can escape freely from region I. Self-absorption remains important in II, so the radiation effectively comes only from the surface of this region, (b) corresponds to a later time than (a), and is drawn to the same scale. The flux from I is calculated by integrating the contributions from elements with different values of D. One such element is illustrated in (b).*

In calculating the flux density during phase 2 we shall make the following approximation: if the optical depth is > 1 we take it to be infinity, and if it is < 1 we assume it to be zero. This does not introduce any great errors, but enables us to separate the contributions to the flux into two components—an integration over the surface of region II and an integration, neglecting self-absorption, throughout region I.

We take D as our integration variable, as illustrated in the figure. The thickness of the element corresponding to the range of Doppler shifts $D-(D+dD)$ is $Dct\, dD$ (the factor D is cancelled out by the relativistic volume contraction factor when we integrate). The elements have a projected area perpendicular to SO of

$$\pi c^2 t^2 (2\gamma_0 D - 1 - D^2). \tag{10}$$

The flux at frequency ν_s emitted from unit volume of the source at time t_s (the measurements being made in a frame moving with the local velocity) is

$$C\nu^{-\alpha} t_s^{-(4\alpha+5)}$$

where the constant C depends on the value of H_0 and the density of relativistic electrons. So the flux density due to I is

$$C\pi \frac{c^3 \nu^{-\alpha} t^{-2(2\alpha+1)}}{d^2} \int_{D'}^{\gamma_0(1+v_0/c)} (2\gamma_0 D - 1 - D^2) D^{-(3\alpha+2)} dD. \tag{11}$$

This integral can easily be evaluated, and, since D' is related to t by (9) we can obtain the flux in terms of t.

The contribution to the flux from II is given by slightly different expressions according to whether $D' \lessgtr \gamma_0$. If $D' > \gamma_0$ there is not only a contribution from the surface of the paraboloid $D = D'$, given by

$$K\pi \frac{c^2 t^3 \nu^{2\cdot 5}}{d^2} D'^{3/2} H_0^{-1/2} (2\gamma_0 D' - 1 - D'^2) \tag{12}$$

(the constants being defined as in (4)),

but also a contribution from the visible part of the surface of the spheroid, which is represented by an integral similar to (5) between the limits γ_0 and D'. If $D' \leqslant \gamma_0$ (as illustrated in Fig. 2 (b)) this contribution disappears and we only get expression (12). We simply add the contributions from I and II to get the flux from the source during phase 2.

Phase 3

When $t > t_2(\nu)$ the flux is given by (11), except that the lower limit of the integral must be replaced by $\gamma_0(1 - v_0/c)$.

The time variations of $S(\nu)$ through all three phases of the expansion are plotted in Figs. 3 and 4. It is only necessary to plot $S(\nu)$ for one value of ν since the graphs have the same shape for all frequencies. The curves appropriate to the frequency $\beta\nu$ are obtained by expanding the t scale by $\beta^{[(\alpha+2\cdot 5)/(4\alpha+5)]}$ and the S-scale by $\beta^{[(7\alpha+5)(\alpha+2\cdot 5)/(4\alpha+5)^2]}$. Curves are drawn for three models with $\gamma_0 = 3$, 4 and 6, assuming the same value of H_0 and the same electron density at $t_s = 1$. S increases like t^3 initially, until phase 2 begins. The rate of increase $\propto \gamma_0^{7/2}$, (approximately), and S attains a higher maximum for the larger values of γ_0 despite the fact that phase 1 is shorter. During phase 2, when the flux is given by the sum of the expressions (11) and (12), which each comprise 3 terms depending on different powers of t, $S(\nu)$ is not accurately proportional to any single power of t, though the dominant term $\propto t^{-2(5+\alpha)/(5+6\alpha)}$ $(\alpha \geqslant 0)$ or $t^{-2(2\alpha+1)}$ $(\alpha \leqslant 0)$. During phase 3, $S(\nu) \propto t^{-2(2\alpha+1)}$.

The duration of phase 2 relative to that of phase 1 is longer for the large values of γ_0. If $\gamma_0 \sim 1$ (and $v_0/c \sim 0$) this phase disappears, and the model reduces to the non-relativistic case discussed by Shklovsky and Van der Laan. For $\alpha = 0$ (Fig. 3) $S(\nu)$ is roughly proportional to t^{-2} throughout phases 2 and 3. For $\alpha = 1$

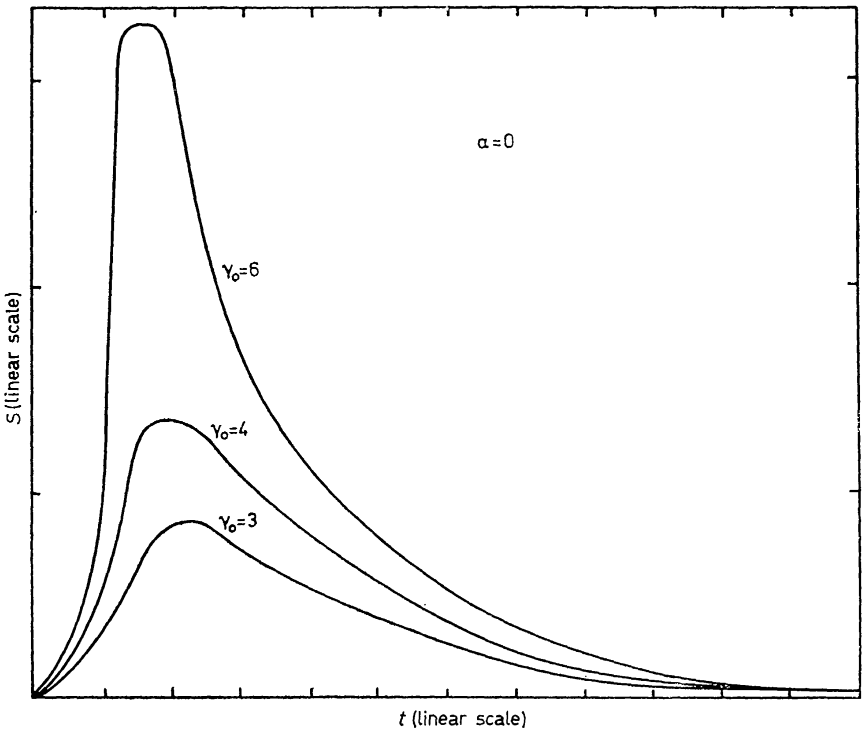

FIG. 3. *Time dependence of the flux density $S(\nu)$ at a fixed frequency from source models with $\alpha = 0$ and various values of γ_0. The values of H_0 and the electron density when $t_8 = 1$ are the same in each case. The scales are linear on both axes, and the curves appropriate to different values of ν are obtained by scaling (see text).*

(Fig. 4) the decrease in flux is less rapid (approximately $\propto t^{-12/11}$) during phase 2, but falls off like t^{-6} during phase 3. Figs. 5 and 6 show the spectrum of the radiation from the model at time t. The spectrum has the same shape at all times, though

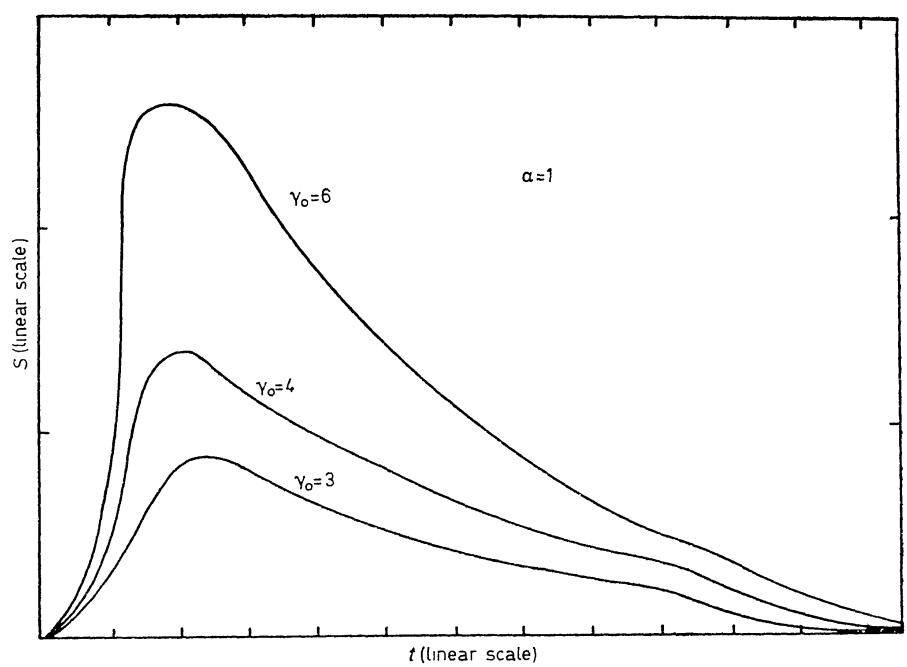

FIG. 4. *The same as Fig. 3, except that $\alpha = 1$.*

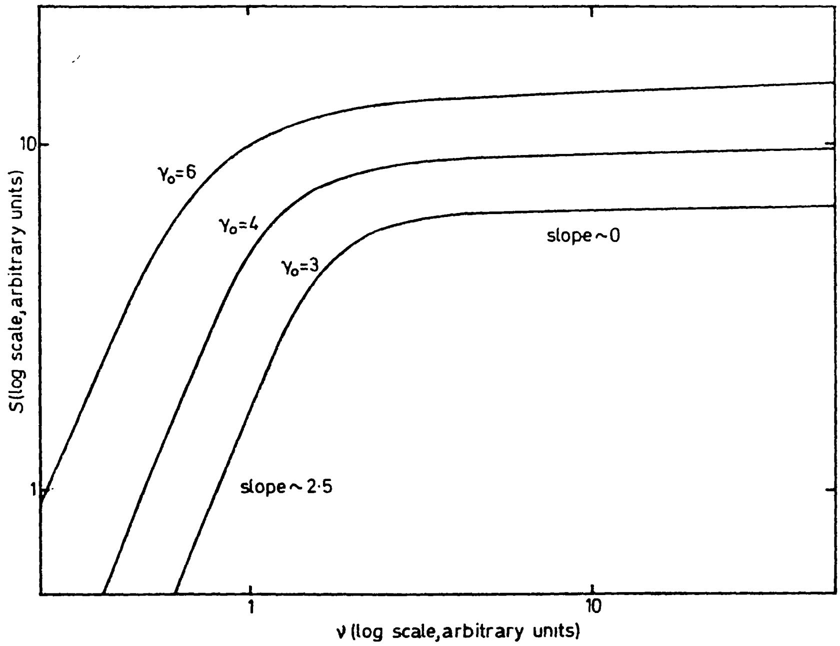

FIG. 5. *$S(\nu)$ at a fixed time for models with $\alpha=0$ and the same values of γ_0 as Figs.* 3 *and* 4. *S and ν are plotted logarithmically. The spectra at other times can be obtained by scaling (see text).*

the features shift towards lower frequencies as *t* increases. If *t* is replaced by βt the ν-scale must be expanded by $\beta^{(4\alpha+5)/(\alpha+2\cdot5)}$ and the S-scale by $\beta^{(7\alpha+5)/(4\alpha+5)}$. At low frequencies the slope of the spectrum has the value 2·5 characteristic of a self-absorbing source. At intermediate frequencies, for which the source is in phase 2, the slope is approximately $5\alpha/(3\alpha+2\cdot5)$ (if $\alpha\geqslant0$) or $-\alpha$ (if $\alpha\leqslant0$), and at high frequencies the slope is $-\alpha$.

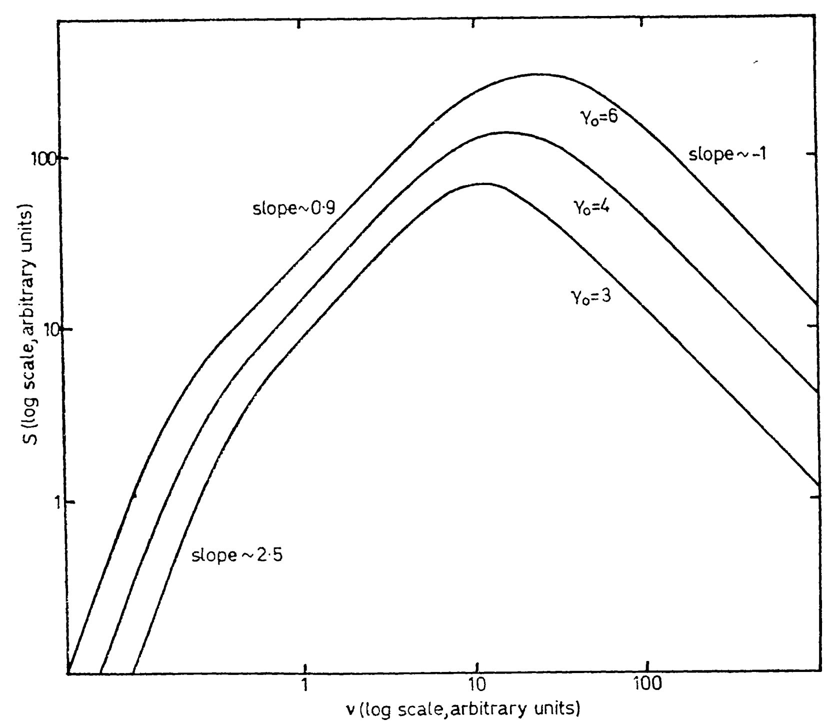

FIG. 6. *The same as Fig.* 5 *except that $\alpha=1$.*

The reason for the different exponents according as $\alpha \gtrless 0$ in the ν and t dependence of S during phase 2 is that when $\alpha > 0$ most of the flux comes from region II (see Fig. 2), whereas if $\alpha < 0$ the contribution from I dominates. If the source has a red-shift z relative to o, ν must be replaced by $(\nu/1+z)$ and t by $(t/1+z)$ in the foregoing.

The work of this section clearly shows that many types of intensity fluctuation can be simulated by a uniform expanding source model, if appropriate values for α and γ_0 are chosen. In particular, the model can exhibit arbitrarily rapid flux increases and decreases if γ_0 is taken to be sufficiently large. In practice an upper limit will be set to permissible values of γ_0 by the fact that phase 1 tends to become very short if γ_0 is large unless the magnetic and particle energies are very high.

We next consider, as an example, how this model can account for the variations observed in one particular source, and show that the required value of γ_0 and the magnetic and particle energies in the model are quite moderate.

4. *The variations of 3C 273.* The first reported variations of 3C 273 were those at 3·75 cm ($\sim$8000 Mc/s) discovered by Dent (1965). He found that there had been an increase of $\sim$12 flux units in the intensity over a three-year period. This flux could arise from a spherical source of diameter $\geqslant 4H^{1/4}$ pc ($\gtrsim 13H^{1/4}$ light years) emitting synchrotron radiation (allowing for the red shift of 0·158 (Schmidt 1963) and calculating the distance from the Hubble law), and the variability can be accounted for if $H \lesssim 10^{-1}$ gauss without appealing to ultra-relativistic effects of the kind described in this paper. We have discussed possible models in an earlier article (Rees & Sciama 1965).

However 3C 273 is now known to vary at other wavelengths: 1 mm, 3·2 mm, (Epstein 1965, Low 1965), 2 cm, 10 cm, and 21·2 cm, (Maltby & Moffett 1965, Pauliny-Toth & Kellermann 1966). Since the minimum radius of a synchrotron source emitting a given flux at wavelength λ is proportional to $\lambda^{5/4}$, it is in general the long wavelength variations which present the greatest problem. We therefore concentrate on the data at 21·2 cm ($\sim$1400 Mc/s). At this wavelength the flux density appears to have increased by $\sim$6 flux units in $\sim$3 years. This flux, if it is synchrotron radiation, must come from a source at least $\sim 17\,H^{1/4}$ pc in diameter. If we adopt the usual criterion that the dimensions of the source can at most be comparable with $c \times$ the 'time scale' of the variations, the observations can only be understood if $H \lesssim 10^{-4}$ gauss. The observed flux would then imply the presence of relativistic electrons with a total energy $\gtrsim 10^{58}$ erg. Since the magnetic energy in the volume of the source would amount to $\lesssim 10^{47}$ erg, the magnetic field would be quite unable to restrain the expansion of the cloud of relativistic electrons. A disadvantage of this model is the fact that it entails somewhat high energy requirements if the source undergoes repeated increases in flux. Each flux increase must correspond to the injection of a new burst of electrons, since the ones which originated in the preceding bursts would, after only a few years, occupy too large a volume to contribute to the variable component of the flux. A single burst corresponds to the release of a quantity of energy $\geqslant 10^{58}$ erg (allowing for the 'inefficiency factor' of the acceleration mechanism) so the total energy required would be exorbitant if (as seems likely from observations of the phenomenon) fluctuations occur for a significant fraction of a quasi-stellar source's lifetime Our relativistic model can account for the behaviour of 3C 273 without these extreme energy requirements.

We note that in any case it would be inconsistent to neglect the results of the previous section in any model where the pressure of the relativistic electrons dominates the dynamics, since under these circumstances relativistic velocities arise automatically.

We take as an example a model with $\gamma_0 = 4$. If the expansion is observed to start at $t = 0$ the apparent diameter after 3 years is ~ 24 light years (~ 7 pc). We specify the model further by taking H to be ~ 50 gauss when $t_s = 1$ year. If this source were at the distance of 3C 273, the flux density observed at 1400 Mc/s 3 years after the expansion started is given by (5) if the model is still in phase 1, and is ~ 6 flux units. A model with these parameters can therefore increase its flux-density as fast as 3C 273 has done in three years. The condition that phase 1 of the expansion should last at least three years requires that the whole source must remain opaque for this time. The maximum value of D is ~ 8 so we require the mean free path of a photon emitted at $\sim 1480/8 = 175$ Mc/s when $t_s \sim 8 \times 3 = 24$ years to be $\lesssim 24$ light years. The value of H corresponding to this value of t_s is $50/24^2 \sim 0 \cdot 1$ gauss, and the electrons which radiate (and absorb) predominantly at this frequency in this magnetic field have $(E/mc)^2 \sim 40$. Their density has to be $\gtrsim 3$ cm^{-3}, which corresponds to an energy density $\gtrsim 5 \,.\, 10^{-5}$ erg cm^{-3}. The total electron energy density exceeds this number by a factor depending on the energy spectrum of the electrons and the cut-offs. We take this factor to be ~ 10; it may (but need not) be very much larger. The magnetic energy density when $t_s \sim 24$ years is $\sim 2 \,.\, 10^{-4}$ erg cm^{-3}, which may be compared with the value $\gtrsim 5 \,.\, 10^{-4}$ erg cm^{-3} for the electron energy density. The ratio of these energies is unaltered by the adiabatic expansion of the source. The inverse Compton effect would be negligible in a source with these parameters. The total energy in such a model, even including the energy of the protons needed to maintain charge neutrality in the expanding source, need not be more than $\sim 10^{56}$ erg. Models with higher values of γ_0 would give even faster increases in intensity (see Figs. 3 and 4).

It does not seem worthwhile to develop more elaborate models for variable radio sources until the observational data is far more extensive than at present. However we enumerate some points that would need to be taken into account in such models.

(i) A proper treatment of the dynamics of the explosion would certainly modify our assumption that the source is uniform. The departures from uniformity would be especially significant at late stages in the expansion when the effect of the external medium becomes important.

(ii) The effect of cut-offs in the electron energy spectrum, and of energy losses other than adiabatic losses, should be allowed for.

(iii) The size of the volume where the electrons are accelerated, and where the energy released in the explosion is stored, cannot be neglected in the early stages of the expansion. Our model, like Shklovsky's original model, becomes unrealistic for small t_s, since the energy losses (Compton, collisional, etc.), would then degrade the electrons in a time shorter than t_s. We have to assume that some acceleration process continues to operate at least until these losses cease to be catastrophic.

(iv) The assumption of spherical symmetry is unlikely to be realized, particularly if the magnetic field has a large scale structure permitting electrons to move preferentially in some directions. The electron velocity distribution need not then remain isotropic, and the source would emit radiation anisotropically.

Clearly models which are not spherically symmetric can show an even wider range of behaviour than those which we have considered.

(v) Observed radio sources almost certainly consist of several unresolved components. The total flux will not therefore have the same spectrum as the variable component.

5. *Possibility of periodic variations.* Records of the radio flux density from quasars do not, as yet, extend over a long enough period to show whether cyclic variations occur. It will be interesting to know whether the apparent periodic optical variations (Smith 1965, Ozernoy & Chertoprud 1966) of 3C 273 are correlated with the radio variations. We would certainly expect this to be the case if the optical continuum is synchrotron radiation, or if it arises from inverse Compton scattering of the radio photons.

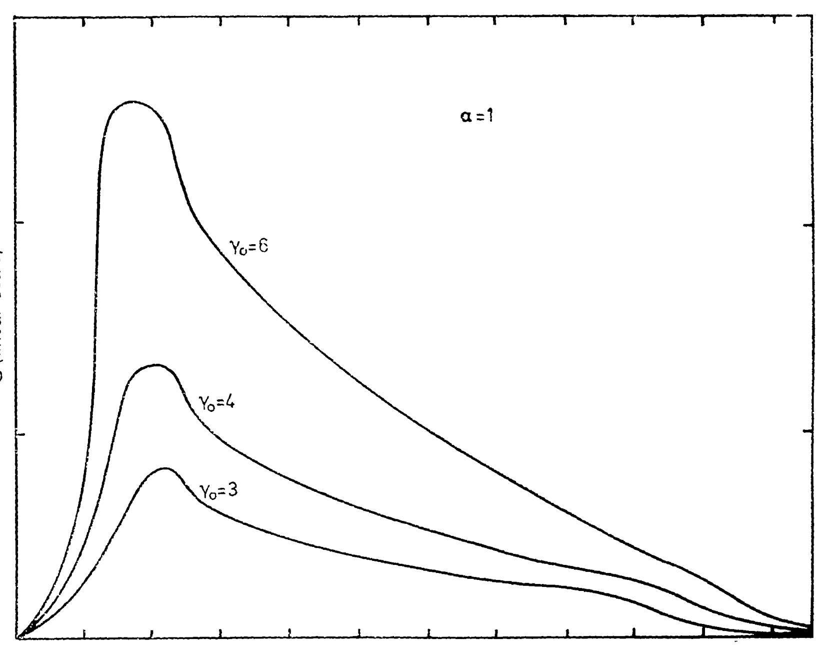

FIG. 7. *$S(\nu)$ as a function of t for a thin spherical shell which expands relativistically with velocity v_0. The magnetic field within the shell is assumed to expand isotropically, and $\alpha = 1$.*

Models which can explain an isolated rapid increase or decrease in flux density are not necessarily capable of explaining periodic variations, but we can easily show that our proposed model *could* account for such behaviour. Suppose that, instead of observing one expanding source whose flux at frequency ν_0 increases for a time $t_1(\nu_0)$ and then decreases (in the manner illustrated in Figs. 3 and 4), we observe the total flux from a series of similar sources which explode at intervals $T \sim 2t_1(\nu_0)$. We would observe large fractional oscillations with period T at frequencies $\gtrsim \nu_0$ unless the flux from the youngest source were greatly exceeded by the total contribution from the older sources. Consequently oscillations will be observed if the flux from the old sources falls off faster than t^{-1} when $t \gtrsim T$, and this in fact happens in our model provided that $\alpha < 5/4$.

However this result is only strictly valid if the explosions occur at different places so that the clouds of particles ejected at different times do not overlap

If all the bursts originate at the same point, we must consider a slightly different model. Suppose that each explosion resulted in the ejection of a spherical shell of relativistic particles and magnetic field (rather than a uniform sphere). The particles and field ejected in successive bursts would remain separated if the shells were sufficiently thin. We have carried out calculations similar to those of Section 3 for an expanding spherical shell, instead of a uniform sphere, and find that the flux decreases even more rapidly during phase 2 than our original model (for the same values of α and γ_0), though the qualitative behaviour is similar. Fig. 7 shows the time dependence of the flux from a model of this kind with $\alpha = 1$. If shells are periodically ejected from S with relativistic velocity, Fig. 8 shows the apparent position of successive shells at a typical instant. Large amplitude variation would be observed provided that each shell starts to become transparent before the next one is ejected.

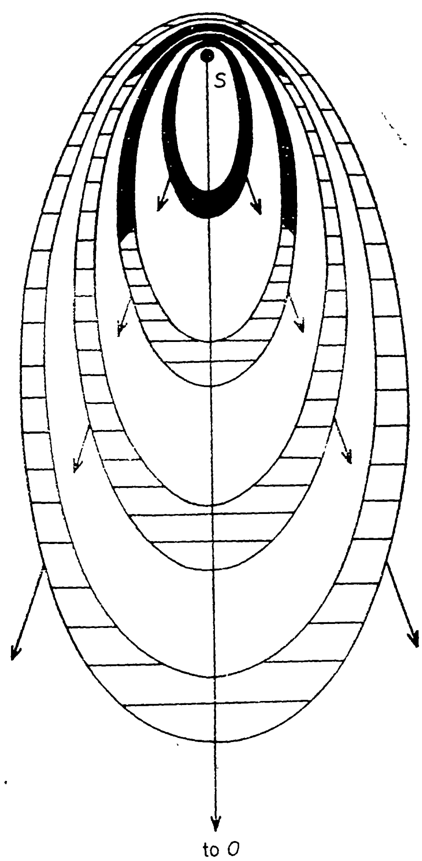

FIG. 8. *A possible model for a source exhibiting rapid periodic fluctuations. The flux comes from thin shells ejected periodically from S, and the figure shows the apparent positions of successive shells at a typical instant. Each shell would separately give rise to the time-dependent flux shown in Fig. 7. The regions where self-absorption occurs are heavily shaded.*

6. *Conclusions.* The main conclusion of this paper is that, if relativistic expansion velocities occur in radio sources, the observed flux variations can be much more rapid than one would deduce by putting $v_0 \sim c$ in the simple expressions valid for expansion velocities $v_0 \ll c$. We have argued that relativistic velocities might be expected to occur in actual radio sources, at least when they occupy a comparatively small volume and the density of relativistic particles is high. The model which we have discussed is a generalization to the case when the expansion

velocity is relativistic of Shklovsky's well-known model for a homogeneous adiabatically expanding source. Even this simple spherically symmetric model can explain a wide variety of flux variations if appropriate values are taken for v_0 and α. However it is obvious that no actual source would conform in detail to such an idealized model, so it does not seem worthwhile attempting to choose the parameters to match the observed fluctuations in individual sources. Periodic flux variations can be accounted for by a similar model (Section 5).

Future observations may indicate that the radio flux from quasi-stellar sources is not synchrotron radiation, but is instead due to plasma oscillations or some ill-understood process. The rapidity of the observed variations might not then pose a special problem. However the mere existence of rapid radio variations from quasars does not rule out the conventional view that they emit synchrotron radiation, since we can explain them by synchrotron models of the kind described here without postulating unreasonably low magnetic fields and exorbitant total energies.

7. *Acknowledgment.* The author is grateful to Dr D. W. Sciama for his interest in this work, and for many helpful comments on the manuscript.

Department of Applied Mathematics and Theoretical Physics,
Cambridge.

1966 September.

References

Dent, W. A., 1965. *Science*, **148**, 1458.
Epstein, E. E., 1965. *Astrophys. J.*, **142**, 1285.
Faulkner, J., Gunn, J. E. & Peterson, B. A., 1966. *Nature, Lond.*, **211**, 502.
Ginsburg, V. L. & Ozernoy, L. M., 1966. *Astrophys. J.*, **144**, 599.
Hewish, A. & Okoye, S. E., 1965. *Nature, Lond.*, **207**, 59.
Hoyle, F. & Burbidge, G. R., 1966. *Astrophys. J.*, **144**, 534.
LeRoux, E., 1961. *Annls Astrophys.*, **24**, 1.
Low, F. J., 1965. *Astrophys. J.*, **142**, 1287.
Maltby, P. & Moffett, A. T., 1965. *Science*, **150**, 63.
Ozernoy, L. M. & Chertoprud, V. E., 1966. *Astr. Zh.*, **43**, 20.
Pauliny-Toth, I. I. K. & Kellermann, K. I., 1966. *Astrophys. J.*, **146**, 634.
Rees, M. J. & Sciama, D. W., 1965. *Nature, Lond.*, **208**, 371.
Schmidt, M., 1963. *Nature, Lond.*, **197**, 1040.
Shklovsky, I. S., 1960. *Astr. Zh.*, **37**, 256.
Shklovsky, I. S., 1965. *Astr. Zh.*, **42**, 30.
Slish, V. I., 1963. *Nature, Lond.*, **199**, 682.
Smith, H. J., 1965. *Quasi-Stellar Sources and Gravitational Collapse* (ed. by Robinson, I., Schild, A. and Schucking, E. L.), p. 221. University of Chicago Press.
Terrell, J., 1964. *Science*, **145**, 918.
Van der Laan, H., 1966. *Nature, Lond.*, **211**, 1131.
Williams, P. J. S., 1963. *Nature, Lond.*, **200**, 56.