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RESEARCH MEMORANDUM

STUDIES IN THE ECONOMICS OF TRANSPORTATION [*]

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Introduction

by Tjalling C. Koopmans

The exploratory studies presented in this report are addressed to analysts in various professions, including economists, traffic and railroad engineers. management scientists, operations researchers and mathematicians, who are interested in assessing the capabilities and studying the efficient operation of transportation systems. Studies relating to two systems, highway traffic and railroad transportation, are offered.

The tasks indicated -- assessing capabilities and appraising efficiency of operation -- are as vast and complicated as the transportation systems themselves. The purpose of the present studies is to develop and illustrate certain concepts, methods and models that may have usefulness as points of departure for the execution of these tasks. While the aim of these studies is thus both modest and provisional, they are offered in the hope of stimulating further factual, conceptual, mathematical and computational research into the efficient utilization of transportation systems.

The method employed consists in the construction of simple models. The word "model", frequently used in engineering studies to mean a physical model, that is, an accurate copy of the system studied (often with the scale altered), is here used in the more general meaning attached to it by physicists and social scientists alike. A model here means a simplified conceptual counterpart of the system studied. In such a model the most important variables of the system studied are enumerated and defined, and the relevant relationships between them specified. Variables and relationships thus express the most essential aspects of the system in question but leave out many other aspects, so that an opening wedge for analysis may be provided.

This analysis is often mathematical in character. However, the mathematical underpinning of the studies here presented has been set off in separate sections, marked by an asterisk(*). These sections can be passed over by the reader interested mainly in their results, which are fully described in sections without that symbol.

    *   *   *

The application of the method of model construction to operating or business problems has increased substantially in recent years. To name first a few examples unrelated to transportation, we refer to two studies on inventory policy by Arrow, Harris and Marschak [1951] and by Dvoretsky, Kiefer and Wolfowitz [1952], in which models are designed to help balance inventory costs against losses from stock depletion. Another family of models constitute the general field of "linear programming", a technique to compute programs for the interdependent activities of a large organization. Besides the original presentations by Dantzig [1951a] (jointly with M. K. Wood), [1951b, c, and d] we mention an expository discussion by Dorfman [1953] in terms of a problem of automobile production, and an application to gasoline blending by Charnes, Cooper and Mellon [1952].

Similar studies in terms of a very simple linear model of transportation have also been made. Two mathematicians, F. L. Hitchcock [1941] in the United States and L. Kantorovitch [1942]1/ in Russia, and one economist (the author of this introduction) independently of each other formulated the problem of most economical execution of a given transportation program between a number of locations in the case where the per unit cost of transportation between each pair of locations is independent of the amount transported. This model belongs to the class of "linear programming" models and has served as one of

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1/ This reference was brought to our notice by M. M. Flood [1954].

the stock examples of this class. Dantzig [1951d] explored the computational aspects of the above (linear) transportation problem, which were carried further and applied to a military tanker fleet problem by Flood [1953], [1954]. Economists studied various implications of this model, such as the relative costs of alternative possible changes in the transportation program and the relationship of these cost ratios to freight rates formed in competitive markets (Koopmans [1947], Koopmans and Reiter [1951]) as well as the relation of freight rates to interregional price differences and movements of goods (Enke [1951], Samuelson [1952], Fox [1953]). An extension of the model to a situation where places of origin and destination are continuously distributed in a plane was given by Beckmann [1952]. An important general result of the studies mentioned is that, under the circumstances of the simple model described, a competitive market solves the most economical routing problem as efficiently as a centrally directed transportation organization could.

From the point of view of the efficient utilization of the transport systems of road and rail, the applicability of the simple linear ("constant cost") model is rather limited. It ignores all phenomena of congestion, either at terminals, or en route. Perhaps the most important case where this assumption is approximately satisfied is that of ocean or lake shipping between uncongested ports. The linear model further ignores indivisibilities, such as arise from the bunching of a number of railroad cars into a train for which one indivisible engine provides traction. To extend the analysis of transportation systems by model construction, it will therefore be necessary to "forget" the linear model, and take a fresh look at the technological and organizational circumstances of various transportation systems. However, before leaving the linear model, let us point out one practical application of importance for the individual business firm that has plants in several

locations. This application arises if, even though actual cost of trans-
portation on each route actually depends on the amount transported, a public
carrier makes the service in question available at a constant freight rate.
The linear model then suffices to show this firm how to minimize its bill
for transportation from plants to customers, even though the minimum-billed-
cost flows of goods so computed have no necessary relation to the best util-
ization of the transportation system.

<p style="text-align:center">*   *   *</p>

In our study of highway traffic, the main emphasis is on the effects
of traffic congestion. Congestion phenomena have been subjected to mathematical
analysis in the theory of telephone systems by Erlang (in Brockmeyer, et al.
[1948])and Palm [1943], and more recently in a more general analysis of queues
(Kendall [1951]) such as arise in many situations: people waiting for service
in a bank, ships waiting for access to port or repair facilities, airplanes
circling to land, pedestrians waiting for an opportunity to cross a street,
etc.

The main purpose of these studies has been to determine, by means of
the calculus of probability, how the average waiting time, and the extent of
fluctuation in individual waiting times, depend on the opportunities for
servicing (number of servers, average of and fluctuations in service time),
and on the amount and irregularity of the inflow of claimants for service.
In Chapter 1 of this report, queue theory is applied to such traffic situations
as: the flow of cars through an intersection, and the passing of slower cars
by faster cars through the use of gaps occurring in the opposing traffic
stream. In addition, in Chapters 3 and 4, the main results of such studies

are incorporated into the assumptions of a (non-linear) model of highway traffic on a road network.

It was said at the beginning of this introduction that one important objective to which our studies are ultimately directed is the determination of the capabilities of transportation systems -- in the present case of a road network. It will now be clear that this cannot be expressed in a single number like so many vehicle-miles per day. It is an essential aspect of congestion phenomena that -- up to a limit beyond which overloading starts -- more "service" can always be obtained at higher unit cost. Hence, the capability concept to be analyzed has more dimensions than a single number. Even for one individual one-way road, it is represented by a curve, which gives the relationship between the flow of traffic through the road and the cost encountered on it, in terms of the money equivalent of travel time, fuel and depreciation, and possibly risk or other sacrifices measurable in money equivalents. The higher the flow, the greater the cost encountered. The time element alone, probably the most important cost factor, is represented in the "capacity curve" of the traffic engineer. In this curve, flow is set off against average speed, the reciprocal of travel time. This curve and the concepts associated with it, are presented and discussed in Chapter 1, which is based on study of the relevant traffic engineering literature. For a road network, the capabilities are expressed by the relationship between traffic flows on all routes and the costs encountered on each as a result of these flows.

A theory of highway traffic should of course go beyond a description of the capabilities of a network to a study of how these capabilities are utilized. This introduces the concept of demand for transportation. The flows of highway traffic are the result of a great many individual decisions about destinations,

routes and preferred speeds. There is somewhat of an analogy between highway traffic equilibrium and the models used by the economist to explain quantity sold in a market by the interaction of demanders and suppliers. The behavior of demanders is summed up in a "demand curve", which states the amounts taken off the market at various alternative prices. Similarly, the supply curve states amounts offered for the various possible prices. Actual price is then determined, by intersection of the curves, to be that price at which demand equals supply.

On the supply side the analogy is not a close one. Except for toll roads, there is no party in the market selling access to roads for a price. Each piece of road not occupied or endangered by another vehicle is free for use by whomever is near. However, there is a cost of transportation, incurred individually in terms already discussed. If we regard this cost as the "price" in the transportation "market", the economist's notion of a demand curve does become applicable. For a single one-way road it would state what flow of traffic is demanded at any given cost encountered on that road. The higher the cost, the smaller the flow demanded, other things being equal. For a road network, the demand function would state what flows are forthcoming on each route in response to given transportation costs along these routes. Equilibrium is established if the flows on all roads arising in response to given costs are just of the magnitudes that produce these same costs.

The demand concept is developed in Chapter 2. It is applied in Chapter 3 to a study of traffic equilibrium on a highway network. The stability of this equilibrium is also discussed, and some observations are made about the use of the analysis in the prediction of traffic flows.

Mathematical tools used in this analysis (in particular Sections 3.1.2, 3.1.3, and 3.2.1) may also have an interest to the mathematical economist apart from their present application to highway traffic.

The analysis of demand and of equilibrium recognizes that freedom of choice of destination, time of departure, route, and speed, within the traffic laws and general safety considerations, are part of the services rendered by the road network. Completely regulated traffic, such as in the truck convoys of an advancing army, can wrench higher rates of flow out of a given network than a traffic system that secures these choices to the individual because they have value to him. There is, however, one particular aspect of this freedom of choice which does the totality of road users more harm than good. This arises from the fact that the choice of route -- once a trip is decided on -- is quite naturally made so as to minimize cost (in terms of delay, risk, nuisance of congestion, etc.) to the individual driver who makes the choice, without reference to delays caused to other users of the roads in question as a result of his choice. We can illustrate the effect of this circumstance by imagining a completely selfless driver, who (a) is fully aware of all delays and other costs he causes others and (b) gives the same weight to everyone else's cost of transportation as he gives to his own. Let such an ideal driver be faced with a choice between two routes, one congested, the other uncongested but with somewhat higher travel time. Then our Mr. Milquetoast will choose the more time-consuming route if the extra time and other cost of that choice to him amounts to less than the extra delays and other costs he would cause to others by choosing the congested route.

It is argued in Chapter 4 that, if all drivers were of this highly informed and selfless type, more valuable service could be obtained in the aggregate from the road network. Perhaps not much can be done about this particular inefficiency of the traffic system. However, it should be emphasized that the difficulty does not arise from free choice as such, but from the fact that the chooser does not bear the full cost (to others as well

as to himself) of his choice. If there were a way to collect tolls from the users of congested roads, at rates that measure the cost to others caused by the average road user, while the revenue so collected is used to lower gasoline taxes or otherwise benefit all road users, then a better use of the highway system would be obtained. Chapter 4 analyzes as a theoretical proposition how the amounts of such "efficiency toll rates" would be determined. It also contains some observations as to how closely maximum efficiency can be approached by proper choice of the toll rates on roads that are at present toll roads, or by other ways of penalizing additions to traffic congestion.

In this discussion, tolls are looked upon, not as a means of financing road construction, but as a means of bringing about the best utilization of the highway network. This is in keeping with the growing acceptance among modern economists of the proposition that best use of facilities requires methods of pricing the services of these facilities that reflect the incremental cost attributable to each service demanded by an individual user. Because of the non-linearity in the relation between amount of use and cost, such pricing does not necessarily produce revenues equal to the total cost of operating and financing the facility. This same principle has been applied by William S. Vickrey [1952] in formulating proposals for fares in the New York subway which would diminish congestion by providing incentives for traffic to shift from peak to off-peak hours and to encourage fuller use of off-peak service by lower fares at these times when incremental costs are low. It is also basic to contemporary theory of electricity rates (see, e.g. H. S. Houthakker [1951]).

It will be clear that these considerations leave unanswered the question of criteria for extensions or improvements of the network to relieve congestion. Some observations on the latter question that flow from the present analysis are given in Chapter 4, Section 4.4.6, and also in Chapter 5, which makes brief mention of many unsolved problems of traffic theory and analysis. However the main part of the present study concentrates on what the economist calls "short run" problems. The road network, represented by a configuration of roads and a capacity curve for each road, is taken as given. Demand for traffic on each route is represented by a fixed function of current cost; that is, if after considerable fluctuation cost were to return to its former level on each road, then demand would also return to its former level on each route. Thus the more gradual responses of demand to changes in cost that arise from relocation of residences, stores or plants are not taken into account. It is believed, however, that the present analysis can be useful as a starting point in developing a theory of balanced extension of the highway network, concurrently with industrial expansion or relocation. The increased vulnerability of metropolitan areas under modern warfare adds a note of urgency to the development of such a theory, already highly desirable before this complication arose.

*   *   *

Differences between our studies of railroad transportation and those of highway traffic reflect the different characteristics of the two transportation systems. The fact that highway traffic results from the interdependent choices of many decision makers, each with an objective of his own, gives to traffic theory a strong social science flavor. In a railroad system, operations are

at least in principle centrally directed. On the other hand, the technical
aspects of railroad operation are a great deal more complicated than those
of highway use. Hence, our exploratory study of models of certain railroad
operations is somewhat closer to physics or engineering. In order to assess
and describe the capabilities of a railroad network, it is again necessary
to construct simple conceptual models of various parts of railroad operation.
Our study, which makes a start with this task, confines itself almost entirely
to the supply side of railroad services. One exception to this is the dis-
cussion in Chapter 6, Section 6.1 of the value to the shipper of speedy trans-
portation. Another exception is the discussion in Chapter 12 of best routing
patterns for empty cars, to which we return below.

It is probable that congestion phenomena, which hold the center of
attention in our analysis of highway traffic, are important also in railroad
operations. However, it has appeared to the authors that other aspects of
railroad operation require prior attention. One of these arises from the
fact that in most circumstances it is economical to haul cars in trains rather
than individually. This introduces the problem of "accumulation delay", the
car-time spent waiting for enough traffic to accumulate so that a train can
economically be formed. Another is the problem of classification, that is,
the problem of sorting cars that arrive in incoming trains or are delivered
from loading tracks, so as to make up new trains that will take these cars
to the next sorting point, closer to or at their destination. The problem
is how to distribute this sorting work over classification yards in a way
that minimizes cost of classification plus the money equivalent of accumulation
delay.

Basic concepts to make possible an accurate formulation and treatment
of problems of this kind are introduced in Chapter 7. In Chapter 8, the

operations of classification yards are described, and simple approximative formulae are proposed for the dependence of classification cost on the classification task performed. These formulae are used in Chapter 9 in a general discussion of the distribution of classification work over yards.

An earlier version of the material in this chapter and the one preceding it was presented to a meeting of the Railway Systems and Procedures Association, held in Chicago on November 5, 1953, and was published as an appendix to the proceedings of that organization.$\underline{2/}$ The material is here reused and extended with the permission of the R. S. P. A. Chapters 10 and 11 are more detailed studies of specific problems. Each of these chapters ignores what the other concentrates on. Chapter 10 is devoted to the problem of distributing classification work between a hump yard and a flat yard located down the line from the hump yard, when problems of scheduling are ignored. In Chapter 11, classification cost is ignored, and instead the problem of scheduling trains between yards to minimize accumulation delay is discussed in detail for a single-line railroad, and in more general terms for more complicated railroad networks.

From this summary the reader will see that time and resources available for the study did not permit us to construct a model that simultaneously incorporates all the main aspects of railroad operations, on the basis of which one could, for instance, discuss the best dovetailing of classification, scheduling and line-hauling operations. For that purpose, further "partial" models would be needed first, such as a model for the study of track capacity. In addition, considerable complications of a purely mathematical character would arise in the attempt to put the partial models together into a model that would fully express the interdependence of the main elements of railroad operation. It is felt, however, that the first steps on the road to a more

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$\underline{2/}$ Beckmann, Koopmans, McGuire, and Winsten [1953].

integrated model have been made, and that this in itself justifies the publication of these studies.

A more integrated model would have several uses. It would help in computing the capabilities of a given railroad network, and the rolling stock requirements of any transportation program that is within the capabilities of the network. It would also facilitate estimating the "incremental" or "marginal" cost (i.e., the cost increase) occasioned by the rendering of an extra unit of service.

The importance of the latter consideration lies in the fact that freight rates are an important element in business decisions about industrial locations and about modes of transportation used. Only if the rate on each unit of service reflects its incremental cost can we expect that such decisions, taken by profit-seeking entrepreneurs in response to freight rates and geographical price differences, will lead to the most efficient utilization of the nation's resources. It is true that this view held by most economists[2] has not been accepted by regulatory bodies as relevant to rate making. However, the economist's case is likely to remain just a nice point of theory unless the operations of railroads are analyzed to the extent necessary to provide a method of estimating the incremental costs of transportation services rendered.

It may be useful here to recall the main properties of incremental cost in the very simplest linear model of transportation mentioned at the beginning of this introduction, where both congestion and the fact that cars are lumped into trains are ignored. In this model "efficiency freight rates", that is, rates reflecting incremental cost, are relatively low, per mile, in directions in which empty cars proceed regularly, and relatively high in opposite directions. On routes not travelled by empty cars the efficiency freight rates

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[2] See, for instance, J. Dupuit [1844], H. Hotelling [1938].

have intermediate values, which can be determined in the same calculation by which the best routing plan for empty cars is determined.[1] In any more refined model, incremental cost freight rates are likely to exhibit these main features, with the effects of congestion and lumpiness superimposed as modifications. It is therefore worthwhile to examine the pattern of empty car originations and terminations associated with the movement of goods on U.S. railroads, the pattern of best routing of empty cars between origination and destination points, and the stability or variability of this pattern between years. The examination of this question in Chapter 12 reveals a substantial stability of best routing patterns of empty boxcars -- and hence of incremental costs of transportation of goods shipped in these cars -- as between peace-time years, and remarkable changes connected with war-time movements of supplies to Pacific coast ports.

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This report has resulted from a research project carried out by the Cowles Commission for Research in Economics under contract with the RAND Corporation. Tjalling C. Koopmans was the research leader of the project. The several authors came to this study with different skills and backgrounds, and accordingly contributed in different and complementary ways to their common task. Martin Beckmann, a mathematical economist especially interested in linear programming and economic activity analysis, contributed most of the chapters of the highway traffic analysis, with the exception of Chapter 1 on capacity. Christopher Winsten, mathematician and economist with a special interest in applications of probability calculus to industrial phenomena,

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[1]　See Koopmans and Reiter [1951], Dantzig [1951e].

contributed the analysis of queues reported in Chapter 1, and the analysis of the division of sorting work between yards. given in Chapter 10. C. B. McGuire, economist, was assigned primary responsibility for the degree of "realism" of the models developed by the group. For this purpose, he gave most of his time in earlier phases of the project to study of the literature on traffic analysis and on railroad operation, to interviews with traffic specialists and railroad officials, to visits to classification yards, and, assisted first by Marc Nerlove and later by Thomas Goldman, to the analysis of operating records. Most of the chapters on railroad problems are the result of joint work by McGuire and Winsten. Chapter 11 was contributed by Koopmans. Chapter 12 was prepared by McGuire on the basis of earlier work by Kirk Fox, Marc Nerlove, Harlan Suits and Thomas A. Healy. The computations for a simple example of a highway network were prepared, and the report thereon in Section 3.3.2 written, by Goldman.

This dry enumeration of contributions does not indicate the extent to which practically every chapter has been affected by the thinking of all members of the group. A good deal of group discussion has been devoted in particular to choice of concepts and models of railroad operations. In the later stage of preparation of manuscript, the group met in weekly sessions for criticism and evaluation of successive drafts. Final editing of the manuscript was done by McGuire with the assistance of James M. Terrell.

Proper acknowledgment cannot be given in this space to all those who helped the authors in their project. Especially deserving of mention, however, for the patient way they dealt with the authors' questions, are the following men in the railroad industry: C. H. Bremhorst, L. H. Dyer, and

PART I

A STUDY OF

HIGHWAY TRANSPORTATION

Chapter 1

Road and Intersection Capacity

1.1 Introduction.

In most cases where an attempt has been made to measure the capacity
of a road or intersection, this capacity has been taken to be a number,
representing the highest possible flow of traffic through the facility
being studied. Accepting this view for a moment, let us examine a particular
case in some detail and see into what complications we are led. Suppose
a certain unsignalled intersection is used by eastbound and northbound
traffic only. What is its capacity? Obviously no one number will suffice
to describe the capacity of the intersection for northbound flows alone,
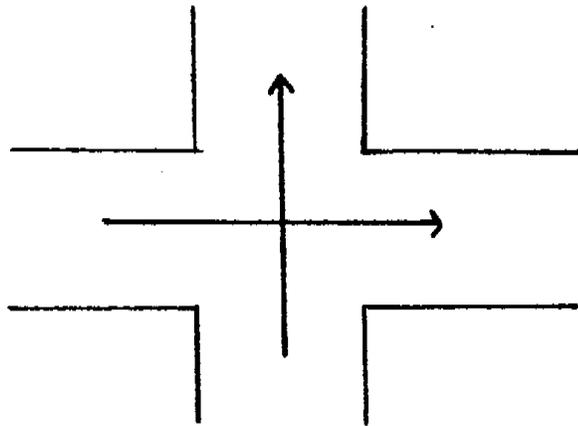
Figure 1

for it is quite clear that the more eastbound traffic there is, the less
will be the amount of northbound traffic that can get through unless this
latter flow in some way dominates the intersection, and we rule this out

for the present. Perhaps, however, it is possible to express the capacity in terms of a number, say 1600 vehicles per hour, which is not to be exceeded by the sum of the two flows. If this is so, then this capacity can be expressed in the following lengthier but more instructive way. The capacity is

	0	vehicles per hour northbound and	1600	vehicles per hour eastbound
or	100	" " " " "	1500	" " " "
or	200	" " " " "	1400	" " " "
	.			
	.			
	.			
	.			
or	1600	" " " " "	0	" " " " .

This set of combinations written out in full says precisely the same thing as the shorter defining sentence preceding it; we can if we wish therefore always describe such capacities in terms of the various highest possible combinations of flows. Since the set of such combinations is rather tedious to write down, it is convenient to describe it graphically in the following way.
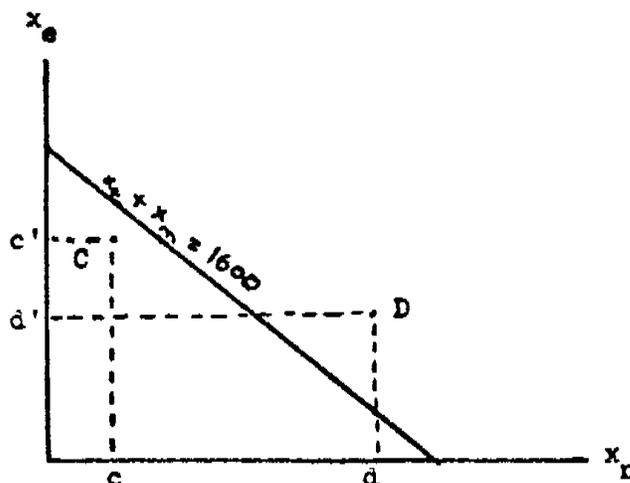
Figure 2

The set of capacity flow combinations can be represented graphically in Figure 2 as all those points with non-negative coordinates x_e, x_n which lie on the downward sloping line whose equation is $x_e + x_n = 1600$. The whole set of combinations is therefore completely described by that line. The capacity expression has thus changed from a simple number to a set - the set of points on the 45° line.

At this point there arises the very natural question of whether the line must always have a 45° slope, or whether in fact it need always be a straight line. Without further information there are no reasons to reject the possibility of capacity curves like those in Figures 3 and 4. Where the capacity curves are of this nature we begin to see some of the advantages of capacity formulations in terms of sets of alternatives rather than single numbers. The capacities described in the last two curves can be represented by sets of combinations just as before, but they cannot readily be represented in a straightforward way by a single number. If we look at the points A and B in Figure 3, we notice that while both represent capacity flow
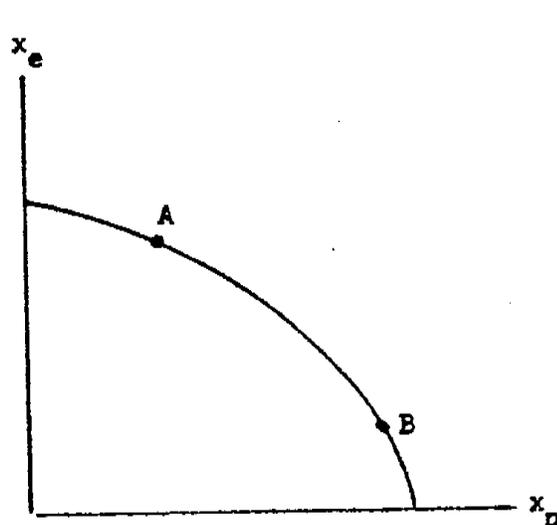
Figure 3
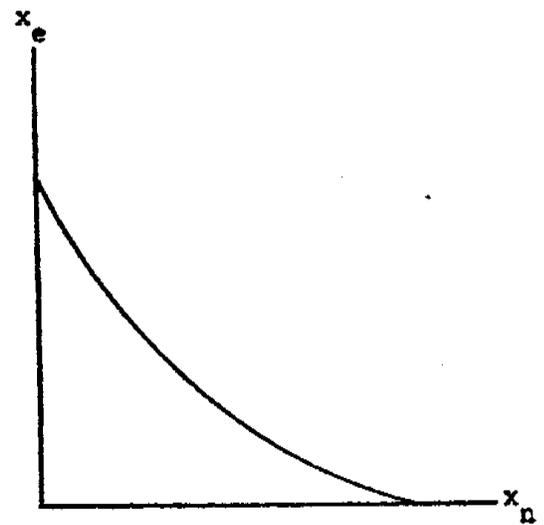
Figure 4

combinations, the total flow represented by A is less than the total flow represented by B. Total flow in these examples is no longer an important element in capacity considerations.

Before leaving this simple intersection example it is worth pointing out that the curve, or set, we have been talking about represents capacity in the sense that it forms a part of the boundary of all those points representing possible flow combinations. Thus in Figure 2, the point C with coordinates c and c' is a possible flow combination since the sum of c and c' is less than 1600, while the point D with coordinates d and d' is an impossible flow combination since the sum of d and d' is greater than 1600. The possible combinations are those represented by points on or below the capacity curve; the impossible ones by points above the curve.

The reader may object at this point that all of this elaboration has been unnecessary, because for most of the important cases the capacity curve does in fact just happen to be a straight line sloped at 45° so that capacity expressed as a limitation on total flow is really all that is needed. If all of the interesting cases were like the ones in Figure 2, the objection would indeed be well taken. We intend to show however that as soon as the capacity notion is complicated somewhat to make it more useful from an economic point of view, the interpretation as a set of possibilities becomes imperative. To make this clear, another simple and rather artificial example will be described. So far we have not considered the driving conditions which the traffic will meet, even when the flows are possible ones. One way of introducing these conditions into a definition of capacity is given in the <u>Highway</u> <u>Capacity</u> <u>Manual</u>,[1] where, very briefly:

[1]   Normann and Walker [1949, pp. 6-7].

basic capacity is the maximum flow under "most nearly ideal

roadway and traffic conditions";

possible capacity is the maximum flow under "prevailing roadway

and traffic conditions";

practical capacity is the maximum flow short of causing "unreason-

able delay, hazard, or restriction to the drivers'

freedom to maneuver."

Thus for each traffic condition specified there is a corresponding maximum

flow.

Alternatively capacity may be defined as a relation showing how traffic

conditions depend on flow. This is to be preferred if the ambiguity of

terms like "unreasonable" is to be avoided. This procedure means that no

two or three arbitrarily selected traffic conditions are concentrated on,

but rather the whole range of traffic conditions is examined, just as the

whole range of flows is.

The word "delay" in the description of the traffic conditions that

define practical capacity covers what is probably the most important element

in traffic conditions. Conditions are good if delay is small; they are bad

if delay is large. In the following example, and in fact in nearly all of

the subsequent discussion in this chapter we shall suppose that "traffic

conditions" are fully described by an assessment of the delays that occur.

Let us imagine a certain vehicle inspection station at which all vehicles

traveling a particular road must stop. Suppose that the inspection takes

exactly two minutes for each vehicle. that only one vehicle can be inspected

at a time and that the arrival of vehicles at the station is known to be

random. What is the capacity of the station?

The example would be completely uninteresting were it not for the fact that as flows become larger, "traffic conditions", that is, delays, get progressively worse. Any particular car has not only to wait the two minutes until its own inspection is completed, but until the inspections of all cars waiting when it arrived are finished as well. The heavier the traffic, the more cars it is likely to find waiting ahead of it. If the average delay that cars suffer is plotted as a function of flow, a curve like ABCD in Figure 5 results. For very small flows the station will seldom be occupied and the average delay will be close to two minutes. As flow approaches 30 vehicles per hour, the station will usually have several vehicles waiting in line and the average delay is likely to be large.
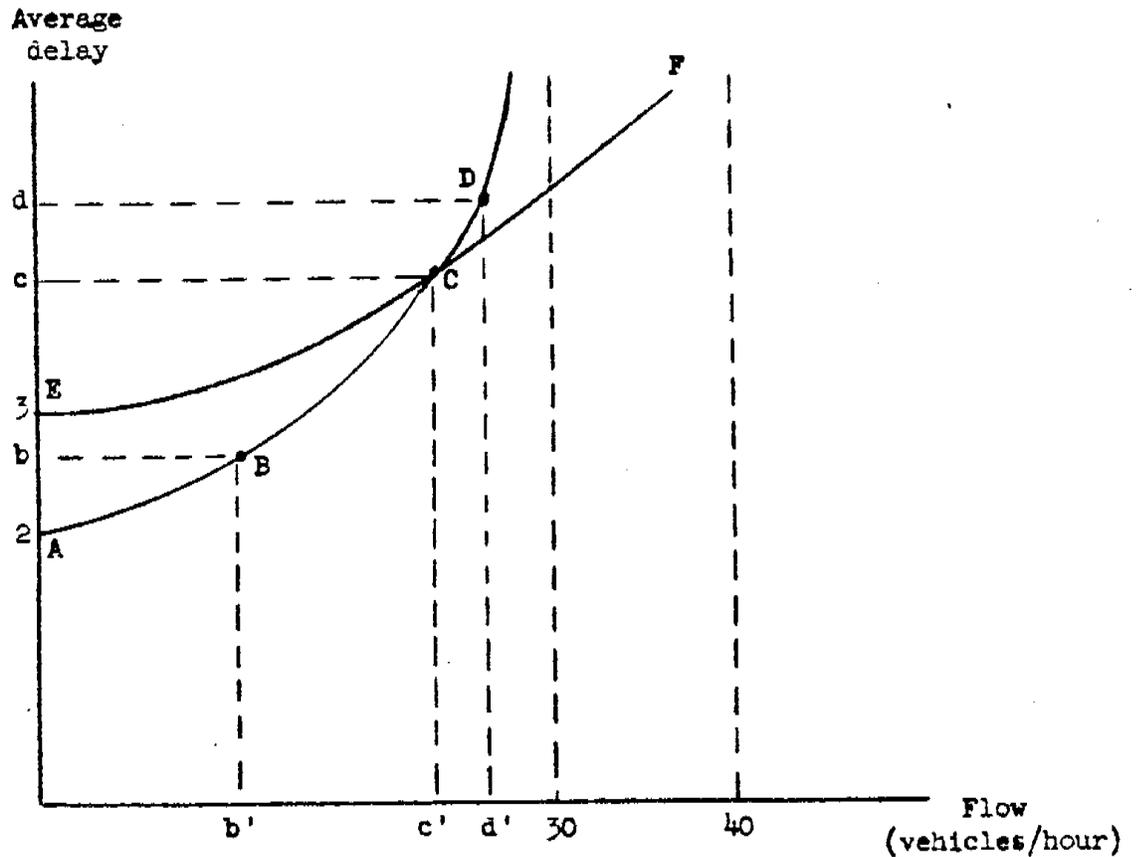
Figure 5

Proceeding just as in the intersection example we can express the capacity as the set of combinations

    0   vehicles per hour and 2 minutes average delay

or b'   "   "   "   and b   "   "   "

or c'   "   "   "   and c   "   "   "

or d'   "   "   "   and d   "   " .   "

etc.

Now imagine another station at which two vehicles can be inspected simultaneously, but where such inspection takes exactly three minutes. The capacity curve for this station is EF in Figure 5. With flows close to zero the average delay is three minutes, a little greater than for the other station. At C where the two curves cross the advantage of the first station in terms of a shorter service time is exactly compensated for by the second station's ability to deal with congestion. Thus, in a sense, for small flows the first station has the greater capacity; for high flows the second station has the greater capacity. Such a capacity comparison would be arbitrary indeed if it were made on the basis of knowledge of delays at the two stations corresponding to one particular flow, or on the basis of knowledge of the flows at the two stations corresponding to one particular value of average delay.

Note that, if we were to plot a similar function for the intersection example given above, then for each combination of flows which the intersection could handle, we would have an average delay. The capacity curve shown in the figure might now be supposed to separate the combinations of flows which had acceptable delays from those where the average delays were too large to be acceptable. When we are considering two flows in this way,

we could break this average delay down into two parts, those experienced by each of the two streams. What kind of functions of the traffic densities these delays are depends especially on the traffic rules governing the intersection. This question will be discussed in Section 1.2 below.

With this introduction, it is time to enter into a more detailed discussion of the capacities of various types of intersections and roads.

1.2 The Capacity of Intersections.

We have sketched above the notion of capacity we intend to use. Before we can develop quantitative expressions for this capacity notion, we must consider in more detail the rules governing traffic at intersections, and we must specify the simplifications inevitably involved in constructing a model of traffic behavior. Some of the previous work on intersection capacity will be discussed, and a new model will be given, which it is hoped, will be useful for quantitative prediction in some situations.

1.2.1 The Stop-Sign Intersection.

In a simple type of such an intersection, a minor road crosses a major road. It is assumed that traffic in the minor road must not interfere with major-road traffic. Hence a car ready to cross the major road must wait until there is a sufficient gap in the major-road traffic. The burden of judging when a gap is sufficient lies on the minor-road driver. It is possible for these judgments to vary considerably, both between one driver and the next, and between different intersections. Drivers may vary because of different degrees of cautiousness, ability to pick up speed, and so on. Intersections vary in visibility, width, and many other factors. It is necessary to consider which of these factors to bring into a quantitative model, and just how they should be brought in.

1.2.2 Intersections Controlled by Traffic Lights

Traffic lights may have various rules of operation, and the delay will depend on which of these is being used. The most common is the fixed repeated cycle. With this rule a green interval of fixed length is followed by a red interval of fixed length (though the length of the red interval may very well differ from the length of the green one). The whole cycle is repeated indefinitely. There may be additional warning intervals between the green and the red intervals or between the red and the green. There are some possible variations; for example, lights can be made to change their cycle according to the densities of the traffic in the two roads. Some types of lights give priority to one of the roads, but we do not discuss these here.

In both the stop-sign case and the traffic-light case, cars turning left or right are an additional cause of variation. Suppose, for example, that at a stop sign a car in the minor road wishes to make a left turn into the major road. It will have to wait for a gap in the major-road traffic, and also for a gap in the minor-road traffic in the opposing stream. Thus it will on the average have to wait longer than a car going straight ahead, and it is likely to delay traffic behind it longer. Thus the delay at the intersection may well depend on the number of cars making left turns, and on the density of traffic in the opposing stream in this situation. Similar considerations apply to the traffic-light case. These complications are mentioned to show that quantitative statements about delays at intersections, and their relations to traffic flows will not be of completely general application, whether they are derived from theory or from experimental observation of actual road conditions. Many intersections may present special features which call for modification of the results.

1.2.3. Related Work in the Literature.

Before we present our contribution to the theory of intersection capacity, we will give some very brief references to related work described in the literature. The discussion has no intention of being exhaustive; it is only meant to call attention to some attempts to develop a quantitative theory of congestion at intersections.

In order to develop a theory of delay, we have to have some way of describing traffic flow. Because of all the various unrelated factors which determine whether there will be a car at a particular point at a particular time, it was suggested by Adams [1936] that a probability model might give a good fit to some classes of traffic. Adams suggested a special type of probability model called the Poisson process as a possible description of the times at which cars pass a particular point. His experimental work confirms that this process gives an adequate description for some roads, and for some purposes. It has since been used in work on traffic congestion by other writers: Garwood, Greenshields, Raff, and Tanner, for example.

A theoretical treatment of stop-sign intersection delays is given in papers by Raff [March, 1951] and Tanner [1952]. Both use the same assumptions and simplifications. It is supposed that cars in the minor road do not interfere with each other. This assumption will be a valid one if traffic in the minor road is sparse. Tanner frames his discussion in terms of pedestrians crossing a road; since pedestrians can usually cross in groups if necessary, all those waiting can cross almost as soon as there is a sufficient gap in the traffic. Thus it seems a good approximation to suppose that the length of time a pedestrian will have to wait to cross does not depend materially on how many other pedestrians are using the

crossing. In other words, the assumption of non-interference seems a valid one in this case too.

It is also necessary to have some more precise definition of "sufficient gap". In the theories developed so far the notion of a fixed critical time gap is used. It is supposed that the driver in the minor road will only cross if there is no car in the major road due at the intersection in the next w seconds, say. Under these circumstances w is called the critical gap.

We can represent the arrivals at the intersection of cars in the major road as points of a time axis, as in Figure 6.

$$\text{———×————— ×—×———— ×———— ×———— time}$$

Figure 6

By using the notion of a fixed critical gap, we can, if we are given a plot of the traffic of this sort, say whether a car in the minor road can or cannot cross at any given time. It is natural to call the time interval during which a car in the minor road is not able to cross a red interval and the time interval in which it can cross a green interval. Thus, if we have a plot of arrivals in the major road, with our assumption of a fixed critical gap, we can divide up the time into a sequence of intervals, alternating red and green. This procedure is illustrated in Figure 7, where the cars on the left end of the time axis are later arrivals than those on the right.
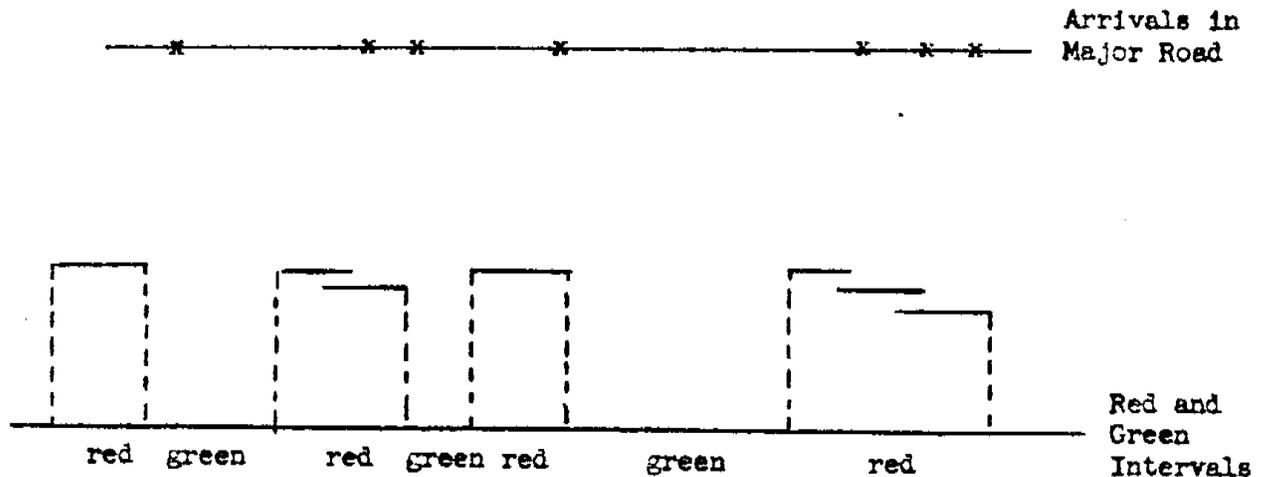
Figure 7

A car arriving in the green interval will not have to wait, but a car arriving in the red interval will have to wait just until the end of that interval. As we are supposing there is no other car in the minor road in front of it to delay it, it will be able to leave at the start of the green interval.

To find the average wait in the minor road, we have to study the distribution of the lengths of red and green intervals. The papers by Raff and Tanner do this for the case in which the arrivals in the major road can be described as a sequence formed by a Poisson process. The assumption that sufficiently large spaces occur between vehicles is the only one necessary for the minor road.

Notice that the reasoning in this case can easily be extended to the case where the critical gap accepted varies from car to car. If we know the relative frequencies of the different critical gaps, we can find the

mean delay for each one, and then average the results using these relative
frequencies as weights. In fact, Raff presents empirical evidence which
shows very clearly that there is considerable variation in the gaps drivers
consider long enough to allow them to cross, so that a refinement of this
sort may be useful. In the case of dense traffic in the minor road, which
we treat later, there is no such easy generalization to the case of a vari-
able critical gap, and we must for the time being be content with the approx-
imation that supposes it to be constant.

Some methods of calculation for the repeated-cycle traffic-light case
are presented by Garwood [1940-1941] and Raff [1950]. In these papers an
attempt is made to deal with the phenomenon of "sluggishness". If a line
of cars is waiting when the light turns green, it will take some time to
clear. Below we give a new method of dealing with this case, which gives
us a useful approximate formula when traffic is sparse, and which can be
used to give results when traffic is heavy.

1.2.4 A Model of Intersection Delays

The results we have mentioned above for the stop-sign intersection
were restricted to the case for which traffic in the minor road was sparse.
In this section we give a method of extending this type of model to the case
where the traffic in the minor road is dense, and cars are delayed not only
by the traffic in the major road, but by cars in front of them in the minor
road too. The type of model we develop turns out to be useful for the re-
peated cycle traffic light case also.

The basic fact we have tried to express in the model is that a queue
of cars waiting to cross a road cannot clear all at once, and the longer
the queue the longer it will take to clear (the phenomenon already described

as sluggishness). The reason for this effect is that the cars as they leave maintain a certain spacing between each other. In our model we also take account of the fact that cars arrive spaced out too.

In constructing the model, a device has been used which may at first sight seem to have a slightly artificial look about it, but which, in fact, expresses the spacing between traffic in a very convenient way. We suppose that events, such as the arrival or departure of a car, can only happen at discrete points in time, rather as if they were being shown by a motion picture film which moved in jerks. Each of the discrete time-points can be considered a frame of the film. At each time-point at most one car can arrive, and at most one car can cross the road. These conditions make sure that traffic is spaced out as it should be. They also imply the restriction that the minimum time spacing between cars arriving at and cars leaving the intersection is the same.

In addition we suppose that each time-point is either a red time-point or a green time-point. A red time-point is a point at which a car cannot cross the stop sign or traffic light. At a green time-point a car will cross, provided that a car has just arrived or that one or more cars have been held over from the last time-point.

With these conditions we can represent a sequence of events at the intersection in a very simple way symbolically. Suppose we represent the arrival of a car at the intersection by the symbol 1, and that the symbol 0 means that no car arrives. Then, for example, a sequence of the type

$$1 \ 1 \ 0 \ 0 \ 1 \ 0 \ 0 \ 0 \ 1 \ 0$$

shows that cars arrived at the first two time-points, but not at the next

two. A car arrived at the fifth time-point, but not at the next three, and so on.

If we represent red time-points by the symbol r, and green time-points by the symbol g, then a sequence of the form:

$$r \quad r \quad r \quad g \quad g \quad g \quad g \quad r \quad r \quad g$$

shows the first three time-points were red, and they were followed by four green ones, and so on.

We are primarily interested in the delay which cars suffer. To find this delay we must calculate another sequence: the queue sequence. The queue at any particular time-point we define, for our purposes, as the number of cars held over from that time-point to the next. With the simple rules of traffic behavior given above, once we know the arrival sequence, the red/green sequence, and the number of cars waiting at the intersection at some initial time-point we can calculate the queues for all later time-points successively. We illustrate such a calculation below.

There is one more sequence of interest: that of departures. How does the process of crossing the intersection alter the spacing of cars? The departure sequence too can be calculated from the others.

We can now illustrate these sequences. The time-points are numbered $t = 1, 2, 3 \ldots$ and we will suppose that at the beginning of the period no cars were waiting to cross.

Time	1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18
Arrivals	0 1 1 0 0 1 0 0 0 1 1 1 0 1 0 0 0 1
Red/green sequence	g g r r r r g g g r r r g g r r r g
Queue held over to next period	0 0 1 1 1 2 1 0 0 1 2 3 2 2 2 2 2 2
Cars waiting to cross	0 1 1 1 2 2 1 0 1 2 3 3 3 2 2 2 3
Departures	0 1 0 0 0 0 1 1 0 0 0 0 1 1 0 0 0 1

The cars waiting to cross at any time-point are those held over from the previous time-point, together with any car that arrives at the time-point considered.

In previous analysis of traffic delays, the assumption was made that the arrivals of cars could be considered as generated by a Poisson process. This assumption, when it was checked against observation, was found quite adequate in many circumstances. Now that, for our present model, we have distorted nature slightly by making time move in jumps, we have to find an appropriate modification of this way of describing the structure of traffic. The modification we use, a very close analogue of the Poisson process for discrete time measurement, can be called the binomial process. The nature of the process can perhaps most easily be seen by the following way of deriving a sequence from it. Suppose that the flow of traffic is such that on the average 37 cars arrive per 100 time-points. Suppose we draw numbers at random from the set of numbers 1 to 100, replacing each one before drawing the next. When a number from 1 to 37 is drawn we write down 1; when a number greater than 37 is drawn we write down 0. If, as above, we consider 1 to represent the arrival of a car, and 0 the non-arrival, then the sequence of 0's and 1's we obtain in this way can be taken to represent

a possible flow of traffic having the specified average density. To obtain a different average density of traffic, a correspondingly different classification of the numbers 1 to 100 would be used. This average density in the minor road we have represented by α. It is defined as the ratio of the number of cars which arrive in a long period of time to the maximum number which could arrive provided the proper time spacing between cars is maintained. The corresponding quantity for the major road we have called ρ.

We have calculated the average delay for the case when "statistical stability" has been reached. This average is the same as the average over a period of time sufficiently long for the particular state of the intersection at the beginning of our observation to become unimportant. However, the calculations involve the assumption that congestion will not tend to increase indefinitely. The probability model we have used has the property that this event will in fact almost certainly not happen provided the average number of green points exceeds the average number of cars arriving at the inter-section in the minor road. If this condition is satisfied, the queue of cars will certainly vanish again and again, even though at times it may build up to a considerable length.

The critical gap also needs redefining for the purposes of our discrete model. We have called it w, and defined it as the number of time-points before the arrival of a car in the major road which are blocked by the close-ness of that car. For example, if $w = 2$, the two time points before the arrival of the car in the major road will be blocked, and also the time-point at which it actually arrives.

The curves in Figures 8, 9, and 10 present the results of the calculations for this model. Based on the mathematical analysis given in Section 1.5, they show the mean delay in the minor road as a function of the traffic

densities α and ρ in the minor road and major road, and the critical gap v. For a given value of ρ and v, as α increases, the mean delay increases, until at a certain value of α the congestion increases indefinitely. The intersection is not able to accommodate a density of traffic corresponding to that value of α or greater for an indefinite length of time.

It is important to note for purposes of analysis in other chapters, that these average delay functions possess a curvature property called convexity.[2]

The same type of analysis has been made in Section 1.5 for the repeated-cycle traffic-light case. The traffic lights are supposed to be red for r time-points, and green for g time-points. The density of arrivals is represented by α as in the stop-sign case. As in that case the average wait over a long time period is calculated. Again congestion will certainly build up indefinitely unless the number of green points exceeds the number of arrivals over a long period. Thus the average delay is calculated on the supposition that this condition is satisfied.

A most striking by-product of our calculations is the adequacy of an approximation which has been used in the past for sparse traffic. The approximation is based on the supposition that the queue of cars has vanished at the beginning of each red period. Though this would seem a drastic assumption, the results calculated by using it turn out to be valid for nearly all the values of α, r and g for which we have carried through the calculations. The stability condition, for queues not to grow indefinitely, is $\alpha < \dfrac{g}{r+g}$, and the approximation is valid even for α quite close to this limiting value.

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[2]   A definition of convexity is given in Section 3.13.
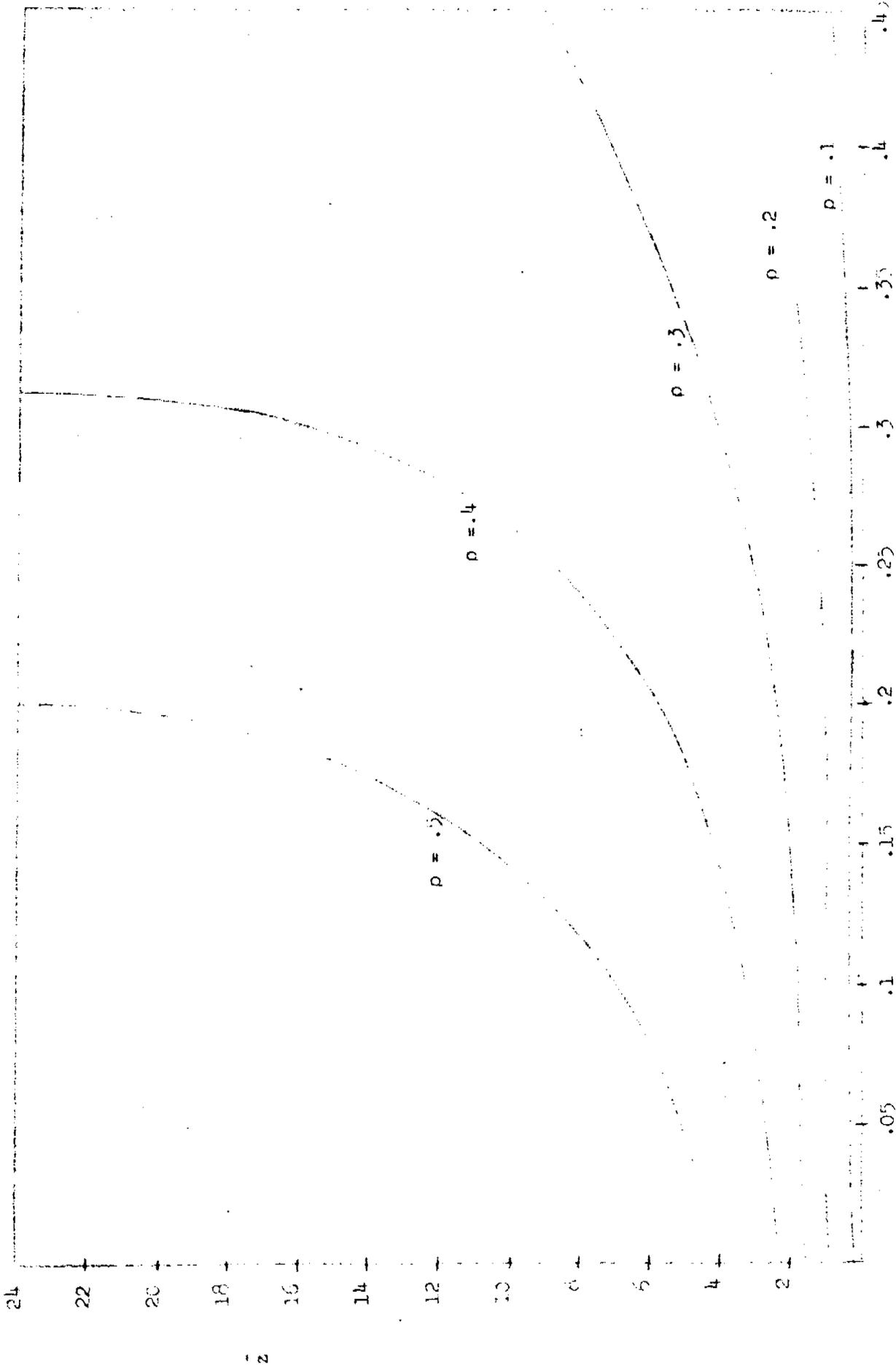
Figure 8. Intersection Delay for w = 2.
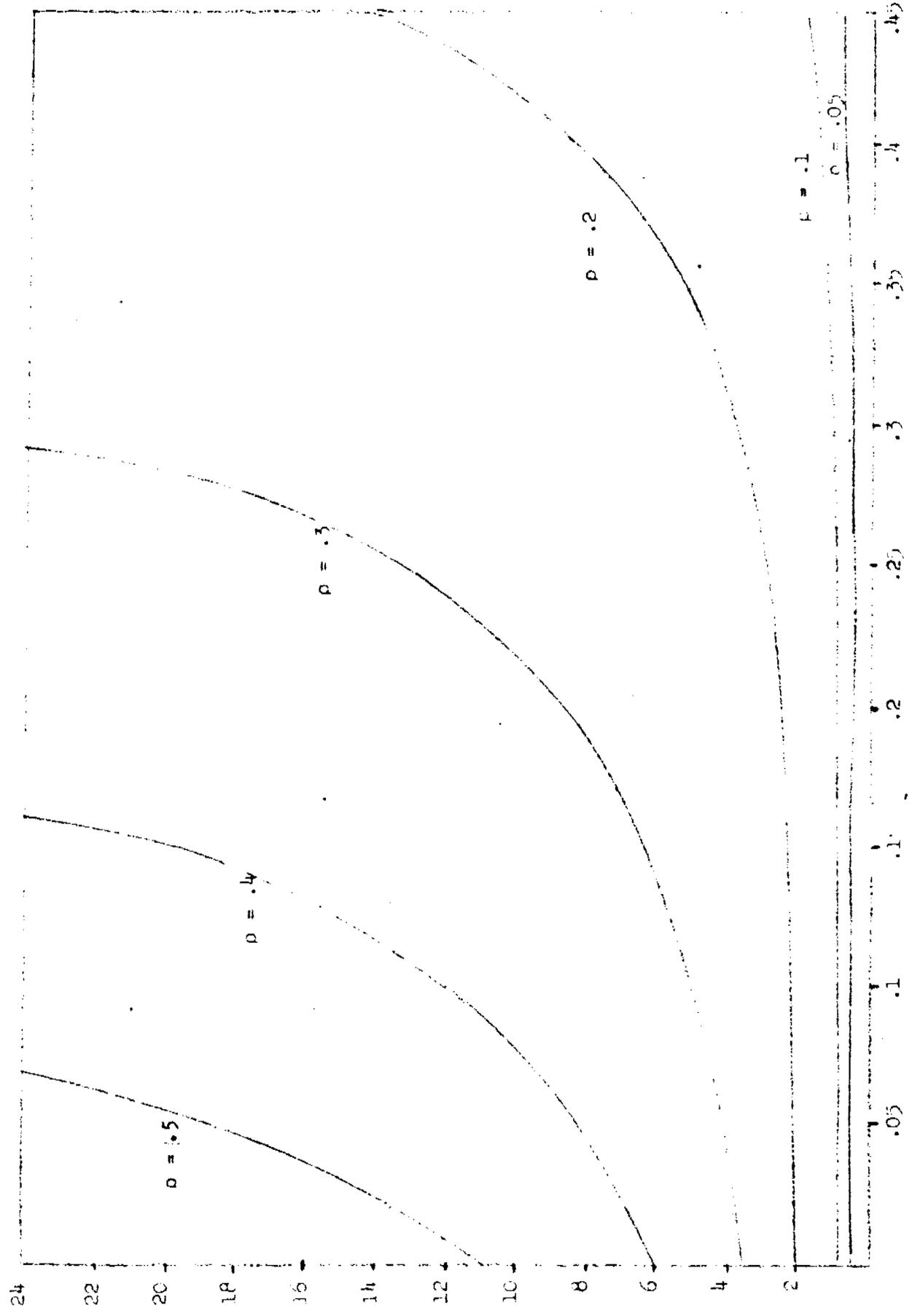
Figure 9.   Intersection Delay for w = 3.
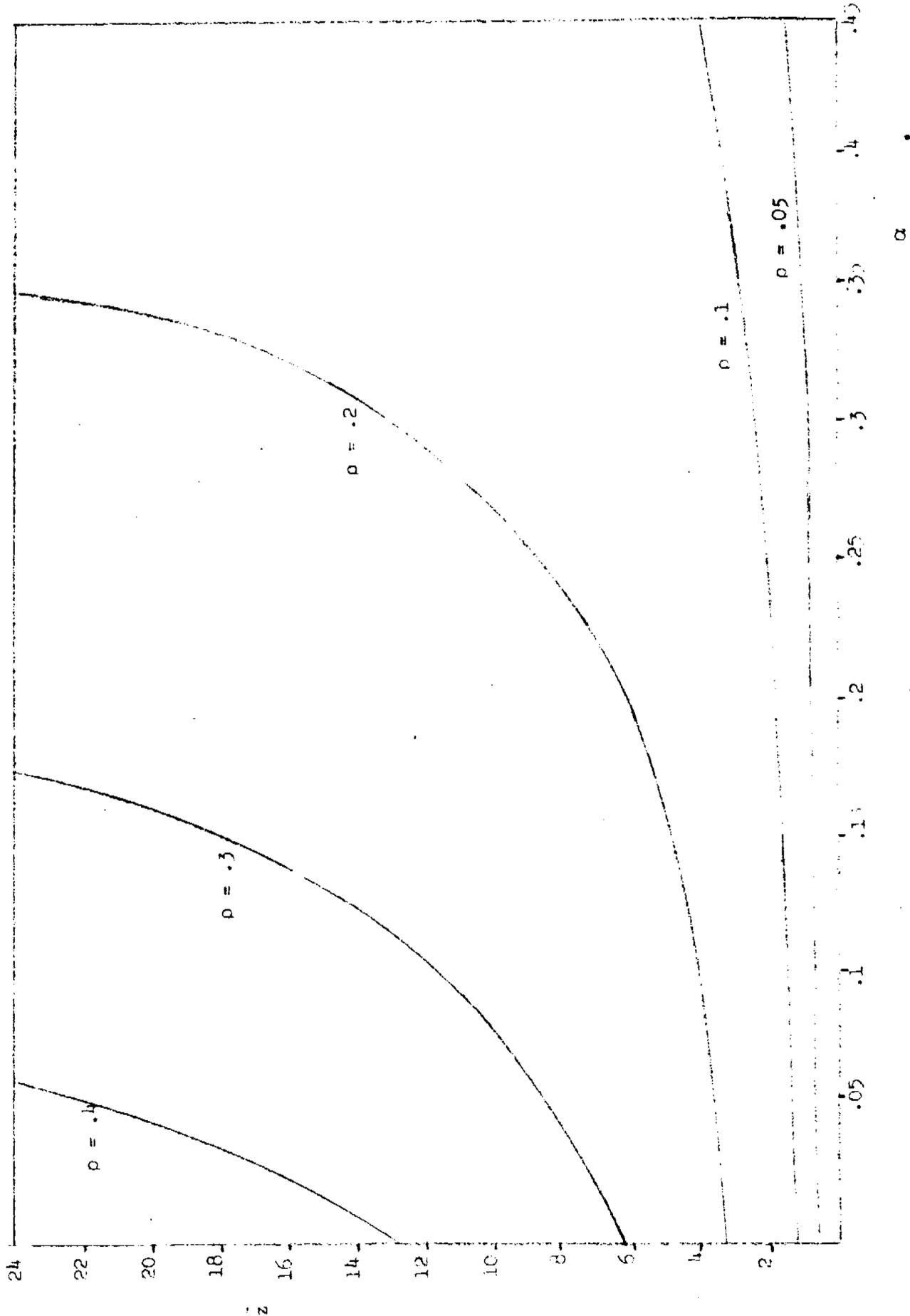
Figure 10. Intersection Delay for w = 4.

1.3. The Uniform-Speed Capacity of a Road

In discussing road capacity we shall first examine a particular type
of capacity curve which will be called the uniform-speed capacity curve.
Among our reasons for giving some attention to this curve are

1) Its possible usefulness in the planning of large _directed_ traffic
   movements such as army convoys, urban evacuations, etc.

2) Its very distinct difference from, and relation to, the more com-
   plicated notion of a capacity curve which is to follow. This differ-
   ence is of some interest in itself for it demonstrates the fact that
   the "capacity" of a facility must be defined with an eye toward the
   use to which the facility is put.

Consider a single-lane road over which traffic moves in one direction.
A _uniform-speed_ will be said to exist when every vehicle moves at the same
speed. At a given point on the road, the _uniform-speed capacity_ at a given
uniform speed is defined as the maximum traffic flow (measured in vehicles
per hour) which can pass that point at that speed under certain assumed con-
ditions. Obviously flow will depend on the distance-spacing between vehicles,
and will be at a maximum for any given speed when spacing is at a minimum,
since flow equals speed divided by spacing.

The assumed conditions are those that affect the minimum spacing between
vehicles at each uniform speed. These include not only such things as the
physical characteristics of the road being examined, but also some character-
istics of the vehicle population being studied, and in addition a statement
of whether the spaces between vehicles shall be determined by reference to
some standard of safety, or by observing the spacing actually maintained by
drivers, or perhaps in some other manner. The main problem involved in
determining these capacities comes from the fact that by any realistic

criterion, the spacing between vehicles must increase with speed. The form this spacing function takes will seriously affect the shape of the curve relating uniform-speed capacity and uniform speed.

Four ways of determining these capacities suggest themselves:

1) Hypothetical spacing functions can be constructed and substituted in the relation

$$\text{flow} = \frac{\text{speed}}{\text{spacing}}$$

to give the maximum flow at each uniform speed.

2) Empirical spacing functions can be determined by observation of existing traffic flows, and they can be substituted in the relation of method (1),

3) Empirical estimates of uniform-speed capacities can be attempted by direct observation of flows, by-passing a determination of spacing.

4) Experiments can be performed.

To our knowledge the experimental method, interesting as it is, has never been carried out. Each of the others will be described below.

1.3.1. Hypothetical Spacing Functions

A minimum "safe" spacing function may be postulated on the basis of information about the distances required to decelerate vehicles from various speeds, and also on the basis of one's beliefs as to what constitutes safety. This has been the most common approach.[3] Without much justification it has usually been assumed that safety requires a distance between vehicles at least equal to that necessary to bring a vehicle to a full stop.[4] The mathematical

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[3] Herrey and Herrey, [1945]
Hess [1950]
Extensive bibliography in Normann [February, 1942, p. 122].

[4] The most notable exception to this statement is the work of Greenshields and Weida [1952, pp. 152-154], where it is argued that full-stop spacing is too conservative an interpretation of safety.

form typically given this spacing function has been

(1.1) $\qquad a + bu + cu^2$

where

(1.2)

a = average length of vehicle

b = perception plus reaction time

$c = \dfrac{1}{2r}$, where r is the maximum rate of deceleration

u = speed

All three values are affected by the composition of traffic (number of trucks, age of drivers, etc.). Time of day and location affect b; and road surface, weather, and grade and curvature of the road affect c. Some of the most careful studies of b have been made by De Sylva,[2] and of c by Moyer.[1] Various values used in constructing these spacing functions are interestingly compared by Normann and Walker.[1] While much disagreement is evident, the values

(1.3)

a = 2.84×10^{-3} miles $\qquad$ (15 feet)

b = 2.78×10^{-4} hours $\qquad$ (1 second)

c = 9.47×10^{-6} hours^2/mile $\qquad$ (i.e., deceleration = 21.6 ft./sec.^{2})

are fairly typical and give an idea of the magnitudes involved. It will be seen that the "safe" trailing distance grows more and more rapidly as speed increases. The resulting uniform speed capacity function which can be written

(1.4) $\qquad \dfrac{u}{a + bu + cu^2}$

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[2] De Sylva [1937].
[1] Moyer [1947].
[1] Normann and Walker [1949, p. 120].

reaches a maximum at some intermediate speed and declines thereafter. This
means that at high levels of speed, the added spacing that is necessary for
a further speed increase more than offsets the direct effects of speed itself
in bringing about an increase in flow. The curve OA'A in Figure 11 is a
uniform-speed capacity curve of this type.

Flow

A'

B

A

O
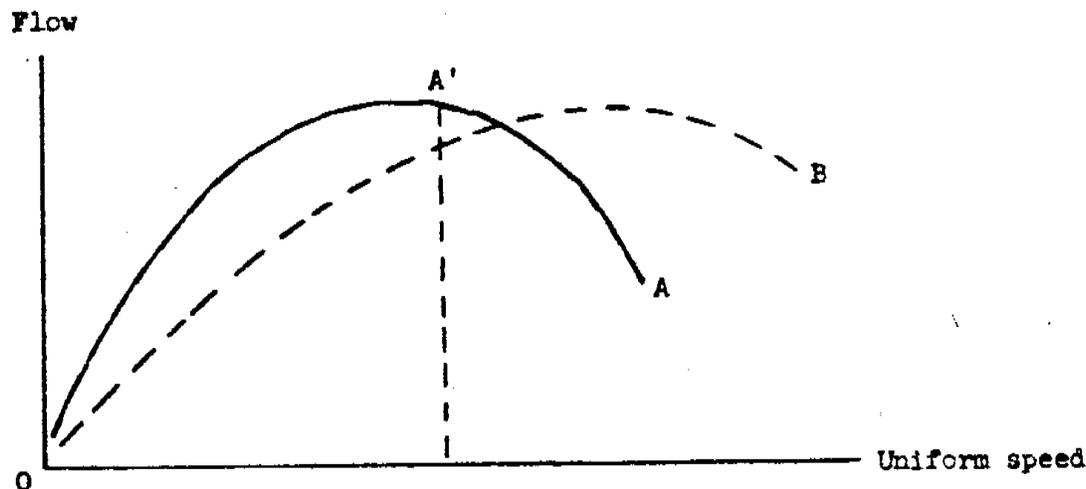
Uniform speed

Figure 11. Uniform-speed Capacity Curves

In most of these studies the uniform speed corresponding to the point
(A' in Figure 11) of maximum flow has been found to be surprisingly low, in
the 11-25 mph range,[8] the maximum flow itself is usually around 1800 vehicles
per lane per hour.[9]

A criticism which might be leveled against this kind of uniform-speed
capacity curve concerns the concept of safety it uses. Minimum safe spacing
is a difficult quantity to define to start with, and furthermore, there is

[8]   Perhaps the feeling that the speed at which flow is maximized must be
greater than 11 or 12 mph accounts for the rather high value of deceleration
cited as typical in (1.3).
[9]   Normann and Walker [1949, p. 120].

ample reason to believe that drivers do not space themselves at distances
which are perfectly safe anyway.[10] While some attempts have been made to
refine this concept of safety in order to secure a better correspondence
between theory and observation, these attempts are more properly classified
under the method to be described next.

1.3.2. Empirical Spacing Functions

An empirical spacing function can be found, and used in the same way
as above to determine the uniform-speed capacity curve. One such procedure,
reported by Normann, was as follows.[11] Of 8,500 vehicles recorded at one
location, about 2,000 were traveling at the same speed as the vehicles just
ahead of them. These were classified into speed groups. The next step was
to find for each speed group the mean spacing between vehicles. To ensure
however that the mean spacing thus derived was an accurate estimate of the
mean minimum spacing for that speed group, it was necessary to exclude from
the sample as many as possible of those vehicles which just happened to be
moving at the same speed as preceding vehicles, without in any way being
impeded by these preceding vehicles. In order to perform this elimination
the time spacing, that is, the space in front of a vehicle divided by its
speed, was calculated for all members of the sample.

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[10]   Normann (1939, pp.226-227).
[11]   Ibid.

Number of
Vehicles
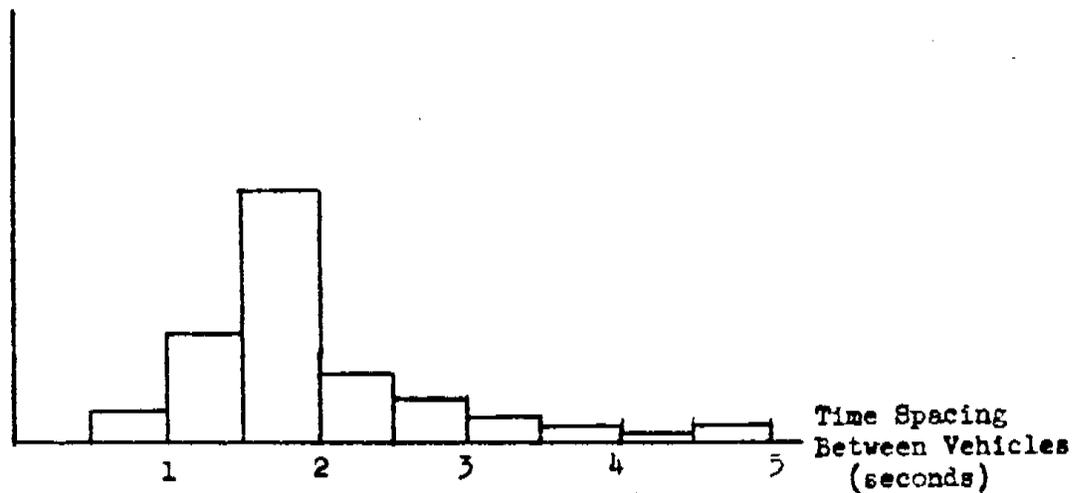
Figure 12.   A Frequency Distribution of Time Spacings
For a Single Speed Group.

The histogram of Figure 12 gives a general indication of the relative fre-
quencies of various time spacings in a single speed group. Since, in the
case pictured, most of the time spacings are concentrated at the low values,
and a very few are to be found at levels greater than four seconds, the in-
vestigators were probably justified in throwing out of the sample those
vehicles with time spacings greater than four seconds. These, they assumed,
were not traveling at minimum spacing. The same procedure was followed for
all the speed groups, and the mean distance spacing of the reduced sample for
each speed group was determined.
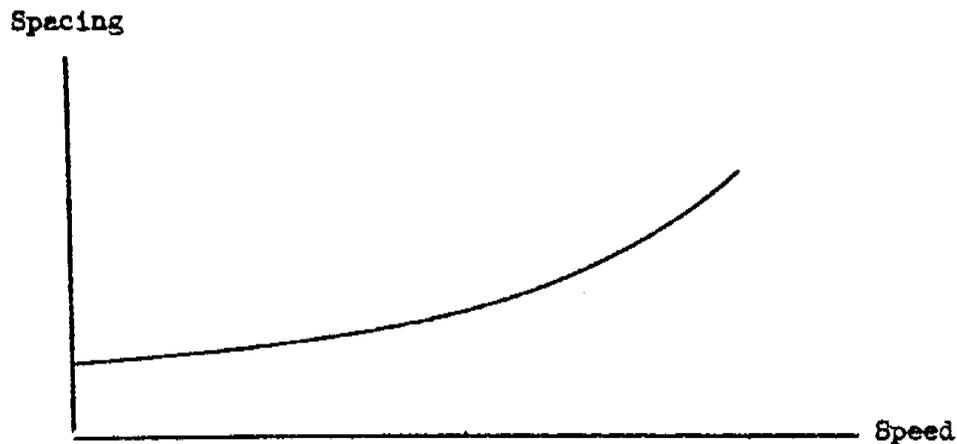
Spacing

Figure 13. Observed Minimum Spacings.

Figure 13 shows, for each speed group, this mean minimum spacing in feet of all vehicles spaced at less than four seconds. Using the data in Figure 13, a uniform-speed capacity curve was constructed which indicates for each speed the maximum flow possible if all traffic moved at just that speed. The resulting curve, OB in Figure 11, differs from a typical theoretical curve, CA'A in Figure 11, mainly in indicating greater flows at the higher speeds. Notice also that the uniform speed at which this curve attains its maximum flow is higher than before.

1.3.3 Direct Observations of Flows

The uniform-speed capacity curve might be estimated directly by fitting a curve to a set of maximums of observed flows for several different uniform speeds. The main difficulty encountered here is that very seldom are the higher uniform speeds observed. When speeds are high some passing nearly always takes place. This difficulty points up one of the rather artificial features of a uniform-speed capacity function. No roads of importance are

one-lane roads, yet the function is defined with the latter as a basis. A multiple-lane road is quite a different thing from a sum of single lanes. The complications introduced by passing will be dealt with in Section 1.4.

One of the reasons for the variety of methods used to determine uniform-speed capacity functions is that congestion often appeared to be present when flows were far short of the maximal levels indicated by the particular curve being used. It was then thought that a more careful procedure would lead to a function without this defect. But the trouble in many cases probably was that an average rather than a uniform speed was being used as the independent variable in the capacity function. The difference between the two cannot be neglected, as we shall see.

Difficulties such as these make it impossible to use a uniform-speed capacity curve alone to explain speed and flow limitations. There are, however, still some reasons for finding realistic uniform-speed capacity curves for typical roads or points. Such information would be useful to an authority (such as the army) which had complete control of a traffic movement, so that all speeds could be specified. These functions also represent useful points of reference in the sense that they show the loss in terms of flow when drivers select a variety of speeds. In addition, they will be found useful below in the discussion of capacity concepts which take into account the fact that uniform speeds are not ordinarily found in practice.

1.4 The Free-Speed Capacity of a Road

For the reasons indicated the uniform-speed capacity curve was never found to be very useful. People do not all drive at the same speed ordinarily, and in those cases where they do some of them at least wish to go faster and will pass the cars ahead of them as soon as the opportunity arises. This

leads to a bunching of traffic which the uniform-speed capacity curve does not take into account.

The capacity notion to be described here is the very different one developed by O. K. Normann and his associates in the Bureau of Public Roads, and described in the Highway Capacity Manual.[12] For a particular road a capacity curve is derived empirically which relates average speed and flow. The curves which have been published show that flow increases only at the expense of a marked reduction of average speed. Further work is necessary before this mechanism can be fully understood, but it seems plausible that at higher flows more passing per car is required to maintain a given distribution of speeds, that limited opportunities for passing prevent the maintenance of the higher speeds, and that flow reaches a maximum when all cars are traveling at very close to the same speed, which then is necessarily a relatively slow one.

1.4.1 The Free-Speed Distribution

Before discussing this matter further, a word should be said about the way drivers' preferences affect the capacity relations we are about to discuss. Nearly every kind of capacity curve imaginable depends in some way on the characteristics of the vehicle and driver population it pretends to describe.

When this population changes, then the capacity curve also changes and the problem is to relate the capacity curve to the underlying vehicle population in some simple way. The uniform-speed capacity curve depended on the minimum between-vehicle spacing required at different speeds by drivers, and it probably sufficed in this case to define "different vehicle populations" to mean "different spacing functions". While the populations may have differed

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[12]   Normann and Walker [1949].

in other respects as well, these were not important so far as uniform-speed
capacity curves were concerned.

For the free-speed capacity curve the characteristic we have chosen to
describe the vehicle population is (as the name of the curve indicates) its
free-speed distribution. A driver's _desired_ or _free_ speed on a given road is
that speed at which he chooses to travel when "alone" on the road, that is,
when the preceding car is so far ahead as to have no influence whatsoever on
the speed of the car in question. Since the matter is discussed again in
Section 2.2.2 we shall only say here that the free speeds drivers select
probably depend on the physical characteristics of the road and on their
estimates of the risks and costs involved and the value they place on time.
The "distribution" of free-speeds over a given road for a specified group
of drivers is simply a summary description of the free speeds of the group;
it tells us what fraction of the drivers have free speeds between any two
given limits. A graphic representation of a free-speed distribution is given
in Figure 14 where the total area under the curve is equal to one and the
fraction of drivers with free speeds between 30 mph and 40 mph is represented
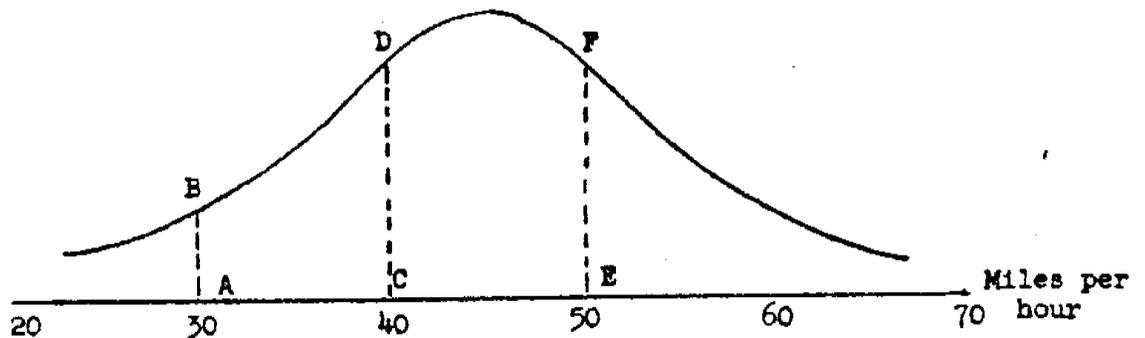
Figure 14. Graphic Representation of a Speed Distribution

by the area ABDC, the fraction with free speeds between 40 mph and 50 mph
by the area CDFE, etc. The reader who is interested in the actual free-speed
distributions found to prevail in certain instances is referred to the Highway
Capacity Manual.13/

To describe, as we are doing, a vehicle population by its free-speed
distribution is very obviously an oversimplification; for certainly there
are many other real differences which may be quite important, such as the
proportion of trucks in the population being considered, or the lengths of
opposing-lane gaps demanded for passing by the drivers. However the simple
description will probably do for our purposes. Whether or not more of the
characteristics that distinguish different groups of drivers should be used
is a question that can only be decided in practice. Once the reader is con-
vinced of the usefulness of description by means of free-speed distributions,
it should not be difficult for him to imagine how other more complicated
descriptions might be handled.

1.4.2 The Realized Speed Distribution

We shall now attempt both to summarize and to enlarge somewhat upon the
work of Normann and Walker in examining highway capacity in terms of speed
reductions. It should be made clear that while much of what follows is based
directly on material in the Highway Capacity Manual (and related publications14/)
the authors of that report are not to be held responsible for possible misinter-
pretations on our part, nor for additions we have made with which they may not
agree. The description will refer to a two-lane road which carries traffic in
both directions. The free-speed distributions of the two streams of traffic

13/   Normann and Walker [1949, p. 32, Figure 6].
14/   Normann [1939], [1941], [June, 1942].

will be assumed known, and we shall suppose that a vehicle will move at its free speed whenever traffic conditions permit.

The distribution of desired speeds will ordinarily not be the same as the distribution of actual speeds. For a fast car to pass a slow car some empty space in the opposing lane is necessary. If the density of vehicles (i.e., the number of vehicles per mile of road at any instant) in this opposing lane is high, then such spaces will be seldom available, and the passing maneuver will be delayed or prevented. As a result, the actual speed distribution will differ from the free-speed distribution, the lower speeds being relatively more frequent in the former.

Another way of looking at this effect that heavy traffic has on actual speeds is to concentrate our attention for a moment on a one-mile stretch of road with unrestricted visibility. Suppose that the southbound traffic flow is very low, so low in fact that for northbound traffic the free-speed distribution is the same as the actual speed distribution. Now let us ask ourselves how the total number of passing maneuvers per hour carried out by northbound vehicles on this one-mile stretch varies with the level of northbound flow. Suppose northbound flow doubles. A 50 mph driver will now find it necessary to complete twice as many passing maneuvers over the one-mile stretch as before, due to the doubling of the flow of vehicles moving at less than 50 mph. But since the flow of 50 mph vehicles has also doubled, this means the hourly number of passing maneuvers by 50 mph vehicles has increased by a factor of four. The same holds for all speeds, and it is only a step further to the conclusion that the total number of passes per mile of road per hour increases with the square of flow if the free-speed distribution remains unchanged.

If now the assumption of negligible southbound flow is dropped, it becomes clear that at some level of northbound flow, the number of northbound passing maneuvers required to maintain the free-speed distribution will be more than the gaps in the opposing traffic stream permit. As a result, some vehicles will be impeded and the realized speed distribution will no longer be the same as the free-speed distribution. Investigations have revealed roughly the way speed distributions change with increases in flow for some cases, and have shown that the actual total number of passings per mile of road per hour increases with flow up to a certain point, and then decreases gradually down to zero as flow becomes so great that all vehicles are forced to move at the same speed.[15]

The realized speed distribution for northbound traffic is clearly influenced by the flows of traffic in both directions on the road. If northbound flow increases, maintaining the same speed distribution requires more passing opportunities than before; if southbound flow increases, fewer passing opportunities present themselves than before. Both phenomena cause a change in the realized speed distribution. An elegant theory of road capacity would be one which described just this relationship. Given the two flows and the two free-speed distributions we could determine the two corresponding realized speed distributions. With this information we would be able to assess the effect of congestion on vehicles of each speed-class in the two traffic streams. However, this is obviously asking too much, at least in our present stage of developing a body of information on the behavior of traffic. A speed distribution is a rather large assembly of information. We shall settle for a simpler, substitute measure of the degree to which speeds have been affected by flows. The measure we use is the mean realized speed. Normann's group has in fact

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[15]   Normann [June 1942, p. 70].

attempted more than this by making use to some extent of the standard deviation of realized speed, in addition to mean speed.[1] In subsequent discussion we shall for the most part ignore these more sophisticated descriptions of the speed-flow mechanism. It should of course be borne in mind that here is a possible weakness in our theory. Practical work may well indicate that mean speed is too gross a measure of the consequences of increased flow, in which case it will be necessary to incorporate additional measures such as standard deviation of mean speed, or one of the others suggested by Normann.

1.4.3 Mean Speeds and the Free-Speed Capacity Function

The question now becomes, how do northbound and southbound mean speeds depend on the two flows? When we move in the direction of further simplification we are faced with a situation much like adding together the two intersection flows in the example in Section 1.1. Northbound mean speed depends on southbound as well as northbound flow; is it going too far to suppose that northbound mean speed depends only on the sum of the two flows? Or, indeed, can we drop the final complication and say that speed averaged over both north- and southbound traffic depends only on the sum of the two flows?

Of course these progressively simpler representations are at most approximations and a final answer as to which of them is best for the purpose cannot be given here. In practice the choice will be dictated by the cost of obtaining data, the reliability of the resulting flow-speed relationships in making predictions, and the consequences of errors when the estimated relationships are applied to specific problems.

In any case, such questions as these, important as they are, need not be decided here. The analysis that follows may be applied to any of the

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[1]   Normann (1939, p. 228].

various representations of the flow-speed relation. It is to make this last point clear that we emphasize these differences.

In one of the earliest articles[17] on the subject Normann chose one of the more complicated relationships for one particular road and by the method of least squares fitted a plane to a set of observations on northbound mean speed, northbound flow, and southbound flow. Since the coefficients he estimated are of interest we reproduce his result:

$$(1.5) \qquad \bar{u}_n = 44.92 - .01044x_n - .00719x_s$$

where $\bar{u}_n$ is in miles per hour and x_n and x_s in vehicles per hour. This plane, shown graphically in Figure 15 is an example of what we shall call a
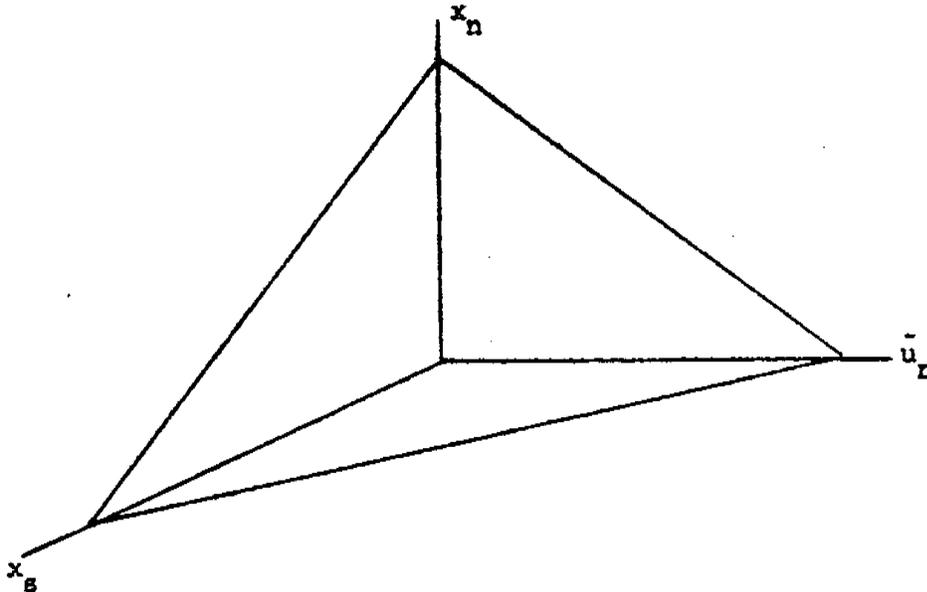
Figure 15

17/   Normann [1939, Figure 11, p. 229].

free-speed capacity function or, where no confusion is likely to result, simply a capacity function (or plane or curve, depending on the number of dimensions involved). The term free speed in this use does not mean (as in the case of the uniform-speed capacity function) that the function applies only to vehicles traveling at free speeds; we retain the term here only to emphasize the fact that the function is based on a free-speed distribution.

More generally, a free-speed capacity function is the function relating flows and mean realized speeds on a particular road for a particular population of vehicles. Other and simpler examples are presented in the Highway Capacity Manual[18] where mean speed of all traffic on the road is made to depend on the total flow on the road. Graphs of these curves (which in fact have usually been assumed to be straight lines) are quite a bit easier to draw and to visualize, the number of dimensions being less. Figure 16 is one example. For reasons to be discussed below, the curve has not been extended all the
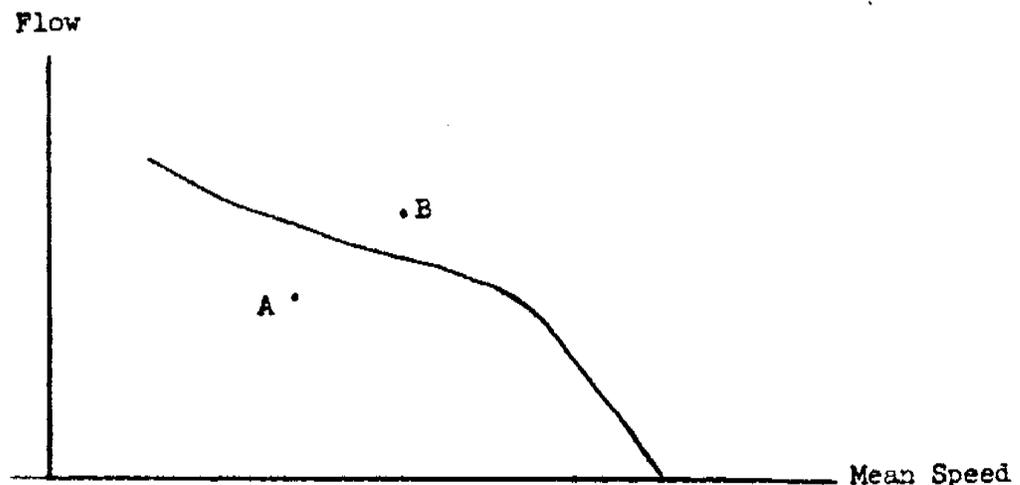
Figure 16. An Example of a Free-Speed Capacity Function

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[18] Op. cit., Figure 5, p. 31; Curve 2 in Figure 7, p. 33. Other similar examples are to be found in: Normann [June 1942, Figure 6, p. 61; Figure 8, p. 63].
Normann [1939, Figure 12, p. 229].
Wardrop [Feb. and Mar. 1952, Figure 1, p. 2].
Glanville [1949, Figure 9, p. 41].
Forbes [1952, Figures 5, 6, pp. 6, 7 respectively].

way to the flow axis. Notice also that in this example the curve is not
a straight line; we have drawn it the way it is merely to emphasize the
fact that from a theoretical point of view there is no reason to believe
it to be so. We shall, nevertheless, often draw it as though it were, if
only for convenience. Most investigators have found that a straight line
provides a fairly good fit to observations.

The important characteristic of all these curves, no matter what their
degree of complication, is that they are negatively sloped; that is, mean
speed decreases as flow increases. It should be emphasized that, apart from
sampling errors and errors of observation, only points on the curve are possible
flow-speed combinations for the vehicle population with the assumed free-speed
distribution. Points below the curve, such as A in Figure 16, would imply
that some drivers are not going as fast as they both can and wish to, and the
width and condition of the road and the presence of traffic prevent points
such as B from occurring.

1.4.4 The Relation Between the Free-Speed Capacity Curve and the Free-Speed Distribution.

The position of a capacity curve at levels of flow approaching zero will
vary with the vehicle population that the curve describes. When very little
congestion is present nearly every vehicle will be able to travel at its free
speed. At zero flow therefore mean realized speed is equal to mean free speed.
If for a certain road it is known that Sunday drivers have a lower mean free
speed than weekday drivers, then we should expect the capacity curves for the
two days to differ in a fashion like that indicated in Figure 17 (assuming
the curves to be of the same simple functional form as the one in Figure 16).
Here the points A and B,
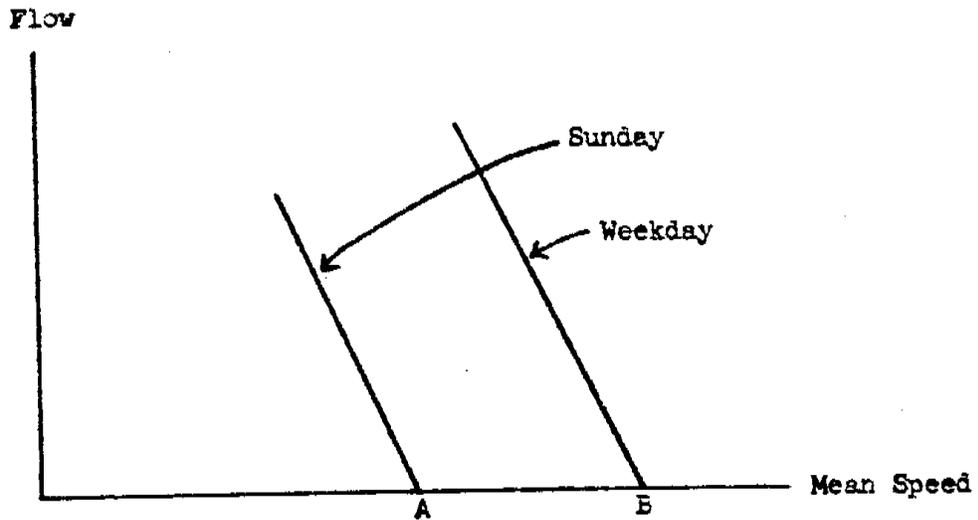
Flow

Figure 17. Capacity Curves Corresponding to Different Free-Speed
Distributions

the mean speeds corresponding to zero flows for the Sunday and weekday traffic respectively, are also equal to the respective mean free speeds of those two days. The capacity curves both slope upward and to the left from these points.

The difference in capacities between wide roads and narrow roads should show up in these curves more as a difference in slope, than intercept. Suppose an improvement program is contemplated for a certain road whose capacity is represented by the curve AB in Figure 18. If the proposed improvements are
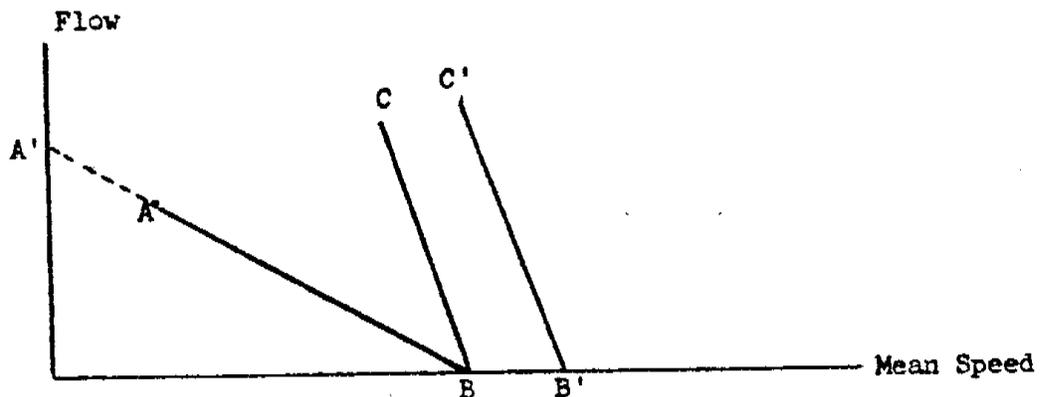
Flow

Figure 18. The Effect of Road Improvement on a Capacity Curve.

effective in reducing congestion, a new capacity curve CB will result which will indicate for each level of flow a higher mean speed than before. As we should expect, the difference is greatest where congestion is high, that is at high levels of flow. When flows approach zero, mean free speed prevails for both curves by the preceding arguments, and if we have reason to believe that people's free speeds will not be altered by the road improvement then these two points will be the same. The type of shift described seems to agree very well with some of the results of highway studies carried out by England's Road Research Laboratory.[12]

It is quite conceivable of course that free speeds will change with the improvement. Wider roads are very likely to cause an upward shift in the whole free-speed distribution, with a consequent shift in the mean free speed from B to B' in Figure 18. The capacity curve in this case might look something like C'B', where the road improvement has changed 1) the slope of the curve through its direct effect on, say, passing opportunities, and 2) the speed intercept of the curve through its indirect effect on drivers' choices as to speeds.

Before any further discussion of the influence of free speeds on the slope of capacity curves we must cover a point we have so far glossed over. In Figures 16, 17, and 18 the curves have not been extended all the way to the flow axis. If in Figure 18 the dotted section AA' were added to the capacity curve AB, then the whole curve would seem to tell us that, for the appropriate road and vehicle population, mean speed will be zero if flow rises to the level A'. But if mean speed is zero, then all speeds are zero and flow cannot be positive. Clearly, the upward extension of the curve must end at some point such as A.

[12]   Wardrop, [February and March, 1952, Figure 1, p. 2].

Generally speaking, as flow increases from very low levels, mean speed declines and the spread of the realized speed distribution grows smaller and smaller. Figure 19 portrays, in a very rough way, this change in the realized speed distribution. If flow continues to increase until the point is just reached where every vehicle is traveling at the same speed, then the endpoint, referred to above, of the free-speed capacity curve will have been attained. The mean speed at this point will, under our assumptions, be the free speed of the slowest vehicle present.[20] It will not be less, for the free-speed capacity curve as we have defined it only describes speed reductions brought about by the absence of passing opportunities. Any reduction in the speed of this slowest vehicle, which has no need to pass, cannot therefore be explained by the capacity curve we have been discussing. It is true that flow-speed combinations above and to the left of the free-speed capacity curve's endpoint are possible, but another mechanism than the free-speed and passing-opportunity one must be called upon to account for them. About this matter we shall have a word to say in a moment when we come to examine the relation between the free-speed and the uniform-speed capacity curves.

In Figure 19 the flow which corresponds to the endpoint of the curve drawn there is attained just above x_3. The mean speed at this point is meant to be equal to the lowest speed in each of the speed distributions pictured. The speed distribution at the endpoint cannot be drawn in the picture; the whole distribution would be lumped at one point.

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[20] If we remember that the capacity curve of Figure 19 is the simplified type which relates the average of the speeds of both north and southbound traffic to the sum of the two flows, then the statement above is not strictly correct. Speeds will be reduced in both lanes by the increase of flow, but not necessarily to the same lower limits. In Figure 19 the mean speed at the endpoint of the curve should really be regarded as the average (weighted by the flows) of these two lower limits, which in turn are the speeds at the low ends of the two free-speed distributions.

Speed distribution
when flow equals x_3

_____ Speed

Speed distribution
when flow equals x_2

_____ Speed

Speed distribution
when flow equals x_1

_____ Speed

Speed distribution
when flow is near zero

_____ Speed

Flow

x_3

x_2

The free-speed
capacity curve

x_1
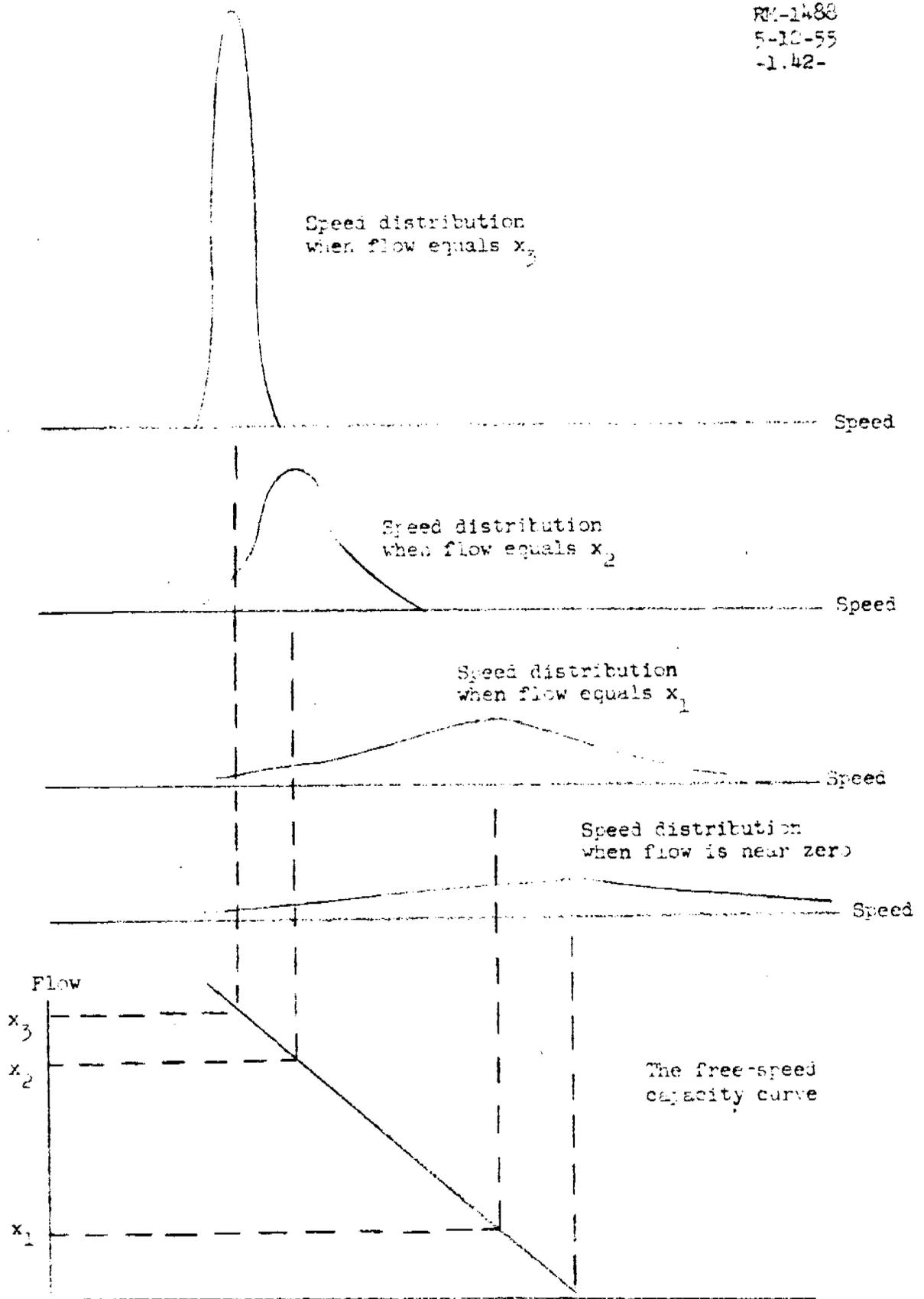
Figure 19. The Effect of Changes in Flow on the Realized Speed
Distribution.

It is clear that the complete specification of a particular free-speed capacity curve must include the location of this endpoint representing the flow and mean speed at which passing becomes impossible.[21] With Figure 19 before us it becomes a little easier to get a rough idea of how the shape of a capacity curve depends on the free-speed distribution behind it. A free-speed distribution with the same mean but less spread will mean a more sharply rising capacity curve. Figure 20 gives such a comparison. A distribution with a higher mean but with the same spread will shift the capacity curve to the right. This effect is portrayed in Figure 21.

1.4.5 Some Shortcomings of the Free-Speed Capacity Concept.

The importance of the effects described above comes from the fact that changes in flow may give rise to changes in the vehicle population and the underlying free-speed distribution. The reasons for such changes and the difficulties they occasion will become clearer in later chapters, but one rather extreme example here may serve to illustrate the point. Suppose on a certain road the usual traffic consists of passenger automobiles with fairly high free-speeds. The capacity curve for this vehicle population on this road we shall suppose is well enough known to enable accurate predictions of the consequences of increases in flow from this population. But if the increment of flow is from another population, one, say, of trucks with low free-speeds, then the capacity curve we started with is no longer the relevant one. A new capacity curve must be determined on the basis of the new free-speed distribution, which now describes the free-speeds of the mixed population of passenger automobiles and trucks. In principle this procedure could be followed with each new increment of flow, but it is clear that much of the simplicity and convenience of the notion of a free-speed capacity curve is gone. In addition

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21/ Normann and Walker [1949 pp. 31-33] indicate how these points may be located in practice by considering mean differences of speeds of successive vehicles.
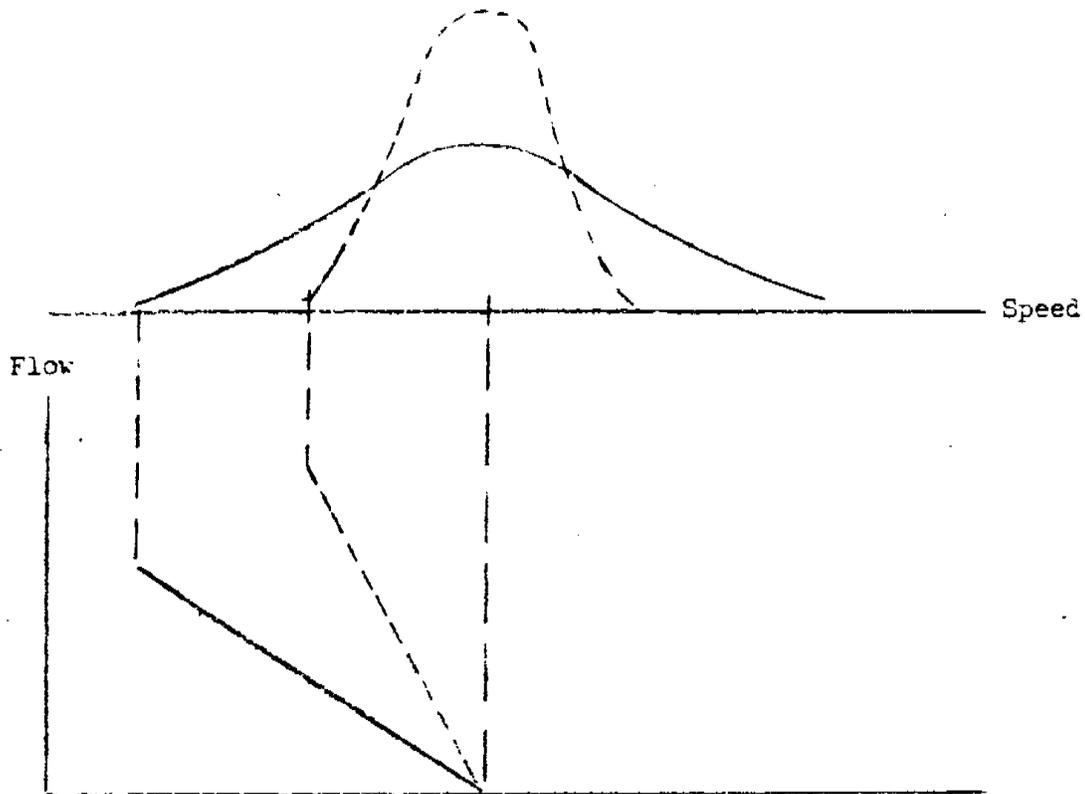
Figure 20.   The Slope of the Capacity Curve is Greater when the
Free-Speed Distribution is More Concentrated.
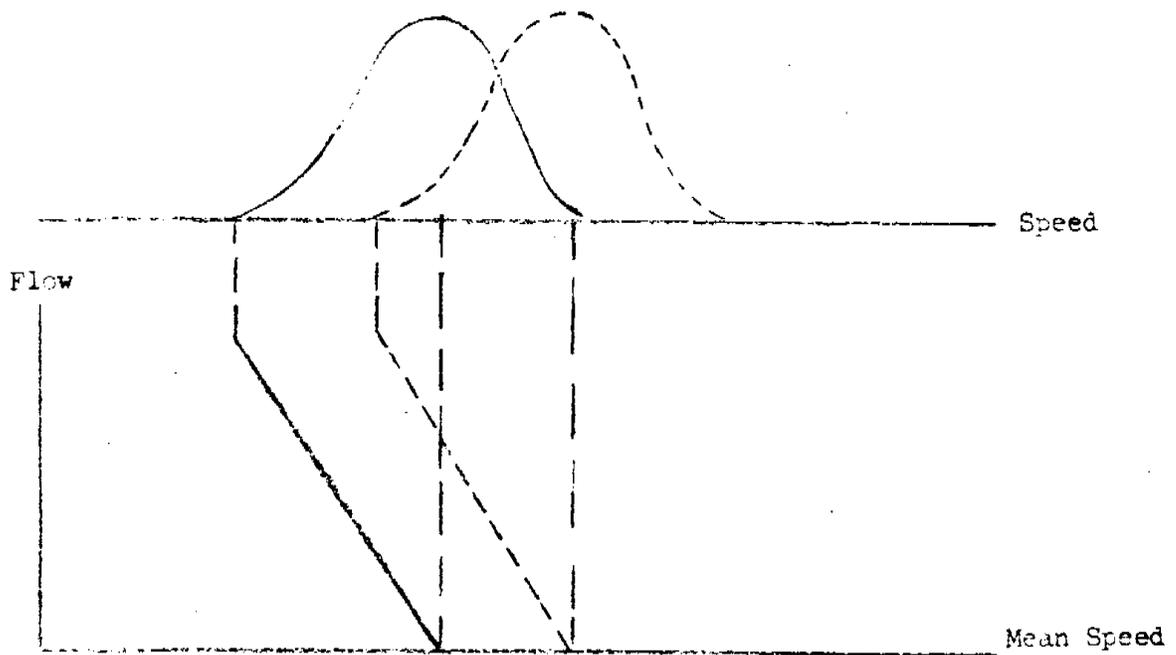
Figure 21.   The Location of the Free-Speed Distribution Affects the
Location of the Capacity Curve.

the procedure introduces the practical difficulties of determining just how the free-speed distribution changes at each point. The analogue of this problem in economic theory is the simultaneous shifting of demand and supply curves.

A quite similar sort of difficulty comes about if the speeds at which drivers choose to travel when not immediately behind other vehicles depend on the level of traffic flow. The way we have made use of the free-speed distribution in defining capacity curves rules this type of phenomenon out of our theory, but it may well be too important to ignore. If increases in flow tend to slow down unimpeded drivers then the effect is the same as in the preceding example; it is just as though an influx of vehicles from a "slower" population had been let into the road.

The difference between these two sources of shifts in the capacity curve with changes in flow is that the second refers to a single population and the first to a change in the population itself. The second difficulty could be eliminated by a more or less complicated redefinition of the capacity curve. We have not attempted to resolve this second difficulty and unless a statement to the contrary is made it will be assumed throughout the rest of this study that changes of this kind in the free-speed distribution are negligible. While for many cases this assumption is quite realistic, enough exceptions exist for it to merit serious consideration.[22]

1.4.6 The Relation Between a Uniform-Speed Capacity Curve and a Free-Speed Capacity Curve.

A moment ago we stated that speed-flow combinations above and to the left of the endpoint of the free-speed capacity curve are sometimes possible. For this to be the case the endpoint must fall below rather than on the

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[22] Cf. Section 2.1 where the concept of the average road user is introduced.

corresponding uniform-speed capacity curve; it cannot be above the curve of course, for the spacing of cars prohibit such speed-flow combinations. At the rather extreme state of congestion represented by this endpoint all cars on the road are moving at very nearly the same speed. However, while a uniform speed can therefore be said to prevail, the flow corresponding to this speed need not be the (uniform-speed) capacity flow indicated by the uniform-speed capacity curve. This can be seen most easily if this congested situation is pictured as series of queues of cars moving along in both directions. At the head of each queue is a car with a relatively low free speed; the cars in each queue wish to pass this slow one but passing opportunities occur only rarely. Thus even though the gaps in the opposing traffic are so few as to cut down passing nearly to zero, they can still be sufficient for the flow to be below the level of the uniform-speed capacity curve.

Traffic from side roads could of course enter these gaps, thus raising flow above that level given by the endpoint of the free-speed capacity curve. In addition, as flows approached uniform-speed capacity, further reductions in mean speed might occur through more complicated mechanisms than the one we have described.[23]

1.5 The Delay At an Intersection

In this section we find an analytic expression for the delay at a stop-sign intersection, and also at a repeated fixed-cycle traffic light. In both cases the model is formulated in terms of discrete time points, as explained in Section 1.2.

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[23] The work of Reuschel [1951] and Pipes [1950] appears to have some relevance to these states of severe congestion.

1.5.1* The Stop-Sign Case

1.5.1.1* With a General Major-Road Arrival Distribution

For the arrival distribution in the minor road the assumptions are as follows. At most one car can arrive at any time-point. The arrivals at successive time-points form a binomial sequence with parameter α. By this statement we mean than the chance of a car arriving at any particular time-point is α, and is independent of what has happened at any previous time-point. The discrete formulation ensures that cars are spaced out. This probability model is very close to one that has been found to give a quite adequate description of traffic.

We suppose at first that the green intervals and red intervals generated by the traffic in the major road are formed as follows:

(i) The distribution of the lengths of green intervals is geometric, that is, the relative frequency of an interval of length x is $\pi^{x-1}(1-\pi)$ $(x = 1,2...)$, where π is the parameter of the geometric distribution.

(ii) The relative frequency of red intervals of length b is $f(b)$, $(b = 1, 2....)$. This distribution is completely unrestricted for the present (except that it will be supposed to have both first and second moments).

(iii) The sequence of red and green intervals is such as would be obtained from a probability model in which red and green intervals were drawn alternately from their respective distributions, each drawing being quite independent of other drawings.

The red/green interval sequence generated by a binomial sequence of traffic in the major road, when there is a constant minimum safe interval in front of each car, is a special case of such a sequence. After finding the mean waiting time for cars in the minor road for the general sequence specified above, we will give the results for that case.

As was shown above, to find the mean wait per car in the minor road
it is necessary to find the mean queue length $E(q_x)$. We will find this
for the case for which the probability mechanism which is supposed to be
generating the queue has settled down to equilibrium. Either the system
will do this after a sufficient time, or else the queues will tend to build
up indefinitely, so that the mean waiting time will become longer and longer,
the later a car arrives. The theory below shows that a necessary and sufficient
condition for equilibrium to be attained is that the average number of green
time-points exceeds the average number of cars arriving. By q_x we will
denote the queue which has to wait from time x to time $x + 1$.

We will find $E(q_x)$ in two stages, first taking averages over green
points, and then over red points.

Let the green time-points be numbered consecutively, $t_1, t_2, \ldots, t_g, \ldots,$
regardless of whether there is a red interval between them or not. Because
green intervals have a geometric distribution with parameter x, the probability
of a red interval occurring between any two consecutive time-points t_g and
t_{g+1} in the green sequence is $(1 - x)$, and is independent of where other
red intervals are interspersed in the sequence. This is the same as saying
that the probability of one green point being followed by another is x.

Define the variables

b_g = the number of consecutive red points immediately following
  t_g, that is, the number of red points between t_g and t_{g+1}

β_g = the number of arrivals at red points between t_g and t_{g+1}
  $(\beta_g \leq b_g)$;

α_g = number of arrivals at t_g (so α_g is either 0 or 1).

If $b_g = 0$, then necessarily $\beta_g = 0$. Generally, $q_{g+1} = \max [(q_g + \alpha_{g+1} + \beta_g - 1), 0],$

which we may also write

(1.6)
$$q_{g+1} = q_g + \alpha_{g+1} + \beta_g - 1 + \delta_g$$

where

$$\delta_g = 0 \quad \text{if} \quad q_g + \alpha_{g+1} + \beta_g > 0$$

(1.7)

$$\delta_g = 1 \quad \text{if} \quad q_g + \alpha_{g+1} + \beta_g = 0 .$$

Thus q_g is generated by a Markoff process of a type discussed in the literature. The necessary and sufficient condition for such a process to settle down to statistical equilibrium is that

$$E (\alpha_{g+1} + \beta_g) < 1 .$$

That is, on the average, less than one car arrives per green point. The necessity is obvious; the sufficiency was shown in Kendall's paper [1951].

The mean queue at green points, $E (q_g)$, can now be found by a device used for a similar problem by Kendall.

When the process has settled down to statistical equilibrium $E(q_{g+1}) = E(q_g)$. Therefore, taking expectations in (1.6),

(1.8)
$$E(\delta_g) = 1 - E(\alpha_{g+1}) - E(\beta_g)$$

and also

(1.9)
$$E(\delta_g^2) = E(\delta_g) \quad \text{since} \quad \delta_g = 0 \text{ or } 1.$$

By (1.7)

(1.10) $\qquad E \delta_g (q_g + \alpha_g + \beta_g) = 0.$

Squaring (1.6) we have

(1.11) $\qquad q_{q+1}^2 = q_g^2 + (\alpha_{g+1} + \beta_g - 1)^2 + \delta_g^2 + 2\delta_g (q_g + \alpha_{g+1} + \beta_g)$

$$- 2\delta_g + 2q_g (\alpha_{g+1} + \beta_g - 1)$$

Taking expectations in (1.11) and using (1.8), (1.9), (1.10), and the further equilibrium condition $E (q_{g+1}^2) = E (q_g^2)$ we obtain:

(1.12) $\qquad E [(2q_g + 1) (1 - \alpha_{g+1} - \beta_g)] = E [(\alpha_{g+1} + \beta_g - 1)^2]$

To find $E(\beta_g)$ notice that the probability that there is a red interval between t_g and t_{g+1} (i.e., that the point $t_g + 1$ is red) is $(1 - \pi)$, and the mean number of arrivals, if there is a red interval, is $\alpha E(b_g)$, $E(b_g)$ being the mean of the distribution $f(b_g)$ from which the red intervals are considered to be drawn. Accordingly,

(1.13) $\qquad E(\beta_g) = (1 - \pi) \alpha E(b_g).$

Similarly, by just taking expectations over red intervals of length b, and then over red intervals of all lengths, we get

(1.14) $\qquad E (\beta_g^2) = (1 - \pi) \alpha^2 E(b_g^2) + \alpha (1 - \alpha) E (b_g).$

Also

(1.15) $E(\alpha_g) = E(\alpha_g^2) = \alpha.$

Using the independence of α_{g+1}, β_g, and q_g, (1.12) can be written

(1.16) $[2 E (q_g) + 1] [1 - E(\alpha_{g+1}) - E(\beta_g)]$

$$= E (\alpha_{g+1}^2) + E(\beta_g^2) + 1 - 2E(\alpha_{g+1}) - 2E(\beta_g) + 2E(\alpha_{g+1}) E(\beta_g)$$

Substituting the values given in (1.13), (1.14), and (1.15) and solving for $E(q_g)$ we get

(1.17) $$E(q_g) = \frac{\alpha^2(1 - \pi)[E(b_g^2) + E(b_g)]}{2[1 - \alpha - \alpha(1-\pi) E(b_g)]}$$

Now we must consider the queue length at red points. Suppose[24] a red interval of length b comes between t_g and t_{g+1}. We can number the points of the interval $t_{g_1}, t_{g_2},\ldots,t_{g_b}$. The queue length just before the start of the interval will be q_g, and these cars will remain throughout the interval. Since b_g and q_g are independent, (1.17) gives the expected size of this "initial queue" over red as well as over green time-points. We require therefore only the expected length of that part of the queue made up of cars that arrive during red intervals.

If there is an arrival at t_{g_1}, it will be in the queue at $t_{g_1}, t_{g_2},\ldots t_{g_b}$. It will thus add b to the sum of the queues over the red interval. Similarly, an arrival at t_{g_2} adds $b-1$ to the sum. Thus the mean addition to queues per red interval of length b caused by arrivals

[24] For simplicity, we shall drop the subscript g in the rest of this section.

during such intervals is $\dfrac{b(b+1)}{2}\,\alpha$, and the mean addition per point in red intervals of length b caused by arrivals in those intervals is $\dfrac{b+1}{2}\,\alpha$.

A proportion $\dfrac{b\ f(b)}{E(b)}$ of red points lie in red intervals of length b. Therefore the mean addition per red point caused by arrivals during red intervals is

$$(1.18) \qquad \sum_b \frac{b(b+1)}{2}\ \frac{f(b)}{E(b)}\ \alpha = \frac{\alpha}{2E(b)}\left[E(b^2) + E(b)\right] .$$

Thus the proportion of red points in the total is

$$(1.20) \qquad \frac{(1-\pi)\ E(b)}{1 + (1-\pi)\ E(b)}$$

and the mean addition to the queue per time-point is

$$\frac{(1-\pi)\alpha}{2} \cdot \frac{E(b^2) + E(b)}{1 + (1-\pi)\ E(b)} .$$

Adding this to (1.17) we get

$$(1.21) \qquad \frac{\alpha(1-\pi)\ [E(b^2) + E(b)]}{2[1 + (1-\pi)\ E(b)]\ [1 - \alpha - \alpha(1-\pi)\ E(b)]}$$

as the mean queue size over all time-points.

1.5.1.2[#] Binomial Arrivals on the Major Road

In this section we consider the case for which the traffic in the major road is also described by a binomial sequence, of parameter ρ. In front of each car in the major road there is a minimum safe interval of w time points. By this statement we mean that if a car arrives in the major road at time t, then no car in the minor road can cross at time $t - w + 1$, $t - w + 2, \ldots, t$. That is, an interval of w points up to and including t red points.

Under these conditions we can examine the structure of the sequence of red and green intervals generated by the traffic in the major road. The length of the green intervals has a geometric distribution, with parameter $1 - \rho$. If we put $1 - \rho = \pi$ in the preceding analysis, this case turns out to be a special case of the more general one treated above. The result for the mean waiting time given by (1.21) shows that for our present purposes we only require the first two moments of the distribution of the lengths of red intervals. We find these below, and are thus enabled to exhibit the mean waiting time per car as a function of the densities in the minor and major roads respectively, and of w, the minimum safe interval. This function is shown in Figures 8, 9, and 10.

1.5.1.3[#] The Distribution of the Lengths of Green Intervals.

We will suppose from now on that the time points are numbered in sequence $t = \ldots, -1, 0, 1, \ldots$. For shortness, if a car in the major road arrives at the intersection at the time point $t = t'$, we will say that there is a car at $t = t'$. The safe interval w is supposed an integer and is defined as follows. Suppose the point $t = 1$ is the first point of a green interval; then there must be a car at $t = 0$, and no cars at $t = 1, 2 \ldots w$. The

point $t = 2$ is also a green point if and only if there is no car at $t = (w + 1)$. Thus, given that $t = 1$ is a green point, the probability that $t = 2$ is a green point is $\pi = (1 - \rho)$. Similarly, the probability that the green interval will continue from $t = 2$ to $t = 3$ is π. By induction, the probabilities of green interval lengths of 1, 2, 3... are proportional to 1, π, π^2,.... and so must be

$$(1.22) \qquad (1 - \pi),\ \pi(1 - \pi),\ \pi^2(1 - \pi),..$$

and these probabilities define a geometric distribution.

Now to derive $E(b)$ and $E(b^2)$, suppose $t = 1$ is the first point of a red interval. This means that there is a car at $t = w$, but none at $t = 0, 1, 2, ...w - 1$. The red interval will end at $t = w$ if there is no car at $t = w + 1, w + 2, ...,2w$. However, if the red interval does go on, suppose the next car after the one at $t = w$ is at $t = w + x_2$, where $1 \leq x_2 \leq w$. We can say that the second car contributes a length x_2 to the length of the interval. We can also say that the first car always contributes a length $x_1 = w$ to the interval. If the red interval continues to a third car, we can consider the contribution, x_3, of the third car to the length of the red interval, and so on.

We will now find the probability that the red interval is formed by just r cars. If there are already $r - 1$ cars in the interval then the probability that there will be no more is $\pi^w = P$, say. Thus the probability that the interval will continue is $1 - P$. Hence the probability that the interval will contain just r cars is $(1 - P)^{r-1} P$.

If the red interval is formed by just r cars, the number of points it covers is $x_1 + ... + x_r$, where $x_1 = w$ and $1 \leq x_i \leq w$, $i = 2, ...,r-1$.

The distribution function of each x_i is the same for $i > 1$. Therefore if the moment generating function (m.g.f.) of x_i is $\phi(u)$, say, ($i = 2, 3 \ldots$), then, since the m.g.f. of $x_1 = w$ is e^{wu}, the m.g.f. of $x_1 + x_2 + \ldots + x_r$ is

$$(1.23) \qquad \phi^{r-1} e^{wu}$$

and the m.g.f. of red-interval length if the red interval can contain any number of cars is

$$(1.24) \qquad X(u) = \sum_{r=1}^{\infty} \phi^{r-1} e^{wu} P(1-P)^{r-1} = \frac{e^{wu} P}{1 - (1-P) \phi} \qquad .$$

It is also necessary to find $\phi(u)$. Since

$$(1.25) \qquad \text{Prob} (x_i = x) = \frac{\pi^{x-1}}{\sum\limits_{x=1}^{w} \pi^{x-1}} = \frac{1-\pi}{1-P} \pi^{x-1} \qquad (i \geq 2)$$

we have

$$(1.26) \qquad \phi(u) = \frac{\sum\limits_{x=1}^{w} e^{ux} \pi^{x-1}}{\sum\limits_{x=1}^{w} \pi^{x-1}} = \sum\limits_{x=1}^{w} e^{ux} \left[\frac{1-\pi}{1-P} \cdot \pi^{x-1} \right]$$

$$= \frac{1-Pe^{wu}}{1-P} \cdot \frac{1-\pi}{1-\pi e^{u}} \cdot e^{u} \qquad .$$

Hence

$$(1.27) \qquad X(u) = \frac{e^{-u} - \pi}{P^{-1} \left[e^{-(w+1)u} - e^{-wu} \right] + 1 - \pi} \qquad .$$

We have to find $E(b) = X'(0)$ and $E(b^2) = X''(0)$. By straight-forward but somewhat tedious evaluation we obtain

$$(1.28) \qquad E(b) = X'(0) = \frac{1-P}{(1-\pi)\,P}$$

and

$$(1.29) \qquad E(b^2) = X''(0) = \frac{1}{1-\pi} - \left[\frac{3-\pi}{(1-\pi)^2} + \frac{2w}{(1-\pi)}\right]\frac{1}{P} + \frac{2}{(1-\pi)^2 P^2}.$$

Putting these values in (1.21), we find the mean queue size per time-point to be

$$(1.30) \qquad E(q) = \frac{\alpha\,[1-P(1+w\rho)]}{\rho\,(P-\alpha)}.$$

Dividing this waiting time by the α cars per time-point we get

$$(1.31) \qquad \frac{1 - P\,(1 + w\rho)}{\rho\,(P-\alpha)}$$

as the mean waiting time per car, where $P = \pi^W = (1-\rho)^W$. The necessary and sufficient condition for the system to settle down to statistical equilibrium is

$$(1.32) \qquad \alpha < (1-\rho)^W.$$

The function (1.14) is the one plotted in Figures 8, 9, and 10 for various values of ρ α and w.

1.5.1.4[*] Two-way Traffic in the Main Road

If there is a two-way stream in the main road, the results still apply with suitable reinterpretation of the parameters, provided both streams are binomial. We assume the safe distance for crossing, w, is the same for both streams. If ρ_1, ρ_2 are the densities in the two streams, the probability that a time-point will be occupied by a car in either of the streams is $\rho_1 + \rho_2 - \rho_1 \rho_2$ and we can set ρ equal to this value in the formulae above.

We have not yet fitted these formulae to actual data, as we have not yet found any data in a suitable form. It is therefore difficult to tell just what the effects of the simplifications we have made will be. However, from the data in the papers by Raff and Greenshields it can be seen that w may be not constant, but may sometimes vary fairly widely from car to car. The formula may still be fairly accurate if we use, in place of w, some parameter of the distribution of w. Previous suggestions as to what parameter to use include the mean of the w distribution (Greenshields) and the median (Raff), but neither of these need necessarily be the best, as the following argument will show.

Consider the mean waiting time when α is small. In this case

$$(1.33) \qquad \bar{z} = \frac{1}{\rho P} (1 - P - w\rho P)$$

approximately. Expressing P in terms of ρ we have

$$(1.34) \qquad \bar{z} = \frac{1}{\rho} (1-\rho)^{-w} - \frac{1}{\rho} - w = \frac{1}{\rho} \left[(1 + w\rho) + \frac{w(w+1)}{2} \rho^2 + \dots \right] - \frac{1}{\rho} - w.$$

approximately. If ρ is also small then

$$(1.35) \qquad \bar{z} = \frac{\rho}{2} \, w \, (w + 1)$$

approximately. When α is small, there is a negligible chance that a queue will form. Thus if w has a distribution, the mean waiting time can be obtained by averaging (1.35) over the distribution of w. Then

$$(1.36) \qquad \bar{z} = \frac{\rho}{2} \, E \, [w \, (w + 1)],$$

suggesting that the constant $\bar{w}$ which gives the best fit is the solution of the equation $\bar{w} \, (\bar{w} + 1) = E \, [w \, (w + 1)]$.

1.5.2[*] Repeated-Cycle Traffic-Lights Case

In this case a red interval of length r is followed by a green interval of length g time points, and then the whole cycle is repeated. To find the mean waiting time, it is necessary to find the average queue length over all time points. First we find the queue length at the start of the xth red interval, q_x say.

Let u_x be the number of cars arriving in the xth cycle consisting of a red interval and the succeeding green interval. If q_{x+1} is the queue at the start of the next red interval, then

$$(1.37) \qquad q_{x+1} = \max \, (q_x + u_x - g, \, 0),$$

where u_x is distributed over $0, 1, 2, \ldots\ldots, r + g$ in a binomial distribution with parameter α, that is,

$$(1.38) \qquad \text{Prob} \, (u_x = n) = \binom{r + g}{n} \, \alpha^n (1 - \alpha)^{r + g - n} \quad .$$

Thus the q_x are generated by a Markoff chain. The condition that this process should settle down to statistical equilibrium is

(1.39) $u_x < g.$

It is simple to find a complete analytic solution for the distribution of q_x when the system has attained statistical equilibrium, as the solution of a finite difference equation. However we will confine this discussion to the calculation of mean queue length over all time-points. We will do this calculation in two parts, first considering the mean car-points of waiting during red intervals, and then during green intervals.

For the red interval the calculation is similar to that in (1.21) above. The queue which is waiting over until the first point of the interval stays throughout the interval, and thus contributes $r E(q_x)$ to the average car-points of waiting during each red interval. A car arriving at the first point of the interval contributes r, and as the probability of a car arriving at that point is α, the mean contribution to the wait is αr. The mean contribution per interval from the second point is $\alpha(r_0 - 1)$, etc., the contribution from arrivals at the last red time-point being α. So the total red-interval waiting of cars arriving in the red interval averages $\dfrac{\alpha\, r(r+1)}{2}$ per red interval and the mean amount of waiting during each red interval by all cars is

(1.40) $r \left[E\left(q_x\right) + \dfrac{\alpha}{2}\left(r + 1\right) \right]$.

Now we must find the mean amount of waiting during green intervals. Suppose the queue at the beginning of the green interval is q_g. Consider

the mean wait of cars in the queue if all subsequent time-points were green. The time taken for the queue to diminish by one (if it is not zero in the first place) is the number of green points to and including the first one in which a car does not arrive; this time is thus distributed geometrically, with parameter α and mean $\frac{1}{1-\alpha}$. The queue during all but the last of these time-points is q_g; during the last time-point it is $q_g - 1$. The mean car-points of waiting during this time is thus $\frac{1}{1-\alpha} q_g - 1$. The time taken for the queue to diminish by one more is distributed in the same way, and the queue during all but the last of these time-points is $q_g - 1$. If all subsequent time-points were green, the process would continue until the queue had vanished altogether. The total wait for that time would have the mean

$$(1.41) \qquad \sum_{x=1}^{q_g} \left[\frac{1}{1-\alpha} x - 1 \right] = \frac{1}{2(1-\alpha)} \left[q_g(q_g + 1) \right] - q_g$$

so that, averaging over the q_g distribution, the mean wait would be

$$(1.42) \qquad \frac{1}{2(1-\alpha)} E\left[q_g^2 + (2\alpha - 1) q_g \right] .$$

However, the green interval is not in fact infinite, and we must subtract the wait that would reach over into the next red interval. To do this we subtract the wait attributable to the queue at the beginning of the next red interval. This will average

$$(1.43) \qquad \frac{1}{2(1-\alpha)} E\left[q_x^2 + (2\alpha - 1) q_x \right] .$$

Thus mean wait over the finite green interval is

(1.44) $\qquad \frac{1}{2(1-\alpha)} \, E \left[q_g^2 - q_x^2 + (2\alpha - 1) \, (q_g - q_x) \right]$.

Suppose v_x is the number of cars arriving in the xth red interval; $v_x = 0, 1, 2, \ldots, r$. Then $q_g = q_x + v_x$ and $E(q_g) = E(q_x) + E(v_x)$. Since v_x is independent of q_x it follows that

(1.45) $\qquad E(q_g^2) = E(q_x^2) + 2 \, E(v_x) \, E(q_x) + E(v_x^2)$.

Cars arrive with frequency α in both red and green intervals, so $E(v_x) = r \, \alpha$. And since v_x is binomially distributed, we have

(1.46) $\qquad E(v_x^2) = r \, \alpha(1 - \alpha) + r^2 \, \alpha^2$.

Thus the mean car-points of waiting during each green interval becomes, by substituting these values,

(1.47) $\qquad \frac{\alpha \, r}{1-\alpha} \left[E(q_x) + \frac{\alpha}{2} \, (r+1) \right]$.

Adding the amounts of waiting given by (1.40) and (1.47) for the red and green intervals, the mean amount of waiting per cycle is

(1.48) $\qquad \frac{r}{1-\alpha} \left[E(q_x) + \frac{\alpha}{2} \, (r + 1) \right]$

and the mean wait per car will be

(1.49) $\qquad \frac{r}{(1-\alpha)(g+r)} \left[\frac{E(q_x)}{\alpha} + \frac{r+1}{2} \right]$.

Thus we only need to find $E(q_x)$. It seems, however, that the simplest way to find $E(q_x)$ is not from the analytic solution. The q_x distribution can be found numerically from the generating equation by a simple iteration. (The matrix is definite, so it has to be truncated. It can be proved that this type of truncation can still give approximations to the ergodic probabilities of the infinite matrix however). The mean wait can then be calculated by the formula given.

1.6 An Application of the Intersection Model to Road Capacity

When fast cars pass a slow car on a two-lane highway. the situation shows some rather close similarities to the case of a car in a minor road crossing a major road at a stop sign. A fast car coming up behind a slow car must wait until there is a sufficient gap in the opposing traffic. When such a gap appears it may be able to pull over into the other lane and pass. If, ahead of it, there are other cars also waiting for an opportunity to pass the slow car. then our car will usually be able to pass only after these other cars have done so. The relation with the wait at a stop sign is clear. Passing the slow car corresponds to crossing the intersection. The opposing traffic, which has the right of way in its own lane, corresponds to the major-road traffic. The fast cars coming up behind the slow car correspond to the minor-road traffic. And we can suppose, as a simplified description of drivers' behavior, that fast cars will only pass when there is no car due to come in the opposing lane for at least some fixed time interval, which we can call the critical gap.

The analogy is close enough to give some hope that the quantitative theory developed for the stop-sign case may also be useful in giving a theoretical basis for the free-speed capacity curve on a two-lane highway.

As a first, and perhaps rather drastic simplification, we might suppose
that in one of the lanes there are just two types of cars, slow and fast,
and that the slow cars are distributed sparsely on the road. In the oppos-
ing stream the traffic all goes at roughly the same speed. If we could
know the average delay to fast cars in passing each of the slow cars as a
function of the densities of fast cars and the density of the opposing
stream, we could then derive a free-speed capacity curve for this simple
case. We show how this can be done in the next section.

1.6.1[*] The Free-Speed Capacity Derived From the Average Delay

Suppose that the speed of the slow cars is v_1, and that they are, on
the average, a distance d apart. The speed of the fast cars is v_2.

We will for simplicity make an extra assumption, that the average delay
of a fast car in passing a slow car is the same for each slow car, W say.
Such an assumption is probably adequate if, for example, the fast traffic
is sparse as well as the slow traffic so that the fast cars do not interfere
with each other in passing a slow car. We have tried to test the validity
for denser fast traffic by means of "Monte Carlo" experiments (that is, numer-
ical experiments designed to simulate traffic behavior). The results of these
experiments are discussed later in this section. W is a function of the
opposing traffic density and of the critical gap. However, we will not show
this dependence in our notation.

We want to find the average speed of the fast cars, taking into account
their delay. The speed of the other traffic is unchanged by the congestion
on the road. To do this we will find the distance covered by a fast car in
a period of time of length T. T can be divided into T_2, the time during
which the car is going at its free speed v_2; and T_1, the time during which
it is delayed behind a slow car travelling at v_1. The average speed v of

a fast car can then be written

$$(1.50) \qquad v = \frac{v_1 T_1 + v_2 T_2}{T_1 + T_2} \ .$$

If by N we denote the average number of slow cars overtaken by one fast car in time T, then we have

$$(1.51) \qquad N = \frac{T_2}{d/(v_2 - v_1)}$$

since on the average a fast car takes a time $d/(v_2 - v_1)$ to travel between one slow car and another at its free speed, and it spends a time T_2 travelling at its free speed. We can now write $T_1 = NW$ and $T_2 = Nd/(v_2 - v_1)$. Substituting for T_1 and T_2 in the expression for v, cancelling the N's, and multiplying numerator and denominator by $(v_2 - v_1)$ we find

$$(1.52) \qquad v_1 = \frac{Wv_1(v_2 - v_1) + dv_2}{W(v_2 - v_1) + d}$$

A similar formula can be derived if we suppose that there is sparse traffic travelling at more than two speeds on the road.

The average speed of both kinds of traffic, fast and slow, taken together, is given by

$$(1.53) \qquad M_1 v_1 + M_2 \frac{W(v_2 - v_1) v_1 + dW_2}{W(v_2 - v_1) + d} ,$$

where M_1 and M_2 are the relative numbers of the fast and slow cars.

This formula gives a theoretical form to the free-speed capacity curve in the case which we consider.

1.6.2   A Monte Carlo Experiment

We have assumed above that the mean delay of traffic in passing each slow car is the same. This assumption needs examination if the fast traffic is dense. Our theory of passing behavior leads us to question it for the following reason. The mean delay of cars crossing an intersection (or passing a slow car) depends on the distribution in time of the arrivals of such cars. In our derivation of a theoretical formula we have assumed a particular form for this distribution; that the arrivals form a binomial sequence. However if cars arrive in a binomial sequence, they do not leave the cause of congestion (the intersection or slow car) in a binomial sequence. Thus their average wait behind this next slow car may be different from the previous one. In the same way the spacing distribution may change again in passing this obstacle so the average wait behind the next slow car may be still different.

So far we are expressing theoretical possibilities. We have not yet succeeded in treating analytically this type of question about passing through successive queues. We therefore decided to resort to "Monte Carlo" experiments, that is, experiments with simulated traffic streams generated by random devices, to get a feel for the problem, and to judge the importance of these effects. These experiments did not have very positive results. However, we will report them very briefly here, as they may give some useful information to others faced with the same sort of problems. When we started the experiments we had two alternative hypotheses, either of which would be useful in deriving a free-speed capacity curve.

The first was that despite the modification of the spacing sequence in passing through each queue, the formulae derived from the binomial assumption were satisfactory in any case.

The second was that, failing the first assumption, the mean delay changes for the first few queues through which the fast car passes, but fairly quickly settles down to be more or less constant for later cars, though this constant might be different from the result derived for the binomial sequence.

To explore these possibilities the experiments we performed were as follows. Initially, a sequence of minor-road (or "fast-car") arrivals was generated over 500 time points by drawings from a binomial distribution. The output from this distribution was passed through a red/green sequence also derived from a binomial sequence, with the assumption of a constant critical gap.

The departure sequence, derived by using the method described in Section 1.2.4, was used as the arrival sequence of an exactly similar type of queueing process. Since all the fast cars would take the same time to cover the distance between slow cars, we could neglect this time in our calculations. The whole process was repeated fifteen times, representing the traffic successively passing fifteen slow cars on the road.

In computing the average waiting time, that is, the average queue length over the period, it was decided to neglect the first time-points, while the process was starting up. An arbitrary rule was used that for each process the waiting time would be computed from a time-point 40 time units after the queue first became non-zero. The mean waiting time was calculated by taking the average queue length over time-points from that point to the last (the 500th).

The sequences fluctuated rather heavily from one process to the next. Part of these fluctuations may be due to the relative shortness of the series. The queue series are necessarily highly autocorrelated and this means that, as a statistical sample, the series are equivalent to a much smaller number of "effective" or "independent" observations than the original 500 time-points. In any case, the experiments we performed give no evidence of stability of mean waiting time.

The work was done on a punched-card machine. The results show that if Monte Carlo methods are to be used in this type of work, the experiments would have to be on a still larger scale, so that an electronic computer would probably be appropriate. But it does not seem that such experiments will give very useful results without more theoretical knowledge of repeated queue processes.

Demand

One of the striking features of traffic is its resemblance to physical flows. The relations between speed and volume and such phenomena as the propagation of waves of stopping and acceleration are understandable in large part from the mechanics of motion[1]. However, appealing as this physical view of the matter is, it neglects the purposive aspect of traffic, which becomes particularly important when attention is shifted from the single road to an entire network, that is to say, when we are interested in understanding the generation and distribution of traffic within a system of interconnected roads.

The point of view to be adopted from here on is that vehicles are operated by people who 1) have a range of choices available to them, and 2) are motivated by economic considerations in their decisions. This element of choice prevails even though, once committed to the road, the movement of vehicles is strongly circumscribed by the traffic conditions around them. For there is always the question why people, anticipating these constraints, and particularly the irritations of congestion, still choose to go when, where and in the manner they go.

The study of rational behavior in the presence of choice leads into the domain of economics. In this and the following three chapters some applications of economic theory to highway transportation will be explored. The purpose is not to exhaust all economic aspects that seem relevant or important. Rather the argument is constructed with reference to two points which may be regarded as but two aspects of the same matter: that it is possible to give some

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[1]   See for instance Herrey and Herrey [1945] and Pipes [1952].

economic explanation of the distribution of traffic in a network; and that
this distribution, as it forms itself spontaneously, falls short of efficiently
utilizing the available capacity. For a fairly systematic development of this
topic, a number of concepts must be introduced and some principles postulated,
which may seem rather trivial if looked at in isolation. But emphasis here
is on the fitting together of all relevant concepts into a theory. The dis-
cussion of fundamental concepts falls for the most part into the present
chapter. It has been labelled Demand, because this seems to be the most
characteristic notion that an economic viewpoint can contribute.

The fundamental question in capacity theory, discussed in the preceding
chapter, is how flow on a road determines the traffic conditions, mainly
average speed. This puts flow into the role of a cause. We can however also
raise the reverse question: how do traffic conditions affect the public's
decisions of which the flow is the actual result. This is the problem of
the current chapter on demand. The actual determination of the flows through
the interaction of demand for transportation and capacity for accommodating
it is deferred to the following chapter on equilibrium. Unless otherwise
noted we shall deal with the problem of passenger traffic only.

2.1. Decisions of Road Users.

A first step toward economic analysis is to disaggregate flow and take
notice of the various road users. These fall into classes that are distin-
guished by origin, by destination, and by behavior characteristics such as
free speed. Whenever we speak of the demand for road transportation we mean
the demand, on the part of road users (or sometimes only the road users in a
particular free-speed class), for transportation from a particular origin to
a particular destination. Whenever we disregard the different free speeds of

road users we shall employ the convenient fiction of an _average road user_.
It presumes that the composition of traffic by free speeds follows some
constant pattern on each road.

With most other means of transportation, once a traveller has chosen
his destination his remaining freedom of choice is usually limited to a
relatively small number of routes and departure times. If he chooses to go
by road, however, he faces decisions about his route, the speed under various
roadway and traffic conditions, and more decisions each time he wants to pass
a vehicle or an intersection or desires to weave into another lane. The
temporal order of the actual decisions is that in which they were mentioned.
But in exercising a rational choice, a road user normally anticipates his
speed and passing behavior under various conditions when choosing a route,
and he also has some notion about the route when deciding whether or not to
travel by road. A situation where the logical order of decisions may run
counter to their sequence in time is not unusual in economics. The location
and size of a power dam, for instance, can be determined optimally only after
the desirable policy of water storage for any given reservoir location has
been specified, which in turn is influenced by the demand for power as
affected by the chosen rate structure. Here again the "natural" sequence
of events is the reverse of the logical dependence of decisions.

In this chapter we shall follow the logical order of decisions; that is,
we shall discuss the choice of individual free speeds first, the selection
of routes (the demand for the use of roads) next, and finally the demand for
road transportation between locations. Before this, however, another concept
which is central to economic thinking has yet to be introduced, namely the
notion of cost. This will be done in the following section.

2.2 Costs

To state that economic considerations influence drivers' decisions is another way of saying that road users are conscious of costs. Cost must be understood here, in the wide sense of economics, to include everything of value that is foregone to attain a certain end.

In this study we are interested in the transportation cost to road users only; that is, we exclude from consideration all cost to public bodies incurred through the maintenance or construction of road capacities. For demand depends on cost to road users only. And we shall not go so far as to analyze the tax component in gasoline costs, say, which would reflect to some extent the cost of maintenance and expansion of the road system. For doing this means going into the "long run".

The term transportation cost needs to be delimited also in spatial respects. Thus a distinction will become necessary between cost of transportation on specific roads and trip cost. In this section we consider the first cost only.

2.2.1 Composition of Transportation Cost

For purposes of the analysis of demand the cost of transportation on a road includes not only the operating cost of a vehicle over the length of the road, but also such things as the travel time and the risks incurred.

Operating cost consists of all outlays directly attributable to vehicle use on a mileage basis. This includes fuel, oil, tire wear, deterioration and maintenance as far as these are dependent on mileage; it excludes vehicle taxes and license fees, depreciation due to time and obsolescence. Certain money costs will be considered separately from operating cost, namely as risk cost.

Time costs will include the times-in-transit of vehicle and driver, and perhaps of passengers and loads. The value of a vehicle-hour is determined by the alternative uses of the vehicle that are possible. In private use the availability of a vehicle at times other than those of its ordinary use may have some value, but typically a low one. In commercial transportation the value of driver time is set by the wage rate of the driver, and the value of vehicle time is the earning power, after all operating cost, of the vehicle per unit of time, multiplied by the average proportion of time during which the vehicle is in use normally. But this earning power is not an independent variable. A brief excursion into the long run is necessary here. In the long run the value of the vehicle hour tends to be equal to depreciation plus interest on the investment in a vehicle. Except insofar as average speed affects the rate of depreciation -- an effect which may well be disregarded here -- the value of the vehicle hour is thus independent of the prevailing speed of traffic.

For commercial transportation it is obvious that there exist definite money equivalents of vehicle, driver, and cargo times. That drivers in private cars tend to mind their time also and at least implicitly assign some money value to it can be seen in their choice of a free speed. For an example, let us accept an estimate in Toll Roads and Free Roads,[2] according to which "Tests on several makes of modern passenger cars show that with steady driving on concrete pavements at a speed of 40 miles per hour the consumption is 0.057 gallon per mile. At 60 miles per hour the consumption becomes 0.082 gallon per mile." It follows that a driver preferring to go at 60 rather than at 40 miles per hour values a saving of one hour by at least the price of 2.9 gallons of gasoline, which is in the neighborhood of 85 cents. Unawareness on the driver's part may blur and possibly diminish this figure somewhat.

[2] Toll Roads and Free Roads, 1939, p. 39.

On the other hand, the fact that we have interpolated linearly a curve
which is bent upward, and that risk and other operating costs are increas-
ing with speed also, is likely to raise the figure again. The estimate
conveys, however, an idea of the order of magnitude of the value of time,
as applied to driving.

Under <u>Risk Cost</u> we shall include the losses from accidents in terms
of life, health, and property, as well as the irritation from the threat
of such accidents which is particularly manifest under conditions of road
congestion.

2.2.2 Transportation Cost as a Function of Flow

It is for the elementary reason that the transportation cost on a road
varies with the flow of traffic that economic considerations of a short-run
character are relevant at all to transportation in highway networks. This
dependence may be described very simply by saying that an individual road
user is the worse off, the more traffic there is on the road he is travelling.
This is so because the presence of other traffic causes delays, added risks,
and extra operating cost as expended in passing maneuvers. The road user
incurs a higher transportation cost the larger the flow, even though a lower
average speed may actually reduce his operating cost and although risks, as
measured in accidents per vehicle mile, may start falling off at a flow level
where passing becomes increasingly rare.

The explanation is of course that these latter cost reductions, if any,
from congestion are more than offset by the accompanying losses of time.
By congestion we mean here traffic conditions which occur at flows that
"substantially" reduce average speed on the road. If it is assumed, for
instance, that below a certain flow level average speed is "practically"
independent of flow so that a vertical section of the capacity curve is

appropriate, then congestion would begin at the flow level where the vertical section of the capacity curve terminates.

On this apparently obvious point that costs increase because delays do, a few remarks are in order. One might ask why road users do not seek compensation for congestion in higher free speeds. While the uniform-speed capacity curve shows that this is physically possible only within certain narrow limits, economic considerations also suggest that it is not feasible for the individual road user to try to stretch his possibilities to this limit. An increase in the free speed is either out of the question because the road user recognizes an absolute speed limit imposed on him by risk considerations or by the law, or else an increase of speed is possible but iself increasingly costly in terms of both risk and operating cost.[3] The conclusion is that both travel time and transportation cost increase or at best remain constant for each road user with increasing flow.

That the average travel time should be increasing (or constant) with respect to flow is just a restatement of the facts described by the free-speed capacity curve. The other relationship, that average cost is also increasing or constant with flow, will prove important enough for our analysis to warrant the introduction here of a corresponding average-cost capacity function. By this we mean the average transportation cost on a road for all road users entering during a unit period of time regarded as a function of the traffic flow. It differs from the previously considered capacity functions only by the inclusion of operating and risk cost.

2.3 The Demand for Road Use

2.3.1 The Choice of Routes

So far we have been concerned with the behavior of road users on a single road. We did not have to consider in detail the economic motivations that

[3] Beakey [1937].

underlie the main decision, that about free speed. Principles of choice' must be considered however, when it comes to the behavior of a road user in a network. The outstanding problem is the determination of choices of routes. Let us introduce some concepts pertaining to networks first.

We start with a set of locations which we shall call points. All origins and destinations will be at these points, and every road will be a direct link between a pair of points. In general of course not every pair of points will be connected by a road, and some points will neither originate nor terminate traffic. By including all intersections in the set of points, we ensure that each road is a distinct segment of our network.

The rate at which traffic originates at a location, i.e., the net outflow to all destinations per unit of time, will be referred to as (net) origination. Similarly (net) termination denotes the rate at which traffic terminates, or the net absorption of traffic at a location per unit of time.

A trip denotes the path of a vehicle starting at one point, the origin, and terminating at another one, the destination. The roads traversed in succession on a trip are said to constitute its route. The traffic flow in both directions on a road is the sum of all traffic flows to the various destinations which enter this road from either direction.

Vehicles are now distinguished not only with regard to their free speed, but more importantly with respect to their destination. Flow and transportation cost remain variables defined with reference to a road. The notion of the cost of a trip between locations is introduced later and set in relation to transportation cost.

The simplest possible assumption about choices of route would be that drivers select the geographically shortest one. But this would disregard roadway and traffic conditions entirely. If traffic conditions are to bear

at all on the choice, an obvious next assumption is that the route should

be chosen so as to minimize total travel time.[4] A completely rational route

in the economic sense would be one that minimizes cost to the road user (in

the broad sense defined above). We shall assume that actual choices of

route approach this ideal to the extent that routes chosen are those of

smallest _average_ cost of transportation (averaged over users of each road).

This amounts to considering a fictitious _average road user_, who incurs on

every road the transportation cost that is indicated by the capacity function

at the prevailing flow. While the errors committed in this way do not exactly

cancel out, it seems a plausible and simple approximation to the actual rout-

ing of traffic flow.

Henceforth, for brevity, we shall call a route that is optimal in terms

of average cost a _shortest_ route. It need not be unique, of course. A

shortest route defined with reference to an origin and a destination contains

within it the shortest route that leads from any intermediate point to the

destination. The notion of a shortest route now permits us to define trans-

portation cost from any origin to any destination, simply called the _trip_

cost, as the average cost along a shortest route. Then, along a shortest

route, this trip cost to a given destination decreases as the origin is moved

from one road endpoint to the next by exactly the average transportation cost

on the road cut out. Along roads not on a shortest route, trip cost to a

given destination decreases by less or even increases. The converse is also

true. If there exist numbers, associated with all conceivable trips, which

decrease by the full amount of transportation cost along roads that belong

to certain routes, and which decrease by less along all other roads, then

the routes so distinguished are shortest routes and the numbers are the trip

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[4]   See Trueblood [1952] where this hypothesis is tested.

costs, provided that they have value zero where origin and destination coincide.

This is best seen graphically. In Figure 1, let 1-2-3-4-5-6 denote the shortest route between 1 and 6. Now the fact that this route continues from 2 via 3 rather than via 7 means that the trip cost from 7 to 6 plus
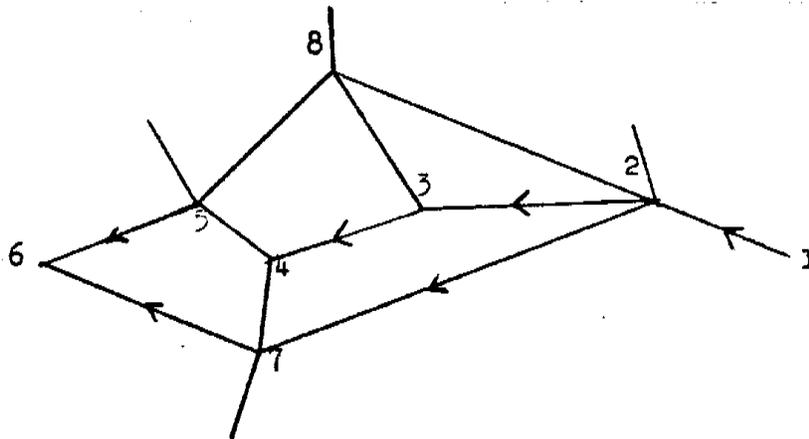
Figure 1

the transportation cost on road 2-7 is more than the trip cost from 2 to 6. In other words the trip cost to 6 decreases along road 2-7 by less than the transportation cost on that road, whereas it decreases by exactly the transportation cost incurred on any road that is part of the shortest route 1-2-3-4-5-6.

Let us assume that the route 1-2-8-5-6 is a shortest route also. Then the cost from 2 to 5 is the same for both of the alternatives, 2-3-4-5 and 2-8-5. Counting transportation cost on roads positive in the direction toward the destination and negative in a direction away from the destination, the sum of costs on roads along the closed circuit 2-3-4-5-8-2 is zero.

A closed cycle of this kind has been called a neutral circuit. Neutral circuits indicate the existence of more than one shortest route.

In the following section we shall formulate mathematically the relationship between shortest routes and trip costs, and demonstrate our assertion that shortest routes are completely characterized by this relationship.

2.3.2* Shortest Routes and Trip Costs

Let locations be indicated by single letters i, j, k, where i usually stands for an origin and k for a destination. Roads are designated by the pair of adjacent locations they connect, ij for instance. (Figure 2)
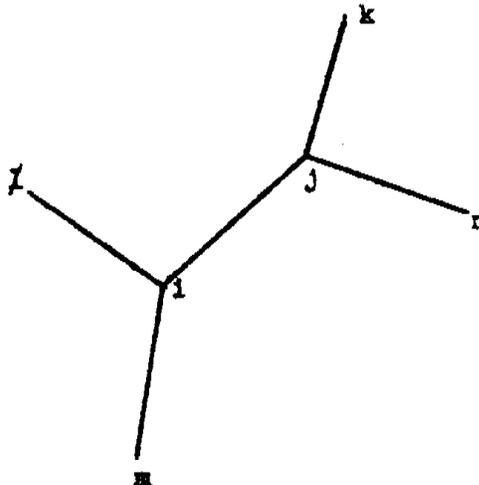
Figure 2

Transportation costs y on roads ij are written y_{ij}. Since we do not distinguish between costs in the two directions, $y_{ij} = y_{ji}$. For points i and j that are not contiguous, y_{ij} is left undefined. Trip costs from origin i to destination k are denoted by $y_{i,k}$. (Notice that the subscripts are separated by a comma.) Now

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5/ Koopmans and Reiter [1951, pp. 247-248].

(2.1) $$y_{1,k} = \text{Min} \; (y_{1j} + y_{jl} + y_{lm} + \dots + y_{nk})$$

that is, the minimum of all chain sums of y_{rs} starting at 1 and terminating at k in which consecutive elements have one subscript in common. In particular $y_{k,k} = 0$. (Since all $y_{rs} > 0$, it follows incidentally that these minimum chain sums do not contain any closed sub-chains.) Consider $y_{m,k}$ for two locations $m = 1$ and $m = j$ connected by a road ij. Extending the minimum chain that leads from j to k by adding y_{1j}, we have a chain from 1 to k, but not necessarily a minimum chain. Thus

(2.2) $$y_{1,k} \leq y_{1j} + y_{j,k}.$$

To summarize: for every pair of points $1,k$ there exists a uniquely determined number $y_{1,k}$ such that

(2.3) $$y_{1,k} - y_{j,k} \leq y_{1j} \quad \text{and} \quad \text{"="} \; \text{for some} \; j; \qquad y_{k,k} = 0.$$

We next show the converse. If for every pair of points i,k there is a unique number $y_{1,k}$ satisfying (2.3), then the $y_{1,k}$ represent shortest distances:

(2.4) $$y_{1,k} = \text{min} \; (y_{1j} + y_{jl} + \dots + y_{nk}) \quad .$$

Let $i-j'-l'-\dots-n'-k$ denote a shortest route. Then

$$y_{i,k} - y_{j',k} \leq y_{ij'}$$

$$y_{j',k} - y_{\chi',k} \leq y_{j'k'}$$

(2.5)
.
.
.

$$y_{n',k} - y_{k,k} \leq y_{n'k} \ .$$

Since $y_{k,k} = 0$ one obtains by addition of these inequalities

(2.6) $\qquad y_{i,k} \leq y_{ij'} + y_{j'\chi'} + \cdots + y_{n'k} \ .$

On the other hand, since the $y_{i,k}$ satisfying (2.3) are assumed to be unique, there must, between every pair of points i and k, exist a sequence of pairs of points starting at i and ending at k for each of which the "$=$" sign holds in the inequalities (2.3). Let one such sequence be denoted $i - j'' - \chi'' - \cdots - n'' - k$. Then

(2.7) $\qquad y_{i,k} = y_{ij''} + y_{j''\chi''} + \cdots + y_{n''k}.$

By definition of a shortest route,

(2.8) $\qquad y_{ij'} + y_{j'\chi'} + \cdots + y_{n'k} \leq y_{ij''} + y_{j''\chi''} + \cdots + y_{n''k}.$

Combining inequalities (2.6), (2.7), and (2.8) we obtain that

(2.9) $\qquad y_{i,k} = y_{ij'} + y_{j'\chi'} + \cdots + y_{n'k} = \text{Min} \ (y_{ij} + y_{j\chi} + \cdots + y_{nk})$

so that the numbers $y_{i,k}$ indeed denote the shortest distances (in terms
of transportation costs).

In the economic theory of industrial location the points i of common
$y_{i,k}$ for a given k are referred to as isodistants if cost is measured in
terms of distance, isochrones if measured in terms of time, and isovectures
if measured in terms of transportation cost.[1]

2.3.3 The Induced Demand for Use of a Road

Once the shortest routes for traffic between each pair of locations are
found at prevailing traffic conditions, flow on each road can be determined.
It is the sum of the numbers of trips per unit of time for all those origin-
destination pairs whose shortest route goes through the road. As explained
more fully in Section 2.4.1 below, we assume that the amount of flow demanded
between any pair of locations depends only on the transportation cost between
them. But whether or not a shortest route goes through any given road depends
on the transportation cost on all roads that might be part of alternative
routes. It is useful as an approximation sometimes to consider the flow on
a given road (resulting from the demands for all trips) as a function of the
transportation cost on that road only. In that case it is implicitly assumed
that transportation costs on all other roads are unchanged, and in particular,
that they remain unaffected by any shifts of flow from the road in question to
alternative routes. This is a plausible assumption only if the flows on the
road considered are relatively small.

The fact that there may be several shortest routes, and that a small
change in transportation cost on the road may alter its flow by the whole
amount of flow on a route, shows that the demand function for road use is

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[1]   Palander [1935, pp. 337-360].

discontinuous under our assumption that route selection depends only on relative transportation costs.[7]

2.4 The Demand for Transportation

A certain amount of driving on a particular road may of course be for pleasure's sake, in which case it represents ultimate consumption. Typically, however, the demand by drivers for the use of a particular road is not immediate, but arises out of the demand for transportation from one location to another.

This need for transportation arises from the divergent locations of economic activities. In personnel transportation, the obvious cause is locational separation of residence, places of work, commodity markets, and cultural facilities. Commodity transportation is promoted ultimately by the geographical division of labor. This in turn is based on differences in the resource endowments of regions and on the greater economies of production on a large scale. Descriptive details on the locational distribution of economic activities fall into the domain of economic geography, and theoretical discussion of these phenomena is the subject matter of location theory.

However, our understanding of locational phenomena is far from sufficient for an accurate prediction of the demand pattern for transportation, given the general economic structure of a city or region. A formula has been advanced by some authors[8] for the intensity of intercity and interregional

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[7] Therefore the result of the general theory of demand that the demand for the mth commodity varies with the price of the nth commodity as the demand for n^{th} commodity does with the price of the mth commodity has no equivalent here. However, if traffic on road ij is decreasing with or unaffected by increasing transportation costs on road mn (i.e., if mn is an alternative to ij), then also traffic on road mn decreases with increasing transportation cost on road ij, always holding other transportation costs constant.

[8] E.g., Stewart [1950].

relations which is modeled after Newton's law of gravitation. Economic relations according to this formula are directly proportional to the algebraic product of population figures and inversely proportional to distance, suitably measured. By implication this formula should also describe the demand for intercity and interregional transportation. This however seems unlikely for no account is taken of the specific economic structure of cities or regions.

Some interesting theoretical suggestions concerning the demand for transportation in cities are also advanced in the recent monograph by Mitchell and Rapkin [1954]. Total origination or termination in an area is related to the character of the residential and commercial establishments in the area. An important observation is, for instance, that origination tends to be proportional to the total floor space devoted to commercial enterprises in a given district. Emphasis is given to the "linkage" in originations of people and of commodities between establishments located in one area.

2.4.1 The Demand for Trips

In a model which attempts to set forth the interdependence of flows in a network, recognition must be given to the fact that the amount of traffic that originates at a given point for a particular destination per unit of time is in general not independent of the traffic conditions on the relevant route or routes.

This obvious principle needs careful limitation. It does not imply that the pattern of originations and destinations could not show considerable rigidity in the short run. In particular it does not dispense with the necessity of studying actual typical patterns. It implies caution, however, in the causal interpretation of such origin-destination patterns. Because

it is true that such and such a ratio of floor space to truck trips prevails
in a given area, [9] this may not prove invariant under changes affecting traffic
conditions. If the output levels of a plant closely determine its origination
and termination intensity for a given technology, traffic conditions may force
a change of technology or of location to maintain profits. A point often
emphasized is that the central business districts of cities may experience a
reduction of their levels of activity to the point where the demand for traffic
can be accommodated by their available road capacity (and parking facilities).

In line with our approach to traffic in terms of its economic motivation,
we shall consider the demand for transportation between a given origin and
destination to be a function of average trip cost between these locations
only.

Demand must be defined with reference to a time period. Now on each
road, traffic shows a characteristic variation over time. This is a reflection
of the fact that the demand for transportation between locations fluctuates
in various ways. For instance, intercity traffic has a pronounced weekly
pattern of fluctuation while commuter traffic shows extreme daily fluctuations
as well. Depending on which types of roads and of road use one is primarily
interested in, a different period is thus appropriate for the definition of
demand. In any case it is a simplification to regard demand in terms of an
average flow per period of given type, as the case may be. It is an even
cruder approximation to have this demand depend on the average traffic con-
ditions during that period only. However, simplification is the price of
analysis.

Let the curve representing the number of trips per period between two
locations which are undertaken at various average cost levels be called the

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[9] Mitchell and Rapkin [1954, p. 172].

demand curve for travel between these locations. The curve in Figure 3 is one example.[10] We assume here that the general price level and the income distribution are fixed. By its nature the demand curve has an upward slope nowhere. In certain circumstances, when the number of trips is independent of traffic conditions, the curve is vertical, and demand can then be described completely by fixed origin-destination figures.

The notion of average transportation cost requires some attention here because the population of road users changes along the demand curve. To be consistent, the average should always be taken over the population of actual road users at any point of the demand curve.

The demand curve may also be interpreted as a ranking of trips, where each trip is labeled by its urgency, that is, by the "critical" cost level at which it would still have been undertaken.
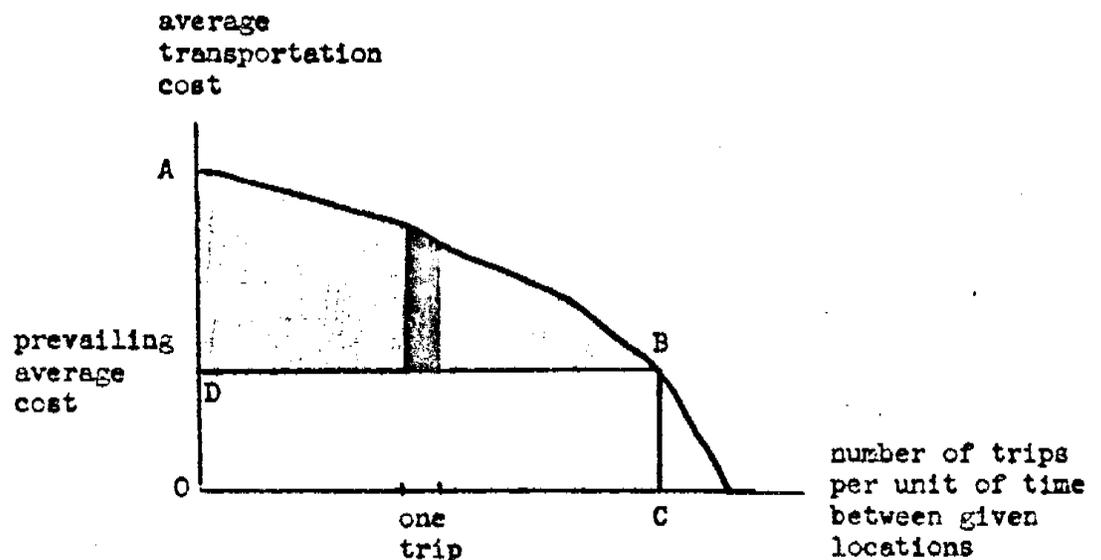
Figure 3

The excess of this critical average cost for only one trip over the prevailing average cost represents a clear gain to the road user in question. It appears in the form of a saving in money cost and corresponds to what is known in the economic theory of market demand as a consumer's surplus. Geometrically it is represented (approximately) by a strip extending from a horizontal line denoting the prevailing average cost to the demand curve with a width corresponding to one trip (crosshatched in Figure 3). While the actual cost for a trip need not be the prevailing average cost, so that the area of the strip may be different from the gain of the particular road user in question, nevertheless the total area between the demand curve and the horizontal line of prevailing average cost indicates exactly the cost saving to all road users at that level of average transportation cost. Here the benefits in money terms enjoyed by various road users have been added up tacitly as homogeneous and commensurable. In particular it is implied that there are no effects of income differences which would render a dollar saved at a high level of cost worth more, or less, than a dollar saved at a low cost level. However, the equilibrium analysis which follows will do without this or any other value judgment. We will refer again to the road user's benefit in Section 4.1.2.

The form of the demand function is dependent, among other things, on the alternative means of transportation available and their rate structure. In general the demand for transportation betwen a given pair of locations would depend also on the costs or travel times to and from other locations. It has been shown, for instance, [11] that most automobile trips within cities are round trips with more than one destination. However, this applies to the movement of the persons rather than those of the vehicle, which appears

[11] Mitchell and Rapkin [1954, pp. 39-53].

often to partake in only the initial and final phase of the journey. If intercity traffic is taken into consideration, a merger of the possibly plural destinations into one would not seem to be a bad approximation.

Similarly in the long run when locational changes are admitted, a case can be made for the dependence of demand on transportation cost from or to various alternative locations. But since our interest is in the short-run phenomena mainly, this complication may perhaps be disregarded here.

2.4.2* Demand Functions and Capacity Functions

Let us now introduce a mathematical notation for demand. The number of trips per unit of time from origin i to destination k will be called $x_{i,k}$ for i not equal to k. The symbol $-x_{k,k}$ we shall define to be the termination at destination k. Since terminations at k must match the sum of originations for k, we have $\sum_i x_{i,k} = 0$. The number of trips as a function of trip cost will be written $x_{i,k} = f_{i,k}(y_{i,k})$. We shall make frequent use of the inverse relationship $y_{i,k} = g_{i,k}(x_{i,k})$. This inverse function $g_{i,k}$ exists because the demand function $f_{i,k}$ is monotonic; as the trip cost goes up, the number of trips decreases, if anything.[12] Stationarity of $g_{i,k}$ over some interval would mean that demand is perfectly elastic there; a small rise in the prevailing transportation cost would induce a substantial decrease in demand. This is not to be expected even in the presence of alternative means of transportation. Without restricting ourselves we may therefore assume $g_{i,k}$ to be strictly decreasing, that is   .

[12] While the demand for certain commodities may increase with price in particular circumstances involving income effects, this is not likely to be the case with the demand for road transportation.

(2.10)
$$\frac{dg_{i,k}}{dx_{i,k}} < 0,$$

wherever $g_{i,k}$ is differentiable. The case of fixed origin-destination figures (perfect inelasticity) means that the graph of $g_{i,k}$ is a vertical line and that the derivative is minus infinity.

We shall now give the notation and summarize the relevant properties of road capacity functions. Let x_{ij} denote flow on road ij. The order of subscripts is immaterial here and in what follows: $x_{ij} = x_{ji}$, $y_{ij} = y_{ji}$, and $h_{ij} = h_{ji}$ (notice the absence of a separating comma in the designation of roads). The capacity function $y_{ij} = h_{ij}(x_{ij})$ is never decreasing:

(2.11)
$$\frac{dh_{ij}}{dx_{ij}} \geq 0.$$

The "=" sign can hold when flow levels are small enough to make interference between elements of traffic negligible. From the considerations in Section 1.4 it appears that delays and hence costs increase indefinitely as a certain flow level $\bar{x}_{ij}$, the absolute capacity limit, is approached:

(2.12)
$$\lim_{x_{ij} \to \bar{x}_{ij}} h_{ij}(x_{ij}) = \infty.$$

The most obvious differences between these two types of elementary data functions, the demand function, and the capacity function, are 1) that in the capacity function flow is the cause and transportation cost the effect, while in the demand function trip cost is the cause and trips (constituting the flow) are the effect; 2) that demand is defined with reference to any

pair of locations while capacity refers only to roads; and 3) that demand
decreases as cost increases, whereas in the capacity curve, cost increases
or remains constant as flow increases.

From the economic point of view, capacity and demand appear to be pairs
of opposites similar to supply and demand. Capacity is the "supply of trip
opportunities at various costs". The analogy, however, is a loose one. In
particular, while under perfectly competitive conditions the supply function
in a market for ordinary commodities tends to be the same as the incremental
cost function, here the capacity function corresponds more nearly to an
average cost function.

Equilibrium

The discussion of capacity and demand in the two preceding chapters
has supplied us with some fundamental notions on traffic flow. In this
chapter we propose to explain the pattern of traffic in a network from
the interaction of the demand for transportation between many pairs of
locations and the capacities of roads in the network. This demand is
distributed over various routes and, on each road, over various free
speeds. It thereby affects the traffic conditions on the roads, and these
react again on the demand for traffic. In an attempt to unravel these re-
lationships of mutual determination, it is useful to have reference to a
situation of equilibrium.

With regard to a road in isolation, equilibrium means the following.
At every level of traffic conditions, as measured by the average cost of
transportation, a certain demand for the use of this road is forthcoming.
If this demand is larger than the prevailing flow , the traffic conditions
get worse and the average cost of transportation increases. This tends to
curb the demand, perhaps to the extent that flows fall below the initially
prevailing level. Then traffic conditions improve again and at the reduced
level of transportation cost an increased demand is forthcoming. There is
however one level of flow at which traffic conditions give rise to a demand
just equal to the prevailing flow. This is what we shall call the equilibrium
flow.

One might think of equilibrium in a network as simply a state in which
flows on every road are in equilibrium. But as we have pointed out, the
notion of the demand for use of any particular road as a function of trans-
portation cost on that road only involves a simplification that seems

inadmissible in the analysis of flows in a network. Such a demand function could be defined only on the assumption of _ceteris paribus_, namely, of unchanged transportation costs on all other roads. But a change of flow on one road is bound to affect flows and hence costs on some other roads. Therefore a somewhat broader approach is called for.

Demand refers to trips and capacity refers to flows on roads. The connecting link is found in the distribution of trips over the network according to the principle that traffic follows shortest routes in terms of average cost. The idea of equilibrium in a network can then be described as follows. The prevailing demand for transportation, that is, the existing pattern of originations and terminations, gives rise to traffic conditions that will maintain that same demand. Or, starting at the other end, the existing traffic conditions are such as to call forth the demand that will sustain the flows that create these conditions.

States of equilibrium may be defined for a short period such as the peak hour in daily traffic, or, if finer fluctuations of demand are disregarded, for a longer period such as a day or a week. In the last case road users presumably go through their routines repeatedly and at regular intervals. The maintenance of the pattern of road use under given demand conditions therefore implies that drivers have not been dissatisfied with their choices. Put another way, this says that the anticipated cost of the various alternative routes or modes of travel on which demand for transportation, choice of route, and choice of free speed are based corresponds to the realized cost.

3.1 Equilibrium in a Network

"Suppose that between two points there are two highways, one of which is broad enough to accomodate without crowding all the traffic which may care to use it but is poorly graded and surfaced, while the other is a much better road, but narrow and quite limited in capacity. If a large number of trucks operate between the two termini and are free to choose either of the two routes, they will tend to distribute themselves between the roads in such proportions that the cost per unit of transportation, or effective return per unit of investment, will be the same for every truck on both routes. As more trucks use the narrower and better road, congestion develops, until at a certain point it becomes equally profitable to use the broader but poorer highway."[1]

This example demonstrates the principle of traffic distribution among alternative routes in equilibrium. 1) If between a given origin and a given destination more than one route is actually travelled, then the cost of transportation to the average road user, as indicated by the average-cost capacity curves, must be equal on all these routes. 2) Since the routes used are the "shortest" ones under prevailing traffic conditions, average cost on all other possible routes cannot be less than that on the route or routes travelled. 3) The amount of traffic originated per unit of time must equal the demand for transportation at the trip cost which prevails.

It is useful to reformulate these conditions in terms of traffic on roads rather than on routes. Making use of the relationship between short-est routes and trip costs we may restate the three principles as follows. At each location there is a well defined trip cost to any destination for the average road user. It decreases toward the destination by the full

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[1] Knight (1924, p. 162).

(average) amount of transportation cost along all roads on which some traffic flows to that destination; along any other road it decreases by less than or at most the same as (average) transportation cost. The amount of traffic originating at any point for any destination is determined by the demand for transportation arising in response to the (average) trip cost to that destination. This origination then equals the excess of outgoing traffic over incoming traffic (to this destination). The amount terminated at a location is, of course, the sum over all origins of traffic flows with this destination.

Total traffic on the various roads fixes the transportation cost on each road and this again determines the trip cost from each origin to the various destinations. Through the demand function this cost regulates the traffic flow between pairs of locations. The distribution of these flows over the various roads follows by means of the relationship between costs on roads and trip costs. Thus the circle of the equilibrium conditions is closed.

But can we be sure that this circular definition is meaningful? Will there always be a well determined equilibrium if demand and capacity curves have the usual properties, that demand decreases as trip costs increase and that transportation costs increase as flows increase on any given road? Is it possible to compute and predict the equilibrium distribution of flows once the demand and capacity data are given? And finally, is the equilibrium stable or will it be thrown off balance by the slightest deviation?

These are all questions that we shall gradually seek answers to in the following sections, not all of them mathematical ones. In these sections we show that the relations described determine an equilibrium, that an equilibrium always exists if demand is a decreasing function of trip cost

and transportation cost is a constant or increasing function of traffic flow, and that the equilibrium is unique whenever the shortest routes between all pairs of locations are unique and cost is strictly increasing with increasing flow. These statements are also true in the less obvious case of commodity transportation.

3.1.1[*] Formulation of the Equilibrium Conditions

The number of vehicles entering road ij from either end per unit of time, briefly called the flow on that road, was denoted by x_{ij}. However, our elementary variable will be the flow on a road in a given direction to a particular destination, written $x_{ij,k}$, where the ordered pair of subscripts ij denotes the direction from i to j on road ij, and k denotes the destination. This flow is distinct from that in the opposite direction and it does not admit of negative values:

$$(3.1) \qquad x_{ij,k} \geq 0 \qquad \text{for all } ij,k.$$

By our previous definition total flow on a road equals

$$(3.2) \qquad x_{ij} = x_{ji} = \sum_k (x_{ij,k} + x_{ji,k}),$$

and of course, (3.1) implies

$$(3.3) \qquad x_{ij} \geq 0.$$

We have already encountered a third flow variable, namely the number of vehicles originating at location i with destination k per unit of

time, $x_{i,k}$. Since this rate of origination is indicated at i by the excess of flow to k on outgoing roads over that on incoming roads we have

$$(3.4) \qquad x_{i,k} = \sum_j (x_{ij,k} - x_{ji,k}).$$

In Section 2.3.2[*] the relationship between transportation cost on roads and trip costs was formulated. We can now express the fact that traffic uses shortest routes by saying that on roads not in a shortest route to a location k flow to that destination is zero:

$$(3.5) \qquad x_{ij,k} = 0 \qquad \text{if } y_{i,k} - y_{j,k} < y_{ij}.$$

The second relationship between flows and costs is that expressed by the capacity function

$$(3.6) \qquad y_{ij} = h_{ij}(x_{ij}).$$

Our third flow variable is related to trip cost through the (inverse) demand function for transportation:

$$(3.7) \qquad y_{i,k} = \varepsilon_{i,k}(x_{i,k}).$$

Finally we restate the definition of $y_{i,k}$ given in Section 2.3.2:[*]

$$y_{i,k} - y_{j,k} \leq y_{ij} \qquad \text{and } "=" \text{ for some } j,$$

$$(3.8)$$

$$y_{k,k} = 0.$$

We now have for each of the variables $y_{i,j}$, $y_{i,k}$, x_{ij}, $x_{ij,k}$, $x_{i,k}$ at least one relation in which it is "explained" in terms of some other variable(s). Is this a complete system of conditions which under suitable assumptions about the functions involved -- $g_{i,k}(y_{i,k})$ and $h_{ij}(x_{ij})$ -- has a unique solution?

To answer this question our system must be brought into a more concise form. As a direct consequence of (3.5) and (3.8) we have

$$(3.9) \qquad y_{i,k} - y_{j,k} \begin{Bmatrix} = \\ \leq \end{Bmatrix} y_{ij} \quad \text{if} \quad x_{ij,k} \begin{Bmatrix} > \\ = \end{Bmatrix} 0.$$

$$y_{k,k} = 0.$$

In all cases where the flow between a pair of points is zero the second alternative of (3.9), which then holds, does not fully determine the values $y_{i,k}$ of trip costs. It merely imposes an upper bound on them. At the same time equation (3.7) affords a lower bound, for it restricts the $y_{i,k}$ to levels above which demand is zero. The slack which is thus introduced whenever flow between two points is zero does no harm, for then the exact value of trip cost is irrelevant economically. Since the notation in the first line of (3.9) will be used frequently in what follows we shall at this first occurrence write out in full the statements involved:

$y_{i,k} - y_{j,k} \leq y_{ij}$, $x_{ij,k} \geq 0$, and $y_{i,k} - y_{j,k} = y_{ij}$ if $x_{ij,k} > 0$.

In particular, (3.9) implies the absence of "cross hauling": if $x_{ij,k} > 0$ then $x_{ji,k} = 0$. As a next step we can substitute for the cost variables the corresponding functions of the flow variables. Our system then reduces to the inequalities

$$(3.10) \qquad \varepsilon_{i,k}(x_{i,k}) - \varepsilon_{j,k}(x_{j,k}) \begin{Bmatrix} = \\ \leqq \end{Bmatrix} h_{ij}(x_{ij}) \quad \text{if} \quad x_{ij,k} \begin{Bmatrix} > \\ = \end{Bmatrix} 0$$

plus the defining equations (3.2) and (3.4).

Incidentally, (3.10) may be used to find two expressions for the prevailing total cost of transportation. Multiplication on both sides by $x_{ij,k}$ and addition yields

$$(3.11) \qquad \sum_{ij,k} [\varepsilon_{i,k}(x_{i,k}) - \varepsilon_{j,k}(x_{j,k})] \; x_{ij,k} = \sum_{ij,k} h_{ij} \cdot x_{ij,k} \cdot$$

Rearranging terms and using (3.4) and (3.2) we obtain

$$(3.12) \qquad \sum_{i,k} \varepsilon_{i,k}(x_{i,k}) \cdot x_{i,k} = \frac{1}{2} \sum_{ij} h_{ij}(x_{ij}) \cdot x_{ij}$$

The left-hand side represents the sum of all trip costs, the right-hand side the sum of all transportation costs on roads.

3.1.2[*] Existence of Solutions to the Equilibrium Conditions

A well known characteristic of the equilibria encountered in theoretical mechanics is that they may be regarded as solutions to certain extremum problems, a fact which has occasionally given rise to some speculation about nature's grand design. Whatever the merits of teleological interpretations, the possibility of formulating an equilibrium in terms of maximization is a useful piece of mathematical information. For instance, the fact that there is a maximum problem associated with the present equilibrium system (as there are extremum problems for many other economic equilibria[2])

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[2]   Cf. Samuelson [1948, pp. 21-23].

will give us a proof that there exist solutions to our system. Consider the function

$$(3.13) \qquad H(\ldots,x_{ij,k},\ldots) = \sum_{i,k} \int_0^{x_{i,k}} a_{i,k}(x)\,dx - \frac{1}{2}\sum_{ij} \int_0^{x_{ij}} b_{ij}(x)\,dx.$$

(Economists are warned that this is not to be interpreted as consumers' surplus! The term on the right has as its kernel the _average_, rather than the _incremental_ cost to users collectively.) Here the factor $\frac{1}{2}$ comes in because we wish to sum over every road but once, while each road is denoted by two pairs of indices, ij and ji. Differentiate with respect to $x_{ij,k}$ after substituting for $x_{i,k}$ and x_{ij} from (3.4) and (3.2). The result is

$$(3.14) \qquad \frac{\partial H}{\partial x_{ij,k}} = a_{i,k}(x_{i,k}) - a_{j,k}(x_{j,k}) - b_{ij}(x_{ij}).$$

The necessary first order conditions for a maximum are that the derivatives of H be non-positive in all directions in which $x_{ij,k}$ can change without becoming negative. In the interior of the positive orthant $(x_{ij,k} > 0$ for all ij,k) we therefore have the usual conditions that the derivatives shall vanish, while at a point of the boundary where some $x_{ij,k} = 0$ the derivative with respect to any vanishing $x_{ij,k}$ must be non-positive:

$$(3.15) \qquad \frac{\partial H}{\partial x_{ij,k}} = 0 \text{ if } x_{ij,k} > 0 \text{ and } \frac{\partial H}{\partial x_{ij,k}} \leq 0 \text{ if } x_{ij,k} = 0.$$

We now recall the existence of absolute capacity limits c_{ij} as expressed in relation (2.12). In the closed set defined by the two conditions, $0 \leq x_{ij,k}$ and $x_{ij} \leq d_{ij} < c_{ij}$ for some suitable d_{ij}, the function

$H(\ldots,x_{ij,k},\ldots)$ is continuous since it is a sum of indefinite integrals of Riemann integrable (because sectionally continuous) functions. We conclude that H assumes its maximum at some point. While this point may still depend on the d_{ij}, it will not do so if d_{ij} is sufficiently close to c_{ij}, because of (2.12). At that point of maximum the necessary conditions (3.13) must be satisfied. But these conditions are identical with (3.10). Therefore there must exist a solution to inequalities (3.10) and the supplementary defining equations (3.2) and (3.4). Of course, E was just so constructed that the first order conditions for a maximum (3.15) become identical with the (necessary) equilibrium condition (3.10).

3.1.3[*] Uniqueness of the Solutions

In preparation for the question of when a solution to our equilibrium conditions is unique, we shall first show that the function H is concave, as defined below.

Consider the quadratic form of its second derivatives (sometimes known as the Hessian) at a point $x_{ij,k}$, with $z_{ij,k}$ denoting the arguments in place of the former $x_{ij,k}$. We shall use the obvious abbreviations

$$(3.16) \quad z_{ij} = \sum_k (z_{ij,k} + z_{ji,k}), \quad z_{i,k} = \sum_j (z_{ij,k} - z_{ji,k}),$$

$$g'_{i,k} = \frac{dg_{i,k}}{dx_{i,k}}, \quad \text{and} \quad h'_{ij} = \frac{dh_{ij}}{dx_{ij}}.$$

We have

$$(3.17) \quad \sum_{\substack{ij,k \\ lm,n}} z_{ij,k} \frac{\partial^2 H}{\partial x_{ij,k} \partial x_{lm,n}} z_{lm,n} =$$

$$\sum_{ij,k} z_{ij,k}(g'_{i,k} \cdot z_{i,k} - g'_{j,k} \cdot z_{j,k}) - \sum_{\substack{ij,k \\ i<j}} (z_{ij,k}+z_{ji,k})h'_{ij} \cdot z_{ij}$$

Rearranging terms in the first sum this becomes the left-hand member of

$$(3.18) \qquad \sum_{i,k} \mathfrak{S}'_{i,k} \cdot z^2_{i,k} - \tfrac{1}{2} \sum_{ij} h'_{ij} \cdot z^2_{ij} \leqq 0.$$

The inequality holds identically in the $z_{ij,k}$ and $x_{ij,k}$ because inverse demand $\mathfrak{S}_{i,k}$ is a decreasing function and capacity h_{ij} an increasing function. This shows that the Hessian is negative semi-definite for all $x_{ij,k}$. A differentiable function whose Hessian is negative semi-definite is known to be a concave function.[3] By this one means the following: for every a, $0 < a < 1$, and every two sets of variables $(x_{ij,k}) = x$, $(\bar{x}_{ij,k}) = \bar{x}$,

$$(3.19) \qquad H \left[ax + (1 - a) \bar{x} \right] \geqq a H(x) + (1 - a) H (\bar{x}).$$

Incidentally, a function is called convex if its negative is concave.

Next we prove the following useful lemma.

Lemma: Let $F(u)$ be a concave function of a set of variables $u = (u_1, \ldots, u_r, \ldots, u_n)$. Sufficient for $F(u^0) = \max\limits_{u_r \geqq 0} F(u)$ is that F be differentiable at $u = u^0$ and that

$$(3.20) \qquad \left(\frac{\partial F}{\partial u_r} \right)_{u = u^0} \begin{Bmatrix} = \\ \leqq \end{Bmatrix} 0 \quad \text{if} \quad u_r^0 \begin{Bmatrix} > \\ = \end{Bmatrix} 0.$$

Proof: Borrowing directly from Kuhn and Tucker [1951], we first prove that for every concave function $F(u)$, differentiable at u^0,

$$(3.21) \qquad F(u) \leqq F(u^0) + \sum_r F_r^0 \cdot (u_r - u_r^0),$$

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[3]   Bonnesen and Fenchel (1934, pp. 18-19).

where F_r^o denotes $\dfrac{\partial F(u^o)}{\partial u_r^o}$. For any a, $0 < a < 1$, we have

$$(3.22) \qquad F(u) - F(u^o) \leq \frac{F[u^o + a(u - u^o)] - F(u^o)}{a}$$

by the definition of a concave function. Hence in the limit, as $a \to 0$,

$$(3.23) \qquad F(u) - F(u^o) \leq \sum_r F_r^o \cdot (u_r - u_r^o)$$

We proceed to the proof of the lemma proper. By (3.23) and (3.20), in that order, we have for all $u_r \geq 0$

$$(3.24) \qquad F(u^o) \geq F(u) - \sum_r F_r^o \cdot (u_r - u_r^o) \geq F(u).$$

It follows that $F(u^o) = \underset{u_r \geq 0}{\text{Max}} \; F(u)$, which proves the lemma.

It is easily verified that (3.20) with F and u_r corresponding to H and $x_{ij,k}$, respectively, is nothing but our old equilibrium condition (3.10). We conclude that every solution of the equilibrium conditions yields a maximum of H. The converse fact, that every maximum (in fact every extremum) of H yields a solution to the equilibrium condition (3.10) is already known to us. From this and the concavity of H it is now seen that for every two solutions $x_{ij,k}$ and $\bar{x}_{ij,k}$ of (3.10) the linear combinations

$$a x_{ij,k} + (1 - a) \bar{x}_{ij,k} \qquad \text{for } 0 < a < 1$$

are likewise solutions of the equilibrium condition (3.10), a fact which is not immediately obvious.

The possibility of a plurality of solutions is inherent in certain properties of the equilibrium. Suppose that two routes are equally attractive between two locations that are transit points for flows from several origins or to several destinations (a so-called neutral circuit[4]). Then it is of course arbitrary in what proportions these flows are allocated to each route as long as the total flows on each route have the proper magnitudes (so as to maintain equality of average cost over the two routes).

On the other hand, one expects equilibrium flows x_{ij} on all roads to be unique when demand functions are strictly decreasing with trip cost and all transportation cost functions are strictly increasing with flow. We are now in a position to prove this assertion. Suppose that there exist two solutions x and $\bar{x}$. By what was said before there must then exist, on the line segment connecting x and $\bar{x}$, a maximum which is located arbitrarily close to x. Call it $x + \delta x$. Expanding H at x

$$(3.25) \qquad H(x + \delta x) = H(x) + \sum_{ij,k} \frac{\partial H}{\partial x_{ij,k}} \delta x_{ij,k} + \sum_{\substack{ij,k \\ lm,n}} \frac{\partial^2 H}{\partial x_{ij,k} \partial x_{lm,n}} \delta x_{ij,k} \delta x_{lm,n} + \cdots .$$

Because x is a point of maximum the second term at the right vanishes. Since $x + \delta x$ is also a maximizer, $H(x + \delta x) = H(x)$, so that the last term of (3.25) vanishes. But this term was shown to be equal to

$$(3.26) \qquad \sum_{i,k} \varepsilon'_{i,k} \cdot (\delta x_{i,k})^2 - \sum_{ij} h'_{ij}(\delta x_{ij})^2 .$$

Our assumptions were

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[4] Cf. end of Section 2.3.1.

(3.27) $g'_{i,k} < 0$ and $h'_{ij} > 0$ for all i,k, ij.

Therefore $\delta x_{i,k} = 0$ and $\delta x_{ij} = 0$ for all i,k,ij. That is the originations and flows on roads are uniquely determined.[5]

Under what conditions does the uniqueness of these variables imply that flows $x_{ij,k}$ distinguished by destinations are also uniquely determined? This is the case when no neutral circuits are present that is, when the network layout and flow conditions are such that no two routes between any single pair of locations are equally attractive.[ ]

3.2 Effects of Changes in Capacity and in Demand

How do the equilibrium flows change in response to shifts in capacity functions or in demand functions? An increase of capacity, brought about by construction or improvement of roads, is understood here to be reflected in a lowering of the capacity curve: to every flow there corresponds a smaller average cost than before. An increase in demand means that the demand curve has shifted upward, so that a given flow comes forth at a higher transportation cost. To be specific, let us assume that all capacity functions are strictly increasing and all demand functions strictly decreasing. Then the following assertions can be made, which are plausible in themselves.

An increase in the capacity on only one road either has no effect on flows at all, or leads to a positive increase of traffic at least on this road. Simultaneous changes in the capacities of several roads either leave all traffic unchanged, or cause a growth of traffic on at least one road of increased capacity, or a decline of traffic on at least one road of decreased

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[5] This proof can easily be extended to show that the two components $\sum_k x_{ij,k}$ and $\sum_k x_{ji,k}$ of each flow are uniquely determined.

[ ] Cf. Section 2.3.1

capacity. More generally one can say that traffic growth tends to accompany capacity increases in the sense that a certain weighted sum of the algebraic products of capacity change and traffic change is positive whenever any change in traffic takes place at all.

With respect to demand, similar statements, obtained by appropriate substitutions, are valid. Although these remarks on changes in equilibrium flows appear quite obvious, they have been mentioned here because they hold equally in the less trivial case of commodity transportation, and also because they illustrate a mathematical principle which will be useful later on.

The next section presents the mathematical theorem whose economic content has been given in this section.

3.2.1* An Inequality on the Effect of Data Changes

In order that capacity and demand be characterized by definite parameters, let us assume both to be given in terms of linear functions. For simplicity it will also be assumed that the absolute capacity limits are nowhere attained. Denote the capacity functions by

$$(3.28) \qquad h_{ij}(x_{ij}) = a_{ij} x_{ij} + b_{ij}$$

and the inverse demand functions by

$$(3.29) \qquad g_{i,k}(x_{i,k}) = e_{i,k} x_{i,k} + f_{i,k}.$$

The maximand H of Section 3.1.2* thereby becomes

$$(3.30) \qquad \sum_{i,k} \left[\frac{1}{2} e_{i,k} x_{i,k}^2 + f_{i,k} x_{i,k} \right] - \sum_{ij} \left[\frac{1}{4} a_{ij} x_{ij}^2 + \frac{1}{2} b_{ij} x_{ij} \right] \cdot$$

In order to evaluate the effect of finite changes δa_{ij}, δb_{ij}, $\delta e_{i,k}$, and $\delta f_{i,k}$ in the corresponding parameters of capacity and demand recourse is had to the following general theorem, which in economic theory falls under the heading of the Le Chatelier Principle.[1]

<u>Theorem</u>: Let q_{nr} be the coefficients of a negative semi-definite quadratic form, $q_{nr} = q_{rn}$. Consider the problem

$$(3.31) \qquad \max_{x_r \geq 0} \left[\sum_{n,r} x_n q_{nr} x_r + \sum_r a_r x_r \right] \text{ subject to } \sum_r b_{sr} x_r \leq c_s \cdot$$

The solution equations are

$$(3.32) \qquad 2 \sum_n q_{nr} x_n + a_r - \sum_s \lambda_s b_{sr} \begin{Bmatrix} = \\ \leq \end{Bmatrix} 0 \quad \text{if} \quad x_r \begin{Bmatrix} > \\ = \end{Bmatrix} 0$$

where $\qquad \lambda_s \begin{Bmatrix} = \\ \geq \end{Bmatrix} 0 \quad \text{if} \quad \sum_r b_{sr} x_r \begin{Bmatrix} < \\ = \end{Bmatrix} c_s \cdot$

Let finite changes in the parameters and variables be denoted $\delta q_{nr}, \ldots, \delta \lambda_s$, δx_r, respectively. We shall show that

$$(3.33) \qquad \sum_r \left\{ \sum_n \left[x_n (\delta q_{nr} + \delta q_{rn}) + \delta x_n q_{nr} + \delta a_r - \sum_s \lambda_s \delta b_{sr} \right] \delta x_r \right\}$$

$$- \sum_s \delta \lambda_s (\delta c_s - \sum \delta b_{sr} x_r) \geq 0.$$

If q_{nr} represents a negative <u>definite</u> form, then $\delta x_r \neq 0$ for some r implies " > " in (3.33).

[1] Samuelson [1946-47], [1948, pp. 36-39]

Abstract of Proof: The Kuhn-Tucker Theorem for concave maximands and linear inequalities as constraints (derived below, Section 4.3.1[*]) asserts that the solution x_r^o and the Lagrangean multipliers λ_s^o can be obtained in terms of unconstrained extremum problems. For shortness, write

$$(3.34) \qquad \sum_{n,r} x_n q_{nr} x_r + \sum_r a_r x_r + \sum_s \lambda_s (c_s - \sum_r b_{sr} x_r) = M(x,\lambda).$$

According to the theorem quoted,

$$(3.35) \qquad M(x_r^o, \lambda^o) = \underset{x_r \geq 0}{\text{Max}} \; M(x,\lambda^o) = \underset{\lambda_s \geq 0}{\text{Min}} \; M(x^o, \lambda).$$

Denote the same function in terms of the modified parameters by $\bar{M}(x,\lambda)$. Then

$$(3.36) \qquad M(x^o, \lambda^o + \delta\lambda^o) - M(x^o + \delta x^o, \lambda^o) \geq 0$$

because, by (3.35), the first term is greater than and the second term less than $M(x^o, \lambda^o)$. The ">" sign applies if some $\delta x_r^o > 0$ and q_{nr} is negative definite, because M is then strictly concave in x_r. In a similar way we find that

$$(3.37) \qquad \bar{M}(x^o + \delta x^o, \lambda^o) - \bar{M}(x^o, \lambda^o + \delta\lambda^o) \geq 0$$

because, $(x^o + \delta x^o, \lambda^o + \delta\lambda^o)$ being the solution to the problem when the parameters are modified, the first term is greater than and the second term less than $\bar{M}(x^o + \delta x^o, \lambda^o + \delta\lambda^o)$. The sum (3.36) and (3.37) yields the desired inequality (3.33).

In the present case constraints (other than the restriction to positive values) do not occur, so the λ_s in (3.33) are zero. Therefore the inequality (3.33) when applied to the function H of (3.30) becomes simply

$$(3.38) \qquad \frac{1}{2} \sum_{i,k} (2x_{i,k} + \delta x_{i,k}) \, \delta e_{i,k} \, \delta x_{i,k}$$

$$- \frac{1}{4} \sum_{ij} (2x_{ij} + \delta x_{ij}) \, \delta a_{ij} \, \delta x_{ij}$$

$$+ \sum_{i,k} \delta f_{i,k} \, \delta x_{i,k} - \frac{1}{2} \sum_{ij} \delta b_{ij} \, \delta x_{ij} \geqq 0$$

where $\sum_{i,k}$ corresponds to $\sum_r$, and we can omit $\sum_n$ and put $n = r$, since now q_{nr} is zero whenever $n \neq r$. The ">" alternative holds if some $\delta x_{ij} \neq 0$ or some $\delta x_{i,k} \neq 0$, provided that all $e_{i,k} < 0$.

If only one capacity is changed and demand remains unchanged, then all the $\delta e_{i,k}$, $\delta f_{i,k}$ are zero and all the δa_{ij} and δb_{ij} are zero except those for some particular ij. In this case (3.38) becomes

$$(3.39) \qquad \left[- \frac{1}{4} (2x_{ij} + \delta x_{ij}) \, \delta a_{ij} - \frac{1}{2} \delta b_{ij} \right] \delta x_{ij} \geqq 0$$

Suppose we have a single capacity increase; a capacity change will be unequivocally an increase only if

$$(3.40) \qquad \delta a_{ij} \leqq 0, \quad \delta b_{ij} \leqq 0, \quad \delta a_{ij} + \delta b_{ij} < 0.$$

Otherwise the traffic required to bring about some particular level of cost would be less than before. Since x_{ij} and $x_{ij} + \delta x_{ij}$ are non-negative, the bracketed expression in (3.39) is non-negative and positive when $\delta x_{ij} \neq 0$.

We conclude that $\delta x_{ij} \geq 0$. (Equality occurs when $x_{ij} = 0$, that is, when the road was useless to begin with.) Thus the reactions of flows to changes in capacity follow in all cases the pattern that one would expect.

3.3 Stability

An equilibrium would be just an extreme state of rare occurrence if it were not stable, that is, if there were no forces which tended to restore equilibrium as soon as small deviations from it occurred.

Besides this stability "in the small" one may consider stability "in the large", that is, the ability of the system to reach an equilibrium from any initial position. This latter type of stability is interesting not only because it concerns the capacity of the system to reach a new equilibrium position after some big change, but also because one may want to use an analog of the adjustment process as a method of computing an equilibrium solution by successive approximations.

3.3.1 Adjustments of Road Users

The study of stability hinges ultimately on the question of how road users adjust themselves to changes, that is, how they adapt the extent of their travel by road and their choice of routes to varying traffic conditions. This however is one of the big unknowns of road-user behavior, so that at the present stage only conjectures are possible. Through a simple and plausible model one can get a rough picture of the minimum of conditions that must be met in order that the adjustment process should converge.

In our model we shall assume simply that those road users who do not just continue in their previous choices will choose their routes and the number of trips by road on the basis of the traffic conditions that prevailed

in the preceding period. (Nothing need be assumed about the length of
these periods, but this type of behavior is most plausible when the periods
are fairly long.) These road users who have or can obtain adequate know-
ledge of the traffic conditions, even if not by first-hand experience,
choose a route which is optimal at the transportation cost of the last per-
iod and set their demand for transportation at levels corresponding to the
average costs of trips during the last period.

The responsive fraction of road users in each period will be regarded
as an independent random sample drawn from the total population of road users.
Its size is assumed to decrease as time proceeds, until a new occasion for
adjustment arises with another change in the network. Traffic which is
actually experimenting to find optimal routes will be disregarded as just
a random disturbance, whose size also decreases in the course of time.
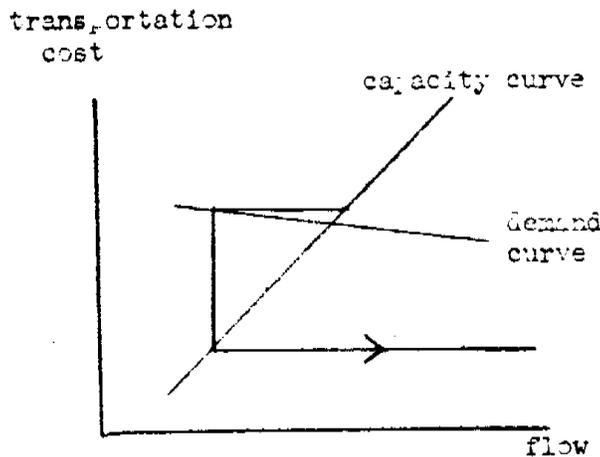
Figure 1. Unstable Case   Figure 2. Stable Case
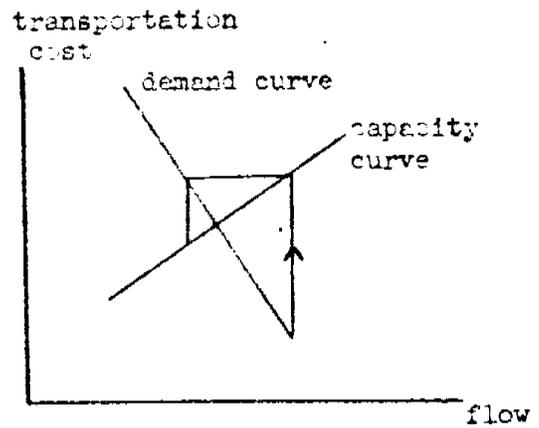
As an extreme case, consider first a single road on which all road users
decide about travel on the basis of the traffic conditions in the preceding
period. This is but an instance of the well known cobweb phenomenon which
occurs in terms of price fluctuation in certain markets (hogs, shipbuilding)

where the production decisions are made considerably in advance of the marketing period.[3] When prices are high production is expanded. After the increased output becomes available, prices drop and as a result production is curtailed. When as a consequence prices go up again, the whole cycle is repeated. Whether these fluctuations continue at a steady level, grow increasingly violent, or ultimately die down depends altogether on the relative slopes of the supply and the demand curves in question. In the present example Figures 1 and 2 show that a dampening of the oscillations takes place whenever the slope of the demand curve is greater in absolute terms than the slope of the capacity curve. This condition is likely to be met in reality, since the demand for road transportation presumably does not respond in a highly elastic way to transportation cost. At any rate the effective demand curve will have a rather steep slope whenever the responsive fraction of road users is small.

By way of contrast consider next a network of two roads between two locations and a fixed demand. To be definite let us assume that just one-fifth of the road users adjust their routes to the last prevailing traffic conditions. Figure 3 gives a graphical description of what happens. Let the entire line segment represent total flow and the numbered points the distribution of flow among the two roads during successive periods. Let 0 be the equilibrium distribution.

Figure 3.

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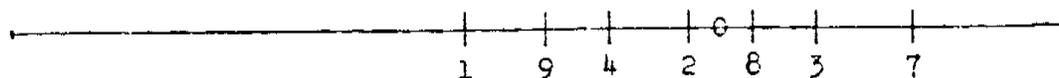
[3] Tinbergen [1951, pp. 143-148].

Consider an arbitrary initial distribution, say point 1. Since the equilibrium point is to its right this means that the cost on the road whose flow is measured by the interval from 1 to the right endpoint is larger. Therefore one-fifth of its flow will shift to the other road. The next distribution point is therefore the point which marks one-fifth of the interval to the right of 1 as measured from 1 on. This is point 2. It still is to the left of 0. And so we obtain 3 from 2 in the same way as 2 was obtained from 1. Point 4 will now fall to the left of 0. Continuing in this way the following becomes apparent. Points keep oscillating around 0 for a while with odd numbered points to the right of 0, even numbered points to the left. At the same time there is a steady drift of even and odd points among themselves to the right. (This is due to the fact that 0 is situated to the right of the center). Finally an even numbered point will fall to the right of and rather close to 0. Say, that this is point 8. But then the next step will carry an odd point (9) rather far to the left but obviously not more than one-fifth of the distance between the left endpoint and 0. It emerges that oscillations will continue to range over one-fifth of the entire flow. Only by an extremely slight chance can a distribution point ever coincide with 0 and thereby arrest any further oscillation.

It may seem paradoxical that equilibrium is approached to within the same proportion of total flow as that which designates the responsive fraction of road users. The smaller the fraction of drivers who take notice of changes in traffic conditions the better is the approximation in the end. Emphasis here is on the phrase, "in the end". For each step by itself in the movement toward equilibrium carries one so much less far.

The adjustment processes in a network may be understood as a superposition

of these two basic types of movement. We have computed two examples which
will demonstrate the approach to equilibrium and the extent and persistence
of fluctuations that our model describes for a network with flexible demand.

3.3.2 Numerical Examples of Approach to Equilibrium

In the following pages two hypothetical examples are presented, solely
for purposes of illustration. They are intended to show a possible method
of computation and to exhibit the process of convergence toward a stable
equilibrium discussed in the preceding section. The parameters were chosen
arbitrarily and any resemblance to an actual road network is accidental.
An attempt was made to select parameters which would result in relatively
inelastic demand functions and so that costs on alternative routes would
be significantly different.

Both examples are based on the same road network, which is represented
in Figure 4.
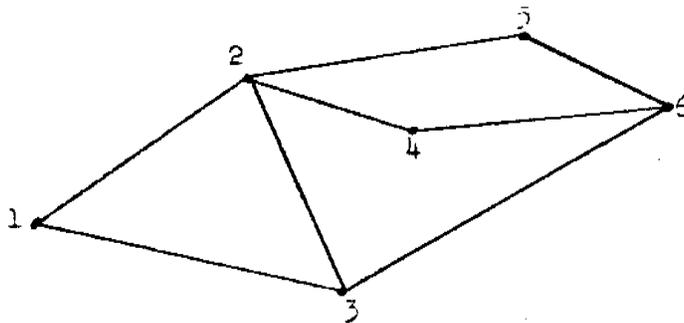
Figure 4.

All traffic is assumed to move in an eastbound direction (i.e. from lower
numbered points to higher numbered points); or, at any rate, westbound
traffic is ignored. All functional relationships. i.e., between flow and
cost and between cost and demand are taken as linear.

In the first example no flows $x_{i,k}$ were assumed between adjacent points. This example was worked first with the fraction a of responsive road users equal to one, and zero flows initially on all roads. With the parameters used, there was no sign of convergence. The example was worked a second time with $a = \frac{1}{2}$, again starting with zero flows on all roads, but there was still little or no indication of convergence. The third attempt was made with $a = \frac{1}{10}$, and beginning flows equal to estimates of the equilibrium flows based on the range of the oscillations observed in the second trial. This approach resulted in apparent convergence.

In the second example, some of the parameters were altered, and flows between adjacent points were included. In this case there was already a tendency to converge with $a = \frac{1}{2}$, starting from zero flow on all roads, but some oscillations persisted. The value of a was then decreased by stages first to $\frac{1}{5}$ and then to $\frac{1}{10}$. The amplitude of the remaining oscillations was thereby decreased.

In both examples it was found that flows from 1 to 6 settled down rapidly to a single route, over which the flows converged asymptotically (if at all). Flow from 2 to 6 oscillated between two routes, namely 2 - 4 - 6 and 2 - 3 - 6. The oscillations were of fairly constant amplitude and period, although the oscillation was asymmetric, as can be seen from the graphs (Figures 5-8). In the first example there was some indication of longer "waves" superimposed on the oscillations, but these were not apparent in the second example.

It can be assumed that oscillations would die out faster if the fraction a of responsive road users were a decreasing function of the cost difference between the alternatives being compared, with a tending to zero as the cost

difference tends to zero. Another factor leading to the same result is the fact that some weight may be given to experience of the more remote past, especially where oscillations have already been experienced.

Example 1

$$y_{12} = 2 + \frac{1}{2} x_{12}$$

$$y_{13} = 2 + \frac{2}{3} x_{13}$$

$$y_{23} = 2 + \frac{1}{3} x_{23}$$

$$y_{24} = 2 + \frac{1}{2} x_{24}$$

$$y_{25} = 3 + \frac{1}{4} x_{25}$$

$$y_{36} = 3 + \frac{1}{2} x_{36}$$

$$y_{45} = 2 + \frac{1}{2} x_{45}$$

$$y_{56} = 3 + \frac{1}{3} x_{56}$$

$$y_{1,4} = y_{12} + y_{24}$$

$$y_{1,5} = y_{12} + y_{25}$$

$$y_{2,6} = \text{Min}(y_{24} + y_{46}, y_{25} + y_{56}, y_{23} + y_{36})$$

$$y_{1,6} = \text{Min}(y_{13} + y_{36}, y_{12} + y_{26})$$

$$\varepsilon_{1,4} = 5 - .4 y_{14}$$

$$\varepsilon_{1,5} = 6 - .5 y_{15}$$

$$\varepsilon_{1,6} = 8 - .3 y_{16}$$

$$\varepsilon_{2,6} = 7 - .5 y_{26}$$
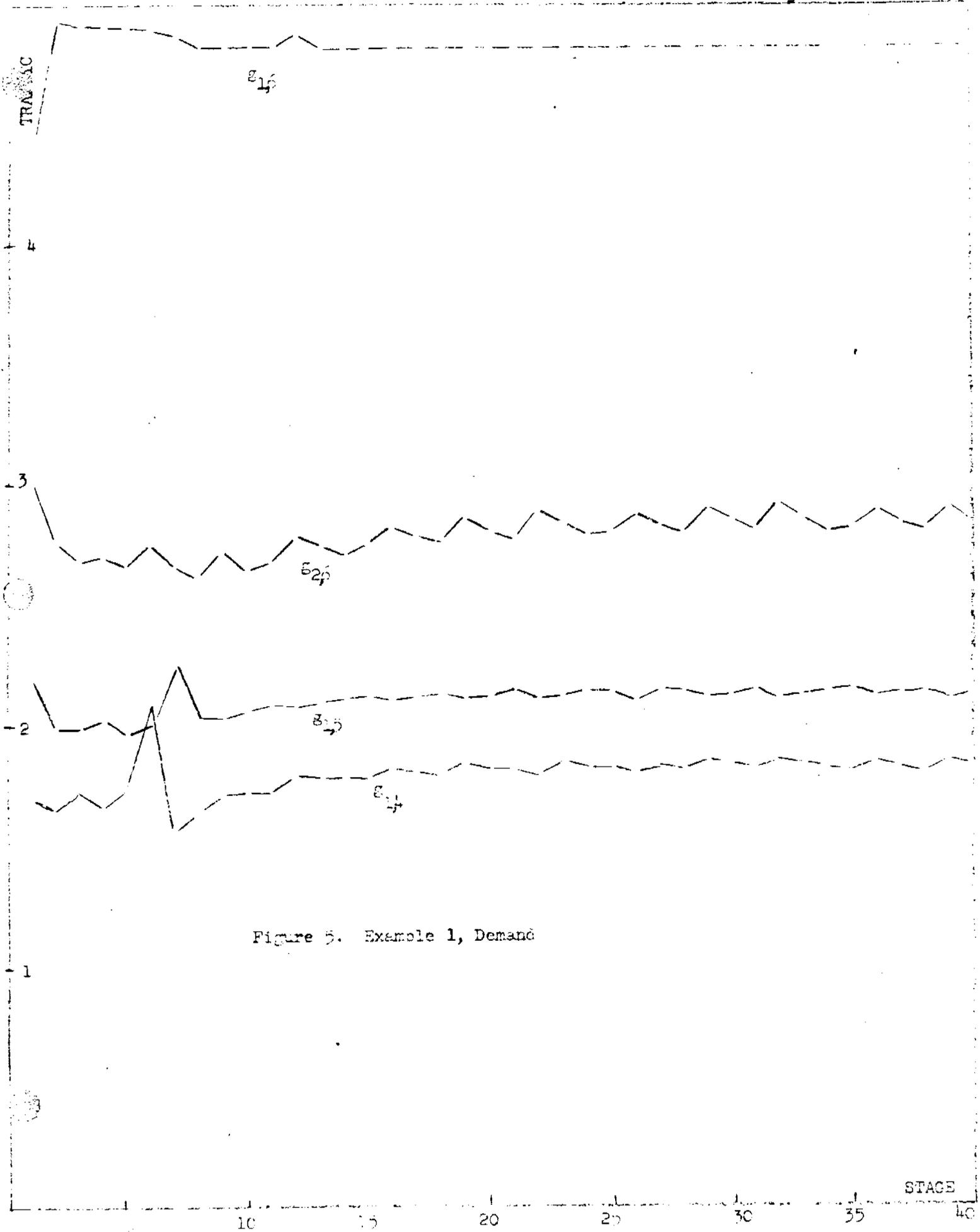
Figure 5. Example 1, Demand

STAGE
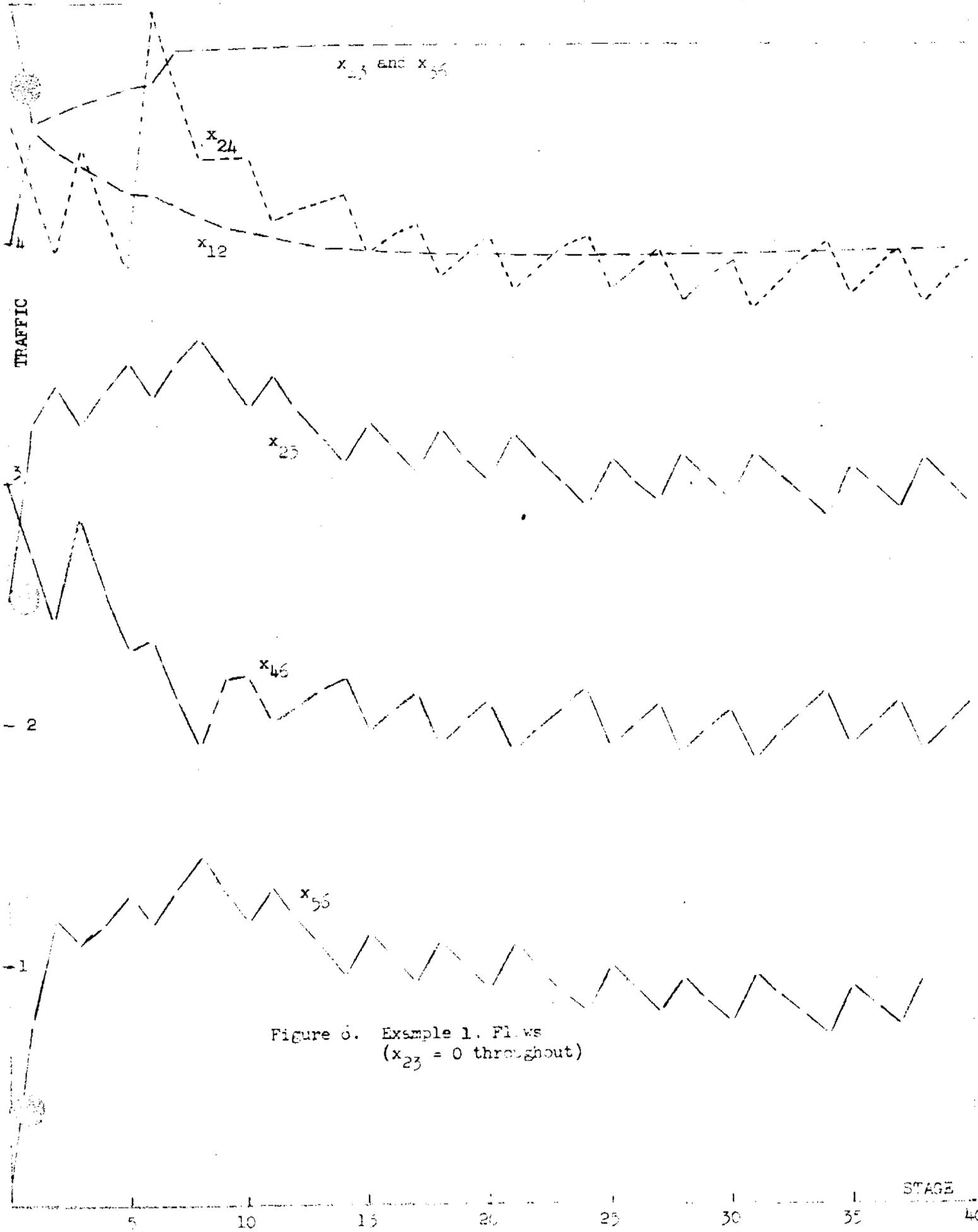
Figure 6. Example 1. Flows
$(x_{23} = 0$ throughout$)$

Example 2

$$y_{12} = 5 + .5 x_{12}$$

$$\xi_{1,2} = 16 - .3 y_{12}$$

$$y_{13} = 15 + .2 x_{13}$$

$$\xi_{1,3} = 14 - .2 y_{13}$$

$$y_{23} = 9 + .6 x_{23}$$

$$\xi_{2,3} = 24 - .2 y_{23}$$

$$y_{24} = 10 + .2 x_{24}$$

$$\xi_{2,4} = 30 - 1 y_{24}$$

$$y_{25} = 11 + .4 x_{25}$$

$$\xi_{2,5} = 28 - .3 y_{25}$$

$$y_{36} = 8 + .3 x_{36}$$

$$\xi_{3,6} = 20 - .6 y_{36}$$

$$y_{45} = 6 + .4 x_{45}$$

$$\xi_{4,6} = 18 - .2 y_{45}$$

$$y_{56} = 7 + .5 x_{56}$$

$$\xi_{5,6} = 12 - .1 y_{56}$$

$$y_{1,4} = y_{12} + y_{24}$$

$$\xi_{1,4} = 50 - .4 y_{14}$$

$$y_{1,5} = y_{12} + y_{25}$$

$$\xi_{1,5} = 60 - .5 y_{15}$$

$$y_{2,6} = \text{Min} \, (y_{24} + y_{45}, \; y_{25} + y_{56}, \; y_{23} + y_{36})$$

$$\xi_{2,6} = 70 - .5 y_{26}$$

$$y_{1,6} = \text{Min} \, (y_{13} + y_{36}, \; y_{12} + y_{2,6})$$

$$\xi_{1,6} = 80 - .3 y_{16}$$

Figure 7. Example 2. Flows
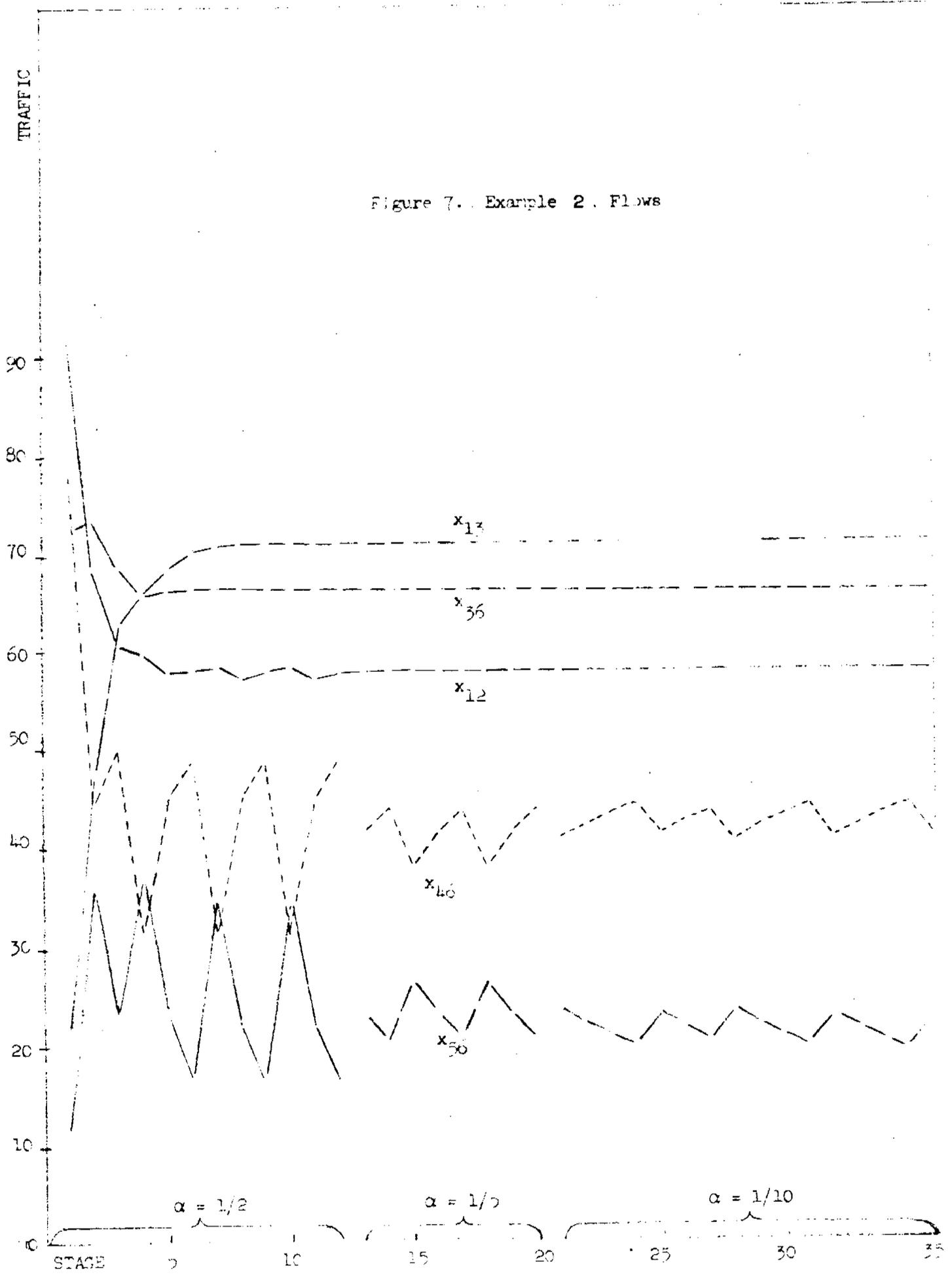
Figure 8. Example 2, Flows (second trial)
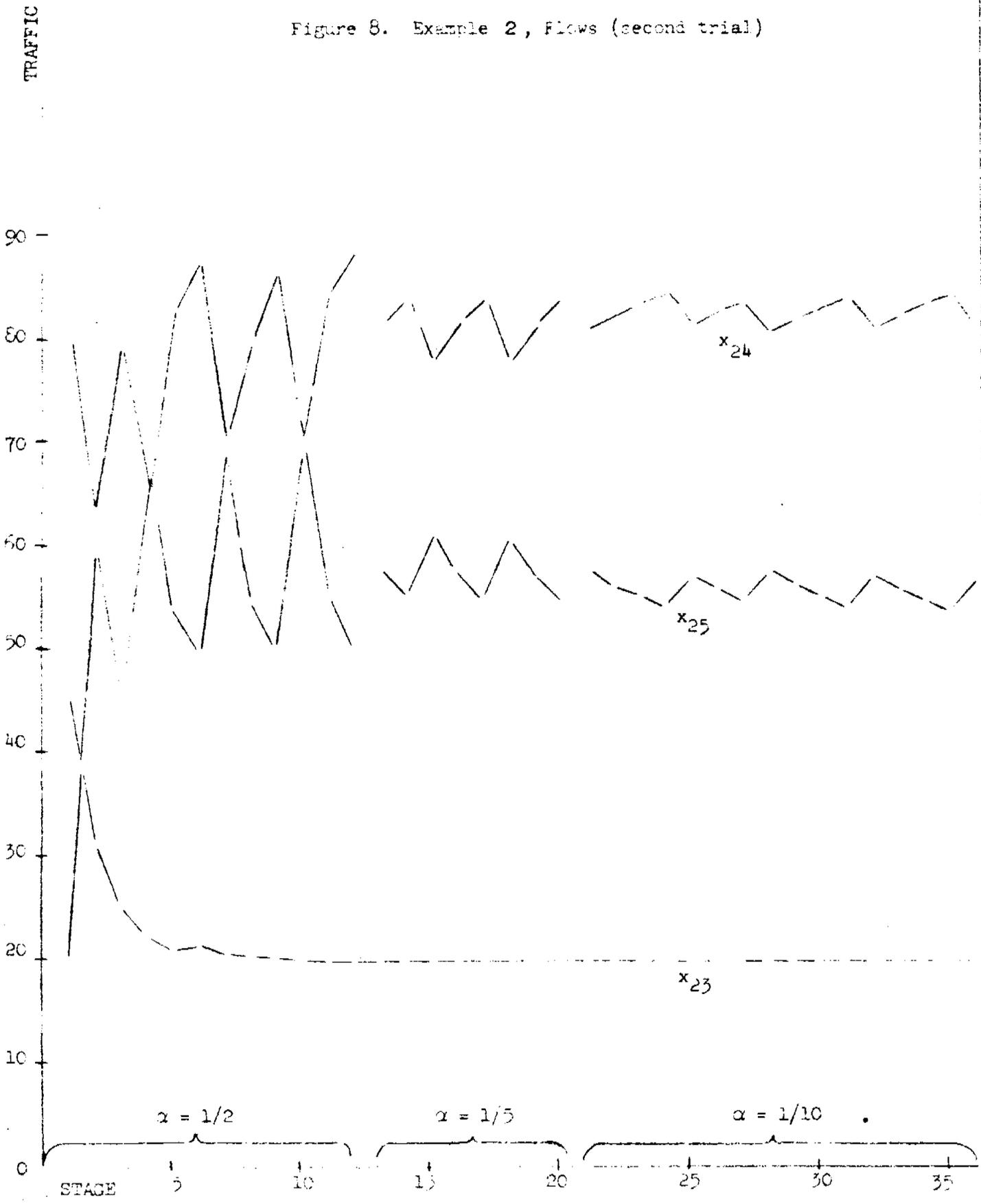
Chapter 4

Efficiency

4.1. The Problem

An economic approach to traffic analysis should not only explain the interaction of many individual decisions in some state of equilibrium, but also provide criteria by which to judge the performance of the system. Since roads are scarce means, their best utilization is an important concern to the community. In the present chapter we attempt to clarify the economic meaning of "best utilization" and to evaluate traffic equilibrium in the light of this interpretation.

4.1.1 The Allocation of Road Capacity

The rules of the road, by determining which traffic is to have priority in crossing an intersection or passing an obstacle, achieve an allocation of the available road capacity to those competing for its use. A similar function is exercised by speed limits and passing restrictions, even if the lawmakers' concern may have been primarily that of "safety". Thus a slow vehicle proceeding on a road where traffic conditions afford little opportunity to pass in effect exercises a power to exclude faster vehicles for extended periods from the road space available in front of it. This allocation of capacity necessarily involves a distribution of time losses and hence of cost.

An allocation of road space also takes place in a more subtle way through the adjustment of route selection to traffic conditions. In the case of two roads between a pair of locations traffic distributes itself so that average transportation cost becomes the same on either road. If one road is shorter but of small capacity, the delays at equilibrium due to the more crowded

conditions on the shorter road would just compensate for the greater operating cost on the longer road. Congestion on the short road, by discouraging further traffic there has led to a diversion of some traffic to the long road. This too may be called an allocation of road space.

There is nothing inevitable about this particular allocation; a different distribution could have been achieved by arbitrarily assigning all vehicles with odd-numbered license plates to the longer route, say. In the case of the equilibrium distribution, as Pigou has pointed out, it would be possible, by shifting a few cars from one road to the other, greatly to lessen the trouble of driving for those left on the congested road, while only slightly increasing the trouble of driving along the less congested road. "In these circumstances a rightly chosen measure of differential taxation against [the congested road] would create an 'artificial' situation superior to the 'natural' one. But the measure of differentiation must be rightly chosen."[1] What is the basis of this economic value judgment?

4.1.2 The Meaning of Efficiency

As long as the demand for transportation between given points or the demand for net transportation of commodities from each location is fixed independently of flow conditions, the choice of a criterion of efficiency seems obvious: the minimization of the aggregate cost to all road users. But the very notion of aggregate cost involves an evaluation of the time losses and dollars spent by various classes of road users. The way in which we have defined our demand and capacity functions implies however that we shall treat everyone like the average road user. There is no scope in this model for a differential evaluation of the costs and benefits to classes of road users.

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[1]   Pigou, [1920, p. 194].

A single exception is to be found at the end of this chapter, in Section
4.5, which explains some of the consequences of differential time valuation.

For our analysis it makes no difference however what the relative weights
are that we attach to the money cost, time, and risk of the average road user.
For, as we shall see, these weights do not affect the relevant properties of
the cost function we use, so that the principle of our conclusion holds quite
independently of the valuations involved.

At this point it may be remarked that an economic valuation of risk is
implied in many decisions that a road user must make, and is equally indis-
pensable in an efficiency analysis from the point of view of society as a
whole. Now any valuation of risk implies in effect the setting of money
values upon human life and health. This is indeed unavoidable whenever
economic activities involve danger to human life and health. Since a strict
adherence to the "safety-first" principle would mean the standstill of many
economic activities without which modern industrial societies could not exist,
everybody makes a compromise, consciously or unconsciously, which implies
certain high, but finite, valuations of human life and health. Highway trans-
portation is no exception to this. In each selection of a free speed, a choice
of the probability of fatal accidents is implied. Since operating cost and
time cost also vary with free speed, and since presumably the individual seeks
his optimum, the increase of the probability of fatal accidents with an in-
crease in free speed must be at least compensated by the simultaneous decrease
of other cost. Knowing how this probability depends on free speed permits us,
in principle, to estimate an upper bound to the value of life as implied in
the individual's driving behavior. There can be no question that individuals
are only vaguely aware of the value implications of their behavior regarding

risk, and that the values revealed by some types of driving appear socially unacceptable.

Let us consider first the case where transportation cost is identified' with travel time. We have mentioned in Section 2.2.2 the fact that for every road user, except possibly the slowest vehicle, travel time increases with flow. This statement can be sharpened with regard to the rate of increase, and we shall need the property we are about to observe as a prerequisite for the applicability of our mathematical analysis. It is clear that the time (per vehicle) spent waiting in queues behind slower cars increases at an increasing rate with flow, at least if the composition of flow by free-speed classes remains reasonably constant. It follows that the weighted sum of all travel times the aggregate travel time must increase with flow at a rate which is itself increasing. A function whose rate of increase is itself increasing or constant is called convex.[2] So we may state that aggregate time cost is an increasing and convex function of flow, regardless of the weights used in the aggregation.

Now whatever the form of the operating cost and risk function, this result remains valid so long as time is valued highly relative to money and risk. But in fact both operating cost and risk cost tend to be increasing and convex functions themselves. For small flows both money cost and risk per vehicle increase approximately in proportion to the increase in the number of passings per vehicle per mile. Risk is, except for a fixed component, mainly associated with proximity to other cars, and passing maneuvers involve the closest and most frequent physical approximation of vehicles to each other. For a straight road with very light traffic passing may take place without any

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2/   Convexity so defined is the same thing as the convexity concept defined in Section 3.1.3* if the latter is applied to a function of one variable.

substantial variation in speed. However, where passing maneuvers are more
numerous, operating cost often goes up because of the fluctuations in speed
usually required in their execution, and the consequent increase in fuel
consumption over the level required at an equivalent steady speed. At small
flows this rate of passing is a convex function of flow. At large flows on
the other hand, the money equivalent of the increase of travel time with re-
spect to flow is of a much higher order of magnitude than any variation of
money cost and risk. Therefore, even if in an intermediate range of flows
the rate of net increase of cost with flow may drop slightly, we should not
commit any serious error in assuming that the aggregate transportation cost
on a road (per mile or for its entire length) is an increasing and convex
function of flow throughout the whole range of flows.

If demand is flexible, that is, if traffic generation depends on traffic
conditions, then cost minimization is not the whole story of efficiency, for
it leaves undetermined the level of demand at which cost is minimized. What
is needed therefore is some measure of the benefits that arise from the satis-
faction of a demand for road use.

In our first discussion of demand, attention was drawn to the meaning
of the area under a demand curve and bounded by the horizontal at the pre-
vailing level of trip cost.[3] This area measures the excess of the maximal
cost that road users would have been willing to spend on the current trans-
portation activities over the amount of cost that is actually incurred. This
expression seems to be an adequate representation of the advantages (often
different for different trips between the same points) that accrue to road
users expressed in comparable terms, money. It is further recommended by
the fact that it is of long standing in economic theory as a measure of

[3] Chapter 2, Figure 3.

consumers' benefit or "surplus" and has been put forward repeatedly in the

context of discussions on the utility of transportation facilities.[4]

All of this is not to say that the consumers' surplus would be easy to

measure in practice, but rather that it is adequate from a conceptual point

of view. However, mention should be made again of an implicit assumption

on which its applicability rests, namely, that there should be no effects

on income which would render the costs saved at various levels of spending

of unequal (per unit) value to the road users.
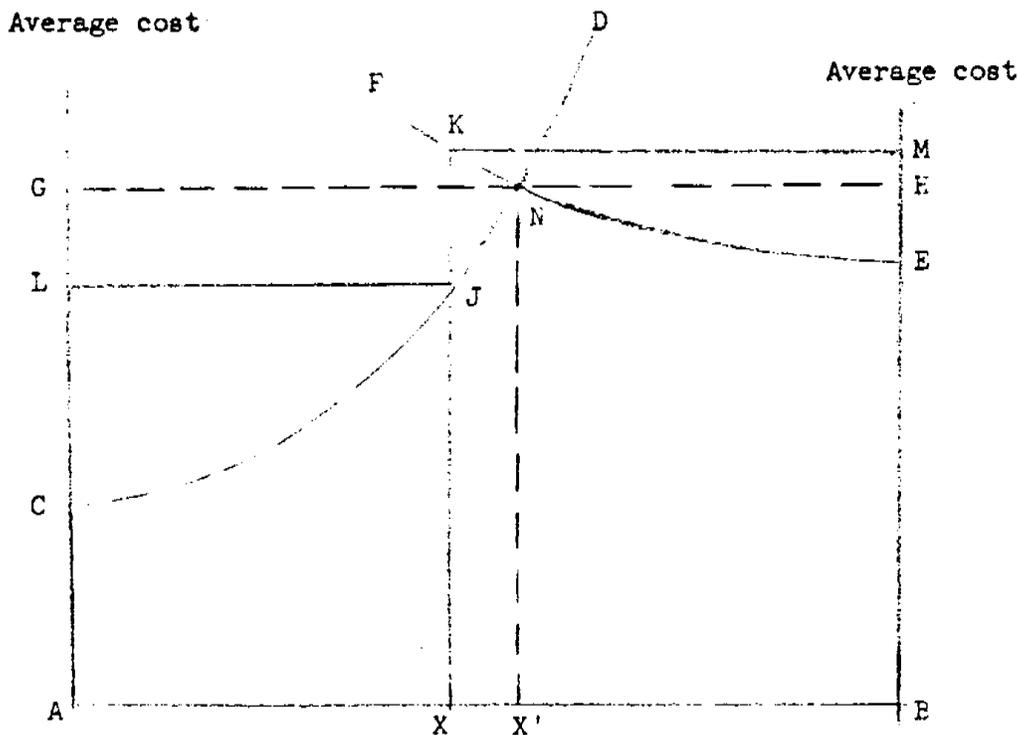
Figure 1

[4]   E.g., Hotelling [1938]. We shall not recite here the various distinctions
possible between consumers' surpluses depending on the assumptions about the
income situation before and after. The fact that we have an "average" curve renders
such subtleties irrelevant here.

4.2 Cost Minimization on Two Roads

4.2.1 Pigou's Problem

The principles of cost minimization in a network may be illustrated by a reconsideration, in terms of a graphical analysis, of the two-road case with fixed total flow. In Figure 1 let AB represent the total flow from a certain origin to a certain destination, and let the way this flow is apportioned to the two roads connecting origin and destination be indicated by some point on the line AB such as X. or X'. Now let the functions relating average cost and flow be represented by the two curves CD and EF. If X represents the particular division of traffic under consideration, then AX is the flow on one road and BX the flow on the other; AL (=XJ) and BM (=XK) are the respective average costs. The total cost of transportation corresponding to the division X is therefore represented by the sum of the areas of the two rectangles AXJL and BXKM. As we have seen earlier, the equilibrium division of traffic will be that which results in equal average costs on the two roads. In this case therefore the point X' with average costs AG and BH (both equal to X'N) represents the equilibrium division; total cost is represented by the sum of the areas AX'NG and BX'NH, which in this case is simply the area ABHG.

Can anything general be said about the division of traffic which results in minimal total cost? This division can be characterized geometrically as that pair of adjacent rectangles with corners on the two cost curves which possesses the smallest joint area. In order to see that in general this will not be the pair of rectangles that have the same height. we may look at a particularly simple pair of average-cost curves, namely, a pair of straight lines (CD and EF in Figure 2). The problem of minimizing the area of

adjacent rectangles with corners sliding along two straight lines boils down
to finding the rates of change of their areas with respect to small shifts
of the common edge.

Consider for instance the left-hand rectangle, designated AXJL in ,
Figure 3. Its area equals that of the trapezoid ACNX which is obtained
by drawing a straight line CN which bisects LJ and every other similar
horizontal from the vertical axis to the average-cost curve CD. The two
triangles, I and II, above and below the bisector CN are congruent.
As the rectangle is expanded along the given line CJ, this bisector CN
retains its position and the area change is given by a vertical strip ex-
tending between the bisector and the horizontal axis (the cross hatched area
in Figure 3). For a narrow strip the area is approximately equal to its width
times the ordinate XN. At X the rate of increase of the rectangular area
AXJL with respect to displacements of the point X therefore equals the
length of the vertical XN.

Let the two bisectors CP and EQ be drawn for both cost curves in
Figure 2. At a point such as X" or X' (the equilibrium point) where one
bisector lies above the other, a reduction of the area representing total
cost can be achieved by diminishing the base of the rectangle for which the
corresponding bisector reaches higher. The minimum is therefore realized by
letting the common edge of the two rectangles be that verticle line XN which
passes through the intersection of the two bisectors in Figure 3.

It may happen that there is no such intersection but that one bisector
reaches the outer edge below the origin of the other bisectors. In that case
all the area is allotted to this one rectangle.

In the linear case just described XN will not pass through the inter-
section of the average-cost curves unless their intercepts with the vertical
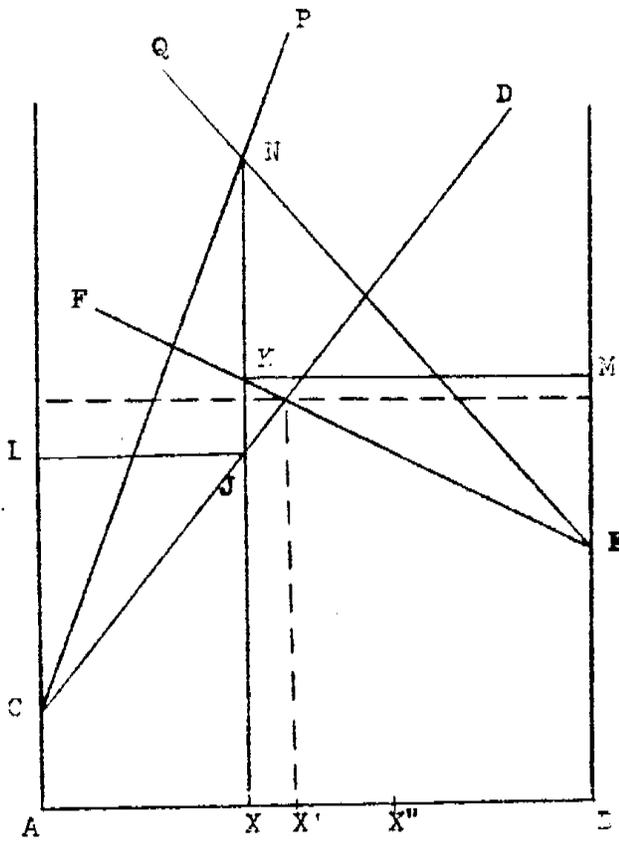
Figure 2
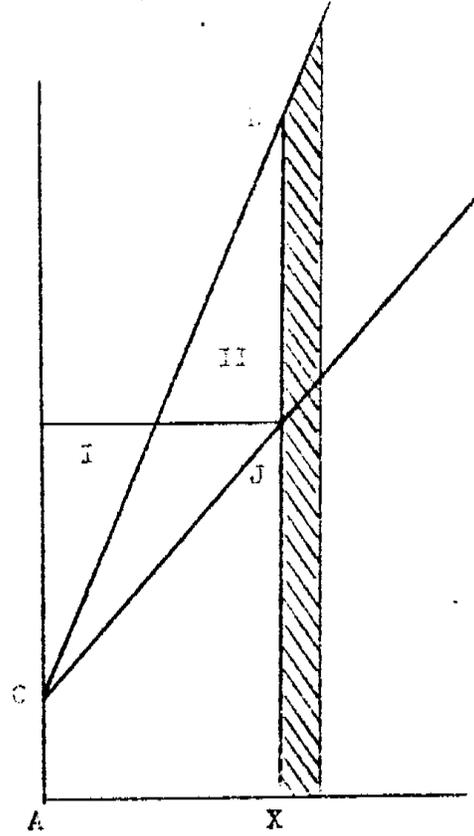
Figure 3
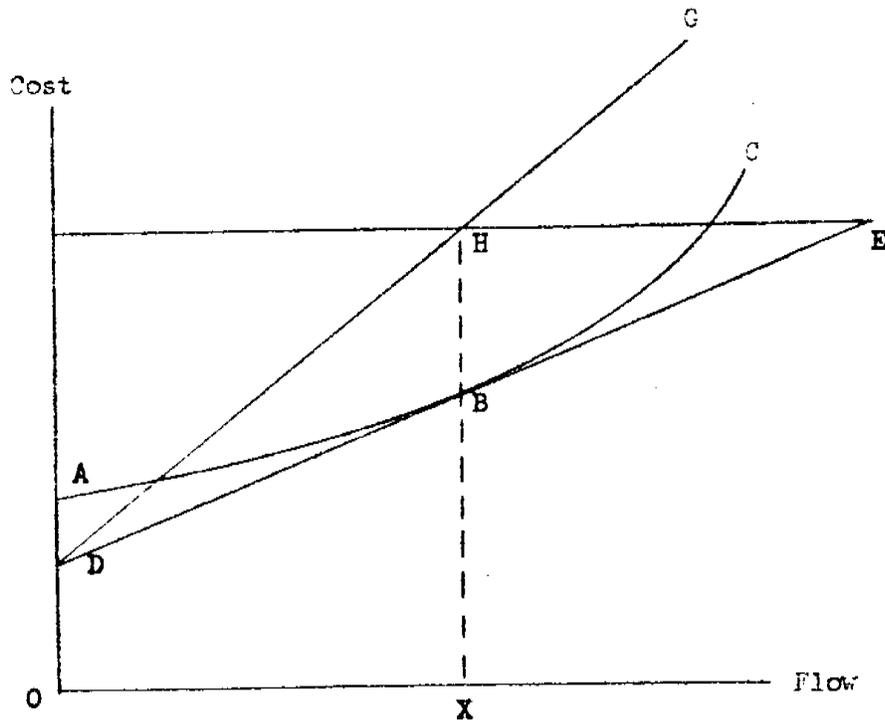
Figure 4

axes are equal. This means that, in general, the equilibrium distribution of flows is not the one that minimizes total costs.

The bisector we have constructed for an average-cost curve represents the rate of increase of total cost with flow. The economic name is an "incremental" or "marginal"-cost curve. These curves can also be constructed for the case of curvilinear average-cost functions as follows. In order to obtain the point of the incremental curve above the point X in Figure 4, draw the tangent to the average-cost curve ABC at the point B, where XB represents the average cost when flow is equal to OX. Extend this tangent DBE to a point E selected so as to make DB and BE the same length. Now if in fact DBE were the average-cost curve the rate of increase of total cost when flow was OX would be given us by the bisector DG which by definition bisects FE at a point H directly above X (note that FH = HE because DB = BE). But since the true average-cost curve and this hypothetical one are tangent at B, the rates of growth of total cost at X must be equal. Therefore the true incremental-cost curve must also pass through the point H. XH then is the incremental-cost when the flow is OX. The curve generated in this way will not generally be a straight line. Nevertheless, with incremental-cost curves our assertions in the previous paragraphs remain valid.

4.2.2 Private Cost and Social Cost

Let us turn back from the geometrical argument to the economic phenomena. What accounts for the failure of an equilibrium brought about by free individual decisions to achieve minimization of total cost? Since every individual minimizes his own cost the failure must lie with the allocation of total cost to the individual road users. To pinpoint the question: does

a road user's share of the cost match his contribution to the total cost
incurred by traffic in the road system?

The answer very simply is no if we look at the average cost of transportation on a road before and after addition of a vehicle. From a certain flow on, each additional vehicle causes some delay and risk to the others present for which it does not bear the cost. To be sure, in his own turn this road user will suffer delays and costs from any further vehicles that might be added. But to the marginal road user, who is just at the point of indifference as to whether to use this road or an alternative one, his own cost is not a sufficient deterrent, because it does not contain the costs he inflicts on the other road users if he should choose the more congested road. As Professor Frank Knight has put it for the case of two alternative highways, "The congestion and interference resulting from the addition of any particular truck to the stream of traffic on the narrow but good road affects in the same way the cost and output of all the trucks using that road. It is evident that if, after equilibrium is established, a few trucks should be arbitrarily transferred to the broad road, the reduction in cost or increase in output to those remaining on the narrow road would be a clear gain[5] to the traffic as a whole. The trucks so transferred would incur no loss, for any one of them on the narrow road is a marginal truck, subject to the same relation between cost and output as any truck using the broad road. Yet whenever there is a difference in the cost, to an additional truck, of using the two roads, the driver of any truck has an incentive to use the narrow road, until the advantage is reduced to zero for all the trucks".[6]

_____

[5] The gain is a clear gain in this case because Professor Knight has assumed that on the broad road an increase of flow will not reduce average speeds. Capacity, that is to say, is very high.

[6] Knight [1952, p. 152].

This situation may also be described in terms of a distinction between "private" and "social" incremental cost. At a given level of traffic the private incremental cost of road use is the operating cost, time, risk, and possibly the toll charges incurred by an additional road user per vehicle mile. Against this one may set the total cost of an extra vehicle mile regardless of who is bearing it and this will be called the social incremental cost of a vehicle mile. If traffic flow is small so that the interference of vehicles with each other is negligible and if traffic does not cause any annoyance to the public at large -- a possibility which we are disregarding on the whole -- then the social and the private incremental costs of a vehicle mile are identical. In general however the presence of an extra vehicle also causes delays and risks to other road users so that the social incremental cost of a vehicle mile exceeds its private incremental cost.

In terms of the capacity function this fact appears as follows. Since average cost increases with flow the cost of a vehicle mile, as averaged, over all road users, is raised by an extra vehicle. This increase of the average cost level times the traffic flow equals the excess of social incremental cost over private incremental cost. If the road user were to bear his full share of cost he would pay a tax or toll equal to this excess. This would make him realize the cost he causes to others and thus provide an incentive to keep social cost down by making the proper choices. This is indeed Pigou's "rightly chosen measure of differential taxation".

4.3 Efficient Transportation in Networks

In the two-road model there was no serious problem of determining which roads would be used under efficient utilization. Traffic on one or the other road was zero whenever the incremental cost of transportation in that road was greater than that on the other road even with all the traffic on it. In a more involved network the answer is less obvious. Which routes must be considered as possible alternatives to a congested route is part of the problem. Also, a single origin-to-destination flow cannot be looked at in isolation, since it competes with other traffic over some or all of its route. (This is clear when it is remembered that a given road can be a part of several routes.) If the analysis were to be in terms of routes, it would thus appear vastly complicated. Fortunately an approach in terms of flows on roads rather than routes is possible. The problem of this chapter thus becomes one of determining for each road the levels of flow in each direction which are compatible with the fixed net originations and which result in the smallest aggregate transportation cost. Since graphical analysis is unpromising in a problem of such complexity, it becomes necessary to apply the formal mathematical apparatus for minimization of a function of many variables that are subject to constraints. Economic intuition may however be called upon in terms of an analogous problem in the theory of the firm.

Let us consider a chemical or metallurgical material which is capable of various stages or modifications, and a firm which undertakes to transform it from certain stages to certain other stages. Not all conceivable transformations may be possible, of course, and it may be that various stages can be reached from a given one only via certain other intermediate stages. Let this firm be given some fixed amounts of this material in various stages and

be asked to produce certain fixed amounts of the material in other stages.
How should the firm choose the transformation processes which will achieve
the production goal at minimal cost if the further assumption is made that
each transformation is subject to per unit costs which increase with volume?

Here the stages of the material correspond to locations the transitions
correspond to roads, and sequences of transformation processes which lead
from a given stage to a desired stage of the material, that is. the production
methods, correspond to routes.

With reference to the production problem economic intuition has been
crystallized in the so-called principle of incremental cost. This principle
says that the last unit of the material at any particular stage should cost
the same in any production method that is used. It also implies that an
additional unit would cost more if it were produced by one of the production
methods that is not used. To spell out the principle in the present case we
may have recourse to an internal price system for the firm. Even though the
production task did not specify any prices of the original or desired stages
of the material, the principle of incremental cost asserts that it should
be possible to associate a suitable price with each of the various stages.
A cost-minimizing selection of production methods is then one where the in-
cremental cost of every transformation process. that is the cost occasioned
by the last unit so transformed. is just covered by the price difference
between the two stages. In particular when nothing is transformed by a
particular process the cost of the first unit transformed by that process
would already exceed the price difference. These internal prices, also
called opportunity costs have a distinct intuitive appeal. But is their
existence really so obvious and can one be sure in all cases that the in-
cremental cost conditions in terms of opportunity costs, ensure the

minimization of total cost?

A substitution of one production method for another changes the costs of all the transformation processes involved in the two methods. But these same processes may also be part of other sequences of processes. In that case their cost structure would be upset, too. Is it always possible to satisfy the incremental cost conditions with respect to all methods simultaneously? Moreover, might not the far-reaching cost changes involved in any substitution of methods permit the possibility that the incremental cost conditions are satisfied before and after a change, while the total costs are different? In such a case the incremental cost conditions might still be necessary but would certainly not be sufficient for cost minimization.

What these questions amount to is really this: can the choice of the best production methods and of the extent of their application always be reduced to comparisons "in the small", that is, to considerations of the effect that the shift of one unit produced in this or that way will have on cost? Or in ascertaining the optimality of a certain combination of processes, are comparisons with combinations very different from the one under consideration indispensable?

To meet these and similar objections that might be raised, a mathematical analysis is called for. Such an analysis will be presented in the next section. In Section 4.4 we shall return to the economic interpretation of the incremental cost conditions for the road allocation problem, which in that context are called the efficiency conditions.

4.2.1* Maximization Subject to Linear Inequalities as Constraints

The fact that the constraints of our maximum problem include inequalities makes it different from the standard extremization problems in the calculus. Fortunately, the theory of linear inequalities has been well developed, and in recent years several theorems for convenient use in so-called _linear_ and _non-linear programming_ have been advanced. For our purposes the following theorem from Kuhn and Tucker [1951] is adequate.

Theorem: Let $f(u)$ be differentiable and concave.[1] Necessary and sufficient that u^o be a solution of

$$(4.1) \qquad \underset{u}{\text{Max}}\ f(u) \ \text{subject to}\ u_r \geq 0 \ \text{and}\ \sum_r b_{sr} u_r \leq c_s$$

is the existence of multipliers v_s such that

$$(4.2) \qquad v_s \geq 0 \ \text{and}\ "="\ \text{if}\ \sum_r b_{sr} u_r^o < c_s,$$

and such that

$$(4.3) \qquad \frac{\partial}{\partial u_r}\left[f(u) + \sum_s v_s (c_s - \sum_r b_{sr} u_r)\right]_{u=u^o} \begin{Bmatrix} = \\ \leq \end{Bmatrix} 0 \ \text{if}\ u_r^o \begin{Bmatrix} > \\ = \end{Bmatrix} 0$$

Proof: Necessity. Let u^o be a point where a constrained maximum of $f(u)$ is reached. At this point some of the constraints are effective (i.e. the limits are reached) and some are not. Without loss of generality we can suppose that $u_r^o = 0$ for $r = 1,\ldots,R'$ and $u_r^o > 0$ for $r = R'+1,\ldots,R;$

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[1] Concavity is defined in (3.17), Section 3.1.3*.

and similarly that $\sum_{r} b_{sr} u_r^0 = c_s$ for $s = 1,\ldots,S'$ and

$\sum_{r} b_{sr} u_r^0 < c_s$ for $s = S' + 1,\ldots,S$. Let u be any other point such that

$$\sum_{r} b_{sr} u_r \leqq c_s \qquad\qquad\qquad \text{for } s = 1,\ldots,S'$$
(4.4)
$$u_r \geqq 0 \qquad\qquad\qquad \text{for } r = 1,\ldots,R'.$$

Then $(1 - a) u^0 + a u$ $0 \leqq a \leqq 1$ satisfies the same constraints and for sufficiently small $\underline{a}$ leaves all those constraints ineffective that are ineffective for u^0. Define $z_r = u_r - u_r^0$ and

$$(4.5) \qquad \beta_{jr} = \begin{cases} b_{jr} & \text{for} \quad j = 1,\ldots,S' \\ 0 & \text{for} \quad j = S' + 1,\ldots,S \\ -1 & \text{for} \quad j = S + r \text{ and } r = 1,\ldots,R' \\ 0 & \text{otherwise.} \end{cases}$$

Then $u^0 + az$ satisfies the original constraints provided that

$$(4.6) \qquad \sum_{r} \beta_{jr} z_r \leqq 0 \qquad \text{for } j = 1,\ldots,S + R$$

and $0 < a < 1$, $\underline{a}$ sufficiently small. Because $f(u)$ is maximized at u^0 subject to the constraints in (4.1) we have therefore that

$$(4.7) \qquad \frac{f[u^0 + a(u - u^0)] - f(u^0)}{a} \leqq 0$$

for all z satisfying (4.6) and for sufficiently small a. But the limit of this expression as $a \to 0$ is

$$(4.8) \qquad \sum_r \left(\frac{\partial f}{\partial u_r}\right)_{u=u_0} \cdot (u_r - u_r^0).$$

Writing f_r for $\left(\frac{\partial f}{\partial u_r}\right)_{u=u_0}$ and z_r for $u_r - u_r^0$ we obtain that

$$\sum_r f_r z_r \leq 0 \quad \text{for all } z_r \text{ such that } \sum_r \beta_{jr} z_r \leq 0, \quad j = 1, \ldots, S + R.$$

We now have recourse to the fundamental

Farkas Lemma: [8] In order that $\sum_r a_r u_r \leq 0$ for all u_r such that

$$\sum_r b_{jr} u_r \leq 0 \quad \text{it is necessary and sufficient that}$$

$$(4.9) \qquad a_r = \sum_j b_{jr} w_j \qquad \text{with some } w_j \geq 0.$$

The Farkas lemma now tells us that there exist $w_j \geq 0$, $j = 1, \ldots, S + R$, such that

$$(4.10) \qquad f_r = \sum_j w_j \beta_{jr}$$

Since $\beta_{jr} \equiv 0$ for $j = S' + 1, \ldots, S$ and $S + R' + 1, \ldots, S + R$ the corresponding w_j can take on arbitrary non-negative values. Let us set these w_j equal to zero. Then, translating back into the b_{sr} we have

$$(4.11) \qquad f_r = \sum_s w_s b_{sr} - w_{S+r} \quad \text{for } r = 1, \ldots, R.$$

By the definition of w_{S+r} it follows that

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[8] Farkas [1901].

(4.12)
$$f_r - \sum_s w_s b_{sr} \begin{Bmatrix} = \\ \le \\ = \end{Bmatrix} 0 \quad \text{if} \quad u_r^o \begin{Bmatrix} > \\ = \end{Bmatrix} 0.$$

In other words

(4.13)
$$\left(\frac{\partial}{\partial u_r}\right) \left[f(u^o) + \sum_{s=1}^S w_s \left(c_s - \sum_r b_{sr} u_r^o\right) \right] \begin{Bmatrix} = \\ \le \end{Bmatrix} 0 \quad \text{if} \quad u_r^o \begin{Bmatrix} > \\ = \end{Bmatrix} 0.$$

<u>Sufficiency</u>. Suppose that there exist v_s for which (4.2) and (4.3) are satisfied. Then, since $f(u)$ is concave, we have by the lemma of Section 3.1.3

(4.14)
$$f(u^o) \ge f(u) + \sum_s v_s \left(c_s - \sum_r b_{sr} u_r\right)$$

for all u with $u_r \ge 0$.

By (4.2), for all u with $\sum_r b_{sr} u_r \le c_s$, $\sum_s v_s \left(c_r - \sum_r b_{sr} u_r\right) \ge 0$, so that

(4.15)
$$f(u^o) \ge f(u) \quad \text{for all} \quad u \quad \text{with} \quad \sum_r b_{sr} u_r \le c_s.$$

4.3.2[*] Minimization of Transportation Cost Subject to a Fixed Program

We now apply the theorem of the last section to the problem of minimizing transportation cost. We first consider the case of fixed originations $f_{i,k}$ and terminations $-f_{k,k}$. The maximand is the negative of total cost:

(4.16)
$$-\frac{1}{2} \sum_{ij} h_{ij}(x_{ij}) \, x_{ij} .$$

The factor $\frac{1}{2}$ is needed because every road appears twice in the sum, once as ij and once as ji, while x_{ij} already denotes the total flow in both directions. Applying the sufficient criterion for concavity used in Section

3.1.3[*], that the Hessian be negative semi-definite, we see that every term $-\frac{1}{2} h_{ij}(x_{ij}) x_{ij}$ is concave in x_{ij} provided

$$(4.17) \qquad \frac{d^2}{dx_{ij}^2} [h_{ij}(x_{ij}) x_{ij}] \geqq 0.$$

The left hand side equals

$$(4.18) \qquad \frac{d^2 h_{ij}}{dx_{ij}^2} x_{ij} + 2 \frac{dh_{ij}}{dx_{ij}}$$

and each of the two terms is non-negative as already argued in Section 4.1.2. It follows that the maximand is concave in x.

As constraints of the minimum problem we have the "program conditions"

$$(4.19) \qquad \sum_j (x_{ij,k} - x_{ji,k}) = f_{ik}$$

and

$$(4.20) \qquad x_{ij\,k} \geqq 0.$$

Since consistency of the program implies $\sum_i f_{i,k} = 0$, we can replace the "$=$" sign in (4.19) by "$\geqq$", for the "$>$" sign cannot be attained. Labeling the multipliers $\lambda_{i,k}$ we obtain the solution equation (4.3) in the form

$$(4.21) \qquad \frac{\partial}{\partial x_{ij,k}} \left[-\frac{1}{2} \sum_{ij} h_{ij}(x_{ij}) x_{ij} + \sum_{i,k} h_{i,k} \left(\sum_j [x_{ij,k} - x_{ji,k}] - f_{i,k} \right) \right] \begin{Bmatrix} = \\ \leqq \end{Bmatrix} 0$$

$$\text{if } x_{ij,k} \begin{Bmatrix} > \\ = \end{Bmatrix} 0.$$

In explicit form,

$$(4.22) \qquad \lambda_{i,k} - \lambda_{j,k} \begin{Bmatrix} = \\ \leqq \end{Bmatrix} h_{ij} + h'_{ij} x_{ij} \quad \text{if} \quad x_{ij,k} \begin{Bmatrix} > \\ = \end{Bmatrix} 0$$

with suitable numbers $\lambda_{i,k}$. We notice that (4.22) leaves the $\lambda_{i,k}$ undetermined up to additive constants c_k. We may fix these such that $\lambda_{k,k} = 0$. The relations (4.22) are then reminiscent of the equilibrium conditions (3.10). They suggest that the $\lambda_{i,k}$ be interpreted as trip costs, based however on transportation charges different from simple transportation costs on roads.

4.3.3* Maximization of Consumers' Surplus

Before discussing this result it is convenient to derive the corresponding conditions for the case of flexible demand. In Section 2.4.1 the consumers' surplus was defined geometrically as the area between the demand curve and a horizontal line representing the prevailing average cost (Figure 3 in Chapter 2). It is now necessary to find an analytic expression for this. Let $x_{i,k}$ be the prevailing flow between locations. Then the area OABC in Figure 3 is given by $\int_0^{x_{i,k}} g_{i,k}(x)\, dx$. The area OCBD representing the prevailing trip cost equals $g_{i,k}(x_{i,k}) \cdot x_{i,k}$. The sum of all consumers' surpluses would seem to be therefore

$$(4.23) \qquad \sum_{i,k} \left[\int_0^{x_{i,k}} g_{i,k}(x)\, dx - g_{i,k}(x_{i,k}) \cdot x_{i,k} \right]$$

However the sum of all trip costs to road users need not be identical with the total transportation costs. An equality of the two, as it was derived at the end of Section 3.1.1*, depends on the particular incidence of cost that we have under conditions of ordinary traffic equilibrium. If, for

instance, tolls are charged, then these are part of trip costs but they should not be included in total transportation costs since they become available again, and can be used for such purposes as tax reduction. On the other hand public expenditures for highway control and maintenance may depend on traffic but as this is a further variable it will be excluded by making the analysis a "short-run" one in which such public expenditures are held constant. Under these assumptions, we arrive at the proper expression for the consumers' surplus if we replace the sum of trip costs to individuals, $\sum_{i,k} g_{i,k}(x_{i,k}) \cdot x_{i,k}$, by $\frac{1}{2} \sum_{ij} h_{ij}(x_{ij}) \cdot x_{ij}$, the sum of operating cost, risk costs and time costs on roads.

$$(4.24) \quad \text{Consumers' Surplus} = \sum_{i,k} \int_{0}^{x_{i,k}} g_{i,k}(x)\, dx - \frac{1}{2} \sum_{ij} h_{ij}(x_{ij}) \cdot x_{ij} .$$

It differs from the function H of (3.11) in the second term only.

In Section 3.1.3[*] we have shown that the first term represents a concave function of the $x_{ij,k}$. In Section 4.3.2[*] the same was proved for the second term. Therefore the expression (4.24) is a concave function. Its maximum over the positive orthant $x_{ij,k} \geq 0$ is therefore characterized according to the theorem proved in Section 4.3.1[*], by the necessary and sufficient condition

$$(4.25) \quad g_{i,k}(x_{i,k}) - g_{j,k}(x_{j,k}) - h_{ij}(x_{ij}) - h'_{ij}(x_{ij})\, x_{ij} \left\{ \begin{matrix} = \\ \leq \end{matrix} \right\} 0$$

$$\text{if } x_{ij,k} \left\{ \begin{matrix} > \\ = \end{matrix} \right\} 0 .$$

This may be written out in terms of two relations

$$(4.26) \quad y_{i,k} - y_{j,k} \begin{Bmatrix} = \\ \leq \end{Bmatrix} h_{ij}(x_{ij}) + h'_{ij}(x_{ij}) \, x_{ij} \quad \text{if} \quad x_{ij} \begin{Bmatrix} > \\ = \end{Bmatrix} 0$$

$$x_{i,k} = f_{i,k}(y_{i,k}).$$

We thus obtain an efficiency condition of the same form as that in (4.22) which related to a fixed program.

4.4 Efficiency Tolls

4.4.1 Interpretation of the Efficiency Conditions

The efficiency conditions suggest that efficient utilization can, at least in theory, be achieved through a state of equilibrium in which suitable taxes or tolls are levied on the use of all congested roads. These tolls are to express the excess of social over private cost caused by an additional road user. The "toll" term added in (4.22) to the private cost of transportation equals the increase in the private cost to the average road user caused by a unit increase in traffic, multiplied by the total flow of traffic. If this term is added to the private cost of transportation one obtains what was called in Section 4.2.2 the social cost of transportation at the prevailing flow on a road. In the absence of congestion, as indicated by constant average cost independent of the amount of traffic, no toll would arise.

If such "efficiency tolls" could be levied they would restore the power of competitive equilibrium to achieve an efficient utilization of resources. By charging everyone a toll equal to his contribution to the

total cost of others, road users can be induced to make an efficient use
of the available capacity. It is clear that this toll or tax does not dis-
criminate between destinations or commodities. On the other hand, for
accurate effects it would have to vary with the traffic conditions and hence
be different at different times. In a more detailed analysis based on a
model which (unlike the one given above) takes more aspects of traffic into
account, it would also turn out that the best charges are different for
vehicles of different free speeds and of different congestion characteristics,
e.g.,trucks as compared with passenger vehicles. We will not pursue this
possibility here.

Under present conditions congestion is the main obstacle to more traffic.
While traffic is not an end in itself, cheap transportation is a contributing
factor to the economic division of labor. Ready accessibility of centrally
located markets is of particular importance, because it determines the size
of markets and the extent to which economies of large scale production and
distribution can be reaped. Congestion, by setting a limit and a premium
upon the movement of persons and commodities restricts the effectiveness
with which the functions of centrally located markets can be performed. It
seems a paradoxical conclusion then that the answer to congestion should be
less traffic -- even less, that is, than the existence of congestion at
present permits.

The point to be made is that traffic becomes uneconomical if its social
cost exceeds the value of the advantages. And if a full charge were made
for the social cost caused by each road user, traffic would keep by itself
within the economically warranted limits. There is no avoiding the con-
clusion that in the short run at least, free access to congested roads just
permits too much traffic. Congestion is not selective in any proper way.

because it spreads the cost of the least important or marginal traffic over all other traffic. It may still seem odd that every individual should pay more for transportation, as must be the case when demand is reduced on the whole. The clue is of course that since transportation costs other than tolls tend to go down with a decrease in flow, they decrease as a result of the imposition of tolls. The remaining part of the expense becomes available again as toll revenue so that the community can gain where every individual seems to lose.

The preceding analysis of efficiency in the case of elastic demand has been based on an equal weighting of everyone's money cost and on uniform money equivalents for time and risk. This is of course only a special way of evaluating the costs and benefits to various classes of road users, but one which has the distinction of simplicity and of *prima facie* egalitarianism.

One can convince oneself that the charging of suitable discriminatory tolls, depending either on the free speed or the destination or any other characteristic of road users, if feasible would result in an efficient utilization of roads in terms of certain non-uniform weights assigned to the benefits to various road-user classes. But the proper choice of such tolls is a formidable problem.[9]

By discriminating against the use of congested roads the imposition of efficiency tolls gives rise to additional shortest routes between given origins and destinations. For through the added costs, part of the traffic on these congested routes is diverted. In this way the availability of

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[9] It may be mentioned incidentally, that almost every distribution of traffic over roads represents a (Pareto) optimum in the sense that it is impossible to make any road user better off without making someone else worse off, as long as no compensation is payable, either directly or via tolls or taxes. For as a result of the change, traffic is almost bound to increase somewhere. It is for this reason that a weighting of benefits becomes indispensable.

these uncongested alternative routes has the effect of placing a ceiling on the efficiency tolls.

When demand is elastic, tolls have the obvious additional effect of reducing not only the demand for the use of roads with heavy tolls, but also the total demand for transportation on all roads. This implies that the level of tolls, which rises with increasing flows, must be less than the level that a calculation of social costs at the ordinary equilibrium levels of traffic would suggest. One of the reasons for this is, of course the utilization of uncongested alternative routes noted above. From the fact that flows are decreased, on the whole, we must conclude that the cost of transportation to road users has risen on the average.

For any particular road, of course, the diversionary effect of tolls on parallel routes may overshadow the decline in the demand for transportation to locations, so that traffic may actually increase there as a result of tolls. This is quite obvious for a two-road network. What can be shown however, is that flows tend to decrease on roads relatively congested. We shall do this in the next section for the case in which demand and capacity functions are linear, and the absolute capacity limits are unattained.

4.4.2[*] Equilibrium and Efficiency Flows

With linear demand and capacity functions the expression (4.24) for the consumers' surplus is, using the notation of (3.28) and (3.29),

$$(4.27) \qquad \sum_{i,k} \left[\frac{1}{2} e_{i,k} x_{i,k}^2 + f_{i,k} x_{i,k} \right] - \sum_{ij} \left[\frac{1}{2} a_{ij} x_{ij}^2 + \frac{1}{2} b_{ij} x_{ij} \right].$$

This is less than the function H of (3.30) by an amount $\sum_{ij} \frac{1}{4} a_{ij} x_{ij}^2$.

Viewing this difference as one brought about by parameter changes

$\delta q_{nr} = -\frac{1}{4} a_{ij}$ we can apply the inequality (3.33) of Section 3.2.1[*] in order to learn something about the differences between the ordinary equilibrium flows and the efficient flows. In this case the inequality assumes the simple form

$$(4.28) \qquad -\frac{1}{4} \sum_{ij} a_{ij} (2x_{ij} + \delta x_{ij}) \delta x_{ij} \geq 0.$$

Since $a_{ij} > 0$, $x_{ij} \geq 0$, and $x_{ij} + \delta x_{ij} \geq 0$, we conclude that δx_{ij} tends to be negative for those ij with relatively high values of a_{ij} or x_{ij}. In other words on congested roads, efficient flows tend to be smaller than flows at ordinary equilibrium. Changing the sign of (4.28) and adding $\sum_{ij} a_{ij} x_{ij}^2$ to both sides we have

$$(4.29) \qquad \sum_{ij} a_{ij} (x_{ij} + \delta x_{ij})^2 \leq \sum_{ij} a_{ij} x_{ij}^2.$$

Since the $a_{ij}(x_{ij} + \delta x_{ij})$ are the toll rates when flows are efficient, and the $a_{ij} x_{ij}$ the efficiency toll rates indicated by conditions at ordinary equilibrium, we see that equilibrium flows and the corresponding cost and demand conditions tend, if used naively, to produce an overestimate of the revenues that would be collected from efficient flows at efficiency-toll rates.

4.4.3 A Limited Toll System

At the present state of technology it is impractical to charge tolls on every congested road. However ways other than the present ones of collecting tolls may be found which at the same time are cheap and do not add to congestion. Or it may be found possible to levy the corresponding

toll charges in indirect ways. While there are psychological resistances against out-of-pocket charges even where the price paid is less than the cost incurred for alternative means to the same end, the case for toll charges to relieve congestion is economically unassailable. The problem of how to make the road user sensitive to the social cost he causes is a real one and the means to achieve it are worth further attention on the technological level.

But even when there are only a limited number of toll roads or bridges, the question arises how the tolls on these should be set so as to achieve, with the obtainable means, a maximum of economic benefit to all road users. In discussing this problem we shall assume that payment of the construction cost from the toll revenue is no consideration. A situation like this may be found in practice after the debt on a toll road has been paid off.

This limited efficiency problem can be approached in terms of the maximization of our consumers' surplus function subject to the additional constraints that on each free road the difference of trip costs between the two end-points of the road should not exceed the average transportation cost on the road. Solution of this modified problem shows that best tolls on the toll roads are different from what they would be for the same roads in a general toll-road system in which efficiency tolls are charged on each congested road. They are higher where congested free roads are predominant as feeder roads, and are less where free roads compete with a toll road as alternatives.

4.4.4   The Value of a Road

The fact shown in Section 3.2 that an upward shift of the cost schedule for one road decreases traffic on that road implies that road use decreases or at most remains constant with added fixed costs (independent of flow) such as tolls. The demand for use of a road as a function of tolls on that road alone has thus a normal (downward or level) slope whatever the capacities of or flows on other roads. This demand function is really a description of the path followed by the equilibrium as one particular constant (the toll on the road under consideration) is varied. Its steepness is of course dependent primarily on the number and capacities of alternative roads and on the demand for transportation to locations reached via this road.

This demand function also defines the amount of toll revenue forthcoming at different toll levels. Obviously the maximum amount of tolls collectable is not above the value of the road to its users. Therefore any road which can be paid for by tolls (at the market rate of interest) is worth its construction cost under given road and traffic conditions on all other roads. We also obtain a lower-bound estimate of the value of a road in the amount of toll that is actually collected or that should be collected in order to achieve efficient utilization.

However there may still be a wide margin between what the road user pays in tolls and what the trip is actually worth to him over and above other transportation costs. Therefore a road will have a higher value in general than what its possible toll income amounts to. It follows that the "pay for itself" criterion of the usefulness of a toll road is too conservative for determination of the proper extent of a highway network.

4.5 Toll Roads Reconsidered

The analysis of an efficient allocation of flows to routes has led in
its own course to the notion of a road toll as an instrument of controlling
access to roads. The common meaning of a toll road is however that of a
special route (an expressway) which offers superior roadway and traffic
conditions at an extra charge. Inherent here is an element of economic
discrimination in making extra service available at a price. To bring out
this significant feature of toll roads, as understood in contemporary dis-
cussion, let us resort again to our two-road example. Suppose that the
alternative roads are of equal capacity and are equivalent also as far as
all other roadway conditions are concerned. Deviating now from our previous
notion of aggregate transportation cost (Section 4.1.2) let the per-unit
value of time vary between the various road users. If the total amount of
flow is again taken to be fixed the ordinary equilibrium distribution of
equal flows on the two roads is also the efficient distribution, and there
is no place for any compensating tolls in our previous sense. It may be
asked, however whether total welfare could not be improved by the creation
of an artificial difference between the two roads through the levying of a
toll on one of them. To the extent that this discourages traffic, it improves
the traffic conditions on that road and in particular it reduces delays to
faster vehicles. Is it possible that the time saving to road users who value
time highly more than compensates for the toll paid, and that conversely the
money collected is worth more than the added inconvenience to drivers un-
willing to pay the toll? Other things equal, a high money value of time,
goes with a high free speed. But of course, other things are never equal;
road users differ in their operating cost at given speeds and in their

valuation of risk. Differences in the free speed on a long-distance highway tend to be relatively small. Therefore in a first formulation of the problem we may disregard any correlation between money value of time and free speed and use the same average-speed capacity curve for both roads regardless of whether or not the traffic with a high valuation of time has been separated out.

In the following mathematical note it is shown that the total benefits to road users can in fact be improved by giving them, through a suitable toll, a choice between faster travel at higher money cost and slower travel at smaller money cost. This is still on the assumption that everybody's dollar is valued the same by the community. If the roadway conditions differ also . between the two roads the case for a separation of high and low speed traffic by a toll on one road is even stronger.[10] For simplicity we shall disregard the dependence of operating cost and risk on speed and consider time and toll costs only.

4.5.1[*] Analysis

Let there be N road users entering the two roads per unit of time. We shall consider them arranged in decreasing order of the money value which they assign to time. Thus let

$m(n)$ be   the money value of time to the nth road user;

x the index of the road user with the smallest money value of time who still uses the toll road (at the same time x is a measure of flow);

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[10]   If the effect on demand for transportation is taken into consideration, the case for toll roads emerges even better, but the determination of the optimal toll becomes more involved. An integrated network raises problems of still greater complexity. The two road example presented here gives however a rough idea of the considerations involved.

$t_1(x)$ the average travel time on the toll road;

$t_2(N-x)$ the average travel time on the free road;

p the toll rate.

For the user x of the toll road who is indifferent between the two alternatives (the marginal road user)

$$(4.30) \qquad m(x)\, t_1(x) + p \; = \; m(x)\, t_2(N-x) .$$

This may be regarded a definition of toll as a function of the flow called forth by it on the toll road (the inverse of the demand function for use of the toll road):

$$(4.31) \qquad p(x) \; = \; m(x)\, [t_2(N-x) - t_1(x)] .$$

Total cost of transportation to road users is now (using the integral notation for convenience)

$$(4.32) \qquad t_1(x) \int_0^x m(n)\, dn + xp(x) + t_2(N-x) \int_x^N m(n)\, dn.$$

From the point of view of the community the tolls do not constitute costs but are available again for redistribution or use in road construction and maintenance. The object of minimization is therefore

$$(4.33) \qquad t_1(x) \int_0^x m(n)\, dn + t_2(N-x) \int_x^N m(n)\, dn.$$

In our example of two roads of equal length and capacity this minimand is

$$(4.34) \qquad G = t(x) \int_0^x m(n)\,dn + t(N-x) \int_x^N m(n)\,dn .$$

All we shall show is that the minimizing flow x must lie in the open interval $0 < x < \frac{N}{2}$. Let $x = \frac{N}{2} + z$ with $z > 0$. Now

$$(4.35) \qquad \left[t\left(\frac{N}{2} + z\right) - t\left(\frac{N}{2} - z\right) \right] \left[\int_0^{\frac{N}{2}+z} m\,dn - \int_{\frac{N}{2}+z}^{N} m\,dn \right] > 0,$$

since by assumption t is an increasing and m is a decreasing function. Multiplying out and substituting for z,

$$(4.36) \qquad t(x) \int_0^x m\,dn + t(N-x) \int_x^N m\,dn > t(N-x) \int_0^x m\,dn + t(x) \int_x^N m\,dn.$$

This means that $G(N-x) < G(x)$ for $x > \frac{N}{2}$.

Therefore the minimizing x must be contained in the interval $0 \leq x \leq \frac{N}{2}$. That it does not fall into an interval endpoint is seen from

$$(4.37) \qquad G'\left(\frac{N}{2}\right) = t'\left(\frac{N}{2}\right) \left(\int_0^{\frac{N}{2}} m\,dn - \int_{\frac{N}{2}}^{N} m\,dn \right) > 0$$

and

$$(4.38) \qquad G'(0) = [t(0) - t(N)]\, m(0) - t'(N) \int_0^N m\,dn < 0.$$

The first-order minimum conditions are

(4.39) $\quad t'(x) \int_0^x m(n)\, dn + t(x)\, m(x) - t'(N-x) \int_x^N m(n)\, dn - t(N-x)\, m(x) = 0.$

But for $\quad x = \dfrac{N}{2}$,

(4.40) $\qquad t'(\dfrac{N}{2}) \int_0^{\frac{N}{2}} m(n)\, dn \; > \; t'(\dfrac{N}{2}) \int_{\frac{N}{2}}^N m(n)\, dn.$

As this shows, the derivative of the minimand at $x = \dfrac{N}{2}$ is positive. We conclude that the minimum is obtained for a flow $x < \dfrac{N}{2}$ on the toll road.

If different weights are attached to the economic benefits for various road user classes, say $w(n)$ to the value of money benefits of the nth class and w_o to the availability of public funds, the minimand is

(4.41) $\quad \int_0^x \left[t_1(x)\, m(n) + p(x) \right] w(n)\, dn + \int_x^N t_2(N-x)\, m(n)\, w(n)\, dn - w_o\, p(x)\, x$

which after substitution for $p(x)$ becomes

(4.42)
$$t_1(x) \int_0^x m(n)\, w(n)\, dn + t_2(N-x) \int_0^N m(n)\, w(n)\, dn$$
$$+ m(x) \left[t_1(x) - t_2(N-x)\left(w_o - \int_0^x w(n)\, dn\right) \right].$$

Except for rather special weight functions w our conclusion remains valid.

Chapter 5

Conclusion; Some Unsolved Problems

The preceding models had to employ many simplifications. Some of the more obvious shortcomings will be pointed out in this chapter on unsolved problems.

5.1 Theoretical Capacity Functions

Although one has come to rely more on empirical capacity measurement, the study of theoretical capacity functions can still be useful. For the design, the interpretation, and the use of measurements always presuppose some theoretical model, however simple.

The regression line of average speed on flow that is, the empirical capacity function, cannot be held to give a universal relationship valid for all roads with the same roadway conditions, because the influence of the driver population is disregarded.

The effect of the driver population and in particular the free-speed distribution on capacity raises many problems both for theory and measurement. The outstanding mathematical problem is perhaps that of deriving the mean delays on a road to vehicles of a given free speed from the free-speed distribution, from the arrival sequences of cars at entrances to the road, and from other features (such as the timing of traffic lights) which may have a bearing on it. Such a model if it were not to run into unmanageable complications, would have to use some average concepts again, although it should allow for a distinction between classes of road users. The possibility and fruitfulness of such an analysis is intimately tied up with the question of whether an equilibrium in the particular sense of queuing theory can be realized on a road or even in a network. Equilibrium of this kind on a road

implies that the flow remains constant and that the free-speed distribution
is preserved. The question arises whether these two traffic characteristics
together will always maintain the same traffic conditions, i.e., the same
mean delays to vehicles of given free speeds.

Delays to traffic may be understood in terms of queuing as shown in
Section 1.2. Now a queue of vehicles behind a slower vehicle may show two
kinds of behavior. After an initial phase of building up, it may either
fluctuate around a fixed mean value or continue to grow indefinitely, accord-
ing as the mean time between passing opportunities for successive cars in a
queue is or is not less than the mean time between arrivals at the queue.
It is the first case that is labelled the stationary case in queue theory.
However, on a road of finite length, the second process may take place to
some extent, even without ultimately causing a tie-up at the entrance of
the road. The effect would merely be to slow down traffic more and more as
it proceeds along the road. As this is compensated by a higher density of
traffic, the same flow can be maintained throughout. In this case a station-
ary flow does not imply a constant free-speed distribution along the length
of the road; the queues that grow as they move on absorb an increasing share
of high-speed traffic. Therefore, the number of road users with high free
speeds represents a larger and larger fraction of the total number of road
users per mile as one moves down the road.

Considering both types of queuing process at the same time a more
interesting question is the behavior of queue lengths over time. Is there
any tendency for queue lengths in a particular road section to approach a
fixed distribution after a sufficient lapse of time? An affirmative answer
would imply that the travel times for each speed class also would approach
a definite probability distribution. Such a situation could represent a

maximum of predictability. For it would mean that the past history of
delays is irrelevant for the mean delay which prevails after a sufficiently
long period. when the system has settled down to equilibrium.

The attainment of such an equilibrium does not exclude altogether the
occurrence of fluctuations around the equilibrium points. These persist in
all queuing processes. All that equilibrium means is that there is some
constant time average of delays, and that this mean value is all one can
predict about traffic conditions in the future.

For instance, there may under some conditions arise prolonged deviations
in the average queue lengths of the two lanes of a two-lane road even when a
steady and balanced total flow prevails. The reason for this is as follows.
To the extent that queues are built up from a given flow. passing opportun-
ities for traffic in the opposite direction increase. This in turn leads to
a more dispersed distribution of cars over that lane. thus reducing again
the passing opportunities for traffic in the first lane. In this way traffic
in the second lane can gain the advantage of the bunched-up condition in the
first lane and maintain or even aggravate this condition. Thus an even dis-
tribution of queuing between the two directions may be an unstable situation
tending to break away in one direction or the other. After a given time has
elapsed from a given initial state. the average queue length may therefore
have a bimodal distribution. Once the situation has shifted to the neighbor-
hood of one mode. the probability of it shifting to the other may in some
cases be large enough to produce frequent shifting back and forth or in other
cases so small as to lock the situation almost completely at one mode or the
other.

The Highway Capacity Manual refers to a related but slightly different
cause of fluctuations in the following illuminating passage: "At this traffic

volume [1,000 passenger cars per hour] spaces occur ahead of the slow-moving vehicles which cannot be filled by other vehicles performing passing maneuvers. In effect traffic in both directions tends to form in queues which continue to increase in length until the spaces between the queues become sufficiently long to permit the performance of passing maneuvers. As soon as a few passing maneuvers are performed the spaces between the queues become partially occupied and are no longer of sufficient length for the performance of passing maneuvers, and the queues immediately start forming again. This accordion effect, with all but a very limited number of vehicles traveling at the same speed as the vehicle immediately ahead, occurs at the same total traffic volume regardless of the distribution by directions."[1]

While it is clear that on a road of finite length the possible fluctuations are definitely limited in magnitude, they may be significant enough to constitute a qualification to an analysis that ignores them. In some cases these fluctuations may warrant a study of the desirability and means of controlling them.

Any treatment in terms of probability of the delays caused by passing runs up against the problem of repeated queues discussed in Section 1.6. Since this problem requires [2] some new theoretical ideas, it may provide a challenge to statisticians and mathematicians.

5.2 Alternative Notions of Capacity for Economic Analysis

The average-cost capacity function as used in Chapters 3 and 4 gives only an approximate expression of the impact of traffic conditions on transportation cost. Under any given conditions a vehicle causes a different amount of delay and affects various classes of road users differently depending

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[1]   Normann and Walker [1949, p. 37].
[2]   Except for the trivial case of service times with exponential distributions.

on its free speed and the direction in which it uses the road. A more detailed model would specify a set of capacity functions, one for each class of users of the given road, which gives the average cost to this class as a function of the traffic that falls into every single class. We notice that in this way some account would be taken of the effect of the driver population on capacity.

Disaggregation of the capacity concept in this way would also permit a refinement of the analysis of traffic distribution over alternative routes -- the problem of "shortest routes". Two routes that appear to be of equal transportation cost in terms of averages may in fact have different attractions to high- and low-speed traffic. If one route once has a certain advantage over an alternative for high-speed traffic, this will in itself lead to a relative concentration of high-speed traffic on this route and in this way further encourage the differentiation of traffic by speeds. Segregation of high- from low-speed traffic seems to have some advantages for both, and an efficiency analysis should determine whether this segregation goes far enough under conditions of ordinary spontaneous equilibrium.

Whether the demand function should also be disaggregated by free-speed classes is less obvious. For it may turn out that there is no essential difference in the flexibility of the demand of high-speed or low-speed road users. Some other differences in user characteristics, such as risk preferences, are perhaps more relevant here.

5.3* Commodity Transportation Models

The principal difference between passenger transportation as discussed in this study and commodity transportation lies in the substitutability of commodity shipments from different origins to a given destination. This makes the demand for trips between two given points a function not only of

the cost of this particular trip but also of all the costs of other trips
that might compete with the one considered. These other trips would include
those from the given supply point to other markets, and those from other
supply points to the given market. A convenient framework for dealing with
this type of demand for transportation is provided by the introduction at
each point of a function relating price and net or excess supply of the com-
modity in question. For any given price the excess supply of a commodity
at a certain location is the amount locally supplied at that price minus
the amount locally demanded at that price. Excess supply, which of course
can be negative, may therefore be regarded as a function of the local price
of the commodity.

For a proper explanation of the demand for commodity transportation it
is fundamental that these local prices are not themselves treated as pre-
determined data but are regarded as interdependent and are so balanced
against each other that all markets of a commodity taken together are in
equilibrium. This means that no profits should be possible through arbitrag-
ing commodity movements between any pair of locations. Prices of the same
commodity in different locations cannot differ by more than the transportation
cost. The price differences are exactly equal to transportation cost for all
pairs of markets between which shipments of the commodity take place.

The mathematical formulation of equilibrium in a model of commodity
transportation is now obvious. Let a subscript m in lieu of the previous
k denote a commodity.

Let $x_{ij,m}$ be the flow of commodity m from point i to point j on road ij;

$p_{i,m}$ the price of m at point i; and

$q_{i,m}(p_{i,m})$ the excess supply of m at point i.

Then $q_{i,m}$ is a non-decreasing function,

(5.1) $$\frac{dq_{i,m}}{dp_{i,m}} \geq 0 \qquad \text{wherever it is differentiable.}$$

We assume for the moment that all traffic flows are of commodities:

(5.2) $$x_{ij} = \sum_{m} (x_{ij,m} + x_{ji,m}).$$

Also we shall again use the notation

(5.3) $$\sum_{j} (x_{ij,m} - x_{ji,m}) = x_{i,m}.$$

The equilibrium conditions, in addition to the identities (5.2) and (5.3), are

(5.4) $$p_{j,m} - p_{i,m} \left\{ \begin{matrix} = \\ \leq \end{matrix} \right\} b_{ij}(x_{ij}), \qquad \text{if } x_{ij,m} \left\{ \begin{matrix} > \\ = \end{matrix} \right\} 0 \ ;$$

and

(5.5) $$x_{i,m} = q_{i,m}(p_{i,m}).$$

Notice that the prices increase along routes of transportation by the amount of transportation cost, while trip costs decrease by the same amount.

Passenger transportation may in fact be included as a special case of commodity transportation. This is the case of a "commodity" whose subscript is really a substitute for two indices, the origin and the destination, and for which the excess supply and demand are localized respectively, in these two points. Here the prices are determined only up to an additive constant

and may be set equal to zero at the destination thus making trip cost appear to be simply a negative commodity price.

The system (5.2) - (5.5) may be regarded as a more general formulation of the equilibrium conditions (3.2), (3.4), and (3.10). It turns out that our analysis of equilibrium in terms of an associated maximand carries over to this more general model. It follows that the inequality (3.35) on changes in the data applies and that the assertion about the adjustments of flows to changes in capacity remains valid.

We also obtain the efficiency condition in a form corresponding to (4.11):

$$p_{j,m} - p_{i,m} \begin{Bmatrix} = \\ \leq \\ = \end{Bmatrix} h_{ij}(x_{ij}) + h'_{ij}(x_{ij}) \, x_{ij}, \quad \text{if} \quad x_{ij,m} \begin{Bmatrix} > \\ = \end{Bmatrix} 0;$$

(5.6)   and

$$x_{i,m} = q_{i,k}(p_{i,k}).$$

Nothing is changed in the role of efficiency tolls the relation between equilibrium and efficient flows, and the relationship between tolls and the value of a road. In short our analysis can be repeated with appropriate changes and some increase in analytical complexity for a model of commodity transportation.

The unsolved problems concern the application of this model to particular cases. A one-commodity network would be of special interest. In particular, the problem of the generation and distribution of electric energy in a network comes to mind. The task of verifying the assumptions or modifying the model might be rewarding, but would call for considerable technical knowledge of electrical engineering.

5.4 Dynamic Equilibrium Models

The notion of a static equilibrium of flow in a network may be thought somewhat limited because of the noted periodicity of traffic during the day, week, year, and perhaps the business cycle. While the equilibrium mechanism is operative during the relatively short periods of a constant load one would like to see a more comprehensive model which contributes to our understanding of the time pattern itself. For instance, there are certain forces which tend to smooth the fluctuations. Since the off-peak hours offer better traffic conditions, some of the demand which is flexible with regard to time is diverted to these periods. On the other hand, the availability of parking space depends on the accumulation and dispersal of stocks of cars at various locations. This puts a premium on early arrival during rush hour periods and might either aggravate, or merely advance, or even spread the peaks of traffic. The generation and the economics of traffic peaks are subjects for further inquiry.

While it is not difficult, by attaching time subscripts to the flow variables, to write down formally the equilibrium conditions of Chapter 4 for a dynamic model, this merely makes the analysis more complicated without explaining much that is new. An understanding of the dynamic aspects of traffic really depends on an understanding of demand substitution over time. To what extent are traffic conditions, and to what extent are parking opportunities the effective considerations? Is differential pricing of parking in response to the daily fluctuations of traffic an equilibrating factor? Are there any economic forces at work or could any such forces be made operative which would induce a staggering of working hours while preserving the essential economies of the simultaneous availability of personnel in business activities depending on mutual contacts? This last problem has

also arisen with respect to the proper pricing of transportation services
of the New York subway system. [3]

5.5 Problems of the Long Run

In assuming the capacities of roads as given, this study has taken the
short-run point of view. Interesting and important problems of a long-run
character are met if one recognizes that demand functions will shift and
that road capacity will be adjusted to demand to some extent. One of the
objectives of a short-run analysis is precisely to supply some of the pre-
requisites for the study of long-run problems. Still the utilization of
capacity by traffic is in large part a short-run phenomenon for the economic
advantages of road construction can be appraised only in the light of the
uses to which this capacity is put at various times of the day, week, and
year, and of the rates of growth of these uses.

Some of the difficulties of a long-run analysis lie in the long-run
responses of demand through locational decisions. The problem here is to
determine and explain how the location of economic activities is influenced
by traffic conditions. Location theory tells us in general terms how trans-
portation costs affect the locational pattern. But further insights could
be gained from an analysis of the differential impact of time losses through
congestion on various industries. Most important perhaps is the question
of how the location of people (their residential distribution) depends on
traffic conditions. Without a clearer insight into the factors determining
residential choices, the prediction of long-run traffic trends on urban
expressways, for instance. must be hazardous.

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[3] Vickrey [1953].

In the long run the network layout does not remain fixed, and this
calls for consideration of construction and maintenance costs as functions
of capacity created and maintained. It would be important to derive capacity-
cost schedules of broad applicability from the technological information
available. This would help answer questions as to how the (publicly incurred)
cost of construction and maintenance can best be balanced against the (privately
sustained) costs of congestion.

Here we encounter what is perhaps the most fundamental problem in traffic
economics: determining the proper extent and layout of a road network. The
difficulty of this problem springs from the fact that choices between a great
number of all-or-nothing alternatives are involved. Adding a particular road
to a network may not be remotely worthwhile unless it is built to at least
some reasonably good standard, which would involve a considerable fixed cost.
But the flows needed to justify this investment depend on what other roads
may also have been added to the network. Thus to get the most out of an
integrated program of road construction one would have to compare a huge
number of possible combinations of road capacities that might be added. The
principles of economics have not been developed far enough to permit short-
cutting such a vast analytical problem. In the economics of indivisible
resources we touch one of the problem areas in economic theory where results
of a general character are urgently needed but seem very hard to come by.

Still other problems arise in situations where the predominant part of
transportation cost is delay at intersections. To give an indication of
this kind of problem, let us consider an example of two alternative roads
intersecting with two other alternative roads. The efficiency problem re-
duces to the question of which is more efficient, to cross the entire cross
traffic at once, or to cross one-half of it at each of two consecutive

intersections? Suppose that the cross traffic has priority in each case.
Since delays increase more than proportionally with cross flows, crossing
one-half of the traffic on two occasions involves less total waiting. In
the case that the one road to be crossed has twice the width of each of the
two alternative roads, the result is plausible on physical grounds, for cross-
ing one-half of the traffic twice amounts to finding a safety island in the
middle of the road. The economics and technology of the synchronization of
traffic lights is still another problem area in its own right, which we can
only just mention here.

5.6 Tolls and Finance

Finally, the problem of benefit evaluation is important for the assess-
ment of the financial burden to be borne by various classes of road users.
This is again illustrated by the case of toll roads. The necessity of fin-
ancing the construction and maintenance of a toll road largely or entirely
from toll revenue imposes a lower bound on the toll charges, which we have
not taken into account here.

From the point of view of the best overall expansion and utilization
of a network, the constraint that each toll road should pay for itself must
be dropped. Instead the problem becomes one of finding the most "equitable"
allocation of road costs to sources of tax revenue. To what extent should
finance be sought in the form of taxes on vehicle ownership at particular
locations on gasoline consumption (and hence total mileage, approximately),
and on the use of particular roads (tolls)? While of course, congestion is
incident on particular roads at particular times, charges which tend to
alleviate traffic need not always be levied in the form of tolls. For in-
stance a general tax on gasoline consumption renders traffic in general more

expensive and thus tends to reduce the total amount of traffic. On the other hand it puts a premium on the use of the geographically shortest routes and therefore induces congestion there. The net effect may therefore be an increase in congestion losses.

It may also be possible to relieve congestion by taxes on business activities. or on car ownership at residences. of which the locations are such as to be mainly responsible for the congestion. However. a charge in this form will discourage traffic along any roads from these locations, congested or not. so that the welfare of road-user classes who are not responsible for congestion on the routes in question is affected. Only a road toll proper is specific enough to be free from this objection. On the other hand, the mere collection of a road toll may itself add to congestion. It is therefore an interesting problem to what extent efficiency can be approximated by other forms of taxation.

Even a specific toll of one of the types currently in use is subject to the drawback that it is usually thought necessary to keep the toll charge constant for considerable periods of time. The best that can then be done is to set the rate at some average level, even though congestion may fluctuate widely. In many cases it would be markedly more efficient to reduce the toll during certain periods of lower traffic or even to suspend it entirely. (This problem has an obvious analogue in electricity tariffs). In addition there are differences between various classes of road users in their contribution to delays and to the deterioration of the road. Thus slow or heavy vehicles tend to contribute more heavily. To some extent this may be allowed for by differentiating between tolls on trucks and passenger cars for example, and by requiring vehicles to be at least capable of maintaining a certain minimum speed, especially in climbing long grades.

However, the difficulties of toll collection as well as a general desire
for "freedom of the roads" limit the amount of revenue that can be collected
in the form of tolls. Indeed toll facilities are often such as to be a
limiting bottleneck to the flow of traffic at peak periods to the extent
that in some cases efficiency would be best served by suspending the collec-
tion of tolls at such peak periods so as to eliminate excessive queues at
toll booths. Other taxes are likely to be needed to sustain the roads. While
a case can be made for some use of general funds for this purpose, since an
efficient road system contributes to the general welfare, the intensity of
communication, and the speed of emergency help, etc., it would seem to be a
point of justice that the bulk of the money should come from the road users
in a form connected with road use. This would leave a considerable share to
both general vehicle and gasoline taxes. The optimal apportionment -- optimal,
that is, with respect to the combined standards of equity and efficiency --
poses an interesting problem which will be the subject of discussion for a
long time to come.

PART II

A STUDY OF

RAILROAD TRANSPORTATION

Chapter 6

The Time Element in Railroad Transportation

In our study of highway transportation we emphasized the importance of including time as a dimension of the output of a transportation system. In assessing the capacity of a system of highways, it did not suffice to talk simply about the traffic flows that did, or could conceivably, take place. It was also necessary to have some information about the speeds at which these flows did, or could, take place. We found that these time considerations were all the more important because there existed interactions between flows and speeds. Greater flows could usually occur only with reductions in speed and, conversely, speed increases had to be paid for in lesser flows unless the road system was improved.

The factor of time is no less important a consideration in a study of railway transportation. As far as passenger traffic goes, the parallel with road traffic is clear. Despite the current advertising slogan, "Getting there is half the fun!" it is probably safe to say that travelers on the whole prefer fast transportation to slow, other things being equal. Thus from this point of view a given distribution of passenger traffic over a railway system represents a more valuable product when the average time of travel is less.

For freight traffic the parallel with passenger traffic on roads is not quite so clear. How important is the time consumed in carrying out a freight shipment? In the chapters that follow we shall quite often have occasion to refer to the delays a freight shipment encounters at various stages in its journey. We shall attempt to determine some of the factors influencing these delays, and also show how delays and other kinds of costs are related. In this chapter we shall try to justify our concern with time-costs in freight

transportation. First. we shall discuss the transit time of a shipment from
the point of view of the consumer of railway services. The later discussion
will be concerned with the relation between transit time and the stock of
freight cars. While the stock of cars has, through this relationship, an
important influence on the capacity of a railway system, we shall not, except
in this chapter, deal with it explicitly but rather focus our attention on
factors affecting transit time itself. In addition, it is hoped that in the
process of this brief discussion the reader will get some idea of the mag-
nitudes of the various kinds of delays involved.

6.1 Transit Time and Inventory Costs

One obvious advantage of faster freight transportation is the resulting
reduction of in-transit inventories that it makes possible. That this phenomenon
is a familiar one is evidenced in the popular use of a transportation term in
discussion of inventories of another sort. "Filling the pipelines" has come
to mean the building up of inventories of many kinds to some required level.
The implication is that the inventories made necessary by the time lag in-
volved in transportation are more easily explained in an intuitive way than
other kinds of inventories.

The pipeline analogy is worth carrying another step to make clear the
relation between speed and inventory requirements. A pipeline half as big
in cross-section area as another could carry the same flow of, say, gasoline
only if its pumps were capable of sending the liquid through the pipe at twice
the speed. The gasoline inventory required to fill the smaller pipeline is
just half that required to fill the larger, and this reduction of inventory
is permitted by the higher speed of transport. Approximately the same phenomenon
takes place in any other form of transportation.

So far we have only discussed the effect of speed on the levels of inventories of goods in transit. There is still another effect of speed on inventory requirements in general. First we must ask why inventories, aside from the in-transit type just described. are desired at all. One reason is that commodity price fluctuations cause buyers to stock up beyond their immediate needs when they believe the price to be a favorable one. Another reason is that commodity traffic flows in practice do not often occur smoothly but rather in lumps. It is more practicable to send a carload of wheat once a day, say. than to send a ton of wheat every hour. This lumpiness occurs even in the one case where one might not expect it. that of pipeline transportation, where one pipe often serves for the transmission of several different liquids. Since these cannot be pumped through simultaneously and still preserve their identity, the pipe is given over to each liquid alone for a certain period of time.

Still another reason for inventories, and perhaps one of the most important from our point of view. lies in the uncertainty involved in predicting just how much of a commodity will be needed in a particular place at some future date. This uncertainty on the part of a consumer of a commodity (or a producer who uses this commodity as an input), together with the time involved in transportation oblige this consumer or his merchant to keep on hand stocks sufficient to cover errors in predicting his needs.[1] If transportation took no time at all, no prediction would be necessary; he could order more of the commodity as the need arose. The slower transportation is. however. the greater the range of time the prediction must cover; consequently. the greater are errors likely to be and the greater the inventories required to insure against these errors.

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[1] Where costs of storage are high relative to costs of short supply this rule must be modified.

It is clear therefore that the time involved in transportation affects the levels both of inventories of goods in transit and of other inventories. To the extent that inventory holding is costly, the people concerned, sometimes shippers, sometimes consignees, are better off with faster and more dependable freight transportation.

For a few special commodities the advantages of fast transportation are especially clear. Livestock requires special handling (watering, etc.) every few hours by Federal Law. This treatment is expensive and especially so if performed at locations without the proper facilities. The faster this livestock movement can be made, the less of this costly unloading and reloading procedure will there be.

Perishable commodities, such as fresh fruit and vegetables, must be shipped in refrigerator cars. These perishable shipments are commonly made over long distances, and call for re-icing along the way. Fast shipment reduces the number of such treatments that are necessary, and, as in the livestock case, reduces the investment in special facilities required to provide this service.

The data we might examine to see how far the foregoing considerations go in explaining the relative transit times for various commodities are very sparse indeed. For 1933 however we do have the results of one very small sample of traffic. In the Freight Traffic Report of the Federal Coordinator of Transportation an extensive tabulation of information on all traffic terminating in the U.S. on December 13, 1933, is given. Figure 1 is derived from this report.[2] On the vertical axis is measured the average transit time. This includes all the time from the moment a car is given to a shipper for loading until it is returned empty from the consignee. On the horizontal axis is measured the average length of haul. Points are plotted for several

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2/   Federal Coordinator of Transportation [1933, Appendix I, pp. 120-173].

Transit
Time
(Days)

8

o coal

7

o forest products
o appliances

6
o grain

o perishables

5
o sand
o petroleum

4

3
o iron and steel

2
o animals
o l.c.l.

1

2   4   6   8   10   12   14   Length of Haul
(Hundreds of miles)
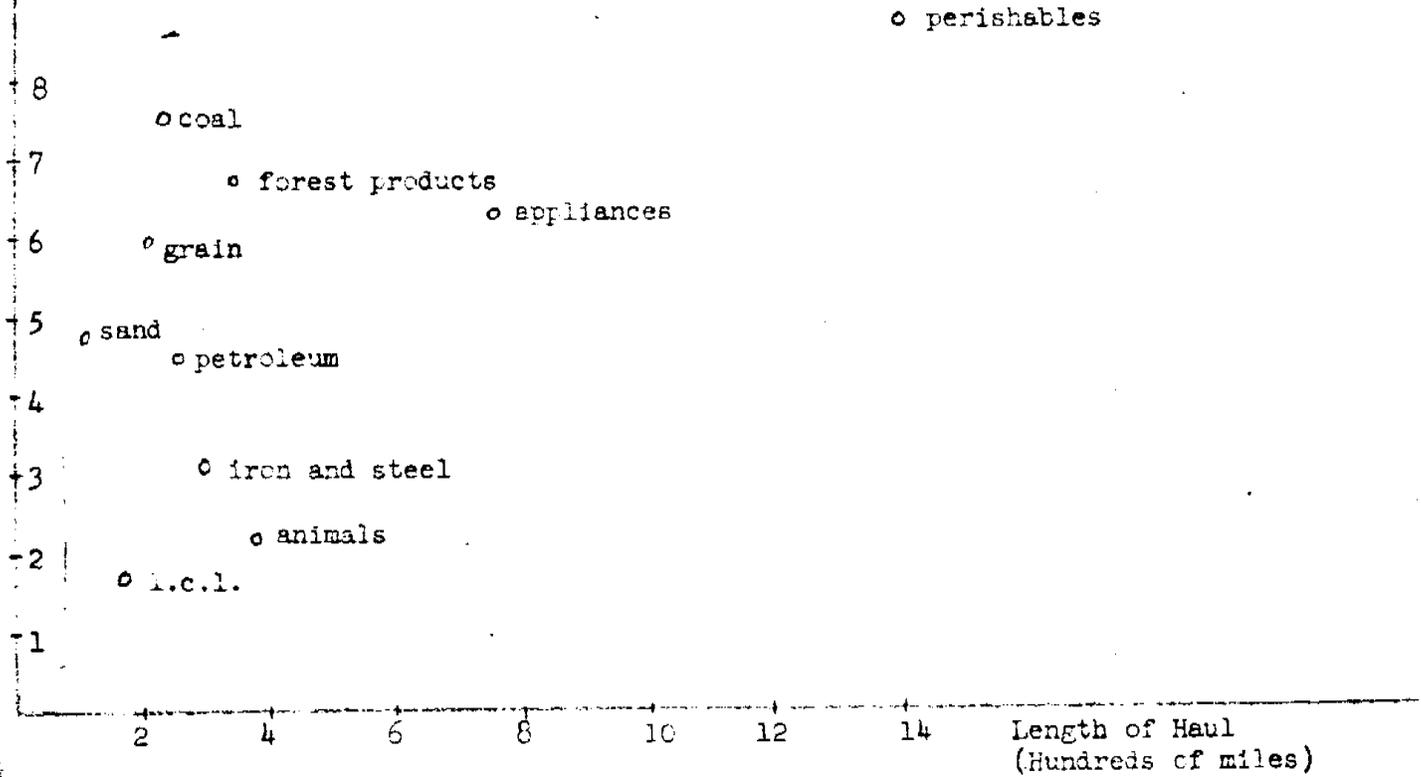
Figure 1.   Average Transit Times for
Several Commodities

of the commodity groups in the Report. While we see from the figure that

perishables take the longest time for shipment and l.c.l.[3]/ the shortest

time, we also see that their relative hauls also bear the same relationship.

If for each commodity we divide haul by time we get an estimate of the miles

traveled per loaded-car day on the average.

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3/   "Less-than-carload lots"

Live Animals	188
Perishables	159
Appliances	119
L.C.L.	113
Iron & Steel	104
Petroleum	61
Forest Products	57
Bulk Grain	38
Coal & Coke	33
Sand & Gravel	27

Since "time" here includes loading and unloading time these figures should not be thought of as measuring the distances traveled per day when actually moving in trains.

This ranking does seem to give some support to the hypothesis that the effect of speed on inventories in turn influences the speed of railway services provided. The more valuable a commodity is (per ton) the higher the cost of keeping idle stocks of it; in the list we find, as we should expect, the higher priced commodities near the top. There are of course many qualifications to such a conclusion. The relative values of carloads of different commodities are quite different from their relative prices per ton, because cars are loaded heavily with some commodities and lightly with others. Unfortunately for our argument the last two on the list, coal & coke, and sand & gravel, are among the heavy loaders. Nevertheless it is difficult to believe that this difference would alter our conclusion. The ranking by "miles per day" rather than simply "days", although obvious enough, requires some explanation in order for us to show more clearly the shortcomings of such a procedure. In a railroad system ideally organized from the economic point of view we should expect the transit time of a particular service to be cut down whenever the consequent reduction of inventory costs exceeds the costs of providing this faster service. One of the reasons that on the average a carload of appliances requires almost two more days[1] in shipment than a carload

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[1] See Figure 1.

of sand & gravel, even though we believe the first carload to represent a greater investment than the second, is that it costs _very_ much more to provide four-day service for appliances than it does to provide four-day service for sand & gravel. The difference in lengths of haul is of course the obvious reason for these cost discrepancies, and it is the one we have taken account of in our ranking of the various commodities by "miles per day". However there are other, and perhaps just as important, reasons for these differences in the costs of providing fast service for particular goods. Our ranking must accordingly be viewed with some suspicion for this reason. We shall say no more at this point about these other influences on cost since much of the rest of this study is devoted to them. Our purpose here has only been to demonstrate the fact that speed of shipment is a valuable characteristic of freight transportation.

6.2 Transit Time and the Stock of Freight Cars

So far we have shown that, just as in passenger travel on highways, the time used up in rail transportation of freight must be considered part of the cost of a railway system because time is a costly commodity to the consumers of railway services. It remains to be shown that speed (understood in the general sense of being the inverse of transit time) is costly to produce. There are obvious costs associated with speed increases if we think of these increases as requiring more or better equipment and physical facilities. Of more interest to us at present however are the increases of speed brought about as a matter of choice by different use of a given physical plant. In the case of highways both these types of speed increase also exist. Roads can be straightened and overpasses built to replace intersections, or, on the other hand, traffic flows at certain points can be reduced, and traffic

regulations and signals can be changed. Both procedures can lead to speed
increases; the one by a change of the physical facilities, and the other by
a change in operating procedures alone.

Is this second method of reducing the time of freight shipments costly?
Without a careful investigation, it is by no means clear that it is. In our
highway investigations we found that the penalty paid for an increase of
speed on a given road was a reduction of flow on that road and perhaps a
decrease of speed on the competing road to which this excluded flow was
forced to turn. The road capacity curves discussed were the relationships
which gave us quantitative estimates of these effects. Somewhat similar
relationships between traffic flow and speed of service will be discussed
in the chapters to come.

There is, however, an important type of relationship between speed and
traffic flow in railroad transportation which was relatively uninteresting
in the case of passenger traffic on highways. It arises from the fact that
the longer the average shipment takes to complete, the greater is the supply
of freight cars necessary to carry out a given program of transportation.
Putting this another way we can say that if the stock of freight cars is
fixed, then the traffic flows that can take place are limited in size by
the transit times of shipments. Considerations such as these help to ex-
plain why it is the case, as we shall see below, that a freight car is idle
so much of the time, and also why the stock of freight cars is what it is.
This relation between the stock of cars, the transit times of shipments, and
the number of carloadings in a year is sometimes summed up in a railroad rule
of thumb: "To find the number of cars necessary to protect loading, multiply
weekly carloadings by two".[5/] The number wo in this case represents an opinion

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[5/]   Parmelee [1940, p. 635].

that the average time of transit, including the time spent in empty movement, is two weeks.

The most interesting aspect of this relationship presents itself when peak traffic periods are examined. Rail carloadings are by no means even over the year, a high peak usually occurring between May and October with very sharp dips taking place on the Fourth of July and Labor Day. To meet these mid-year peaks without severe car shortages a much greater number of freight cars is needed than would be if the same level of annual carloadings were spread out evenly over the year. This means that when the relationship connecting the freight car stock with carloadings and transit times is applied to a period of a year it appears to represent a constraint which is unimportant because it never seems to pinch. In this context an increase of speed would seem to provide no saving in the number of freight cars necessary. As we shall see below, an examination of peak periods dispels this illusion.

Returning to the case of road transportation for a moment, it is instructive to ask how the same considerations apply there. To start with let us suppose that we know how many automobile trips are to take place in a certain year, and we also know the average distance of each trip and the average time necessary to complete it. If these trips are spread out evenly over the year a very simple calculation will tell us how many serviceable vehicles are necessary to accomplish the job. If this minimum number of vehicles were the number actually in use, it would mean that every serviceable vehicle was being used at every moment. The nationwide system of car pools represented by this description has little correspondence to the real situation in transportation by automobile, for we know that most automobiles stand idle a great part of the time. The fact that the actual stock of vehicles is so much greater than the minimum number derived above can in part be explained by

the great variations in the number of trips in process at different times. While it has been suggested that a careful examination of peak traffic periods for railroads is worthwhile, it is not as likely to be so in the case of auto transportation for several reasons. First, since the average length of trip by automobile is short, the periods we must examine to find a peak period are also short. The information requirements are therefore great. Second, the peak periods in different locations will surely occur at different times. While this is also true of rail traffic the trouble can be at least partly eliminated there by taking explicit account of the time spent in empty travel. In the absence of this shortcut it would be necessary to do a separate analysis of each location. Third, we already know to a large degree what the result of such an analysis would be in the case of passenger automobiles. The cost of automobiles appears to be such that people are quite willing to pay to rid themselves of the inconveniences of part ownership in a carpool. Where this institution does exist (taxicabs are one example) it is always surrounded by special conditions such as the use of a chauffeur, the identity of trips, or the common residence of members of the pool (family cars). In any case it is clear that many more considerations enter than in the case of rail transportation.

Before we enter into a more detailed examination of the influence of speed on the freight car supply required, it will be useful to discuss first the available information on the time per shipment actually used at various stages of the transportation operation.

For this purpose we refer again to the 1933 Freight Traffic Report of the Federal Coordinator of Transportation. This material covers the 65,769 cars that terminated on December 13, 1933. From it can be derived, for several commodity groups and for the group as a whole, the average times

spent by a car at various stages in its journey from shipper to consignee. Table I presents these averages and Table II translates them into percentages of total time of shipment for each commodity group. The labels, along the left edges of both tables require some explanation. Loading and unloading time represents the time a car spends in the hands of its shipper plus the time in the hands of its consignee. Carrier termini time measures the delay a car suffers at both ends of its journey while still in the hands of the carrier. Carrier road time tells us how long a car is in a train during its trip. This time is not necessarily consecutive for the car may leave one train, spend some time in a classification yard, and then join another train for more road time. It should also be pointed out that road time is not quite the same as time spent rolling. Trains are subject to many kinds of delays; these also count as road time for the cars of a delayed train. Intermediate handling and interchange times cover the period a car spends waiting in classification yards along the way. Interchange time represents that part of this wait which occurs at points where the car is transferred from one railroad to another, that is, "interchanged". Total time therefore represents the time from the moment an empty car is supplied to the shipper until the time the car is returned empty by the consignee (perhaps, of course, to a different railroad).

One of the striking things about the entries in these two tables is the relatively small amount of time a car spends in trains. Thus carrier road time amounts only to about a sixth of the total time of shipment, or about a third of the shipment time less the loading-unloading period. Also, if these carrier road times for the different commodities are plotted against lengths of haul we notice that there is a fairly good straight-line relationship between the two (Figure 2).

Table I. The Average Number of Days Spent By a Car Between Delivery For Loading and Release From Unloading - Cars Terminated December 13, 1933.

	All Commod- ities	LCL	Coal & Coke	Perish- ables	Live Animals	Sand, Stone & Gravel	Appli- ances	Petroleum & Products	Forest Products	Iron & Steel	Bulk Grain & Beets	All Carload Commod- ities
Loading - Unloading	2.21	.36	3.96	3.08	.54	2.52	2.49	2.09	2.99	1.53	2.33	2.84
Interchange	.29	.05	.42	.71	.16	.23	.51	.34	.33	.20	.20	.37
Intermediate Handling	.61	.17	1.02	1.29	.36	.51	.80	.53	.86	.36	.75	.75
Carrier Termini	1.02	.73	1.25	.83	.26	.90	.87	.80	1.39	.56	2.01	1.10
Carrier Road	.83	.45	.81	2.88	.84	.49	1.57	.71	1.04	.40	.65	.96
Total	4.95	1.77	7.47	8.79	2.16	4.64	6.25	4.47	6.60	3.05	5.94	6.02
No. Cars in Sample	65769	16664	14804	2199	2696	1592	840	5180	3448	2432	3085	49105
Avg. Haul	331	201	249	1400	405	127	746	272	376	316	224	376
Miles Per Loaded Car Day	74	113	33	159	188	27	119	61	57	104	38	63

Table II.   Distribution of Active Car Time - Cars Terminated Dec. 13, 1933.

	All Commodities	LCL	Coal & Coke	Perishables	Live Animals	Sand, Stone & Gravel	Appliances	Petroleum & Products	Forest Products	Iron & Steel	Bulk Grain & Beets	All Carload Commodities
Loading-Unloading	44.8%	20.5%	53.1%	35.1%	25.2%	54.2%	39.9%	46.8%	45.3%	50.2%	39.3%	47.3%
Interchange	5.8	2.8	5.6	8.1	7.4	5.0	8.2	7.6	5.0	6.5	3.3	6.1
Intermediate Handling	12.3	9.7	13.7	14.7	16.7	11.0	12.8	11.8	13.0	11.8	12.6	12.5
Carrier Termini	20.5	41.9	16.7	9.5	11.8	19.4	14.0	18.0	21.1	18.4	33.8	18.3
Carrier Road	16.8	25.2	10.9	32.7	38.9	10.4	25.1	15.8	15.7	13.2	11.0	15.9
Total	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%

Source: Federal Coordinator of Transportation, Freight Traffic Report, Appendix I, pp. 120-173.

Of the 4.95 days average transit time, 1.92 days or 38.6 percent are used up at the three stages interchange, intermediate handling, and carrier termini. This is 70 percent of the total shipment time less loading-unloading time. We throw these three types of delay together for reasons that will be better understood in later chapters; it is the time used at these three stages with which we shall be most concerned. In Figure 2 we saw that road time could be fairly well explained in terms of the length of haul of a shipment. In Figure 3, where the vertical axis now measures the sum of interchange,

o perishables

o appliances

o forest products

coal o   o animals

grain o   o petroleum

l.c.l o   o iron and steel

o sand

2   4   6   8   10   12   14   Length f Haul
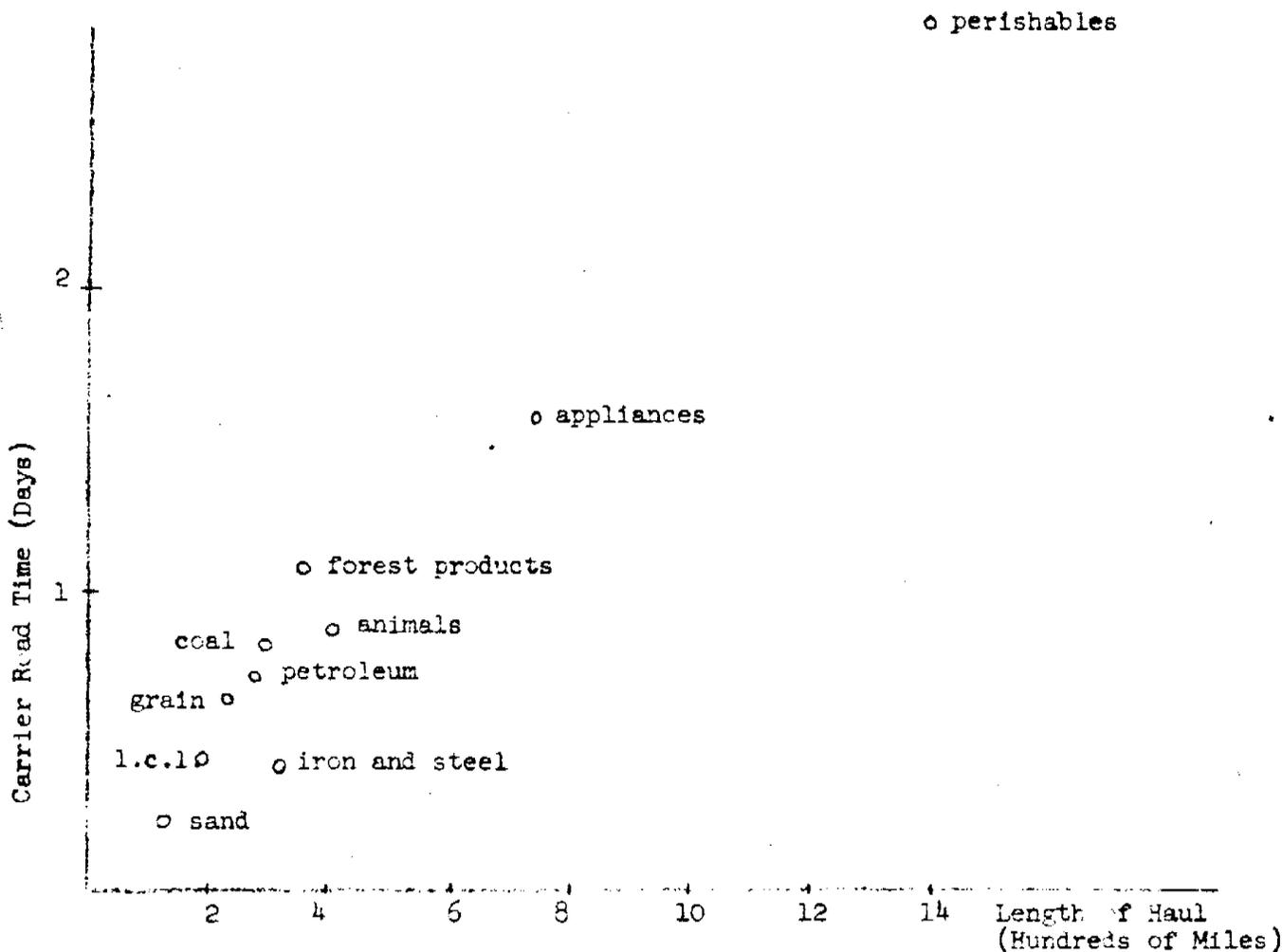(Hundreds of Miles)

Carrier Road Time (Days)

Figure 2. Carrier Road Times for Several
Commodities of Different Average
Hauls

intermediate handling, and carrier termini times, we find that no such simple explanation of the variation suffices in this case. There seems to be little if any relationship between the two variables plotted.

Except in this chapter we shall not say any more about loading-unloading time despite its very substantial magnitude. Part of the reason for the size of this component of transit time undoubtedly lies in the fact that at current rates of demurrage consignees find it cheaper to use freight cars instead of warehouses for storage purposes.

These time estimates from a single day in 1933 seem to be about the only publicly available source of such information. Individual railroads, it is true, have conducted investigations of the transit times of various shipments over their own lines, but we do not know how extensive these have been, nor quite how the results compare with those of the Freight Traffic Report. Since we have so few other data with which to compare these figures, it is difficult to estimate the changes that have taken place in them since 1933. The regular statistics of the I.C.C. bear on only one of these components of transit time -- carrier road time. By making some rather daring assumptions we shall attempt to use these latter statistics, together with the results of the Freight Traffic Report, to estimate the transit times for years other than 1933. We shall then try to extend this attempt to cover the time consumed in the average empty trip. Finally a comparison of the sum of these two times will be made with the actual freight car time available in each of several years, in order to see to what degree the stock of freight cars is utilized.[1]

If for each year we knew the average length of haul of freight cars and their average speed when in trains then the average time in trains during a trip could be found by dividing haul by speed. The result could be compared

[1]   The discussion that follows has been greatly stimulated by a reading of Chapter 4 of Hultgren [1948].
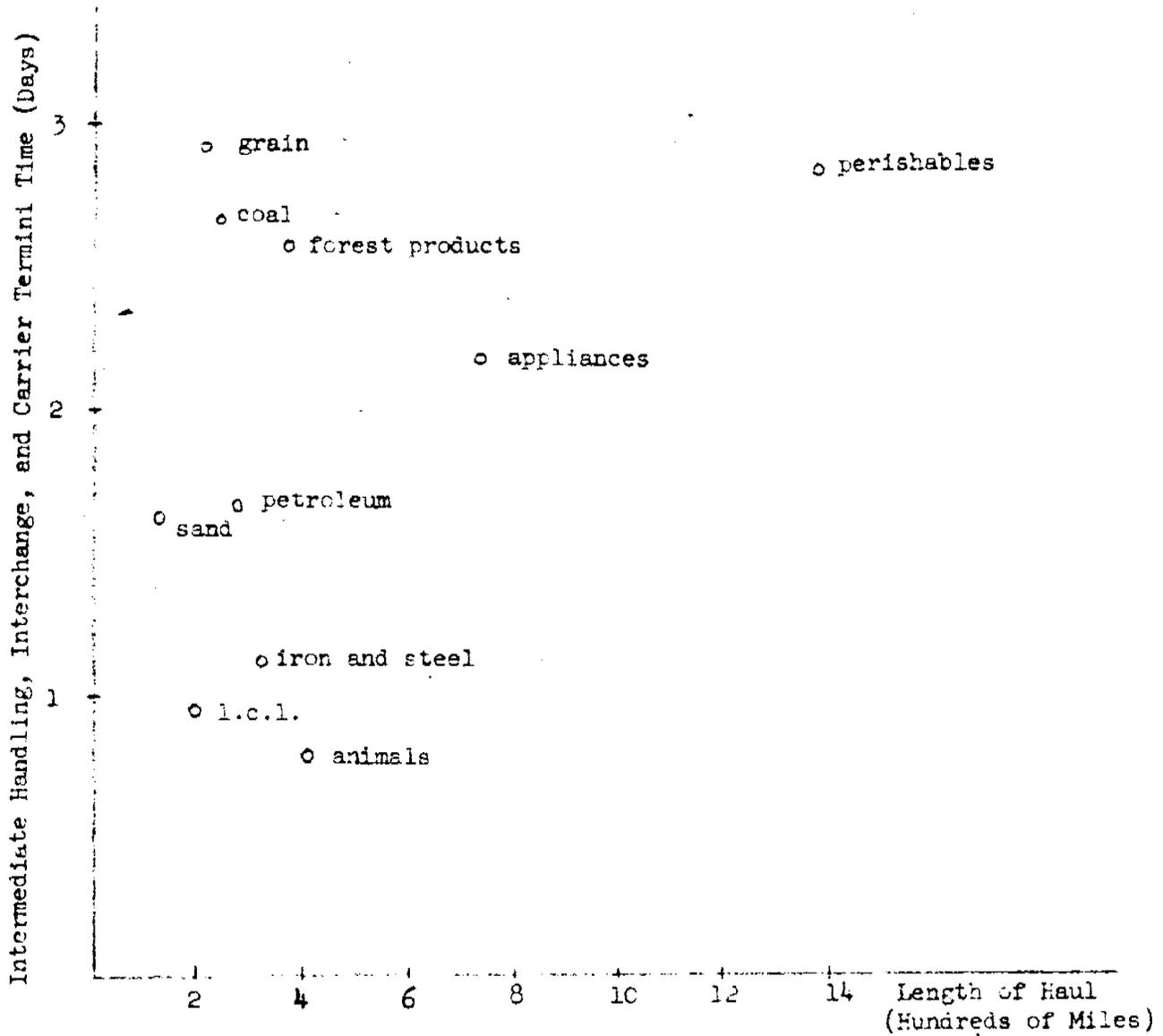
Figure 3. Intermediate Handling, Plus Interchange,
Plus Carrier Termini Time for Several
Commodities of Different Average Hauls

with the estimate of carrier time derived from the 1933 data. While we are

not quite so fortunate as this, some very closely related data are regularly

reported by the I.C.C. and the Association of American Railroads. Using

these we have the ratios

car-miles[7]/carloadings[8] and train-miles[9]/train-hours[10] as estimates, respectively, of the average length of haul of a car and the average speed of a train. If now we assume that the average speed of a train is the same as the average speed of a car then we have the relation

$$(6.1) \qquad \frac{\text{car-miles} \times \text{train-hours}}{\text{carloadings} \times \text{train-miles}} = \text{average carrier road time of a car.}$$

For the year 1933 this estimate of carrier road time is 23.4 hours as compared with the estimate of (.83 x 24 =) 19.8 hours in the Freight Traffic Report (Table I). These new estimates of carrier road time will err on the low side if, on the average, longer trains are slower than shorter trains, and on the high side if longer trains are faster than shorter trains.

To obtain estimates of transit times for the different years we have assumed that carrier road time comprises a constant proportion of transit time from year to year, so that for 1940 for instance

$$(6.2) \qquad \frac{\text{transit time}}{\text{for 1940}} = \frac{\text{transit time}}{\text{for 1933}} \times \frac{\text{carrier road time for 1940}}{\text{carrier road time for 1933}},$$

where, on the right hand side, the 1933 figures are from Table I and the 1940 figure is derived from the relation (.1). The resulting graph is shown as the curve DD' in Figures 4 and 5.

The curve CC' in Figures 4 and 5 graphs for the different years our estimates of the total transit time plus the average time spent by a car in

7/ I.C.C., Statistics of Railways in the U.S., Table 158, Item No. 457.
8/ Association of American Railroads, Revenue Freight Loaded and Received From Connections, Statement CS-54A
9/ I.C.C., Statistics of Railways in the U.S., Table 158, Item No. 429.
10/ Ibid., Item No. 488.

completing its empty trip to the next shipper. The distance between the
two curves is simply

$$(6.3) \qquad \text{time of the empty trip} \ = \ \text{transit time} \ \times \ \frac{\text{empty car-miles}}{\text{loaded car-miles}} \ .$$

The curve CC' therefore represents the number of days on the average that
a car uses to complete a whole cycle from one loading to another.

The average time that is available for this cycle depends on the level
of traffic and on the stock of freight cars in serviceable condition. Curve
AA' in Figure 4 represents

$$(6.4) \qquad \frac{365 \text{ days}}{\dfrac{\text{annual carloadings}}{\text{stock of freight cars}}}$$

and tells us the length in days of the average turnaround period of a freight
car. Curve BB' is derived in the same way except that the stock of service-
able cars is substituted for stock of cars. The difference between AA' and
BB' tells us what part of each cycle is devoted to repair work on the average.
Notice that in the early thirties this difference is markedly higher than in
other years. This does not mean that more repairs had to be made, but rather
that, since business was slack, "bad-order" cars were not repaired immediately.

The difference between curves BB' and CC' represents freight-car
time so far unaccounted for. The size of this unexplained part is surprising
until we recall the fact that peak traffic periods have not entered the analysis.
In Figure 5 the Curve BB' has been changed to represent only the two-week
period of each year with the highest average number of carloadings. If every
year's two-week peak persisted throughout the whole twelve months Figure 5

would tell us what proportion of a freight car's time was utilized. As it
is it refers only to the utilization in these peak periods (which do not
always occur on the same, or even nearly the same, date).

Another reason for the unexplained part of a car's time is very similar
to the reason, discussed earlier, for holding idle stocks of any commodity.
Quite often at some location a number of empty cars larger than that which
subsequently turns out to be needed may be kept on hand in order to cover
errors in forecasting carloading requirements. To the extent that the move-
ment of empties from one location to another can be speeded up these empty-
car stocks at various places can be reduced, just as the speeding up of
commodity movements enables a reduction of ordinary commodity inventories.

Figure 4. Estimated Turnaround Time and Transit Time for the Average Freight Car.

AA' = Average number of car-days available for a single shipment

BB' = Average number of serviceable car-days available for a single shipment.

CC' = Average number of days from start of one loading to start of next loading.

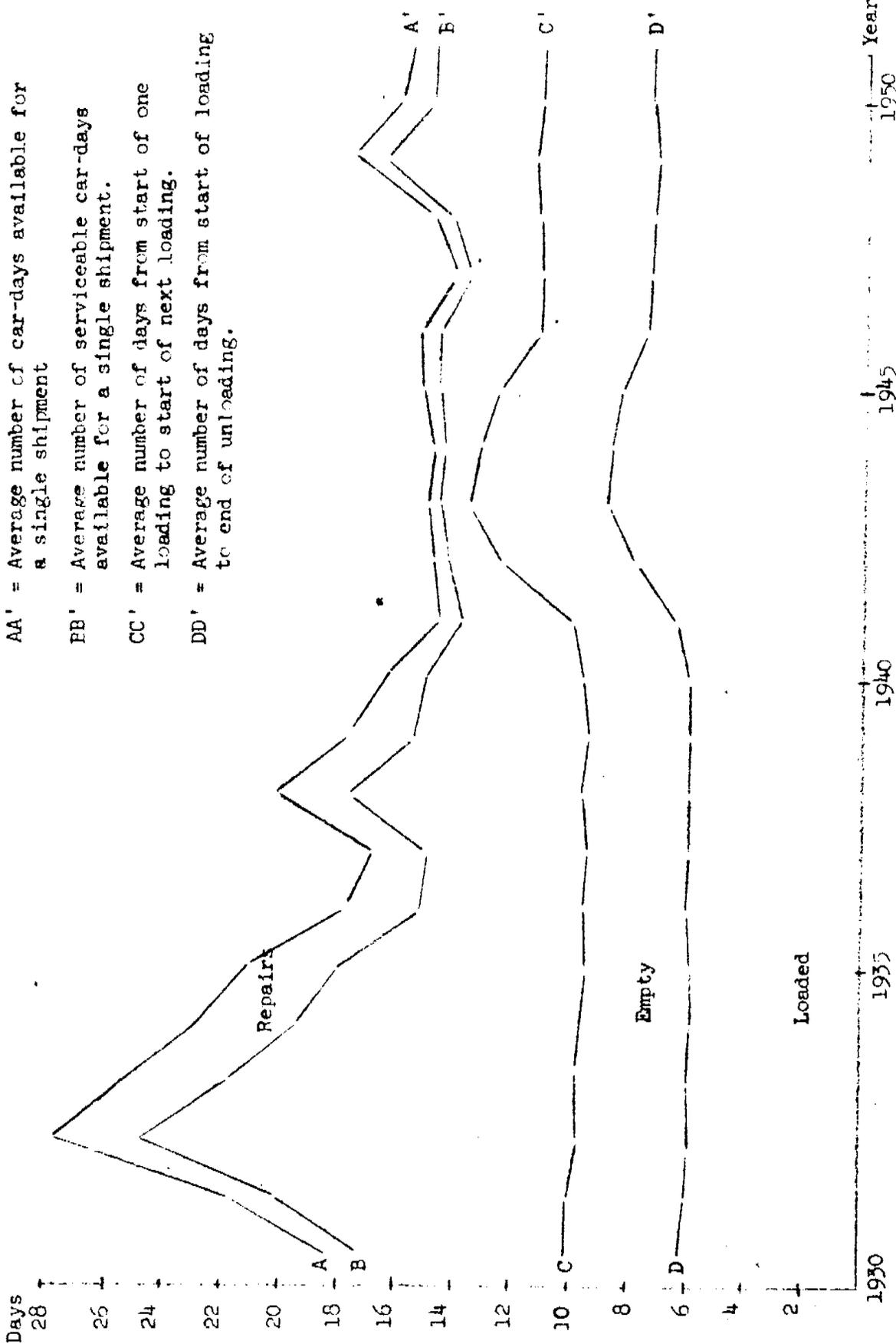DD' = Average number of days from start of loading to end of unloading.

Figure 5. Estimated Peak Period Turnaround Time and Transit Time for the Average Freight Car.

AA' = Average number of car-days available for a single shipment during each year's peak period.

BB' = Average number of serviceable car-days available for a single shipment during each year's peak period.

CC' - Same as in Figure 4.

DD' = Same as in Figure 4.
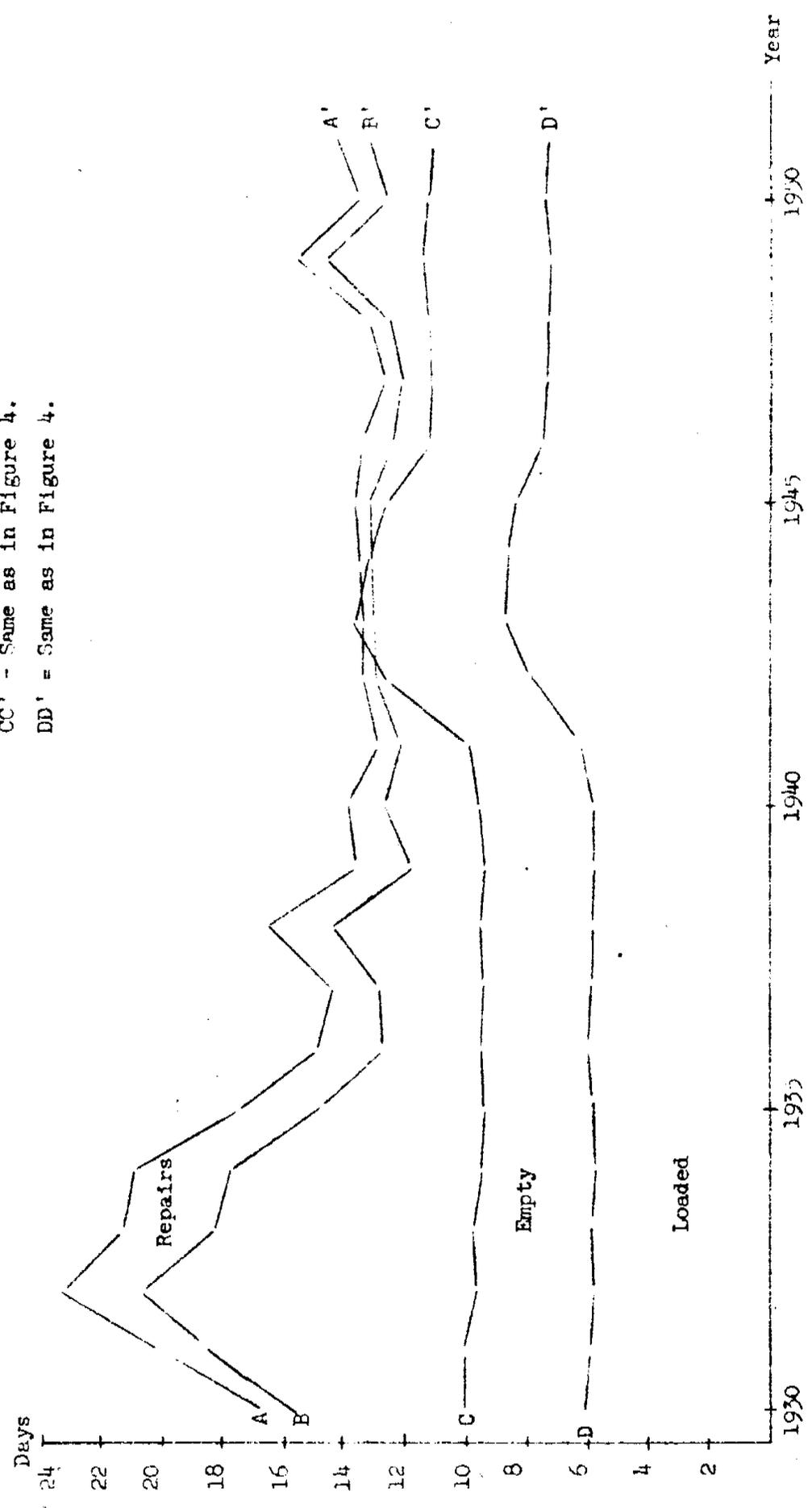
Chapter 7

Freight Operations and the Classification Policy

7.1 A General Description of Freight Operations

In this section some aspects of railroad operations are sketched. The object is not to give any detailed account of them, but rather to focus attention on those aspects of the operations with which we are concerned in later chapters, and to introduce some phrases and definitions which we will find useful. Attention will be concentrated on the workings of a freight system.

It may be useful to follow a carload of freight on a highly simplified railroad system. Figure 1 shows such a system.
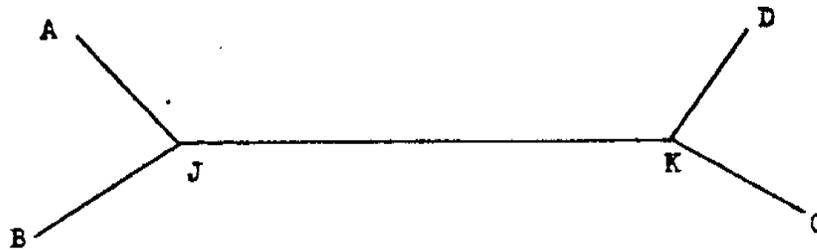
Figure 1

Suppose a load is being dispatched from A to D. At A it will be loaded into an appropriate car. However it may have to wait at A for some time either until a sufficient number of other cars are ready to travel in the same general direction to warrant running a train, or perhaps until a train is scheduled to leave. Delays of this type arise fundamentally because it is not usually considered economical by the railroads to haul trains with

only a few cars. It is an example of what we call an accumulation delay.

After a time enough cars will have collected to go in the direction of J. Some of these may be going to B, C, or K or perhaps only as far as J itself. In any case the order of the cars for all five destinations will probably be well mixed up in the train. Thus at J the cars in the train will have to be uncoupled and sorted. Cars going on to K and beyond will be sorted together. Again they may then have to wait until more cars destined for K or beyond (that is to C or D) arrive from B; they may therefore suffer more accumulation delay. Then the car we are following will be hauled out in a train to K.

At K the same sequence may be followed. The train will be uncoupled and sorted into cars going to C and to D, and cars remaining at K. Again there may be a wait until the train for D leaves, and the car is taken on the last lap of its journey.

At points such as J and K there must be facilities for sorting out the cars, and storing them until they are taken out in a train. Such facilities are called classification yards. These yards may be large and complicated, with a considerable amount of expensive equipment. They usually involve a large investment for the railroads.

We have suggested already that there are two principal ways of arranging the times at which trains are dispatched from ne yard to a second. One possible method is to wait until enough cars have been collected which are traveling as far as the destination yard or farther. How many cars are considered "enough" to make up a train should presumably depend on a balancing of various economic factors; for example the extra delay involved in waiting for sufficient cars for a long train as against the increased costs in fuel, labor, etc., of running several short trains instead of one long one. Such

a method of operation can be flexible, since those responsible for dis-
patching trains can change their plans to meet different types of flows.
However this method is not well adapted to synchronizing the various move-
ments of trains to ensure that they meet at junctions -- the classification
yards -- in such a way as to hold down accumulation delays. For this to be
accomplished, the trains must be dispatched according to timetable. This
method of operation is becoming increasingly common for freight trains on
American railroads. The disadvantage is that, on occasions when traffic
fluctuates, trains may have to run either very short, or so long that they
must be run in two sections in order to keep to the scheduled times, so that
the system may not be working most economically. With timetable operation,
accumulation delays occur in slightly different ways. After a car has been
sorted it has to wait until the train on which it has to travel is scheduled
to leave. We still call this an accumulation delay, however, to keep to a
single term. For fairly steady flows of traffic. it may be possible to adjust
the timetable to ensure that delays and the costs of running trains are properly
balanced. This question is taken up in later chapters.

Various modifications of the method of operation outlined above are used.
One of these is "maintracking." If sufficient traffic from A is going to K
or beyond, it may be a sound policy to allow traffic to accumulate at A for
a train to be dispatched from there consisting entirely of this type of traffic.
Such a train need not have any work done on it at J; indeed it can pass by
J on the main line. But such a policy may cause traffic for J or B to
be delayed longer at A, or trains from A to be run shorter. It may also
have the same effect on traffic leaving J. Thus the question of whether or
not to run a maintracker can be a difficult one. In practice there may be a
quite intricate pattern of maintrackers, some trains, for example, running

straight from A to C, others from J to C, others from A to K,
as well as trains running only between adjacent stations.

Preclassification is another frequently used device. It is primarily
a method of transferring the sorting work from one yard to another. Such a
transfer may be necessary if one yard gets overloaded, for example. so that
delays become too heavy. Or it may be useful if one yard is better equipped
than another, so that sorting work can more easily be carried out there. The
idea of preclassification can again be explained in terms of Figure 1. Suppose
a train from A arriving at J contains traffic for all the other stations.
At J it will have to be broken up and sorted. One train is to leave for
B, and another, containing traffic for C, K and D is to leave for K.
It would be possible to leave the actual sorting out of cars for C, K and
D from each other until the train reached K. However, the yard J may do
this work for the yard at K. In that case the train from J to K will
have all the cars for J grouped together, all the cars for C grouped
together and all the cars for D grouped together. It will then be completely
preclassified. The only sorting work left to the yard at K will be the
separation of the three groups associated with the three different destinations.
It is also possible to preclassify the train less completely. It may, for
example, be decided to sort out only the traffic bound for K at J and
leave traffic for C and D mixed up, to be finally sorted out when the
train gets to K. In later chapters an attempt is made to develop a rational
approach to the problem of how much preclassification should be done in some
situations.

7.2 The Classification Policy of a Yard

In the last section the main functions of a classification yard were described as the sorting of cars into different groups, the storage of cars up to the time of departure, and the making up of trains. Clearly there are many different ways in which these functions can be carried out. Before entering into a discussion of the advantages and disadvantages of these different methods, it is necessary to have some language in which to discuss them. In the rest of this chapter we shall try to develop this terminology; having done this we shall devote some time to describing some instances of its use.

"In the early days of railroading", according to E. W. Coughlin. "freight trains generally were made up of any cars on hand for movement to or beyond the next terminal, without regard to final destination or class of freight. Thus each successive terminal or yard switched out from their chance locations in each arriving train the cars belonging in its environs as well as those cars for local stations short of the next terminal..... The balance, plus such loads and empties as were 'made' at the terminal or from incoming local movements, were let go in the same chance order into succeeding trains run to the next terminal."[1] If we add to this description some such rule as "A train will be dispatched when 85 cars are available", then we have one instance of what we shall call a classification policy.

More generally, a classification policy is a set of rules, for each yard, which completely specifies the manner in which incoming traffic is to be reconstituted into outbound trains. This set of rules need not be explicit, nor indeed even consciously in the minds of the men responsible for the various

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[1] Coughlin, [1952, p. 44].

operations involved. The formal itemizing of rules may be more a theoretical than a practical necessity. In setting out to construct a model of yard operations, we cannot rely as the president of a railroad does on the expert judgments of yardmasters, division superintendents, dispatchers, etc.; these must be supplanted in our theory by formal rules whose consequences can be studied by logical and mathematical means. This is not to say of course that where in practice such formal rules are found to be lacking, things would necessarily be improved by the application of the same simple rules. Our theory, to be useful, must be a great deal simpler than its real counterpart and while it may be possible to formulate theoretical rules which are in some sense "good", the same formalization in practice may prove to be difficult. In such a case a combination both of "formal" rules and "informal" judgment will always be used.

Our problem here is to determine what alternative sets of rules adequately describe the range of policies available to a railroad in its conduct of yard operations. We cannot of course say anything as yet about what makes one set of rules better or worse than another; these policy choices will be based on the cost considerations studied in later chapters. At this point we only wish to point out the different questions relating to methods of operation to which any classification policy, good or bad, must give definite answers.

7.3 Typical Yard Operations

Some of these questions will be made clear by a brief examination of the operations in a typical classification yard. Although yards differ in many important respects, the classification operations carried on in all of them are basically the same. Usually a yard is made up of several sub-yards, each of which has a track pattern essentially like that in Figure 2, where

the arrows indicate the direction of traffic through the yard. The picture abstracts from several important differences between yards, but for the
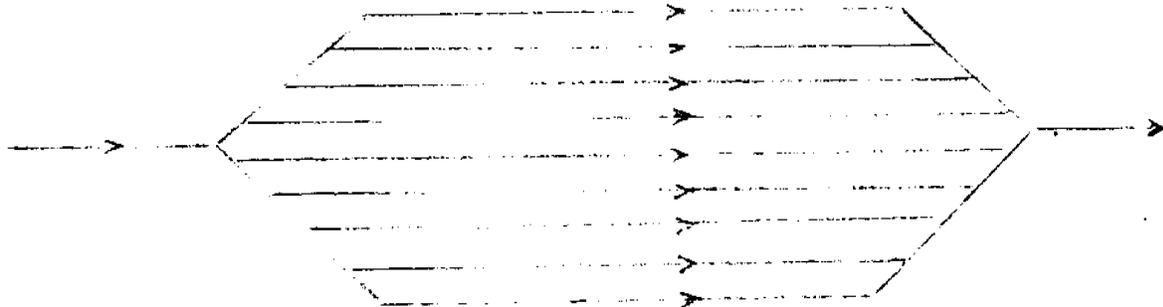
Figure 2. The Basic Track Pattern of a Sub-Yard

moment it will serve our purposes. Suppose now we consider a yard handling only one direction of traffic and consisting of two sub-yards as in Figure 3. The first sub-yard the traffic enters is usually called the receiving yard or receiving tracks; the second, the classification yard[2] or classification
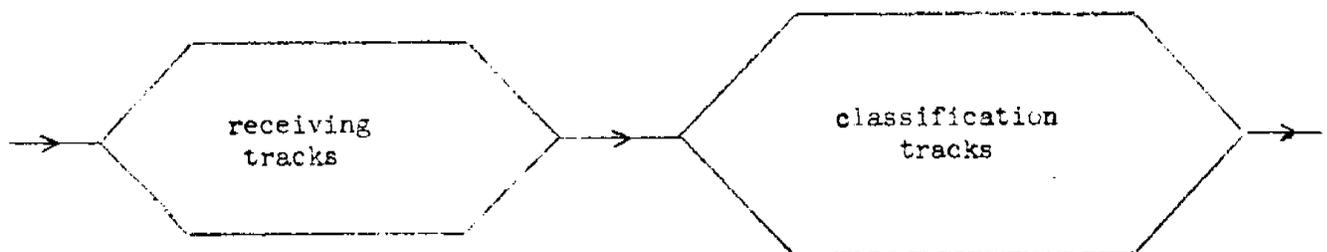
Figure 3

2/ We shall usually avoid this term, since it is the same as the term for the yard as a whole. The word sub-yard is useful but, so far as we know, not popular.

tracks. Incoming trains entering Figure 3 from the left are run as a whole into one or another of the receiving tracks to await sorting. Since only one string of cars can be sorted at a time these receiving tracks are sometimes necessary as storage space for trains that arrive while another is being sorted.

When its turn comes for sorting, or classification, this trainload of cars is pushed out of the receiving track, through the neck of the yard between the two sub-yards, and by various means each car of the train is made to run into the classification track to which it is assigned. Usually these classification tracks will already have other cars standing on them, to which the cars of the train we are examining will couple automatically. When the classification process is finished the cars which at the start were all on one track in the receiving yard will generally be scattered over several of the classification tracks. The choice of the classification track into which a particular car is moved will usually be made on the basis of the car's destination and perhaps its contents, or condition, the idea being to group similar traffic for departure in outbound trains and to separate out cars needing attention.

As this sorting procedure is being carried on in a fairly continuous manner at one end (the left in Figure 3) of the classification tracks, the process of making up outbound trains is also carried on at the other end of the classification tracks. This make-up process consists in drawing the strings of cars out of specified classification tracks and assembling them into trains.

This description of the classification operation leaves out many yard activities that will require our attention at a later point. However, such things as the inspection of cars and the re-icing of refrigerator cars are

not essential parts of the classification process even though, for reasons

of convenience, they are usually carried out at the same time and place.

The description therefore brings out most of the important questions which

a classification policy must answer.

7.4 The Requisites of a Policy

7.4.1 Receiving

The first question is one of priority. If the sorting facilities are

prepared to receive another string of cars, and if there are several trains

waiting on the receiving tracks, then which should be handled first? In very

small yards this problem may never arise since only one train at a time will

be available. In the usual case however, the arrivals of incoming traffic

are not always spaced out evenly over the day and it is quite likely that

at some moment, after a period of bunched arrivals, several trains will be

standing on the receiving tracks while an earlier train is being sorted. In

some yards the problem is more complicated. If the receiving tracks are not

long enough to accommodate the full length of an incoming train, then this

train will have to be broken up into two or three (usually not more) pieces,

and each put onto a separate receiving track. Often too, it is desired to

give high priority to one part of a train and low priority to the other, in

which case the question of just how to break up the train for storage on the

receiving tracks creates another policy problem. The rule or set of rules

dealing with these problems we can designate as the yard's receiving policy,

which is part of its general classification policy. The simplest possible

receiving policy is exemplified by a "first come, first served" rule. This

is the one we shall usually assume to exist, since this element of policy

is not the one we shall find most interesting.

7.4.2 Grouping

The second policy question has to do with the sorting of cars. Into which of the classification tracks is any given car to be switched? Since a given classification track is at some yards used for different kinds of cars at different times, the question is perhaps better phrased as, to which _classification_ does a car belong? The latter question is substantially equivalent to the former since at any given time a single classification track is devoted to the cars of only one classification. The practice in larger yards, especially those with an ample number of classification tracks, is to assign one or more tracks to each classification on a permanent basis. Since the number of classifications is a matter of policy, it becomes a maxim of yard operation that the number of classifications should not exceed the number of classification tracks. Departures from this rule are however not uncommon, and if it ever happens that cars of all these classifications are present at the same time, then extra switching is called for.

Destination, contents, type and condition of a car are the most common bases of its classification. Thus among the classifications used by the Southern Pacific's Taylor Yard at Los Angeles we find:

Track	2	Fresno to and including Planehaven
	3	Edison to but not including Fresno
	4	San Pedro -- Los Angeles Harbor
	.	.
	.	.
	7	King City to but not including San Jose
	8	San Luis Obispo to but not including King City
	9	Colton to but not including Indio
	10	All points Roseville and beyond including Ogden
	.	.
	12	Weighers
	.	.
	20	Commodity cars to clean
	21	39th St. to and including Firestone Park

Track	22	Empty gondolas, etc.
	.	.
	.	.
	24	Dirty reefers
	25	Santa Fe R.R.
	26	Union Pacific R.R.
	.	.
	.	.
	38	Icers
	39	Bad order empties

Henceforth we refer to the list of classifications as the grouping policy, to signify that cars in a given classification are to be grouped in outbound trains.

7.4.3 Make-Up

The third policy question refers to the make-up of trains. Let us identify each outbound train by some code name, such as "No. 97", etc., and let us suppose that we know its time of departure. Then, calling the string of cars standing on any one classification track a group, we ask what groups are to go into any given train. If traffic conditions vary from day to day, then the exact number of cars in any one group at any particular moment will not be known in advance. The total of cars in the groups designated to be made up into "No. 97" may therefore be sometimes more than the train can haul; at other times the number may be so small as to render the movement of this train quite unprofitable. Although these matters somewhat encroach upon the discussion of later chapters. their mention is necessary here, because the make-up policy described above does not, as it stands, facilitate consideration of train length. One way of avoiding this difficulty is to define the make-up policy to be a set of numbered lists, each one of which ranks the groups that are to go into some outbound train. Each list is to be interpreted as an assignment of priorities to the groups of cars on different classification tracks. An outbound train will be filled by first taking

the group at the head of that train's make-up list, next the group second in the list, and so on until the length of train thought in the circumstances to be best has been reached.

7.4.4 Scheduling

Before saying any more about make-up, let us go on to the fourth and last policy question we shall examine. This pertains to the scheduling of outbound trains. The scheduling policy for a yard may consist simply of a list of trains with a departure time and destination for each, or it may be somewhat more complicated. If the make-up policy is not of the priority-list type then the scheduling must be of a somewhat different form in order to avoid the train-length troubles mentioned earlier. Suppose train "No. 97" is to be made up of groups A, B, and C; all of the cars in these three groups are to go in "No. 97", and no other cars are to be allowed in the same train. If traffic in these groups is quite variable from day to day, so that at 10 A.M., say, on one day there are 200 cars for "No. 97" and on another day only 25, then the schedule for "No. 97" must take one of the following forms:[3]

"No. 97 will depart when (say) 100 cars are available".

or perhaps

"No. 97 will depart at 10 AM if at least 100 cars are available at that time,
     " 11 AM if "   "   80   "   "   "   "   "   " ,
     " 12 AM if "   "   60   "   "   "   "   "   " ,
     "   1 PM if "   "   40   "   "   "   "   "   " ,
     "   2 PM if "   "   20   "   "   "   "   "   " .

---

[3] The corresponding rules in actual practice will not of course be capable of such simple representation. It is felt however that the scheduling rules presented here do take into account the more important considerations upon which practical decisions on this matter are made.

This last rule lends itself very conveniently to a graphical representation. The curve DD' in Figure 4 represents this rule with the addition that points of time intermediate between 10 AM and 11 AM, for example, are also assigned numbers of cars between 100 and 80, etc. The curve EE', representing the availability experience of one particular day, shows how may cars of groups A, B, and C, taken together, have accumulated on the classification tracks at any one time. On this day, according to the scheduling rule, "No. 97" will depart at about 11:50 AM with about 64 cars, these figures being the coordinates of the intersection of curves EE' and DD'.
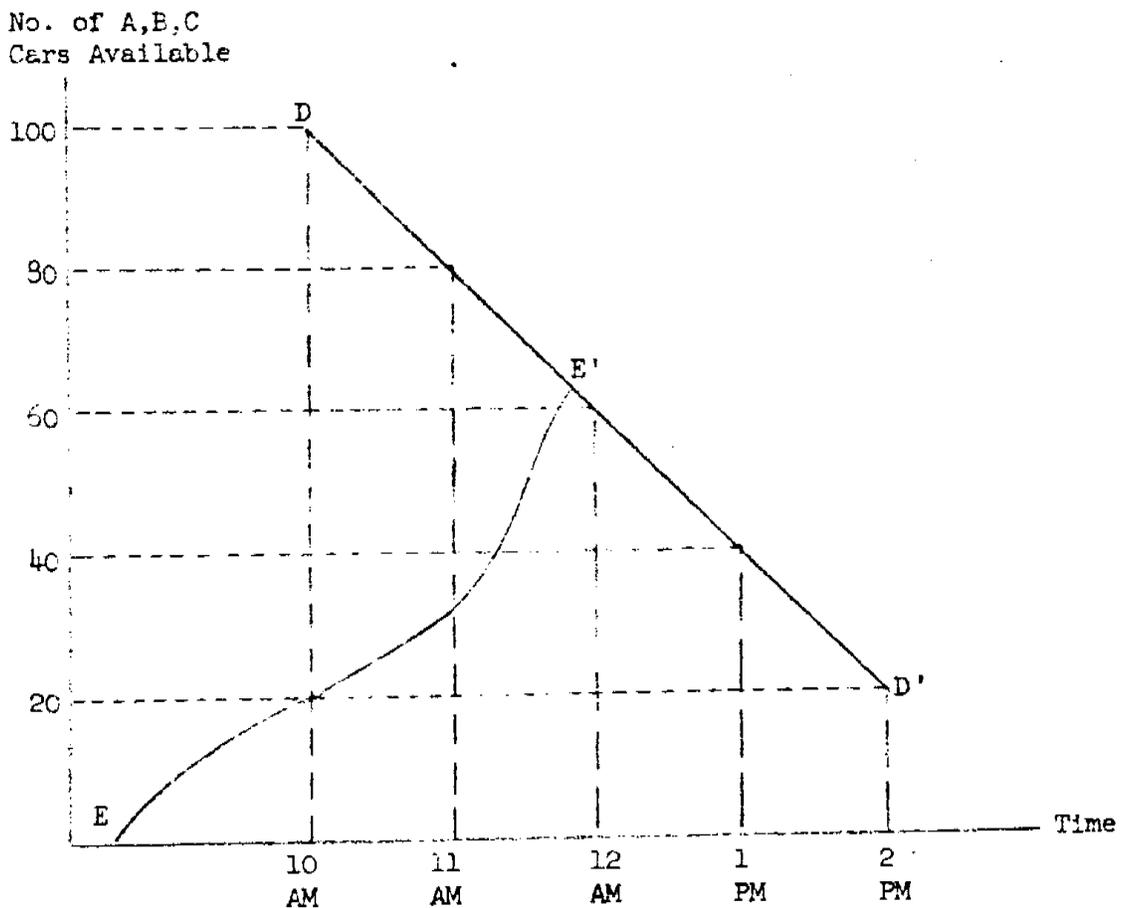

Figure 4. The Graph of a Departure Schedule

The important point in this discussion is not why the curve DD' should have such a slope, but only that it is possible to insure against absurdly long trains (by making the schedule of departure times dependent on the numbers of cars of the proper kinds which have accumulated), and at the same time not completely abandon the timetable type of scheduling. Notice in Figure 4 that if the curve DD' were vertical we should have a pure timetable schedule; if DD' were horizontal we should have what might be called a "leave when filled" type of scheduling. The curve as drawn represents a scheduling policy for "No. 97" which is intermediate between these two extremes. It must be emphasized of course that a schedule can be represented sensibly by a curve only when the make-up policy for the train specifies certain groups of cars all of which must go on the train. The make-up policy that assigns priorities to different groups is not of this kind. Make-up and scheduling policies which represent mixtures of the two variants we have described are conceivable and probably can be found in practice, but we shall not attempt to discuss them.

Of the four components of classification policy discussed, we shall be concerned mainly with the last three: grouping, make-up, and scheduling. Briefly, these tell us, respectively, what cars go into what groups, what groups go into what trains, and when each train departs.

7.5 Some Examples

To return to Coughlin's description of early railroading practices, we can now restate in our new terminology the classification policy used there. In Figure 5, let the points denoted by capital letters be what Coughlin refers to as terminals; at each, we shall assume, there exists a classification yard. Let the points denoted by lower case letters be local stations. At Yard A the grouping will be
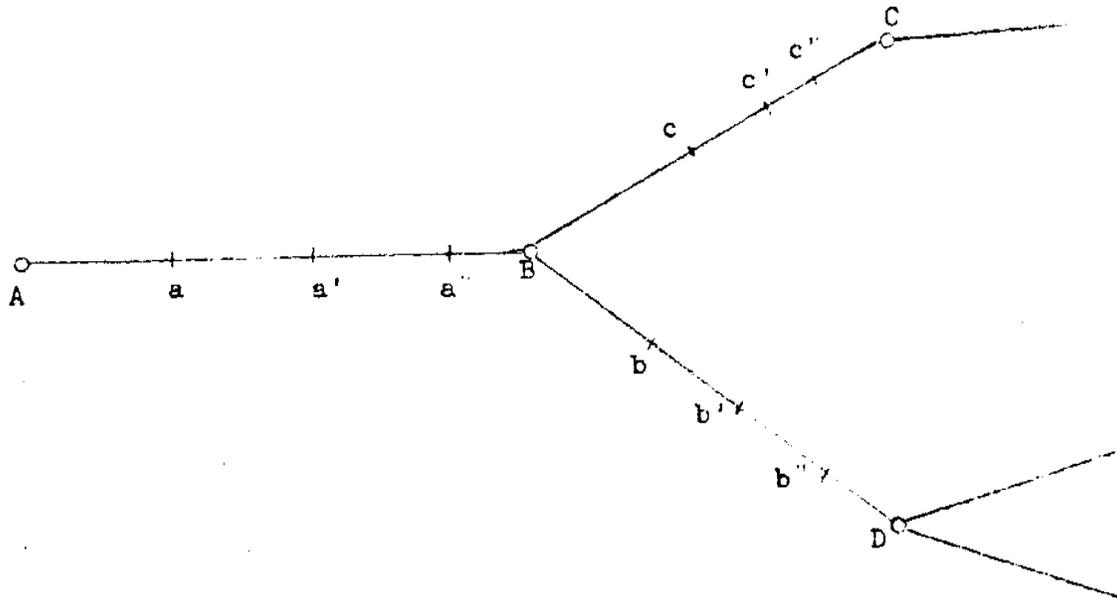
Figure 5. Classifications in a Rail Network

      Cars for a, a', a"

      All other cars.

At Yard B the grouping will be

      Cars for b. b', b"

      Cars for D and beyond

      Cars for c, c', c"

      Cars for C and beyond

The make-up in both cases is particularly simple. Each outbound train will consist of one group only, local trains from A carrying only the single group [a, a', a"] and through trains carrying only the single group [all other cars].

    At B, one local train will be dispatched with the single group [b, b',b"], another local with the group [c, c', c"], a through train with the group [D and beyond], and another through train with the group [C and beyond].

Nothing explicit was said in the example about scheduling.

Another example is provided by a scheme proposed by M. L. Byers in 1908.[4] Interesting in itself, it fits in here as the plan next in order of simplicity to the one described by Coughlin. Since we shall have occasion to refer to Byers' plan again in Chapter 9 we shall quote it at length. The reader should be aware that Byers uses the word "cut" where we have used "group"; we shall reserve the former term for a somewhat different use. The plan is as follows:

1. Normally, each yard will classify through trains into two cuts each, viz.:

    (a) Cars for the first yard in advance, including cars for points between the first and second advance yards and which consequently, are not handled in through trains beyond the first advance yard.

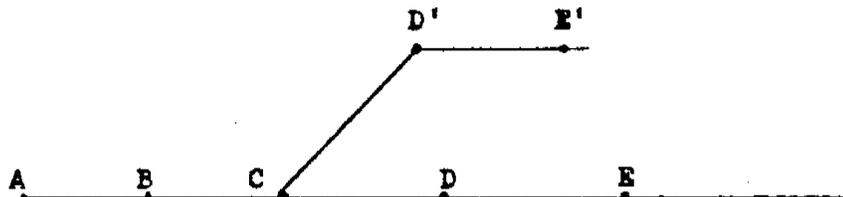   (b) All other cars

Figure 6. Showing Locations of Yards

Example. -- Yard A (Figure 6) will make up through train for yard B with all cars for Yard B (including local cars for points between B and C exclusive) in one cut, and all cars for Yard C and beyond in a second cut.

Yard B will receive this train, remove the first cut and classify the second cut into:

4/   Byers [1908, pp. 539-540].

A₁. Cars for Yard D and local beyond D;

A₂. Cars for Yard E and beyond.

It will add any cars held in the yard for these destinations to these cuts, and the train is then ready to proceed to Yard D.

In like manner, it will classify other cars of the second cut into:

B₁. Cars for Yard D' and local beyond D';

B₂. Cars for Yard E' and beyond.

It will add any cars held in the yard for these destinations and the branch train is then ready to proceed. Then, at its leisure, it will dispose of the cut set off and switch the cars into the locals, or to points in the yard as the destinations of the cars may indicate.

2. If any yard, after performing the normal work above indicated, has available yard-engine time before the departure time of the through train, it will perform such of the work of the next succeeding yard as may be done without delay or increased cost.

Example. -- Yard A, having completed its work in time to do further switching on the through train for B, will separate the second cut (for points beyond C) into:

A. -- Cars for yard D and beyond;

B. -- Cars for yard D' and beyond.

If further time is available, it will separate cut A into cuts A₁ and A₂, and cut B into cuts B₁ and B₂, before mentioned, thus reducing the delay at Yard C.

In spite of the omissions with respect to the disposition of cars bound for C and local points beyond C, the general principle of classification is clear. If we list, one on a line, the groups formed at a yard, and if we use brackets to indicate the groups that are made up together into trains, then Byers' policies for the various yards can be written as follows:

For A   { local beyond A   }

{ B and local beyond B
C and beyond }

For B$^{5/}$ { local beyond B   }

{ C and local beyond C
D and beyond }

{ D' and beyond }

For C   { local between C and D }

{ local between C and D' }

{ D and local beyond D
E and beyond }

{ D' and local beyond D'
E' and beyond }

et cetera.

If a yard has spare time at its disposal its policy is, according to Byers, to be modified. The first such modification for Yard A is described as follows:

{ local beyond A }

{ B and local beyond B
C and local beyond C
D and beyond
D' and beyond }

The second modification, to be carried out if A still has time available, is:

5/ We have altered the grouping at B so as both to bring in the C traffic which Byers inadvertently omitted, and at the same time make B's policy conform to the principle implied. The C traffic could of course equally well have been put on the third train.

$$\begin{cases} \text{local beyond} \quad A \end{cases}$$

$$\begin{cases} B \quad \text{and local beyond} \quad B \\ C \quad \text{and local beyond} \quad C \\ D \quad \text{and local beyond} \quad D \\ E \quad \text{and beyond} \\ D' \quad \text{and local beyond} \quad D' \\ E' \quad \text{and beyond} \end{cases}$$

et cetera.

Both of the above examples are of course extremely simple ones. The classification policies in actual use today are much more complicated. While each yard's policy is still quite capable of being expressed in our terminology, it is true that the policies of a system of yards cannot be easily expressed in a general form as we have done with the plan of Byers. Indeed, there is little reason to expect such a uniform pattern of policies. For an example of a system of policies the reader is referred to the Baltimore and Ohio's Working Book No. 19, which devotes a page to the grouping, make-up, and scheduling of each train.

_/ Baltimore and Ohio Railroad Company [1951].

Chapter 8

A Single Classification Yard

In this chapter we shall attempt to give some more precise meaning to
the concept of the "work" involved in classification operations. For the
moment we shall confine our discussion to the workings of a single yard in
isolation; we shall ask how costs at this yard depend on the yard's physical
facilities, on the characteristics of the traffic handled, and on the classi-
fication policy. Cost will be thought of as being made up of two components,
money cost and time cost, where the latter represents the delays suffered by
cars and freight in undergoing the classification process. Most of the de-
scription will be of a qualitative rather than quantitative nature; that is
to say, we shall usually have to be satisfied in knowing whether costs go up
or go down with certain changes in yard facilities, traffic, or policy. The
exact extent of these changes, except in a few cases, must remain a subject
for further inquiry. It should also be emphasized that the relationships we
shall describe are tentative ones, open to criticism and perhaps refutation
by subsequent studies of a similar nature. While it is felt that most railroad
men would be in agreement with each one of the cost relationships taken singly,
many different opinions can undoubtedly be found on the relative importance of
each.

To begin with it is convenient to break down yard costs into three cate-
gories -- the cost of handling, the cost of accumulation, and the cost of
congestion. Under the title handling are included all those yard operations
having to do with the movement and servicing of cars in the yard. Handling
costs will obviously contain both money costs, in the form of switch-engine
and labor costs, and time costs corresponding to the periods of time necessary

to carry out the classification operations. <u>Accumulation</u> and <u>congestion</u> costs are composed entirely of time costs. The first of these, which we shall usually refer to as accumulation delay, pertains to the delay suffered by a car in waiting either for its scheduled departure or for a train length of the "right" kind of traffic to accumulate in the yard. What kind of traffic is "right" and what the train length shall be will of course be specified by the make-up and scheduling policies of the yard. The reason for the name we have given to this element of delay is clear in the case where the scheduling policy is of the "leave-when-filled" type but not so clear when it is of the timetable type. As soon, however, as we realize that one of the ruling considerations in the selection of a particular time-table is the average train length to which it leads, then we see that the delay involved in waiting for a timetable departure can also be considered a waiting for cars to accumulate.

Congestion delays refer to the periods during which a car must stand idle as it waits to be serviced because the handling of cars serviced ahead of it is not yet finished. It should be pointed out that this definition excludes some delays that are also caused by congestion, such as those brought about by a possible slowing down of handling operations in times of heavy traffic. These latter delays (if they exist) would in our terminology still be classed as handling costs.

We shall now enter into a more detailed discussion of each of these costs.

8.1 Handling Cost

It is perhaps simplest first to discuss handling cost entirely in terms of its time-cost component. We shall usually find in fact that both the time- and the money-cost components respond in the same way to changes in yard facilities, policy, and traffic. On the important question of the degree to which, under different circumstances, the two types of cost are substitutable, something will be said at a later point.

Following the discussion of yard operations in the preceding chapter we shall consider separately the delays arising in each of the several handling operations -- receiving, sorting, and make-up.

8.1.1 In the Receiving Operation

If, in the receiving operation, an incoming train is broken up into two or more pieces to be stored on the receiving tracks, the delays involved are probably greater than when the whole train is treated in a single section. More important at this point of the operation are the delays brought about by the inspection process. It was stated earlier that the condition of a car was one of the classification criteria. The inspection of cars to determine their condition must therefore precede this sorting operation. If a car's need of immediate repair is only discovered after it has been sorted, extra switching will be necessary to remove it from the group with which it has been placed. Such switching will very likely have to be performed at a part of the yard designed for other work. Unless this work can be done at a slack period, the result is delay, not only to the car in question, but also to the cars coupled to this bad-order car. The inspection therefore is usually carried out in the receiving yard.

A small crew of men goes through a routine checklist inspection of each car in the train. The list includes such things as couplings, journals, and brakes, and is quite often very detailed. The time needed to finish the inspection of a train depends partly of its length, of course, but it also may depend on the number of men available for inspection at any one moment. Large classification yards have as many as 100 men, divided among three shifts, who devote full time to inspection. This means that during slack periods the process can be considerably speeded up. In order that the reader might have some idea of the length of this inspection period, we have plotted in Figure 1 inspection times (vertical axis) against train lengths (horizontal axis) for one particular day's traffic through a large classification yard. The preponderance of short "train" lengths is explained when it is recalled that quite often sections of a train are treated separately. While the scatter of the points in Figure 1 is large, perhaps reflecting variation in the size of the inspection crew, there does seem to be some indication that, on the average, the time required to inspect a train increases with train length. If inspection time per car is, on the average, a constant independent of train length, as seems most likely, it is still true that the inspection delays incurred by cars will depend on the lengths of the trains (or sections of trains) in which they are stored in the receiving tracks. This is so because no single car can be moved until the inspection of the whole string of cars to which it is coupled has been completed. This is perhaps a roundabout way of stating that the inspection delay attaches to the train (or section of train) rather than to the car. However, since the train loses its identity in the sorting process (its cars may end up in several other trains), we prefer to use the car rather than the train as our base.
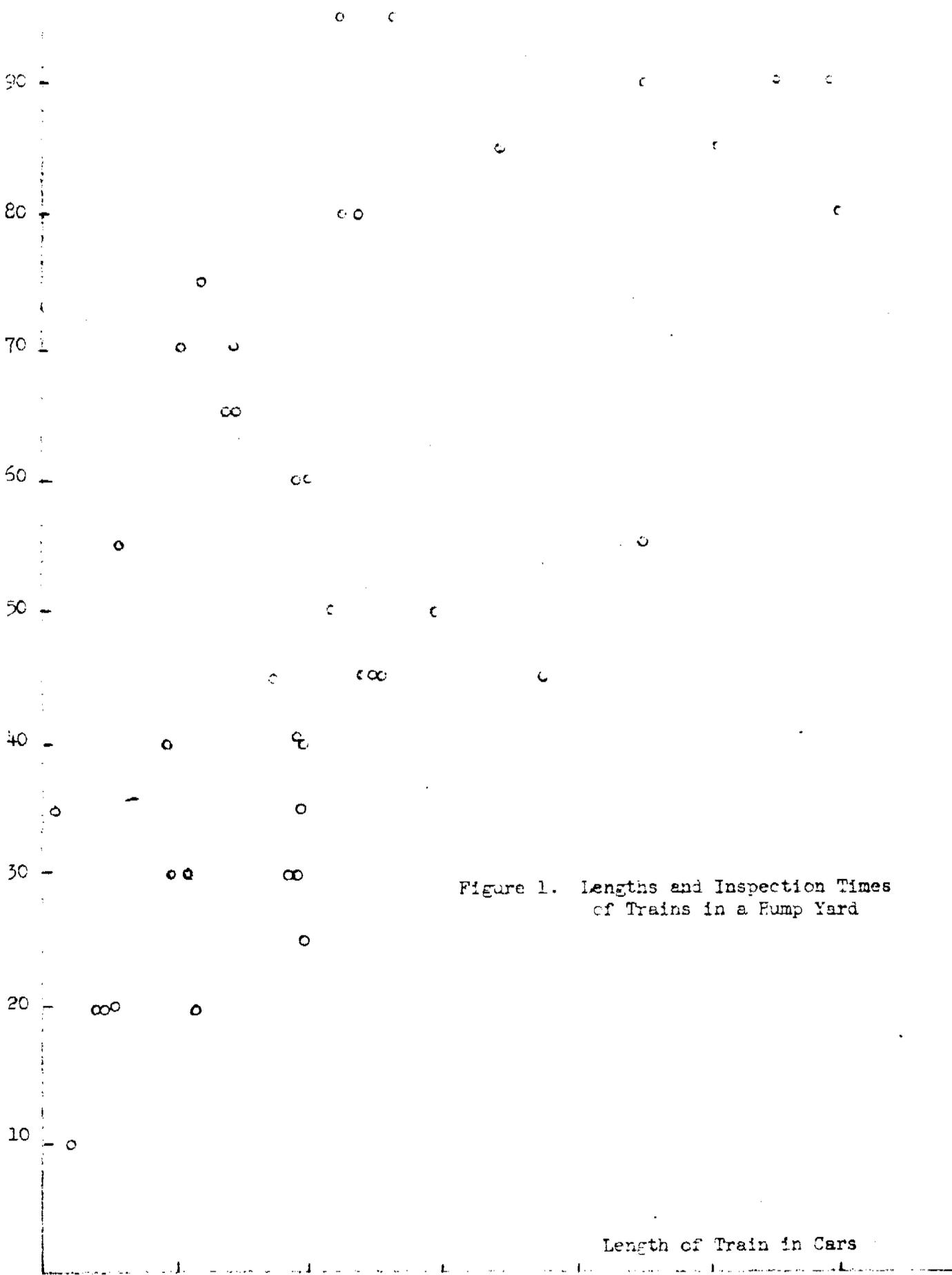
Inspection Time
in Minutes

Figure 1. Lengths and Inspection Times
of Trains in a Hump Yard

Length of Train in Cars

Clearly then the average inspection delay suffered by cars will vary with yard facilities and with receiving policy. The shorter the sections into which incoming trains are broken up the less will be the average inspection delay of a car. Such savings are offset however by the fact, previously mentioned, that a train is more quickly moved in and out of the receiving tracks when it is handled in fewer sections. While the question of how receiving delays are affected by receiving policy is an interesting one, we shall not pursue it further here.

8.1.2 In the Sorting Operation

The handling delays arising in the sorting[1]/operation are of special interest to us because it is here that the possibility of shifting work from one yard to another is most in evidence. According to the means by which they carry out the sorting procedure we shall distinguish two main types of classification yards -- the flat yard and the hump yard. In order to explain the difference, the diagram of the classification track pattern will be slightly elaborated for the two cases.

In Figure 2 we see a typical pattern of classification tracks for a flat yard together with the leads to and from these tracks. To sort a train of cars a switch engine pushes the train in along the entering "lead track" from the receiving yard. From a previously prepared "switch list" the members of the switching crew know into which classification track each car is to go. Cut is the name given to each of the parts into which a train is cut up in the sorting operation; thus a cut is a group of successive cars bound for

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1/   To describe the operation of switching cars into the classification tracks the term "sorting" is used rather than the more commonly used "classification" in order to avoid confusion with the whole yard operation, for which the latter term is reserved.
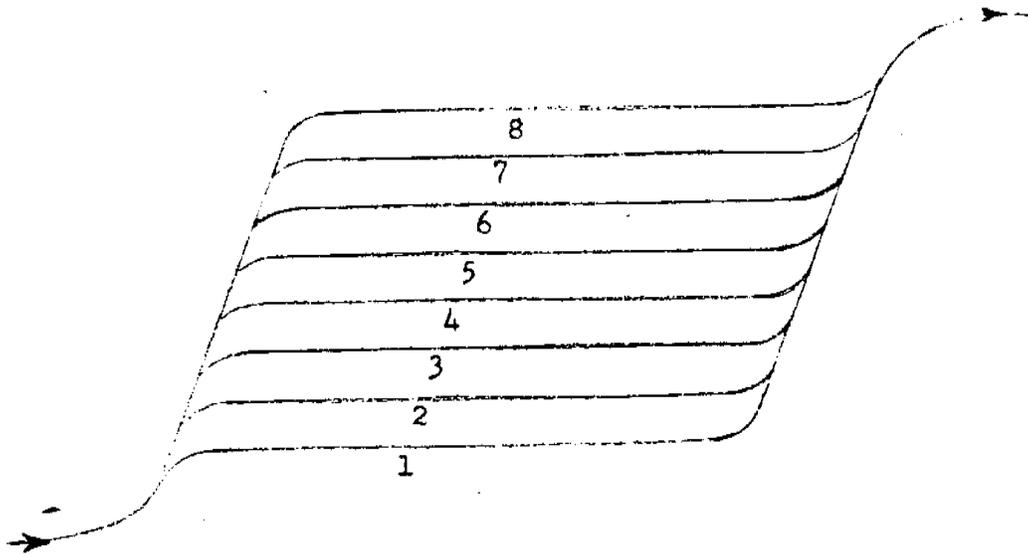
Figure 2.   A Typical Classification-Track Pattern for a
Flat Yard

the same classification track. The switch list therefore defines the cuts

to be made. Let us say that the first two cars of the train are to go into

Track 4; cars 1 and 2, that is to say, constitute the first cut. As the

string of cars moves toward the switch leading into Track 4 one man pulls

the pin that locks the coupling between the second and third car. A switch-

man throws the switch so as to direct the pair of cars into Track 4, and

the engineman brakes the train sharply, thus "kicking" this cut into the

proper track. These two cars roll down Track 4 until they lose momentum,

couple with other cars already standing there, or in some cases are

stopped by a brakeman on the cars. Meanwhile the switchman restores the

switch to Track 4 and sets the switch for the track appropriate to the next

car or cars. The pin-puller separates this next "cut" of cars, and the

engine brakes again, and so it continues.

An experienced crew can perform this operation much more smoothly than it can be described. One who has never watched it would suppose that, after each kick, the engine would have to back up with its string of remaining cars to repeat the process. However this is by no means always the case. Quite often several cuts can be kicked into their appropriate classification tracks without ever completely halting the forward progress of the engine. The only requirement is that there be sufficient space between the cuts to enable the switchman to reach and operate the proper switches. Suppose three successive cuts are destined for Tracks 6, 7 and 4 respectively (Figure 2). The second cut must be far enough behind the first to give the switchman time to restore the switch to Track 6 after the first cut has passed over it and then set the Track 7 switch if he has not already done so. He must then be able to reach the Track 4 switch before the third cut does. Unless the train is very short, backing up will often be necessary, for the string of cars will eventually have passed beyond the turnouts for classification tracks still to be used.
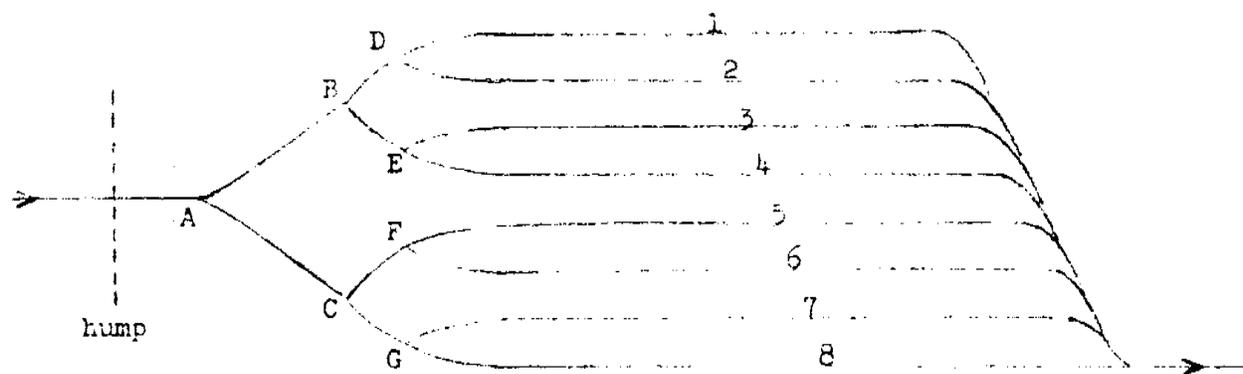

Figure 3.   A Typical Classification-Track Pattern For a Hump Yard

In Figure 3 a typical classification-track pattern of a hump (or
gravity) yard is shown. It will be noticed that the right-hand half of
the diagram is no different from that for a flat yard. The essential dif-
ference between the two types of yard is to be found in the "hump", the
crest of which is indicated in Figure 3 by the dotted line. The track from
the receiving yard slopes gradually up the left side of this hump, and drops
off rather sharply on the right side. The downslope continues through the
switches A, B, C, etc., and, almost imperceptibly, well into the body of
classification tracks. The crest of the hump is usually 12 to 20 feet above
the level. The grade of the tracks varies from as much as 4 percent just
beyond the hump to about 0.2 percent in the classification tracks.[2]

In sorting, or in this case "humping", a train, the switch engine pushes
it up the left-hand side of the hump at about walking speed. Just before a
cut of cars passes the hump a pin-puller unlocks the coupling between this
cut and the following one. As soon, then, as the cut is pushed over the hump
it breaks away from the rest of the train and starts rolling down the sharp
grade toward the switch A in Figure 3. The switches A, B, and E are set so
as to direct this first cut into, let us say, Track 4. Meanwhile the
engine continues pushing the train up the hump, so that perhaps at the
moment the first cut is moving through switch E, the second cut is already
passing over switch C, and the third over switch A. It is not at all un-
usual to see three or four cuts rolling down from the hump simultaneously
(each, of course, a little behind its predecessor), for the whole operation
is a very rapid and continuous one. Except for unusual events, such as
jammed couplings or cars requiring special handling, the movement of the
train over the hump takes place at an almost constant speed.

2/   American Railway Engineering Association [1952, p. 301].

In the earliest hump yards the switches were thrown manually, as in
a flat yard, and the cars were braked as they rolled off the hump and into
the classification tracks by men called "riders", who rode the cars and
operated the hand brakes. Today most hump yards are equipped with elec-
trically controlled switches, operated from desks in control towers that
give their operators a good view of car movements. Braking is also auto-
matically operated from control towers by means of electrically or pneumat-
ically powered "car retarders". These retarders consist of cast-iron shoes,
one on each side of each track, which can be made to pinch the wheels of
cars as they pass over. In most yards a skilled operator, having information
on the weight of a cut of cars and the classification track it is headed for,
applies just enough pressure with the retarders to slow the cut to a speed
that will bring about a gentle coupling with the cars already standing on
that classification track. There are usually several retarders; in Figure 3
there would probably be one between the hump and switch A, another between
switches A and B, another between B and D, and others between successive
points. In some modern yards, retarders can be set to apply automatically
according to the speed and acceleration of the approaching cut.

In discussing the inspection process we assumed quite realistically,
that the inspection period of a car was not finished until the whole train
or section of train to which it was coupled had also been inspected. We
shall make exactly the same assumption with respect to the sorting process.
That is to say, we shall suppose for simplicity that nothing further can be
done with any single car until the whole string of cars containing it has
been sorted. Our first concern therefore is with the time necessary to sort
a train. Once we have this information we can return to the question of the
average delay suffered by a single car.

In both flat yards and hump yards we should expect sorting time to increase with the length of the train being sorted. This is borne out for the case of a hump yard by the observations plotted in Figure 4, where the vertical axis measures the time used to sort a train, and the horizontal axis measures the train length. Each point represents a single train or section of train. The scatter of the points cannot be ignored, but it is clear that train length has an important effect.

From the description it is quite obvious that, at least for a flat yard, the number of cuts in the train must also play an important part in determining the sorting time. In flat switching, the number of times the engine will have to back up will almost certainly be higher if the number of cuts is high. In hump sorting the influence of the number of cuts is less obvious and not nearly as great. Nevertheless the number of cuts has a significant effect. In Figure 5 we have plotted sorting time per car against cuts per car for the same set of observations as were plotted in Figure 4. Because the relationship is not obvious in this case, and because it seems quite reasonable that cuts might have no appreciable effect on sorting time in a hump yard, we have computed the regression of "minutes per car in sorting" on "cuts per car" and "length of train". The resulting equation is

$$(8.1) \qquad \frac{t}{n} = \underset{(.06662)}{.094249} + \underset{(.08407)}{.291214} \; \frac{c}{n} + \underset{(.0002811)}{.001306} \; n \; ,$$

where t is the time used in sorting a train, n is the length of the train in cars, c is the number of cuts into which the train is separated in the sorting process, and the figures in the parentheses are the standard errors of the coefficients above them. Since there were 108 observations,

Sorting Time
of Train
in Minutes

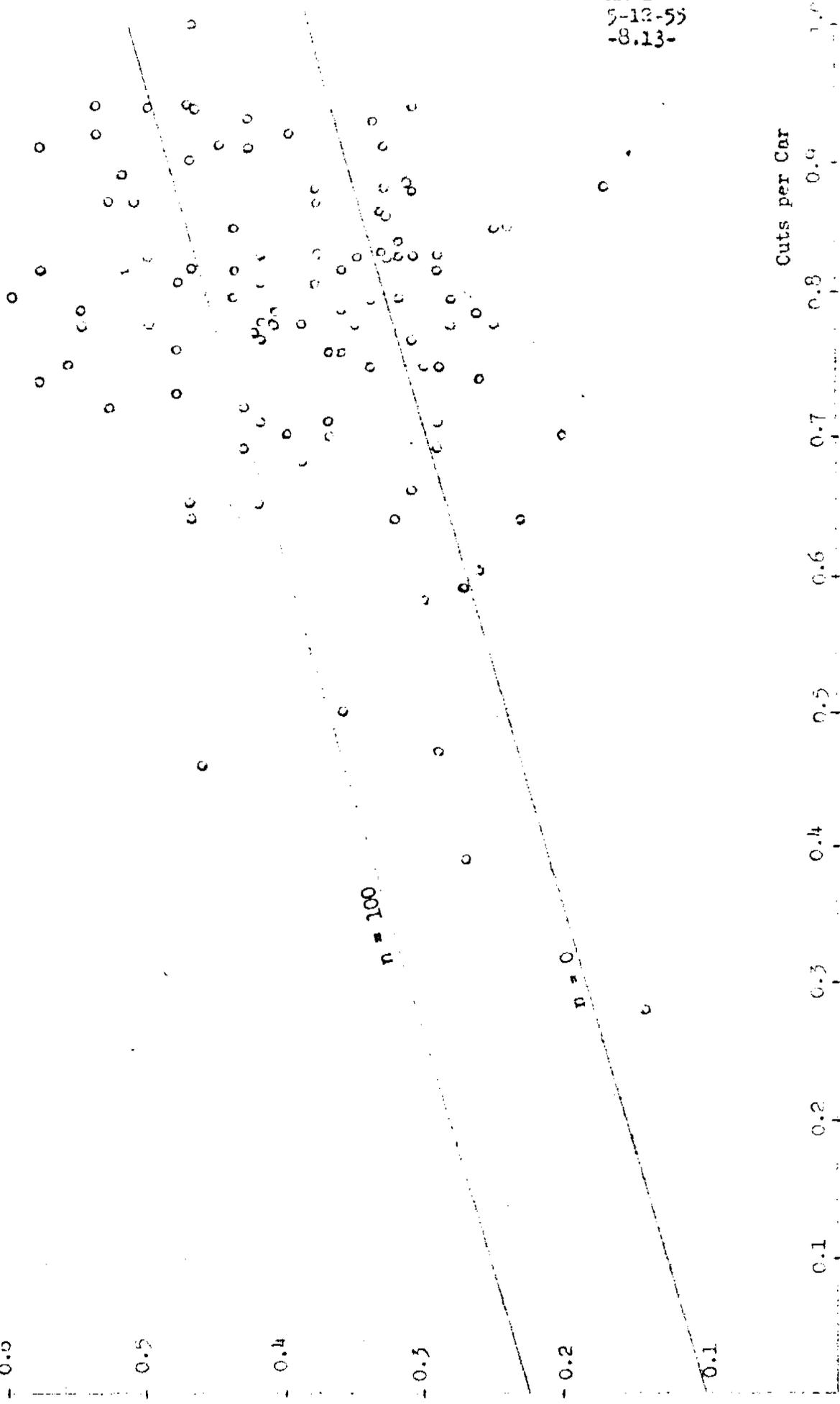Figure 4. Length of Train and Sorting
Time in a Hump Yard

1.0 cut/car

.5 cuts/car

Number of Cars in Train

Figure 5. Minutes per Car and Cuts per Car
in a Hump Yard
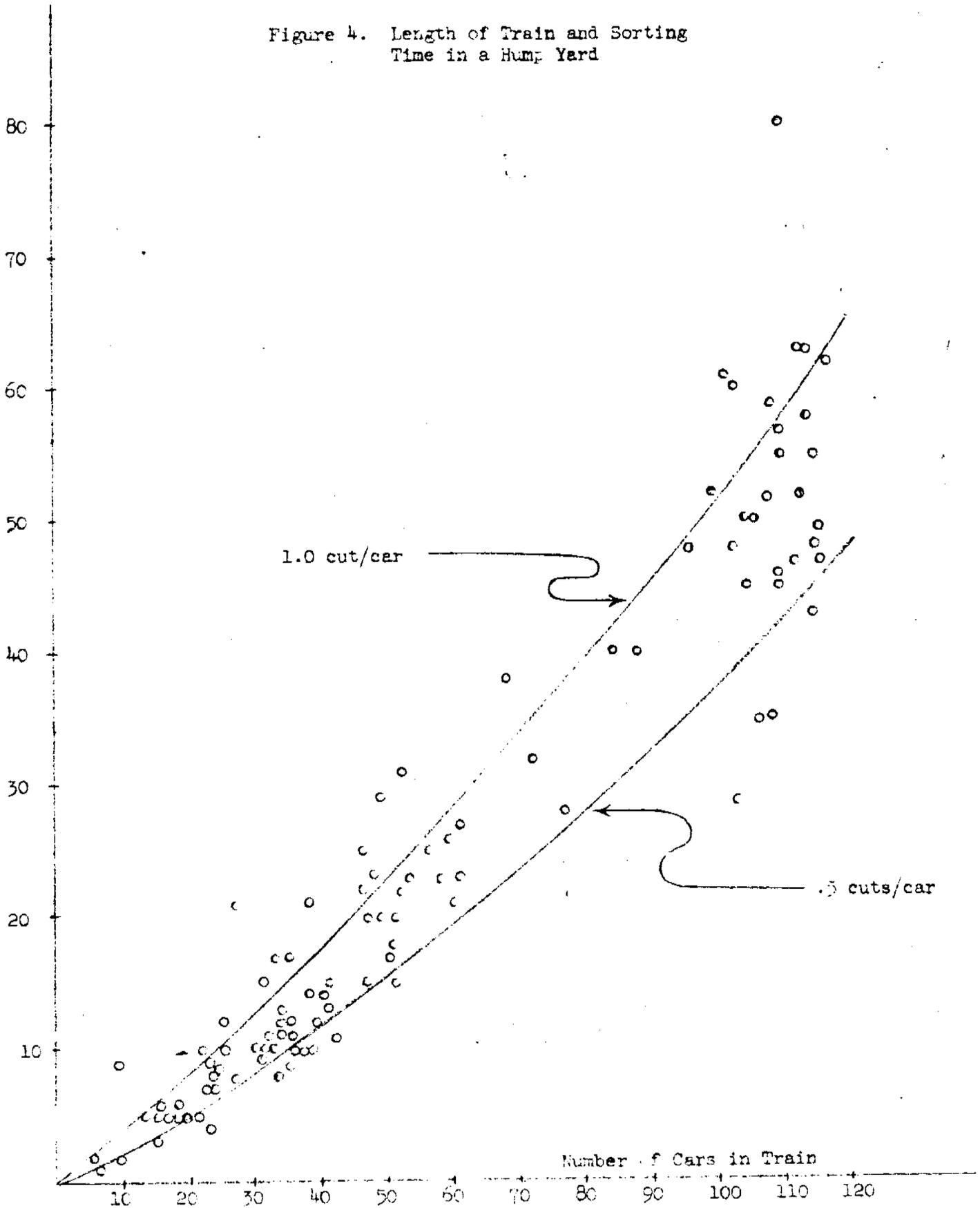
Sorting Time per Car
in Minutes

Cuts per Car

it is seen that all coefficients but the constant term are significantly different from zero. The equation is plotted in Figure 5 for $n = 0$ and $n = 100$. Sorting time per car for train lengths intermediate between zero and 100 can be found by linear interpolation if the cuts-per-car variable is taken to be fixed.

Multiplying through by n the equation (8.1) can be put in the form

$$(8.2) \qquad t = [.094249 + .291214 \frac{c}{n}] n + .001306 n^2$$

which is plotted in Figure 4 for $\frac{c}{n} = 1$ and $\frac{c}{n} = .5$. It must be emphasized at this point that the figure, "sorting time per car", used in Figure 5, is not the same thing as the sorting delay of a car, which, as we have said before is equivalent to the sorting time of the whole train.

Unfortunately we do not have statistics on flat-yard operation that would tell us how sorting time depends on cuts per car. It is clear however from more casual observation than that underlying Figure 5, and from the statements of railroad men, that, for a flat yard, the lines corresponding to those in Figure 5 would have a much sharper slope upward. This difference in the effect of the number of cuts per car is, in fact, the essential differ-ence between a flat yard and a hump yard. The latter is much better equipped to handle sorting when the number of cuts per car is high.

What makes this relationship interesting is the fact that the variable, cuts per car, is almost completely a matter of policy. In order to concentrate our attention on this effect of cuts alone, let us simplify the discussion by supposing that the lengths of trains as they are sorted are fixed. This need not mean that all trains are of the same length, but only that from day to day, let us suppose, the same pattern of train lengths keeps repeating

itself, so that, for instance, the twelfth train through the yard each day is always sorted in two sections of about 50 cars each. We can now examine the effect of grouping policy on sorting time through its influence on the number of cuts per car.

Other things equal, the number of cuts per car will be high if the list of classifications which constitutes the yard's grouping policy has many entries. On the other hand, cuts per car will be low if the grouping policy lists only two or three classifications. Any yard therefore, which wishes to minimize sorting delays incurred in it will maintain as coarse a classi-fication system as possible; local traffic will be sorted into one group, and traffic for the next terminal and beyond will be sorted into another -- nothing more. The minimization of sorting delays at each yard does not at first glance appear to be a foolish aim, and probably goes far toward explaining the early classification practices described by Coughlin.[5/] Even aside from the fact, however, that sorting delay is only one part of total delay, the minimization of sorting delay at each yard individually is not a wise goal. For it may be possible by way of a judicious choice of grouping policies at the various yards of a system to decrease the delay considerably at one yard by increasing the delay to a lesser degree at another; the aggregate result is then better than that achieved by individual minimiza-tion. The way this can take place is as follows.

We have described how the grouping policy influences the number of cuts per car, and how the latter in turn affects the sorting time differently in a flat yard and a hump yard. The third influence on sorting time operates, like grouping policy, through its effect on cuts per car. This is the char-acter of the incoming traffic with respect to the ordering of cars in trains.

3/   Quoted in Section 7.2.

To the extent to which cars in incoming trains are already grouped, the
number of cuts per car will be less and the time consumed in the sorting
operation will be diminished.   If the preceding yard has exactly the same
grouping policy, then in any single train not more than one cut will go into
any classification track; cuts per car will have been reduced as far as possible
by this means. If no grouping has been done by the previous yard (if, that
is to say, it puts all traffic into one group) then cuts per car will, on
the average, be very high. There will usually still remain some two- and
three-car cuts, for no matter how mixed up a train may be, a chance remains
that similarly destined cars will be found next to one another in a train.
Many-carload shipments will also prevent the number of cuts per car from
attaining the level 1.0.   To this interesting problem of just how the group-
ing policy at a preceding yard influences cuts per car we shall return in
Chapter 10, where an attempt will be made to take these chance variations
into account by means of an explicit mathematical treatment.

At this point we only wish to emphasize that since cuts per car have
little effect on sorting delay in a hump yard it becomes, as far as the
sorting delays at this yard alone are concerned, almost a matter of indif-
ference how fine the grouping policy is and how mixed up the trains are as
they enter the yard. On the other hand, in a flat yard each additional classi-
fication in the grouping policy adds noticeably to sorting delays, and whether
or not a train has been already grouped to some extent at a previous yard is
a matter of considerable importance.

8.1.3 In the Make-up Operation

The third type of handling delay we shall consider is that arising in the make-up operation. The make-up policy as described in the previous chapter specifies which of the classification tracks are to be "pulled" in order to assemble any particular outbound train. Suppose that the cars on Tracks 1, 3, 7, and 14 are to be made up into a train. While there are several different ways of doing this it is perhaps simplest to imagine the switch engine entering Track 1 from the departure end of the classification tracks, coupling on to the group of cars in Track 1, and pulling these out onto the ladder track which connects the classification tracks at this end of the yard. This string of cars is then pushed into Track 3 and a coupling is made with the string of cars standing there. This in and out movement is continued to pick up the groups in Tracks 7 and 14, and finally the train as a whole is moved off to a departure track. In large yards there may be several departure tracks side by side forming another subyard much like the receiving subyard.

In a well designed yard it is quite possible that more than one train can be made up at a time. For this to be the case there must be more than one lead track to the departure end of the classification tracks and the groups to be made up into separate trains must be so located on the classification tracks as to permit carrying on the two operations without interference.

The time consumed in making up a train depends for the most part on the number of groups that go into the train, since each group calls for a movement in and out of a classification track on the part of the switch engine. If the number of cars in a group has an influence on the time necessary to pull this group out of its classification track, then perhaps make-up time is best

expressed as a function of number of groups and length of train. Notice that this last relationship is formally identical to the one we have proposed for delay in the sorting operation. This should not, of course, surprise us for make-up is very much like the flat-yard sorting operation in reverse, with the difference that it cannot be performed quite as smoothly. Cuts have been replaced by groups in the relationship; there will usually be only a few groups, but the influence of this number on make-up delay will probably be relatively greater than the influence of the number of cuts on sorting delay, because each group calls for a push-pull action which even in flat-yard sorting was quite often avoidable.

Just as cuts per car was in great part determined by grouping policy, so the number of groups in a train (or perhaps better groups per car, if train length by itself proves to have an important effect) is in great part determined by make-up policy.

8.2 Congestion Cost

From the foregoing discussion and the data that have been presented, one might suppose that there is little reason for the average car to spend more than about three hours in any one classification yard. This would allow one hour for inspection (Figure 1), one hour for sorting (Figure 4), and one hour for make-up. Not all yards, it is true, are like the one for which we have presented data, but it is probably not unrepresentative of a large class of hump yards. In Table I are presented some daily averages of the time spent by some cars in undergoing classification at this yard.

The table presents a rather worse picture than it might have, for perishable traffic, which has highest priority in going through the yard, has unfortunately been omitted in calculation of the averages. Granted, however,

Table I

Some Daily Averages of Delays to Non-Priority Traffic in a Classification Yard (Hours)

	Northbound Traffic		Southbound Traffic	
	Average Time From Arrival To End of Sorting	Average Time From Arrival To Departure	Average Time From Arrival To End of Sorting	Average Time From Arrival To Departure
June 2	5.88	17.63	2.98	10.42
3	6.67	18.73	4.22	11.80
4	6.20	17.73	4.30	12.65
5	5.37	16.65	4.60	11.48
6	5.23	15.83	5.08	11.70
7	6.62	18.90	5.65	12.67
8	6.73	17.35	4.32	12.53
9	6.80	18.43	4.33	13.88
10	6.33	17.68	5.40	12.88
11	4.82	17.57	4.37	12.07
12	3.18	14.93	5.43	12.18
13	4.60	14.35	5.00	12.10
14	6.22	17.25	4.20	11.75
15	7.85	18.68	4.58	11.95

that Table I describes only the delays to slow traffic, it is still clear that a great part of the total delay suffered by a car in the process of classification remains to be explained. At this point it would be naive indeed to assume that the difference between the three or four "explained" hours and the fifteen hours, say, of Table I, represented waste or inefficiency on the part of railroad management. The remainder of this delay falls into categories we have called congestion delay and accumulation delay. We shall first say a few words about congestion delay and its determinants.

If every train took exactly one hour to inspect and one hour to sort, the average time from arrival to the end of sorting would still be greater than two hours, unless no train ever arrived until the inspection of the train preceding it had been completed. Where a yard has only one main track leading into it, there is some reason to expect incoming trains to be spaced out in this manner, for the capacity of the main-line track is limited, and the previous yard is not likely to make up and dispatch trains in very close succession. In large yards however, incoming trains will quite often enter from several directions. Bunched arrivals will not be uncommon, with the result that cars will often be forced to stand idle in the receiving yard while another train is being inspected. We shall call the time spent in such "waiting lines" or "queues" a congestion delay. Such delays are also caused by fluctuations in the lengths of inspection and sorting times.

Table II presents a record of the inspection and sorting operations for part of one particular day. The arrival time of each train is given, along with the beginning and end of the inspection period, and the beginning and end of the sorting period. The numbers in the first column simply denote first train, second train, section a, section b, etc., while the third column shows the lengths in cars of the sections into which each train is broken up for handling purposes. The fourth column, "Paper work", indicates the time at which the switch lists are completed. Sorting cannot begin until both the inspection and the paperwork are finished.

Table II. A Record of the Trains Classified Between Midnight
and Noon on March 15, 1955, in a large Hump Yard

Train	Arrival		Cars	Inspection		Paper work Finished		Classification	
1	3:50	PM	22	6:40-7:10	PM			3:28-3:35	AM
2	5:00	PM	24	5:55-7:10	PM	6:30	PM	12:32-12:38	AM
3	5:55	PM	19	7:10-7:50	PM	7:05	PM	4:27-4:33	AM
4	7:20	PM	53	8:00-9:35	PM	8:25	PM	3:40-3:55	AM
5	9:25	PM	21	10:00-11:10	PM	10:35	PM	3:10-3:22	AM
6	10:00	PM	112	10:35-3:30	AM	11:55	PM	4:42-5:35	AM
7a	11:05	PM	28	1:45-2:50	AM	11:35	PM	6:17-6:25	AM
b			112	3:30-5:00	AM	2:40	AM	6:34-7:25	AM
8a	11:30	PM	60	11:30-12:20	AM	12:35	AM	12:48-1:12	AM
b			4	12:30-12:40	AM	12:57	AM	1:23-1:24	AM
c			46	3:40-5:00	AM	12:57	AM	8:22-8:40	AM
9a	12:10	AM	39	12:10-12:50	AM	1:07	AM	1:25-1:42	AM
b			40	12:50-1:15	AM	1:35	AM	1:46-2:00	AM
c			29	12:40-1:45	AM	1:40	AM	2:08-2:17	AM
10a	1:00	AM	35	1:15-2:00	AM	2:00	AM	2:26-2:40	AM
b			38	2:00-2:30	AM	2:10	AM	2:50-3:03	AM
c			49	5:00-5:45	AM	2:55	AM	9:02-9:17	AM
11a	3:50	AM	10	7:00-7:20	AM	4:00	AM	9:55-9:58	AM
b			118	5:45-7:45	AM	4:50	AM	9:58-10:52	AM
12	5:41	AM	45	5:45-7:20	AM	6:10	AM	9:25-9:42	AM
13	5:50	AM	76	7:15-8:00	AM	6:30	AM	11:17-11:42	AM

Hunting through the table one finds that at 12:20 AM, while 6, 8a, and 9a are being inspected, six train sections are standing in the receiving yard awaiting inspection. These are 7a with 28 cars, 7b with 112 cars, 8b with 4 cars, 8c with 46 cars, 9b with 40 cars, and 9c with 29 cars -- a total of 259 cars in the inspection queue. At 1:10 AM six train sections whose inspection has been completed are waiting to be sorted: 1, 3, 4, 5, 8b, and 9a with a total of 158 cars. The yard is not always this congested, but a short study of Table II does indicate that queues usually form ahead of the inspection and sorting operations, and that the resulting congestion delays are fairly substantial. One writer[4] has reported the following average delays to trains for a one-week period in a hump yard:

Waiting for inspection	65	minutes
Inspection	71	"
Waiting for sorting	160	"
Sorting	25	"

In interpreting both these delays and those in Table II a word of caution is in order. Strictly speaking, the time spent waiting in inspection queues and sorting queues can all be called congestion delay only if traffic is pushed through these operations as fast as possible, given the working force of both labor and switch engines. If, however, the yard manager knows that the cars in a certain inbound train are not scheduled to leave the yard for say twelve hours, then he may very well put off the inspection or the sorting of these cars for eight hours or so, not simply because other traffic is waiting but only because it makes no difference whether the work is done now or later. This difficulty of separating congestion delay and accumulation delay makes the measurement of congestion delay rather difficult and may

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4/   Crane [1953, p. 102].

account in part for the surprisingly large estimates quoted above, as well as the congestion delays in Table II.

In most queuing situations[5] the average congestion delay increases with the level of traffic. To test whether this is the case in a hump yard, linear regressions of daily average time from arrival to and of sorting on number of cars per day were computed for both northbound and southbound non-perishable traffic through a large hump yard over a period of four weeks. The results fail to tell us very much:

$$(8.3) \qquad \text{Northbound:} \quad d = 135 + .31n$$
$$(.11)$$

$$(8.4) \qquad \text{Southbound:} \quad d = 204 + .11n$$
$$(.11)$$

where d = average time in minutes from arrival to end of sorting, n = number of cars per day, and the numbers in parentheses are the standard errors of the coefficients of the n's. For the northbound traffic a significant increase of average delay with number of cars was found; for the southbound traffic the increase was not significant. The validity of these calculations is of course seriously impaired by the omission of the perishable traffic, which, although exact figures are not available, is known to constitute quite a large portion of the total traffic.

Policy choices will affect average congestion delay to the extent that handling times both in inspection and sorting can by this means be made more regular. Such opportunities are probably not very great. Equal in interest to the effect of policy on average congestion delay is the effect of receiving

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5/   Cf. Section 1.2, especially Figures 8, 9, 10 at the end of that section.

policy on the distribution of congestion delays to different classes of traffic. By assigning high priority in handling to one class of traffic the congestion delay of this class is reduced at the expense of the con-gestion delays suffered by all other traffic. Notice in Table II that the first sections of trains 5, 8, 9, and 10 are sorted before trains that arrived earlier. The average delay is unchanged by such priority assignments; only the distribution of the total is affected.

8.3 Accumulation Costs

When the sorting of a train has been completed, the cars that were in it are ready to be made up into new trains. In the case of the usual car this make-up operation does not take place immediately, with the result that the car suffers another loss of time which we shall call an accumulation delay.

The name is derived from the reason for this delay. Enough similarly bound cars to achieve long-train economies must accumulate before an outbound train can be assembled. The decision as to how long the train is to be, and just how similar the destinations of the cars must be, is implicitly expressed in the make-up and scheduling policies, which state, respectively, what groups go into any particular outbound train and when the train is to be made up. If a whole trainload of low-density traffic is to be assembled, the average accumulation delay to the cars involved will be high; it will be low, on the other hand, if a train carries all traffic bound in its general direction. Considerations as simple as these go far in explaining why traffic bound from New York to Seattle is "yarded" at several intermediate points instead of being hauled in one solid train from origin to destination without inter-ruption.

Speaking more precisely, we define the accumulation delay of a car to be the sum of the periods during which the car is waiting for the arrival or handling (or both) of other cars with which it is to be dispatched.[_/] If we assume that the inspection and sorting of cars proceeds as fast as possible, given a certain working force, then the above definition is equivalent to the statement that the accumulation delay of a car is the period of time between the completion of sorting of the train (or section of train) in which it arrived and the beginning of the make-up of the train in which it is to depart. With this assumption, the accumulation delay of a car becomes very nearly the same as the time the car spends on a classification track.

The level of the average accumulation delay suffered by a particular component of traffic will, for the most part, be determined by the make-up and scheduling policies together with the level of flow of that traffic component. To make this clear let us suppose for a moment that the make-up and scheduling policies are of the "leave-when-filled" type (Figure 4) rather than the timetable type. Now let us look at the traffic specified by the make-up policy to go out on one particular train, No. 97 . If this traffic flow is high, then the intervals between departures of "No. 97" will be short and accumulation delay per car for this traffic will be low. If some of this traffic departs on other trains as well as No. 97, the description becomes more complicated, but our conclusion is unchanged. If the make-up policy is altered so as to exclude some of this traffic from No. 97, and if no other new traffic enters the make-up of No. 97, then the accumulation delay per car for No. 97 is increased for it will take a longer time to accumulate a train load from this smaller traffic flow. To find out what happens

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_/   Only in the extreme case of dispatching by timetable will this definition of accumulation delay become somewhat ambiguous; for this one particular case we can define accumulation delay as the waiting time to the beginning of the make-up of the train in which the car is to depart.

to the accumulation delay of the excluded traffic it is of course necessary to look at more than one train at a time. If the scheduling policy is altered so that trains (with the same make-up as before) are dispatched more frequently, then accumulation delay is decreased.

The effects on accumulation delay of simultaneous changes in make-up and scheduling may sometimes cancel out. Thus if the make-up of some trains is restricted but trains are scheduled more frequently, accumulation delays may remain the same.

In the case of timetable scheduling, the average accumulation delay per car of each component of traffic is only affected by make-up and scheduling and not by changes in the level of flow of that component, since a train's departure does not depend on the time required to accumulate a train load. In this case, as we shall see in Chapter 11, the influence of flow levels acts through their effect on the total of accumulation delays for all cars.

8.4 Summary

We shall now attempt to sum up in tabular form our discussions of the effect of yard facilities, traffic input, and policy on the various costs at a single yard and on the traffic output from that yard.

Across the top of Table III are listed the various factors influencing costs and output, and along the left margin are listed the various costs and the output. For the sake of completeness we have also included line-haul cost from this yard to whatever yards the departing traffic goes. The phrase in any one cell of the table is an attempt to describe very briefly the means by which the item at the head of the column influences the item named in the left margin of the row. Where the influence is negligible or indirect the cell has been blacked out.

On	Yard Facilities	Traffic Input	Policy			
			Receiving	Grouping	Make-Up	Scheduling
Yard Cost — Handling: Receiving	size of inspection crew	length of time into which incoming trains are broken	length of sections into which incoming trains are broken			
Handling: Sorting	hump or flatswitching	order of cars		cuts per car	number of groups in train	size of groups
Handling: Make-Up	simultaneous make-up of two trains					
Congestion	through effect on handling time	bunched train arrivals	priority in handling	through effect on handling time	through effect on handling time	through effect on handling time
Accumulation		level of component flows	substitution of receiving delay for accumulation delay		restrictive make-up means less frequent departures	frequency of departures
Line-Haul Cost to Next Yard					length of train	length of train
Traffic Output		order of cars within groups		order of cars	number of groups in train	size of groups

Table III. The Influence of Yard Facilities, Traffic Input and Policy on Cost and Output of a Single Yard.

Chapter 9

The Assignment of Classification Work in a System of Yards

9.1 The Selection of a Set of Policies That Minimizes System Costs

We have seen that various characteristics of the incoming traffic play
a large part in the determination of both money and time costs at any single
classification yard. Since many of these characteristics, such as the bunching
of trains, the lengths of trains, and the grouping of similarly bound cars are
importantly influenced by the classification policies of earlier yards, it is
clear that the cost at any one yard is a function not only of its own policy,
but also of the policies at all those yards through which the traffic of the
yard in question has passed at earlier stages in its journey.

Diagramatically, this causal relationship can be represented as in Figure 1,
where the boxes represent classification yards, and the arrows represent policies,
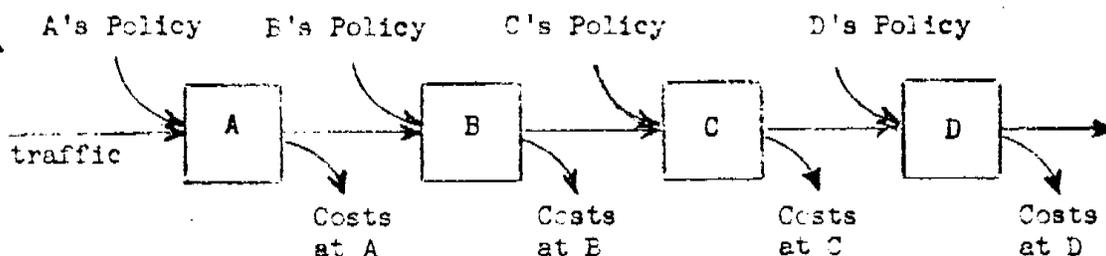costs, and the characteristics of various traffic flows. The arrows leading

Figure 1. Causal Relationships in a System of Yards

into a yard determine the arrows leading out. Thus C's costs and traffic output are jointly determined by C's policy and traffic input, and the latter, being B's output, is in turn determined by B's policy and the input to B, etc. The arrangement depicted, four yards in line, could easily be extended to more complicated systems.

We see therefore that costs at different yards are closely related to each other. A yard which adjusts its policy so as to reduce costs at a yard farther down the line will usually find that its own costs have increased in the process. If we assume the program of traffic (i.e., the traffic flows between various points of origin and destination) handled by a given system of yards to be fixed, then any selection of a set of policies for the different yards is equivalent to a particular assignment to the yards of the classification work involved in handling this program.

The problem of policy selection, therefore, amounts to assigning this work in such a way that, in some sense, the sum of costs over the system is minimized. The system costs to be summed here must include, not only the costs at yards, but also line-haul costs, in order to bring into the problem recognition of the fact that accumulation delays are incurred to gain the hauling economies of long trains. The hauling operation involves time as well as money costs, of course, but for purposes of simplicity we shall ignore these time costs (by assuming them to be constant) on the grounds that yard policy has little influence on them.

Strictly speaking, it is not, of course, quite correct to refer to the problem as one of simple minimization[1] since our cost concept, involving both time and money cost, has more than one dimension. Indeed, if delays to different components of traffic are valued differently -- and very often

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[1] In the mathematician's language, the problem is one of vector minimization.

they are in practice -- our cost concept becomes many dimensional. In
such a case the search for the "best" set of policies for the yards of a
system takes the following form. First those assignments (substituting this
shorter word for "sets of policies") must be found which have the characteristic
that if another assignment results in less of one kind of cost, it inevitably
results in more of another. An assignment which meets this specification will
be called efficient. Thus one efficient assignment might result in fast service
through the system of yards to AB traffic, but slow service to CD traffic.
Since both assignments are efficient there can be no assignment which provides
the same fast service to AB traffic and the same fast service to CD traffic.
In searching for a best assignment we need not consider any inefficient assign-
ment because by definition there is always an efficient assignment which is
less costly in all respects.

The next step is to select from among the set of efficient assignments
the one (or ones) which most nearly reflects the valuations put on the various
time costs. The plural, valuations, is used because of course equal time
losses to different components of traffic will not in general be valued equally.
If we knew from the start how time was to be valued in terms of money, we
should not have had to break down costs into time cost and money cost; instead,
we could have treated time just as we have treated labor and fuel, in terms
of money cost. With such advance knowledge, the problem of policy selection
would simply have consisted in minimizing total money costs, and we should
not have needed to trouble ourselves with the intermediate step of searching
for efficient assignments.

From the economic point of view, however, knowledge of the set of
efficient assignments and the various costs associated with each is of
fundamental importance. For the valuations of time eventually decided upon

(perhaps only implicitly) will be influenced not only by the diverse demands for speed on the part of shippers, but also by the extent to which a change from one efficient assignment to another enables a reduction of delay to one component of traffic at the expense of an increase of delay to another, or perhaps an increase in money cost. This latter information is precisely what we learn from the set of efficient assignments and the costs corresponding to each. The same considerations apply, of course, to labor, fuel, and other inputs, but since markets and prices exist for these commodities we have felt justified in summing them up in money terms.

In the rest of this chapter and in the next two chapters we shall attempt to explore the differences in costs between various assignments. In Chapter 10 we shall examine assignments which differ only in grouping policies, in Chapter 11 those which differ only in scheduling policies. Since it is not immediately apparent that the work assigned to yards changes with changes in scheduling, it should be recalled that, not only does scheduling influence yard costs, directly, but also that some schedules enable grouping and make-up policies not consistent with other schedules. Only the direct influence will be treated in Chapter 11. In this chapter we shall be concerned with an exploratory,and consequently rather general, discussion of these differences. In the next two chapters, simplifying assumptions are made which enable more rigorous discussions.

9.2 Two Extremes in Assignments

One of the earliest writers to give some attention to the problem of selecting the classification policies for a system of yards was M. L. Byers, a Chief Engineer, Maintenance of Way, on the Missouri Pacific Railway. As early as 1908, he pointed out, as other authors have since done, that generally

speaking there are two extremes among policies:

"(1) Where no classification is made until the car has arrived at the last yard which it enters, it being left to this yard to separate such cars from those going to the yards beyond.

"(2) Where each yard brings together all cars for the same destination and then arranges the groups of cars in the train in the order in which the yards follow."

For convenience we shall refer to these as the extreme deferment and extreme anticipation methods. In evaluating the relative virtues of these two policies, he went on to say that "The effect of [the anticipation] method as compared with the [deferment method] is especially noticeable where the railroad system contains one or more very large cities, toward which traffic gravitates from the other points on the system. The work of the outlying yards is increased over that required of them by the [deferment] method, but the work in each succeeding yard approaching the final terminal is reduced, owing to part of the work having been already performed by the outlying yard; the total work performed by all of the yards is, however, considerably increased by the [anticipation] method, and the sole excuse for its adoption should be that the final yards are incapable of and cannot be made capable of doing economically all of the classification work which would be thrown upon them under a modification of the [deferment] method of classification."[2]

Surprisingly enough, the method of anticipation, of which Byers is so critical, comes closer to describing present-day classification systems than does his method of deferment. Indeed, one modern writer, describing the classification policy of the Baltimore and Ohio, states that, "The railroad's aim is to classify cars into destination blocks [i.e., groups] as completely

2/   Byers, [1908. p. 338-339].

as possible near points of origin, so that blocks (and sometimes whole
trains) may be run without break-up through intermediate terminals to
destination." 3/

9.3 A Comparison of Assignments When the Scheduling Policies are Fixed

As a compromise between his two extremes, Byers recommended a method
we have already described in Section 7.5 for the purpose of demonstrating
our policy terminology. At this point the reader is invited to turn back
to the description of this compromise plan, which, in addition to being of
interest in its own right, is noteworthy for its uniqueness in the literature.
Except for the two extremes described above, it is the only explicit and
completely general proposal of a set of policies for a system of yards.

This plan, and especially that part of it which deals with the work a
yard is to do after completing its "normal" work, serves as a good intro-
duction to our discussion. A yard, according to Byers, should do sorting
work on a through train above and beyond the normal amount specified by the
deferment policy whenever it has available yard-engine time before the de-
parture of the train. While nothing explicit is said about the scheduling
policy, it is clear from this statement that "through" trains are dispatched
by timetable, and that the particular time-table to be used has already
been selected. The proposal therefore deals only with part of the problem
of policy selection, and the assumption implicit in the discussion is that
the best selection of scheduling policies has been accomplished. It is true
that these policy suggestions have an interesting dynamic element which we
have so far ignored. That the classification policy of a yard on a day of
heavy traffic should be different from the policy on a day of light traffic

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3/   Coughlin [1952, p. 45]; italics ours.

is a very plausible consideration, but one we have felt justified in over-looking until the easier case of a constant policy has been more fully explored. We shall therefore assume that traffic flows are about the same from day to day, and proceed to examine Byers' suggestions under such conditions. It might be objected that under these conditions no yard would ever have any spare time in which to do extra work, for the schedule of trains would be tightened up throughout the system so as to eliminate the spare time. Such a tightening-up procedure would reduce the system of policies to one very much like the first extreme method; no account would be taken of the superiority, due to facilities or traffic, of one yard over another in the carrying out of classification work. We prefer to interpret the plan literally for the case of traffic flows that are fairly stable. We shall assume that for some unspecified reason the scheduling policies, i.e., the timetables, have been selected so as to leave some spare time at various places in the system. Since the spare time at any particular yard will, under these conditions, be constant, the amount of extra work this yard can do will be constant, and as a result its grouping and make-up policies will be fixed.

(Figure 6 of Chapter 7)

Let us suppose the schedules are arranged so as to allow yard A enough spare time to carry out the first modified policy and the other yards no spare time at all. Then, if the plan is a good one, two implications become clear. The first is that the additional costs[4] incurred at A in carrying out the extra work are more than offset by the reduction in costs at B that result from A's sharing B's work load. The second implication of the

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[4] We are purposely ignoring Byers' stipulation that the extra work much not involve "increased cost", for additional work will always give rise to some cost at the yard that does the work. If we interpret his "cost" as system cost then of course his condition begs the question of policy selection.

plan is that the grouping of the through train at A, namely

$$\left\{\begin{array}{l} B \text{ and local beyond } B \\ C \text{ and local beyond } C \\ D \text{ and beyond} \\ D' \text{ and beyond} \end{array}\right\},$$

is better, in terms of system costs, than some other grouping that could
also be accomplished in the time allowed by the schedule, for instance,

$$\left\{\begin{array}{l} B \\ \text{local beyond } B \\ E' \text{ and local beyond } E' \\ \text{All others} \end{array}\right\}.$$

As we saw in the last chapter, one of the most important influences of
a policy change on yard costs operated through the variable "cuts per car".
Clearly the modification of A's grouping policy increases cuts per car at
that point. At the same time the number of cuts per car at B is reduced,
because with the change in A's policy trains that previously arrived at
B in a rather scrambled fashion are now grouped to a greater extent. Sorting
costs therefore increase at A and decrease at B. Make-up costs will also
increase at A since only two groups were put into the through train before
the change, while now four are. Whether on balance the costs to the system
go up or down depends on the sorting facilities at A and B. If A is a
hump yard, and B a flat yard, then we should have good grounds for believing
that the policy modification is a good one, for we know that an increase in
cuts per car at a hump yard only slightly increases costs, while a reduction[2]
in the same variable for a flat yard lowers costs considerably.

_____

[2]   The increase in cuts per car at A will not in general be the same as
the reduction in cuts per car at B. This point will become clearer in the
discussion of Chapter 10.

But what if B is the hump yard and A the flat yard? In this case Byers' scheme seems to move us away from, rather than toward an optimal assignment of classification work. At this point it could be argued that one of the functions of the scheduling policy is to timetable trains in such a way that only the most efficient yards have spare time remaining after the completion of their normal work. With respect to the considerations introduced so far this seems to be a legitimate argument. As we shall see later however, when we begin to take accumulation delays into more explicit account, this approach to an optimal assignment which involves selecting scheduling policies first and grouping and make-up policies afterward is not satisfactory in general.

The second implication of Byers' plan is more difficult to defend. There seems to be no a priori reason why the grouping he suggests should result in less cost to the system than the arbitrary grouping we have suggested for purposes of comparison.   While a more detailed examination of a problem similar to this one must be deferred to the next chapter, it is tempting to guess at Byers' reasons for selecting the particular grouping policy for A that he did. If traffic is evenly distributed throughout the network, and if long trips occur less frequently than short ones, then the grouping proposed by Byers will cause all of the groups in a train to be of about the same length. Under the same circumstances our arbitrary grouping will bring about groups of very unequal lengths. The Group "E' and local beyond E'", for instance, will be small, and the group "All others" will be large. Now group length is probably not a very important factor in yard costs, but the relative flow magnitudes of the various traffic classifications specified by the grouping policy are, and these are fairly well represented by group lengths. The relative levels of these flows influence

sorting cost through their effect on the number of cuts per car, as we shall see at a later point. Thus whether or not Byers' grouping policy is to be preferred depends on the levels of the various traffic flows, and on the effect of these levels on the cuts-per-car variable.

For purposes of illustration let us suppose that Yard A in Figure 6 of Chapter 7 is a very large and efficient hump yard, while all the other yards in the system are small ill-equipped flat yards. In this case, not only is it conceivable that Byers' third policy modification will be put into effect, but perhaps even a fourth or fifth such modification. A will be doing some of the normal work of all the other yards in Figure 6. Group lengths on trains departing from A will become very much smaller than under a policy of less pre-classification, and the question arises as to whether the relief afforded other yards does not begin to diminish as preclassification is carried beyond a certain point. If A puts cars bound for E' and local points beyond E' into a single group, this will not provide much saving of work at the next yard that groups this traffic, for the group from A will (unless E' is an important destination) be small so that the difference between the number of sorting operations when E' cars are grouped and when they are not is also small.

9.4 Extending Preclassification At the Cost of Additional Accumulation Delay

By changing its make-up or scheduling policy (or both) A has still another means of saving sorting work for yards farther along the line in the direction of traffic. If a train with the E' group in it is scheduled to depart from A somewhat later than before, then at the time of departure more E' cars than before will have accumulated at A, and the E' group in the outbound train will be longer than previously. Longer groups-mean

fewer cuts per car at an advance yard (say P) whose grouping policy does not contain finer classifications than that of A.[6/]

The length of the E' group in an outbound train can also be increased, and this is probably the more important method, by changing the make-up policy at A so as to exclude E' traffic from some of the trains in which it was previously dispatched, and to exclude some of the other traffic from the trains that still contain E'. Thus perhaps before the change the make-up of several trains leaving A was[7/]

$$\left\{ \ldots \mid E \mid E' \mid F \mid \ldots \right\} .$$

If this is changed for some trains to

$$\left\{ \ldots \mid E' \mid \ldots \right\}$$

and for others to

$$\left\{ \ldots \mid E \mid F \mid \ldots \right\} ,$$

then the group lengths of E, E', and F will all be longer than before and cuts per car at the advance yard will again be reduced. The cost at A will of course be in terms of accumulation delay to each of these components of traffic. Carried far enough this process leads to the "solid-train" type of make-up described earlier. The "maintracker" type of train, whose make-up is restricted to an extent that enables it to by-pass some yards entirely, is another example.

6/ The effect of longer groups on cuts per car is one of the "end effects" which for simplicity is assumed away in Chapter 10.

7/ The dots are to be interpreted as groups which are not affected by the change. To save space, we have listed the groups horizontally rather than vertically as in Chapter 7.

Chapter 10

The Division of Sorting Work Between Yards

10.1 Introduction

In this chapter we are concerned with the sorting work of yards. We shall show that because of the different characteristics of hump and flat yards it is sometimes useful to do sorting work at one yard which could possibly be done at another. Our object is to give a quantitative method for deciding when sorting should be transferred in this way. To develop such a method, we must know in more detail what determines the work required to carry out the sorting at each of the two types of yards.

We have seen that flat yards and hump yards differ in the way the cars are actually put onto the classification tracks. The assumption we shall make in the case of a hump yard is that no more work is involved in the humping operation whether there are few cuts in the train or many, that is, whether the order of the cars is scrambled up or not. There is some evidence[1] from actual yard data that if there are many cuts the operation takes longer, but since this effect of cuts in hump-yard operation is small relative to the same effect in flat-yard operation we shall ignore it in the case of a hump yard.

In the case of a flat yard we shall assume that the more scrambled up an incoming train is, that is, the more cuts there are in the train, the more work the switch engine will have to do, and the more time it will take to do it. To start with, it seems justifiable to take as a measure of the work done on a train the number of cuts in the train.

The situation we will consider is represented in Figure 1.

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[1]   Section 8.1.2.
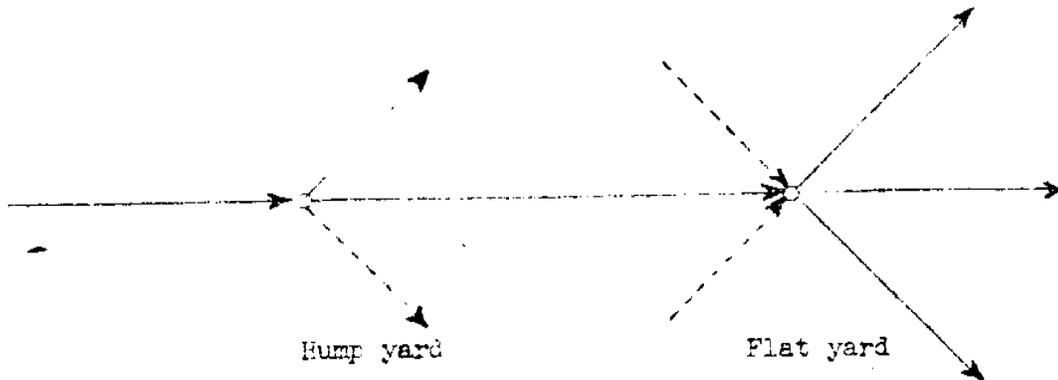
Hump yard   Flat yard

Figure 1

Some of the traffic through the flat yard has previously passed through the hump yard. By recognizing more categories of traffic at the hump yard, that is, by finer sorting, it would be possible to save some of the work at the flat yard. However, if cars are to depart in the same trains as before the change in grouping policy, then this will involve additional make-up work at the hump yard, for the outgoing trains will contain more groups than before. We want to find some approach to the problem of balancing these ·costs. We also want to find which of the categories at the flat yard should be sorted out at the hump yard.

Preclassifying at the hump yard means that instead of all the traffic for the flat yard going on a single classification track at the hump yard, it goes onto two or more tracks. For example, if it goes on to two tracks, then on one of these tracks will be put traffic which goes on one group of tracks in the flat yard, while traffic for the remaining group goes on the other track. Thus the hump yard will do some preliminary sorting for the flat yard. It may be that traffic for a single track at the flat yard is sorted out with a classification track all to itself at the hump yard. We

will show in fact, that under some circumstances this is the best way for
the hump yard to save work for the flat yard.

10.2 A Probability Representation of Trains

The composition of trains varies from train to train, even with trains
coming from the same point. Also, for any given classification policy, the
number of cuts (in a sense, the degree of scrambledness) of trains varies.
We will suppose that, for simplicity of operation, the grouping policy will
remain the same from train to train, that is, the same classifications will
be recognized despite these variations. Thus the amount of work to be done
in sorting at the flat yard will vary from train to train. The work we shall
be considering will be the average over many trains.

We still have to find some useful way of representing the composition
and order of cars in trains coming into the hump yard. There is a strong
analogy between an unsorted train and an unsorted pack of playing cards. A
shuffled pack of cards can best be represented by probability methods, and
this suggests that the order of cars in a train can be supposed to be repre-
sented by a similar type of probability model. However, the representation
which we will use is rather different from that used to represent the order
of a pack of playing cards. Let us suppose that the various suits in a pack
represent the different categories of cars. Then, in the train of 52 cars
whose order the pack is supposed to represent, there will always be exactly
13 of each of the four different types of car. This would be a very unreal-
istic restriction in attempting to describe the order and composition of a
train. Consequently the first and simplest probability description we use
is the following one. Instead of supposing the trains to be represented by
the shuffling of a pack, we suppose they can be represented by drawing a
sequence of cars in order from a pack, replacing each card after it has been

drawn and reshuffling after each drawing. Thus the proportion of cars of different categories can change from train to train. Sequences obtained by this method (they are called _multinomial_ sequences) can easily be studied. We have found them to give adequate representations of the composition of trains in several cases we have examined, provided the proportions of cards of different types are made to correspond with the data. However, the way in which trains are assembled suggests that such a representation will not always be adequate. Many shipments are in more than carload lots, so that, for example, three or four or perhaps more cars will stay together throughout the trip. If there is much of this prior grouping, then cut lengths may well tend to be longer than the previous hypothesis (the multinomial hypothesis) admits. Even if cut lengths are changed, there is still a simple way of describing the cuts themselves as well mixed (as we expect them to be in an unsorted train). Our results extend to the case where the sequence of _cuts_ is the same as the sequence of _cuts_ in a multinomial sequence, though the number of cars in the cuts may be different from those in a multinomial sequence, so that the sequence of _cars_ is not necessarily the same as that of a multinomial sequence.

10.3 The Best Sorting Policy.

Suppose, for the moment, that it has been decided just how many categories to recognize at the hump yard; that is, the grouping policy is known. Suppose as an example we are to give over four of the tracks at the hump yard to pre-classifying for the flat yard. Then we shall see that it would be best to sort out the three categories with the most cuts separately onto three of the tracks, and put all the rest of the cars onto the fourth track. If the number of cuts is roughly the same for all the categories, however, it does not matter

what sort of preclassification policy is used, provided four tracks are actually used.

It should be emphasized that these results hold only if all the cuts come from the same type of sequence. If, as often will happen, the various incoming trains from various origins differ very much in their composition, the work function given below will have to be averaged over the different types of trains.

We should calculate this work function for each possible number of tracks of the hump yard devoted to preclassification. With each increase of this number, the extra work saved the flat yard diminishes, and the work involved in assembling trains at the hump yard increases. Thus at some point it becomes uneconomical to do further preclassification.

10.4* Derivation of the Work Function For a Multinomial Sequence
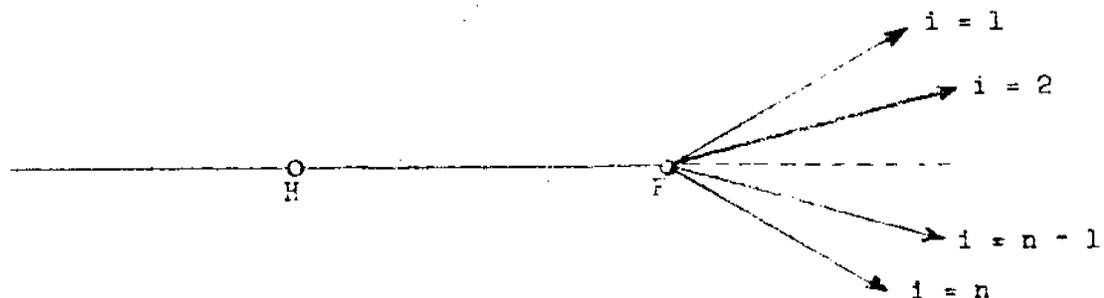
The situation we will consider is represented in Figure 2.

Figure 2.

Traffic for a flat yard F passes through a hump yard H. At the flat yard n different categories are recognized, represented by i = 1, 2....n. The hump yard H is to do some preclassifying for F, and will recognize m different categories of traffic going beyond F. The problem is first to choose the categories to be recognized at H in order to save as much work as possible at F. We also have the problem of determining just how much work this saves at F, so that this work saved can be weighed against increased costs at H. Finally, choice of the best value of m is still another problem.

We take as a measure of work at F the average number of cuts per car for traffic coming in to F. A cut consists of an uninterrupted sequence of cars of the same category. It is the same as a run in probability theory. For example: for the sequence of cars 2 2 1 3 1 1 1 3 3 2 there are 6 cuts, so cuts per car equals 0.6.

Suppose at first that the sequence of cars coming over the hump at H, and bound for F, is a multinomial sequence with probabilities p_i. The sequence is assumed to be long, so that we can neglect the end effects both of the original sequence and the sequences into which it is divided by the preclassification. (As already indicated, we shall later suggest a more general hypothesis, which allows for the fact that shipments are often made in more than carload lots.)

For a multinomial sequence, the probability that any designated car is the first car of a cut of i's is $p_i(1 - p_i)$. Therefore the probability that a car is the first car of any cut is $\sum_{i=1}^{n} p_i(1 - p_i)$. It follows that the mean number of cuts per car equals $1 - \sum_{i=1}^{n} p_i^2$, and this is an index of work at F if H does no preclassifying.

If H preclassifies the traffic bound for F into m classes, these classes are formed by a partition of the categories $1, 2, \ldots, n$ into m mutually exclusive subsets $\mu_1, \mu_2, \ldots, \mu_m$. This partition defines F's grouping policy for traffic bound to H and beyond. The cars now arrive at F in a series of runs, each run consisting of cars belonging to a μ_j $(j = 1, \ldots, m)$.

Ignoring end effects of such runs, the mean number of cuts per car for cars of μ_j is

$$(10.1) \qquad 1 - \sum_{i \in \mu_j} \left(\frac{p_i}{\sum\limits_{i \in \mu_j} p_i} \right)^2$$

But a proportion $\sum\limits_{i \in \mu_j} p_i$ of all the cars belong to μ_j. Therefore the mean number of cuts per car for all cars is

$$(10.2) \qquad \sum_{j=1}^{m} \left[\sum_{i \in \mu_j} p_i \right] \left[1 - \sum_{i \in \mu_j} \left(\frac{p_i}{\sum\limits_{i \in \mu_j} p_i} \right)^2 \right] = 1 - \sum_{j=1}^{m} \frac{\sum\limits_{i \in \mu_j} p_i^2}{\sum\limits_{i \in \mu_j} p_i}$$

The difference between this and $1 - \sum\limits_{i=1}^{n} p_i^2$ is

$$(10.3) \qquad \sum_{j=1}^{m} \frac{\sum\limits_{i \in \mu_j} p_i^2}{\sum\limits_{i \in \mu_j} p_i} - \sum_{i=1}^{n} p_i^2$$

and represents an index of the amount of work the given grouping policy saves for F.

We wish to find what partition $\mu_1, \ldots, \mu_m$ will leave the minimum work to F. Let us renumber the classification so that $0 < p_1 \leq p_2 \leq \cdots \leq p_n$. Then for any partition $\mu_1, \ldots, \mu_m$ of the numbers $1, 2, \ldots, n$, we can show that

$$(10.4) \qquad \sum_{j=1}^{m} \frac{\sum\limits_{i \in \mu_j} p_i^2}{\sum\limits_{i \in \mu_j} p_i} \leq p_n + p_{n-1} + \cdots + p_{n-m+2} + \frac{\sum\limits_{i=1}^{n-m+1} p_i^2}{\sum\limits_{i=1}^{n-m+1} p_i}$$

That is, the partition which maximizes the left-hand sum and thus maximizes the saving to F is $[1, 2, \ldots, n - m + 1]$, $[n - m + 2]$, $\ldots$, $[n - 1]$, $[n]$. First we will prove the result for $m = 2$; that is the n classifications are divided into two groups, μ_1 and μ_2. Let μ_2 be the group that contains n, the most frequent classification, and let μ_2' be the remainder of μ_2. Furthermore, let

$$(10.5) \qquad \sum_{i \in \mu_1} p_i = \alpha \quad \text{and} \quad \sum_{i \in \mu_2'} p_i = \beta$$

and

$$(10.6) \qquad \frac{\sum\limits_{i \in \mu_1} p_i^2}{\sum\limits_{i \in \mu_1} p_i} = a \quad \text{and} \quad \frac{\sum\limits_{i \in \mu_2'} p_i^2}{\sum\limits_{i \in \mu_2'} p_i} = b.$$

Note that

$$(10.7) \qquad a \sum_{i \in \mu_1} p_i = \sum_{i \in \mu_1} p_i^2 \leq \left(\sum_{i \in \mu_1} p_i \right)^2 = \alpha \sum_{i \in \mu_1} p_i,$$

so that $a \leq \alpha$.   Also,

(10.8)
$$\sum_{i \in \mu_1} p_i^2 \leq \sum_{i \in \mu_1} p_n p_i = p_n \sum_{i \in \mu_1} p_i$$

so that $a \leq p_n$ and, similarly, $b \leq p_n$.   Substituting these values in (10.4), we are to show that

(10.9)
$$a + \frac{p_n + b\beta}{p_n + \beta} \leq p_n + \frac{a\alpha + b\beta}{\alpha + \beta} \quad .$$

Multiplying by $\frac{1}{\beta}(p_n + \beta)(\alpha + \beta)$, which is positive, (10.9) becomes

(10.10)
$$ap_n + a\beta + \alpha b \leq bp_n + \alpha p_n + p_n \beta.$$

Adding ab to both sides, this can be written

(10.11)
$$-(p_n - b)(\alpha - a) - (p_n - a)(b + \beta) \leq 0 \quad .$$

All four of the quantities in parentheses are positive. Hence the inequality must hold, proving (10.4) for the case $m = 2$.

Now consider any partition $\mu_1, \mu_2, \ldots, \mu_m$, and let μ_1 be the group containing the most frequent classification n.   Applying the theorem of (10.4), just proved for $m = 2$, to the universe consisting of $\mu_1 + \mu_2$, the expression on the left-hand side of (10.4) will not be diminished if we replace $\mu_1, \mu_2, \ldots, \mu_m$ by $\mu_1', \mu_2', \ldots, \mu_m'$, where $\mu_1' = (n)$, $\mu_2' = \mu_1 + \mu_2 - (n)$, $\mu_3' = \mu_3, \ldots, \mu_m' = \mu_m$.   Again, let μ_1' be the group containing

classification $n - 1$. Replace μ_i' and μ_j' $(i \neq j, i \neq 1, j \neq 1)$ by $(n - 1)$ and $\mu_i' + \mu_j' - (n - 1)$ respectively. The inequality (10.4) follows after m such steps.

From this theorem we see that it is best under these conditions to select the $m - 1$ classes with highest probabilities and preclassify them separately. Note though that if $p_1 = p_2 = \ldots = p_n = \frac{1}{n} = p$ then both sides of (10.4) reduce to mp, and it becomes quite immaterial what pre-classification policy we use.

10.5*   Extension To a More General Type of Sequence

Although the multinomial sequence can often give quite a good representation of the incoming car sequence, it is rather restrictive. This can be seen especially if loads are shipped in more than carload lots. Then a run of cars is kept together throughout a trip, and in fact treated as a single car for sorting purposes. Thus mean cut length will tend to be longer than that expected from the multinomial distribution. However the results of previous sections apply to a more general incoming sequence which will include this case.

This generalization is achieved by thinking of the incoming sequence as a sequence of cuts rather than individual cars. We will call a group of cars which form a cut before sorting a segment to distinguish it from a group of cars forming a cut after sorting. The latter we will still call a cut. Let A_i be a segment of incoming cars of type i. By the definition of a segment, no two segments of the same type can be together in the incoming sequence. Suppose the sequence of segments is generated by a Markoff chain with constant transition probabilities

$$(10.12) \qquad p_{ij} = \frac{p_j}{1 - p_i} \qquad (i \neq j)$$

$$= 0 \qquad (i = j)$$

(as in the case where the incoming cars are a multinomial sequence).
We shall allow the number of cars within segments to be distributed in any
manner whatever. Suppose now that the A_i are classified, as in the previous
sections, by partitioning the i into m groups $\mu_1, \ldots \mu_m$. After sorting,
each cut will contain one or more segments. It is therefore possible to
talk of cuts per segment. Let x be the mean number of cars per segment,
and let y be the mean number of cuts per segment after sorting. Then $\frac{y}{x}$
equals the mean number of cuts per car after sorting.

We can find y as follows. If the original sequence is multinomial,
as in the previous less general case, then the segments before sorting will
be generated by a Markoff chain of type (10.12). In the case of the multi-
nomial sequence, the mean number of cars per segment equals

$$(10.13) \qquad x_o = \frac{1}{\sum\limits_{i=1}^{n} p_i(1-p_i)}$$

and we saw from (10.2) that the mean number of cuts per car after sorting
equals

$$(10.14) \qquad \frac{y_o}{x_o} = 1 - \sum\limits_{j=1}^{m} \frac{\sum\limits_{i \in \mu_j} p_i^2}{\sum\limits_{i \in \mu_j} p_i} .$$

We have used the subscript _zero_ to indicate that the values refer to a multinomial sequence.

For the multinomial sequence we now have

$$(10.14) \qquad y_0 = \frac{y_0}{x_0} = \frac{1}{\sum_{i=1}^{n} p_i(1-p_i)} \left\{ 1 - \sum_{j=1}^{m} \frac{\sum_{i \in \mu_j} p_i^2}{\sum_{i \in \mu_j} p_i} \right\} \quad .$$

But the insertion of additional cars in the various segments (or their removal, provided at least one car is left in each segment) in no way affects the relation of cuts to segments after classification so that this result must hold for the general sequence in which the lengths of the segments may have any distribution. For the general sequence, therefore, the mean number of cuts per car after sorting equals

$$(10.15) \qquad \frac{y}{x} = \frac{y_0}{x} = \frac{1}{x \sum_{i=1}^{n} p_i(1-p_i)} \left\{ 1 - \sum_{j=1}^{m} \frac{\sum_{i \in \mu_j} p_i^2}{\sum_{i \in \mu_j} p_i} \right\} \quad .$$

This new work function considered as a function of the partitioning of the classification, differs only by a constant factor from the function for the simpler case and is therefore minimized for the same partition as before.

Scheduling of Trains to Minimize Accumulation Delay

11.1 Introductory Remarks

In the preceding chapters we have discussed classification policy without much explicit reference to the scheduling of trains between yards. In the present chapter, we shall discuss scheduling policy without much reference to the other aspects of classification policy. In addition, we disregard any restrictions on schedules that may result from limited track capacity. The integration of the study of classification and scheduling under conditions of limited track capacity is a task that surpasses the scope of the present studies.

Implicit in the analysis of this chapter are therefore the assumption that track capacity is sufficient to accommodate without congestion any schedule under consideration and the further assumption that none of the schedules considered will raise classification costs by creating congestion at yards.

In fact, even if the scheduling problem is thus artificially isolated from the other operation problems, it still has great complexity. In this chapter, we shall consider in some detail only the case of a network consisting of a single line on which each train is reclassified at each terminal and on which traffic flows in one direction only. After that, we shall make some more general comments on maintrackers (trains that by-pass intermediate terminals), on two-way traffic, on lines with side branches, and on networks containing circuits.

11.2 A Simple Scheduling Problem Considered

Let us first consider a network consisting of four terminals only, labeled $i = 1, 2, 3, 4$ and arranged along a single line as shown in Figure 1.
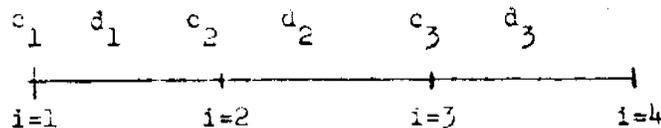

Figure 1. Network Consisting of a Single Line

We shall consider given amounts of traffic (denoted by x_{ij}), expressed in numbers of cars per unit period (day, week), from any terminal i to any terminal j east of it (that is, such that $i < j$). Ignoring westbound traffic in the discussion means assuming that there is no interference of opposite flows on tracks or in classification operations. It also means assuming that there are no "side branches" to our network. The presence of a side branch out of a certain terminal would place a premium on coordinating train departures from that terminal in opposite directions among the main line with the arrival of a train to that terminal from the side branch.

Of course continuing flows of eastbound traffic need to be offset by equal flows of loaded or empty cars in the opposite direction in order to keep the same pool of cars in operation. We shall not include the cost of this return flow of cars in our considerations, but assume that any saving of car-days by greater economy of scheduling eastbound flows, resulting on the average in earlier arrival of cars at destinations, will be a gain that is not offset by delays in the westbound return flow.

We shall assume further that trains run only from any one terminal to the immediately following one. The time required for the trip from i to $i + 1$ will be denoted d_i. It is assumed to be constant and independent of the length of the train. At any terminal, arriving trains are broken up, cars destined for that terminal are removed, and cars originating in that terminal for destinations east are added. The time needed for this operation is again assumed independent of the number of cars involved, and is denoted by c_i, if i labels the terminal in question. If a train is scheduled to leave from terminal i at time t_i, cars can be incorporated in it if and only if delivered for classification, whether from local origins or from incoming traffic, not later than $t_i - c_i$.

We shall discuss only periodic schedules. By this we mean a list of departure times for trains eastbound from each terminal which repeats itself with a constant period. While this might be thought of equally well as a week or as a two-day period, we shall speak of it for simplicity as a "day".

Let us assume that the given program specifies the following daily flows of cars from i to j, where $i < j$,

(11.1)

x_{ij}	$j=2$	3	4
$i=1$	45	15	20
2		110	25
3			30

It is useful also to register for each of the three stretches of line the total daily flows arising from these figures,

$$(11.2) \quad \begin{cases} \text{on } (1,2) & x_{12} + x_{13} + x_{14} & = 80 \\ \text{on } (2,3) & x_{13} + x_{14} + x_{23} + x_{24} & = 170 \\ \text{on } (3,4) & x_{14} + x_{24} + x_{34} & = 95 \, . \end{cases}$$

For simplicity, we shall assume further, against better knowledge, that these cars become available at each originating terminal at an even rate around the clock. This assumption will enable us to consider more effectively other aspects of the scheduling problem. Ultimately, of course it will have to be replaced by a more realistic specification.

The first question arising in making up a schedule is how many trains per day to run on each of the three stretches. On the basis of the figures in (11.3), let us somewhat arbitrarily decide that we shall schedule one train daily on the stretches (1,2) and (3,4), and two trains daily on (2,3). Then the trains on (1,2) and (3,4) will contain 80 and 95 cars, respectively, while the two daily trains on (2,3) will average $\frac{170}{2} = 85$ cars. A possible daily schedule, which has the appeal of simplicity and symmetry, is that represented by Figure 2.
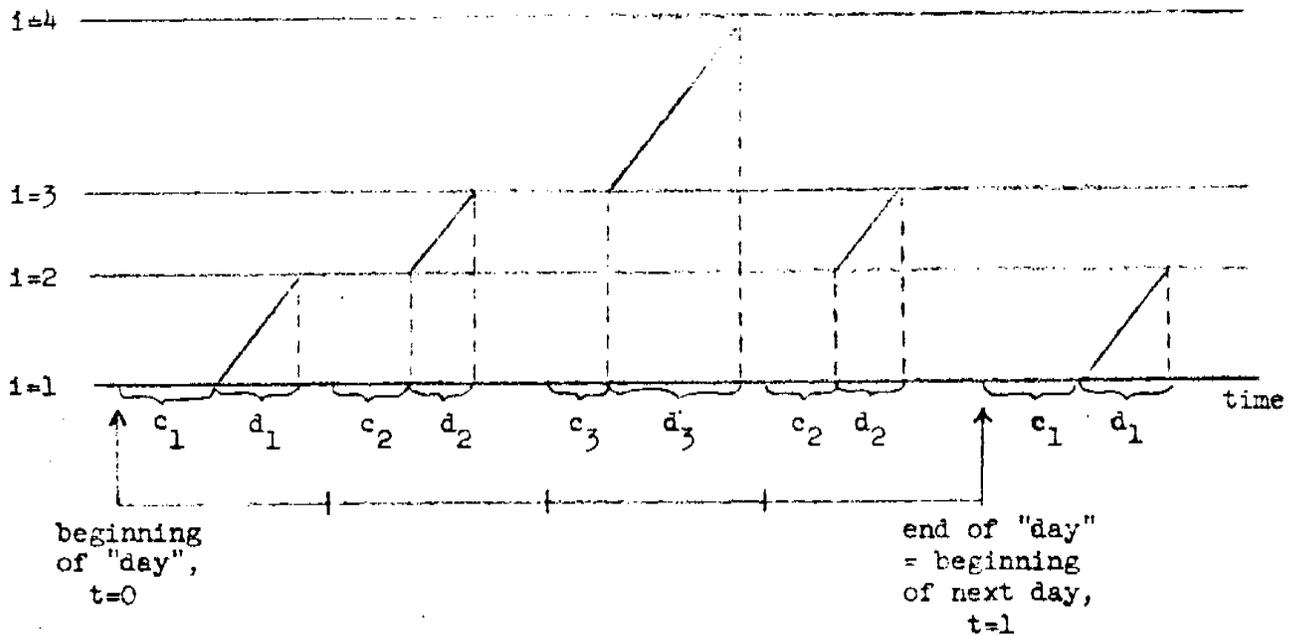
Figure 2. Tentative Schedule for the Program (11.2)

The slanted straight lines represent actual travel of trains at constant "speed." However, the straightness of the lines, and the interpretation of the vertical axis as "distance" from terminal 1 are immaterial. All that matters is that there is a constant travel time d_i on each stretch, and a constant minimum classification time c_i for inclusion in each train.

Before going into the question of the suitability of this schedule for the program (11.2) at hand, it will be useful to introduce a simplification in the treatment of time. For simplicity, let the point marked "beginning of day" at terminal $i = 1$ correspond to midnight. Then we decide to measure time at terminal 2, for the purposes of our analysis, on a clock which shows midnight when it is actually $(c_1 + d_1)$ beyond midnight, that is, at the earliest possible time of arrival at the terminal 2 in question of a train for which classification operations are started at terminal 1 at midnight. Similarly, at terminal 3, we measure time on a clock showing midnight at a

still later time, later by the corresponding interval $(c_2 + d_2)$ between starting classification at terminal 2 and earliest possible arrival at 3, etc. Let us call this artificial time scale (which is different in different terminals) "referred time" and denote it by the symbol s. This is a time measurement "referred" to actual time at terminal 1, in the sense that, at 11 A.M. actual time say, the referred time at some terminal i is measured by the latest actual time at which a car could conceivably enter classification at terminal 1 and by an uninterrupted sequence of classification operations and travel in suitably scheduled trains reach terminal i at 11 A.M. If actual time is denoted by t, actual and referred time are related as follows:

$$(11.3) \quad \begin{cases} \text{at terminal } 1, & s = t, \\ \text{"} \quad \text{"} \quad 2, & s = t - c_1 - d_1, \\ \text{"} \quad \text{"} \quad 3, & s = t - c_1 - d_1 - c_2 - d_2, \\ \text{"} \quad \text{"} \quad 4, & s = t - c_1 - d_1 - c_2 - d_2 - c_3 - d_3. \end{cases}$$

In a single-line network with n terminals, referred time at the i-th yard is similarly given by

$$(11.4) \quad s = t - c_1 - d_1 - c_2 - d_2 - \cdots - c_{i-1} - d_{i-1} .$$

The advantage of the use of referred time lies in the fact that the referred times of the starting of classification of a train at a terminal i and of its arrival at the next terminal $i + 1$ are the same. Hence, in Figure 3 eastbound trains can now be represented by vertical arrows. Each such train is fully defined by its referred "timing" s and by the stretch $(i, i + 1)$ on which it travels. The times required for classification and
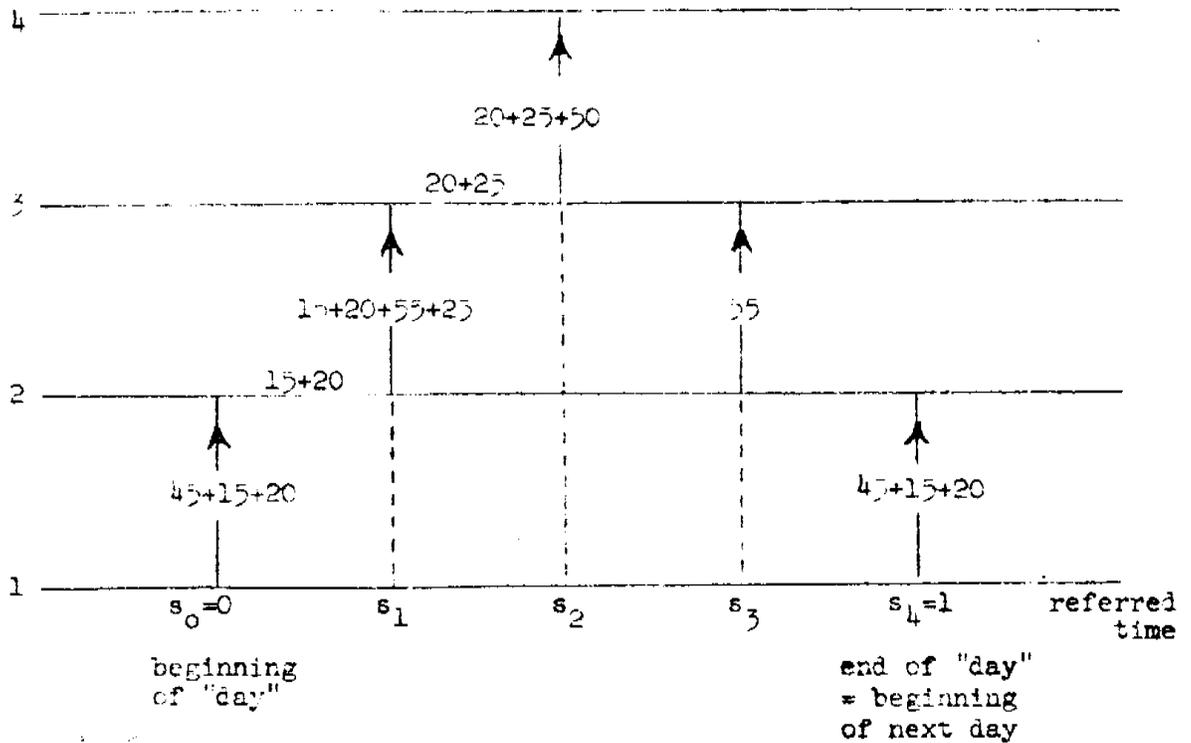
terminals

Figure 3. Schedule of Figure 2 Expressed in Referred Time.

travel (by assumption constant!) no longer clutter up our diagram.
For a car to make a connection between trains at a given terminal, all that
is necessary is that the arrow representing the incoming train is not to the
right of the arrow representing the outgoing train. From here on, we shall
often omit the word "referred" in the phrases "referred time" and "referred
timing."

Let us now have a close look at the particular schedule of Figure 2 repre-
sented in referred time by Figure 3. We shall first ask the question whether this
schedule could be improved without adding or taking out trains, merely by changes
in the timing of trains. If we take the day as the unit of time, we can write
$s_0 = 0$ and $s_4 = 1$ for the timing of the $(1,2)$-train at "beginning" and "end" of

day. The quantities to be chosen then are the timings s_1, s_2, s_3 of the first $(2,3)$-train, the $(3,4)$-train, and the second $(2,3)$-train, to be repeated periodically on subsequent days. The choice is to be made in the light of the effect of changes in these timings on various costs.

In the present simple example, the main effect is on accumulation delay, here represented as the waiting of cars for classification operations to start for the next train, although in actual operation this waiting may occur as well after classification. There are two categories of such cars, which we shall call locally originated cars and cars in transit. At any yard 1 the number of locally originated cars (i.e., originated at 1) that are waiting increases with time, by assumption at a constant rate, until classification starts, at which time the number falls to zero, and thereafter grows again. For schedules with equal time intervals between trains on each stretch, as exhibited in Figure 3, the average number of waiting cars in this category depends, for a given rate of local origination, only on the frequency of trains (one, two, or more per day), but is independent of the timing of the first train in each day. Hence, as long as we keep $s_3 = s_1 + \frac{1}{2}$, we can change s_1 and s_2 without affecting the number of cars waiting in this category.

More interesting is the second category of cars, those waiting for connections between trains. Figure 3 shows with each arrow the number of cars assignable to the corresponding train. It also shows the number of cars waiting in transit (in the second category), written in with the time interval during which they wait. (The figure does not record the variable numbers of waiting cars in the category of locally originated cars).

A moment's reflection will show that in the present simple case, there

is no need for waiting between connections at all. If we make $s_2 = s_1 = 0$

(and keep $s_3 = s_1 + \frac{1}{2} = \frac{1}{2}$) as in Figure 4 we cut out all the accumulation
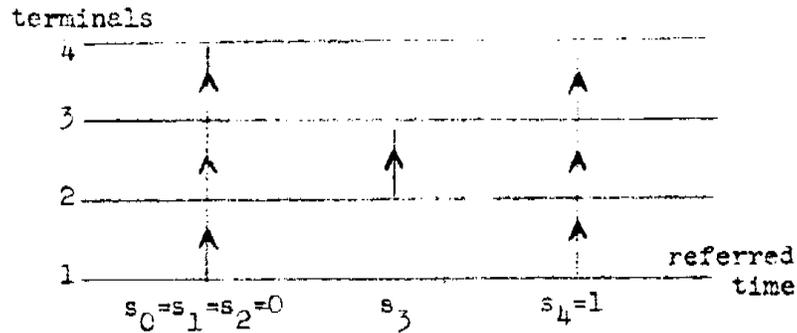
delay between

Figure 4. Improved Schedule for Program (2).

connections at terminals 2 and 3 without incurring any costs elsewhere.

Obviously, then, the schedule of Figure 4 is superior to that of Figure 3.

Of course, s_2 should not be made less than s_1, or s_1 less than zero,

because in that case connections are missed, and accumulation delay greatly

increases.
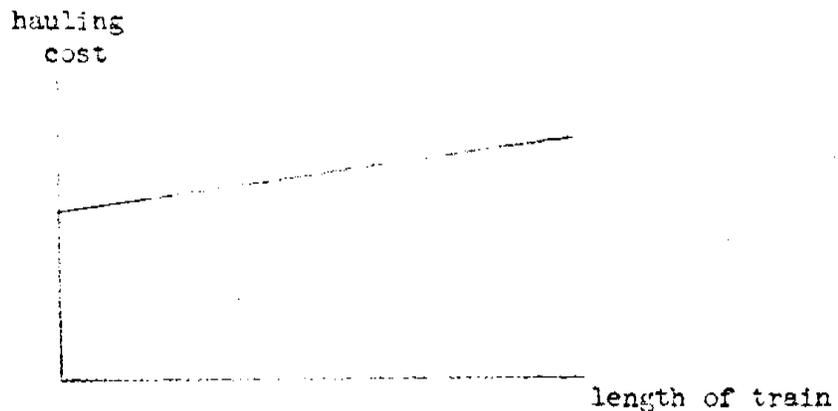
Figure 5. Hauling Cost and Length
of Train

Let us finally ask whether $\frac{1}{2}$ is the best value for s_3. Variation of s_3 affects the accumulation delay of locally originated cars at terminal 2. It may also affect hauling cost on the stretch (2 3). Making s_3 larger shifts cars from the 115-car train timed at $s_y = s_1 + 1$ to the 5-car train timed at s_5. This will save hauling cost if the cost of an extra car to a long train exceeds that of an extra car to a short one. Beyond a certain length of train this will undoubtedly be the case. For the present, we shall assume that on each stretch, within the range of train lengths considered, the hauling cost of an extra car is independent of the length of the train. This is indicated by the constant slope of the hauling cost curve in Figure 5. In that case, changes in s_3 have no consequences for hauling cost.

It remains to trace the effect of a change in s_3 on accumulation delay. In Figure 6 the total of this delay incurred each day is represented by the sum of the areas of two triangles bounded by the horizontal axis and the straight-line segments representing number of cars waiting at any time. This sum equals
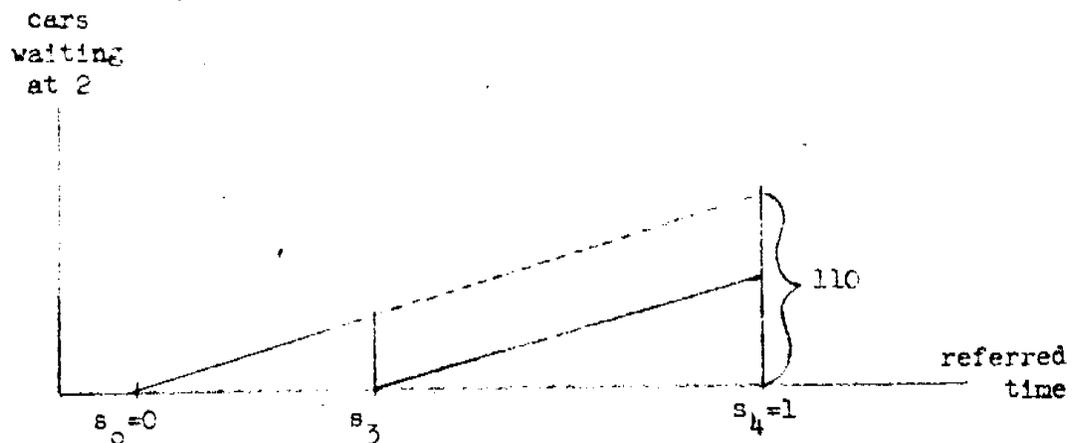

Figure 6. Minimization of Accumulation Delay

$$(11.3) \qquad \frac{1}{2} s_3 \cdot (110 \, s_3) + \frac{1}{2} (1 - s_3) \cdot 110 \, (1 - s_3) = 55 \left[s_3^2 + (1 - s_3)^2 \right] ,$$

where the figure 110 represents the daily flow of (2,3)-cars according to the program (11.2). It is easily seen by means of calculus that this expression is minimized by $s_3 = \frac{1}{2}$, that is, by making the intervals between successive trains equal.

The reader may wonder why the 25 cars originating daily at terminal 2 for terminal 4 have not been included in this calculation. A moment's reflection will show that this would not have affected the outcome. It is readily seen from Figure 4 that cars in this flow could not arrive at terminal 4 any earlier by taking the train timed at s_3 as far as 3. This would merely transfer the location at which (and the "category" in which) a part of their waiting time is incurred, not the total amount of waiting. For this reason, we have in Figure 3 arbitrarily assigned all such cars to the s_1-train to avoid ambiguity in the definition of categories.

In order that the meaning of the "best" schedule in Figure 4 be entirely clear, we translate it back to natural time in Figure 7.

The simple case considered so far suggests that, if on two successive stretches we schedule trains with equal frequency, there is no reason for introducing time intervals between arrival and departure of "connecting" trains beyond the minimum necessary for the classification operations that permit cars to make the connection. It also suggests that if more than one train a day is scheduled for a flow or group of flows in a program that permits us to look at this scheduling problem in isolation, constant rates of origination during the day call for equal time intervals between successive trains.
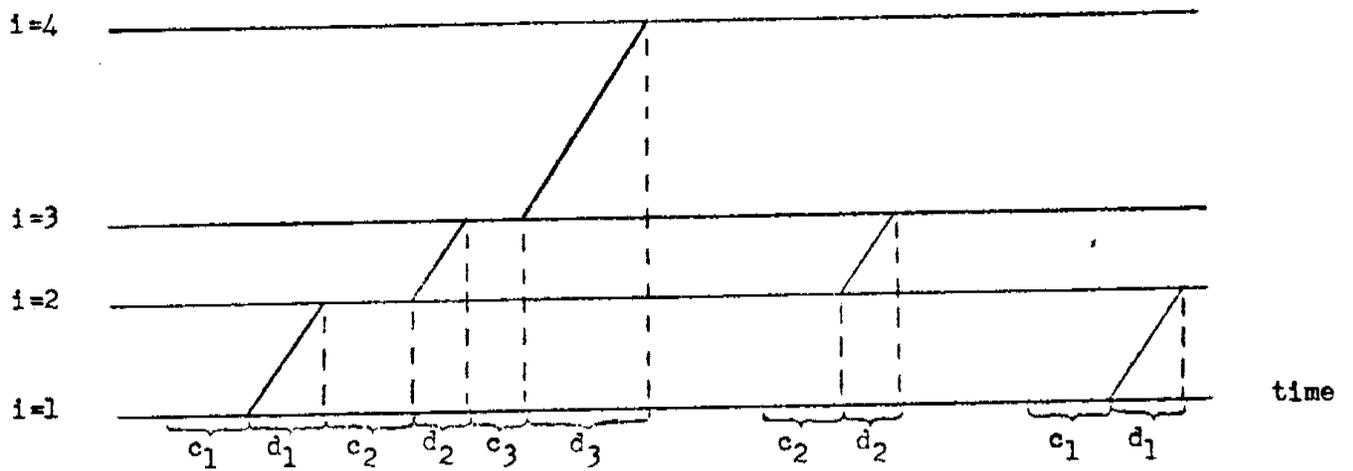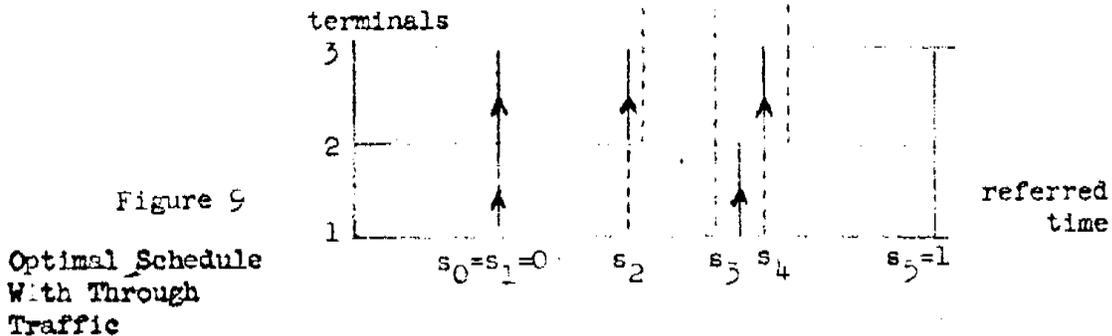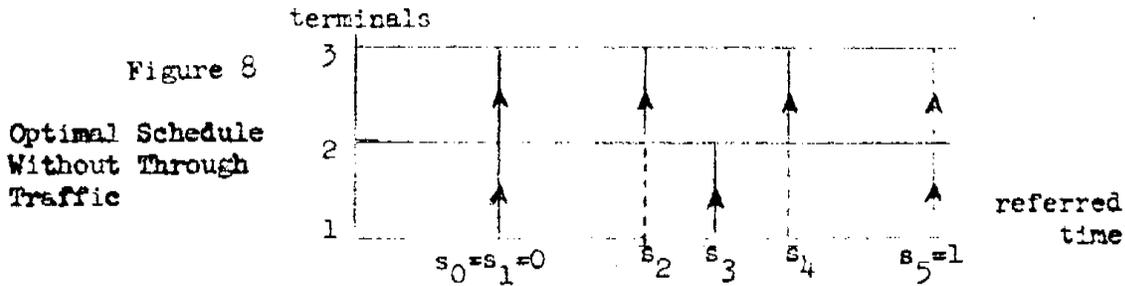
Figure 7. The Schedule of Figure 4
in Natural Time

The latter inference does not extend to cases where traffic flows extend over successive stretches, whenever trains run on these stretches with different frequencies neither one of which is an integral multiple of the other. For instance, while the schedule of Figure 8 would be optimal for a program with three terminals such that $x_{13} = 0$, $x_{12} = 180$, $x_{23} = 270$, the introduction of a positive flow x_{13} straddling both stretches of line would call for s_3 and s_4 to approach each other (see Figure 9), with s_2 remaining midway between s_0 and s_4. Beyond a certain critical amount of $(1,3)$-traffic density s_3 and s_4 would actually coincide.

Figure 8

Optimal Schedule
Without Through
Traffic

Figure 9

Optimal Schedule
With Through
Traffic

The broad outlines of the scheduling problem on a single line with constant rates of origination begin to show up from the preceding discussion. Daily frequency of trains on a given stretch is to be related to the total amount of traffic using that stretch. There is an advantage in running trains on successive stretches as closely timed as possible without causing cars to miss connections. This advantage is stronger, the larger the amount of traffic using several successive stretches. On the other hand, there is an advantage in equal spacing of successive trains on the same stretch. This advantage is stronger, the less the through-flows of traffic tying the scheduling on this stretch up with that on other stretches where different frequencies of trains are more economical. Thus where local traffic is relatively high trains will tend to be spaced out evenly over the day; where through traffic is high at least some trains will be scheduled to meet connections more closely.

In the next section, we illustrate how, through the use of calculus and of computation procedures based on trial and error, the best compromise between these considerations can be found when they pull in different directions. This section can be passed over by the reader who is not interested in mathematical aspects of the problem.

11.3* Schedules of Given Structure that Minimize Accumulation Delay.

In this section we shall maintain the assumptions of a one-way single line network, with terminals $i = 1, 2, \ldots, n$; of constant rates of origination x_{ij} on all routes such that $1 \leq i < j \leq n$; of constant classification times c_i and travel times d_i; of a periodical schedule; and of a hauling cost on each stretch which depends linearly on length of train. The discussion in the preceding section has suggested that it will often be economical to

telescope trains on successive stretches of line. Let us therefore call
a _run_ any (maximal) sequence of trains on successive stretches timed so as
to leave only the minimum interval needed for classification between arrivals
and departures at connecting terminals. In terms of referred time, a run
is a (maximal) sequence of trains on successive stretches with identical
referred arrival times, to be called also the (referred) _timing_ of the run.
A run will be denoted by $(s; i, j)$ where s is the (referred) time of arrival
anywhere, i the terminal of departure of the first train of the run, j
the terminal of arrival of the last one, $1 \leq i$, $i \leq j - 1$, $j \leq n$. These
terminals will be called beginning and end of the run. The word "maximal"
is inserted in the definition because, in the schedule of Figure 4, for
instance, we do not wish to consider the sequence $(s_0; 1, 3)$ as a run, be-
cause it can be extended to a larger sequence $(s_0; 1, 4)$ which also meets
the other specification in the definition. The latter sequence is regarded
as a run because no further such extension is possible. A single train can
also be a run, namely, if there is no connecting train with the same (referred)
timing at either its departure or arrival point.

A (periodic) schedule can now be regarded as a statement of all runs
$(s_r; i_r, j_r)$, $r = 0, \ldots, R - 1$ provided for in one day. The periodicity
can be expressed by the requirements

$$(11.6) \qquad s_{r+R} = s_r + 1, \qquad i_{r+R} = i_r, \qquad j_{r+R} = j_r,$$

and the time-scale can be made definite by requiring

$$(11.7) \qquad s_0 = 0.$$

It will be natural to label runs in order of increasing s_r. However, this would still leave an ambiguity when two runs have identical timing, and anyway it would be more restrictive than is desirable. For purposes that will become clear below, we will permit any labeling of runs such that, within the set of all runs that "contain" any given terminal i, that is, of all runs r such that

(11.8) $$i_r \leq i \leq j_r,$$

the timing s_r shall increase whenever r increases. Under this rule, any of the three schedules in Figure 10 are properly labeled, even though in each of them the labels $r = 2$ and $r = 3$ could as well have been interchanged.
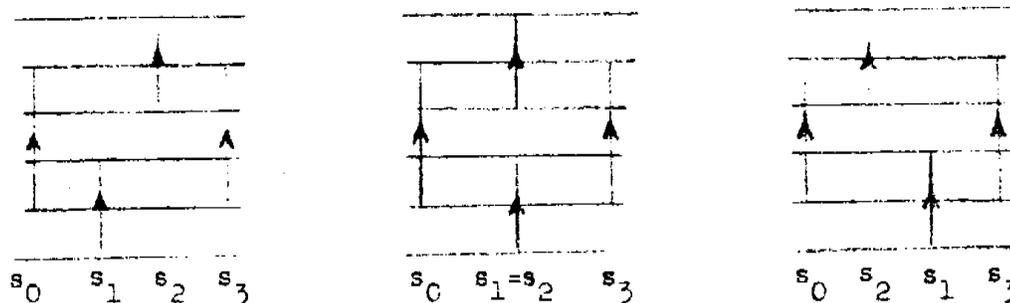

Figure 10. Three Schedules with Identical Structures

Another suggestion implied in previous discussions is that it may be useful to separate the problem of determining the precise timing of the runs in a schedule from the problem of how many runs to have, which terminals to select as their endpoints, and in what order to place these runs on the referred time scale. To express the latter choices we shall introduce the

notion of the structure of a schedule. We shall say that two schedules have the same structure if we can label the runs of each consistently with the above rule of labeling, in such a way that corresponding runs (i.e., runs with the same label) have identical beginning and end points. Under this definition, the three schedules of Figure 10, for instance have identical structures. On the other hand, the two schedules in Figure 11 have different structures.

Figure 11. Two Schedules with Different Structures

Thus, any two schedules of which one is obtainable from the other by horizontal displacements of arrows without any two arrows either touching or overlapping during the displacement, have the same structure. Any two schedules for which this is not possible have different structures.

We now recall our assumption that hauling cost on each stretch is a linear function of train length. Total daily hauling cost therefore depends only on the number of trains per day scheduled for each stretch, and not on how the daily quota of cars on each stretch is distributed over successive trains. In particular, then, hauling cost is the same for all schedules with the same structure, and that irrespective of how cars are assigned to trains within any given schedule.

With hauling cost constant, the problem of finding the most economical schedule of given structure (the most economical timings for its runs) is a problem of minimizing the accumulation delay incurred daily in carrying out the given program. We shall show that this problem is equivalent to solving a system of R-1 linear equations in an equal number of unknowns: the timings of all runs in a day except the first. We shall also indicate how the data of the problem, that is, the given transportation program x_{ij}, and the given structure of the schedule, enter into the coefficients of the equation system. Algebraic expressions will apply to a line with n terminals and an arbitrary schedule structure. The latter is represented by a set of R runs $(s_r; i_r. j_r)$ of which the endpoints (i_r, j_r) are given for $r = 0,...., R - 1$, and of which the timings are restricted, apart from the periodicity requirements (11.6) and (11.7), only by the stipulation that, for any terminal i, the timings s_r of those runs that contain i, as defined by (11.8), form a sequence increasing as r increases.

Before studying the minimization of accumulation delay by proper choice of a schedule from among those of given structure, we must consider for a moment the assignment of cars to trains in a given schedule (that is, with timings of runs given as well). Even here there is a problem, although a trivial one, of minimizing accumulation delay. Waste of car-days would result if cars by being held over to later trains would miss connections and thereby arrive at destinations later than they could have. A simple rule that is sure to avoid this mistake would be to let each car proceed with the first (east-bound) train that leaves. However, for purposes of analysis only, we prefer to specify another assignment rule which likewise avoids the kind of waste we are now considering. This rule has the advantage of giving us readily a simpler formula for the minimum accumulation delay associated with a given

schedule. It specifies that a car originating in i at some time s for destination j should be incorporated at i in the latest train that permits it, through subsequent connections if necessary, to arrive at j at the earliest possible moment.

A situation where this rule differs from the earliest-possible-departure rule was already encountered in Figure 4. A car originating at 2 just before s_3 with destination 4 has as earliest arrival time s_4, and can achieve this by leaving 2 either at time s_3 or at s_4. In practice, of course, such a choice would be determined by other considerations such as the lengths of the trains in question. Where under our simplifying assumptions these considerations are inactive, we choose our assignment rule merely to help the analysis forward. The rule as given does not always completely specify the assignment. At some connection point k between i and j another situation may arise where the passing up of a departing train does not delay earliest possible arrival at j. While it would be easy to extend our assignment rule to create definiteness at all intermediate points, we will not need to do so for our present purpose.

We shall now define, for each traffic flow x_{ij} between given terminals i and j, what we shall call critical departure times v_q^{ij} from i and critical arrival times w_q^{ij} at j, where $q = 1, 2, \ldots q_{ij}$. We shall illustrate the definition with the example of $(i,j) = (1,5)$-traffic in the schedule of Figure 12. Let r_1 be the label r of the lowest-numbered run $(s_r; i_r, j_r)$ in the schedule $r = 0, \ldots R - 1$, containing a train that leaves i. In Figure 12, we find $r_1 = 0$, because the first train in the run labeled $r = 0$ leaves terminal $i = 1$. We define w_1^{ij} as the earliest possible arrival time at j for a car available at i at time s_{r_1}. In Figure 12 we find $w_1^{15} = s_0$ because run $(0; 1, 5)$ reaches as far as terminal

$j = 5$. We also define v_1^{ij} as the latest departure time from i that permits a car to arrive at j at time w_1^{ij}. In Figure 12, this is again $w_1^{15} = s_0$.
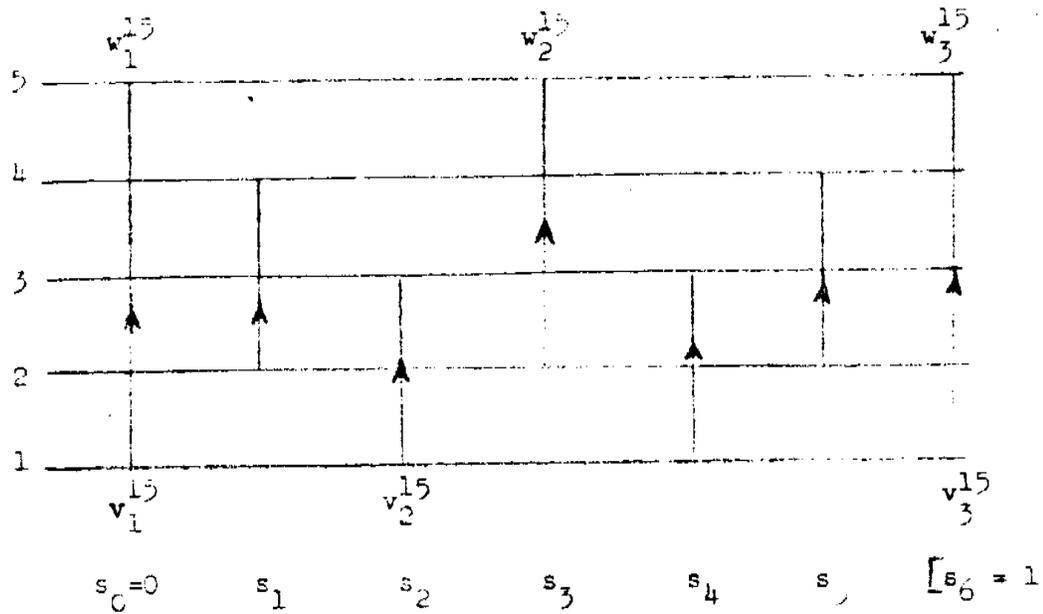

Figure 12. Example of a Schedule.

Next we define w_2^{ij} as the earliest arrival time at j of a car originating at i immediately following the critical time v_1^{ij} for earlier arrival, and v_2^{ij} as the latest departure time from i permitting arrival at j at time w_2^{ij}. In Figure 12, we have $w_2^{15} = s_3$, and $v_2^{15} = s_2$. In this manner we go on until we reach the end of the daily schedule. In Figure 12, $w_3^{15} = w_1^{15} + 1$, and hence we define $Q_{15} = 2$ as the number of critical (departure and arrival) times found within one day's schedule. That is, we choose Q_{ij} so that $w_{Q_{ij}}^{ij} + 1 = w_1^{ij} + 1$. It is clear that the number Q_{ij} and the location in time of critical points can vary with the pair of terminals (i,j) considered. For instance, for $(1,4)$-traffic in Figure 12 we have $Q_{14} = 3$.

The accumulation delay sustained by (i,j)-cars under our assignment rule can now be expressed as follows. During the first "critical interval" at i, that is during the time period

(11.9) $$v_1^{ij} < s \leq v_2^{ij},$$

the number of n of (i,j)-cars waiting at time s is a linear function of s,

(11.10) $$n = x_{ij}(s - v_1^{ij}).$$

Expressed in car-days, the total delay incurred by these cars in this interval is (omitting temporarily the subscripts and superscripts i,j)

(11.11) $$\int_{v_1}^{v_2} x(s - v_1) \, ds = \frac{1}{2} x (v_2 - v_1)^2.$$

At time v_2 a block of $x(v_2 - v_1)$ cars is ready to leave. Before it reaches j at the critical arrival time w_2, each car sustains another delay of length $(w_2 - v_2)$, where the place incurred (but not the length) of this delay may depend on further assignment decisions that we have not bothered to specify. Total delay of this kind, in car-days, therefore equals

(11.12) $$x (v_2 - v_1) (w_2 - v_2).$$

The two types of delays represented by (11.11) and (11.12) have to be added up for all critical intervals in one day's schedule, to give us the accumulation delay y for any one traffic flow

$$(11.13) \qquad y = x \sum_{q=1}^{Q} \left[\frac{1}{2} (v_{q+1} - v_q)^2 + (v_{q+1} - v_q)(w_{q+1} - v_{q+1}) \right].$$

Finally, to obtain total accumulation delay Y, indices i,j designating the traffic flow have to be restored to this expression, whereupon these amounts need to be summed over all traffic flows (i,j).

$$(11.14) \qquad Y = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} x_{ij} A_{ij},$$

where

$$(11.15) \qquad A_{ij} = \sum_{q=1}^{Q_{ij}} \left[\frac{1}{2} (v_{q+1}^{ij} - v_q^{ij})^2 + (v_{q+1}^{ij} - v_q^{ij})(w_{q+1}^{ij} - v_{q+1}^{ij}) \right].$$

The expression for Y we need to minimize is given by (11.14) and (11.15). We note that it is a quadratic function of the unknown timings s_r, which occur in the guise of critical arrival or departure times associated with particular flows (i,j). None of the unknowns is missed, because the timing s_r of any run $(s_r; i_r, r)$ always coincides with a pair of critical departure and arrival times for all flows (i,j) between terminals contained in that run, and we would not include in a proposed schedule any run on which there is no traffic between any two terminals of the run (i.e., any run $(s; i,j)$ for which $x_{m,n} = 0$ for all m,n such that $i \leq m < n \leq j$).

For any given schedule structure, the expression (11.14) can be set up by a graphical procedure which we illustrate in Figure 13, based on a schedule of the structure exhibited in Figure 12. Dotted lines immediately to the left of the vertical lines that represent runs are traced upward, those to the right downward. Dotted lines immediately above the horizontal lines that represent
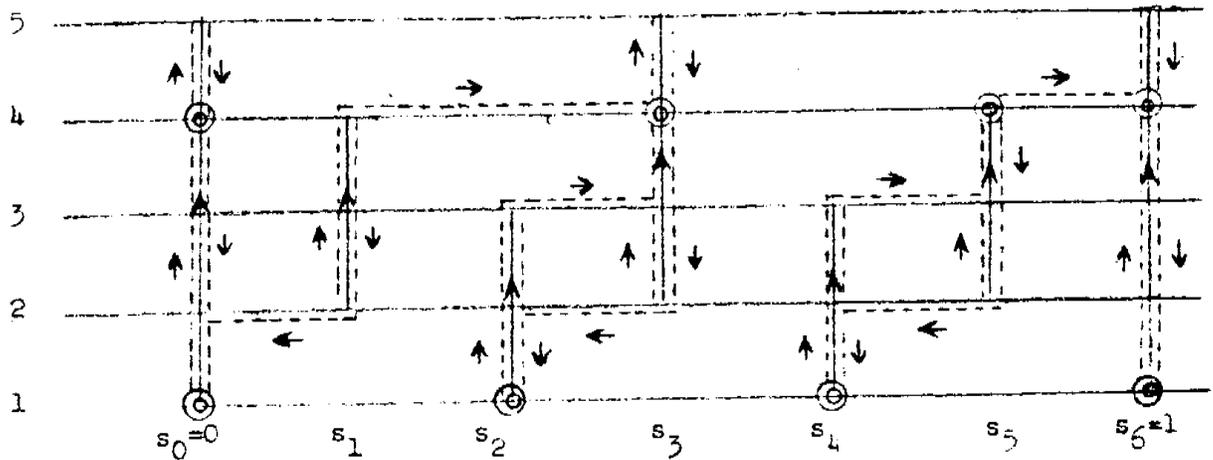
Figure 13. Determination of Critical Times for the Schedule of Figure 12.

terminals start from the endpoint of a run and are traced to the right up to the next run that reaches further. Dotted lines immediately below horizontal lines start from the starting point of a run and are traced to the left back to the nearest preceding run that starts at an earlier terminal. No dotted lines are traced along the horizontals for the first and last terminals $i = 1$ or . For any flow (i,j), the critical times v_q^{ij}, w_q^{ij} are found as pairs of points, at the levels i and j respectively, such that from each point of a pair, the other can be reached by following dotted lines in the direction indicated by the arrows. As an additional example, in Figure 13 these pairs of points have been encircled for $(i,j) = (1,4)$, showing that

$$(11.16) \qquad v_1^{14} = v_1^{14} \quad s_0 = 0, \qquad v_2^{14} = s_2, \qquad w_2^{14} = s_3,$$

$$v_3^{14} = s_4, \qquad w_3^{14} = s_5, \qquad v_4^{14} = w_4^{14} = s_6 = 1, \text{ etc.}$$

The system of dotted lines serves simultaneously for the determination of critical points for all flows (i,j). It is interesting to note that, although we are considering traffic in only one (eastbound) direction, the construction of critical times is entirely symmetrical as between departure and arrival times.

Each of the expressions (11.16) is a quadratic function of the timings s_r in the schedule, and can therefore be written as

$$(11.17) \qquad A_{ij} = \sum_{r,r'=1}^{R} a_{ij,rr'} s_r s_{r'} \quad ,$$

in which we are free to make the matrix $a_{ij,rr'}$ symmetric by specifying

$$(11.18) \qquad a_{ij,rs} = a_{ij,sr} \quad .$$

Because $s_0 = 0$, the value $r = 0$ does not occur in the summations over runs. However, the summation has been extended to include $r = R$, where $s_R = 1$, in order to include in it terms linear in the unknowns $s_1, \ldots, s_{R-1}$ and a constant term $a_{ij,RR}$. These terms arise as follows. Because of the periodicity of the schedule (omitting affixes i,j), we have

$$(11.19) \qquad v_{Q+1} = v_1 + 1 \quad .$$

While both v_1 and v_{Q+1} occur in the summation (11.13), they depend on at most one unknown s_r. The use of (11.19) to express this fact will in general introduce linear and constant terms in (11.17). Other such terms arise if a certain w_q, $q \leq Q$, coincides with a certain s_r, $r \geq R$.

In the notation of (11.17), the expression of total accumulation delay incurred daily, which is to be minimized, is

$$(11.20) \qquad Y = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} x_{ij} \sum_{r,r'=1}^{R} a_{ij,rr'} s_r s_{r'} \; .$$

It is apparent from the method of their derivation that the coefficients $a_{ij,rr'}$ depend only on the structure of the schedule. The problem of minimizing accumulation delay for a schedule of a given structure therefore involves setting up the coefficient system $a_{ij,rr'}$, evaluating the sums

$$(11.21) \qquad \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} x_{ij} a_{ij,rr'} = b_{rr'} = b_{rr'}$$

say, and solving the linear equation system

$$(11.22) \qquad \frac{\partial Y}{\partial s_m} = \frac{\partial}{\partial s_m} \sum_{r,r'=1}^{R} b_{rr'} s_r s_{r'} = 2 \sum_{r=1}^{R} b_{mr} s_r = 0, \quad m = 1,\ldots,R-1.$$

Before a solution $\hat{s}_r$, $r = 1,\ldots, R - 1$, say, to this equation system can be looked upon as a most economical schedule of the given structure, it needs to pass two tests. In the first place, it should represent a minimum of accumulation delay, not a maximum or a saddle-point. The outcome of this test is controlled by the coefficients $b_{rr'}$. If these make up a so-called positive definite matrix, the solution represents a minimum of the function being minimized. If the matrix $[b_{rr'}]$ were to be indefinite, this is an indication that there exists another structure containing fewer runs, and no more trains, permitting a smaller accumulation delay than the structure under estimation.

Secondly, even if $[b_{rr'}]$ is positive definite, the solution $\hat{s}_r$ found represents the minimum accumulation delay achievable with the given structure only if $\hat{s}_r$ defines a schedule that actually has the structure specified. If this fails to be true, the solution has no meaning, because outside the domain of the given structure, the function minimized does not represent accumulation delay. However, if such a meaningless solution is obtained, this again indicates that there is another structure containing fewer runs and no more trains, permitting a smaller accumulation delay.

It is likely that the special way in which the coefficients $b_{rr'}$ are obtained will make it possible to find short cuts in examining the positive definiteness of $[b_r]$ and in solving the equations (11.22). In this exploratory study of the general nature of the problem, we have not attempted to look for such short cuts.

As an illustration, we will tabulate the coefficients of the equation (11.22) for the schedule structure of Figure 12. The first step is to record the critical departure and arrival times for all traffic flows (i,j). In Table I, for each i and j, we tabulate the run labels of the successive pairs of critical points. By this we mean the numbers $r(q)$ and $\bar{r}(q)$, say, which when used as subscripts to s make

$$(11.23) \qquad v_q^{ij} = s_{r(q)}, \qquad w_q^{ij} = s_{\bar{r}(q)}, \qquad q = 1,\ldots, Q_{ij} .$$

To obtain the coefficients b_{mr} of (11.22), we derive from (11.14)

$$(11.24) \qquad \frac{\partial Y}{\partial s_m}\Big] = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} x_{ij} \frac{\partial A_{ij}}{\partial s_m} ,$$

Table I. Run Labels of Critical Departure and Arrival
Times for the Schedule Structure of Figure 12

Terminal of Origin	Terminal of Destination			
	$j=2$	3	4	5
$i = 1$	$q\ \ = 1\ \ 2\ \ 3$ $r(q) = 0\ \ 2\ \ 4$ $\bar{r}(q) = 0\ \ 2\ \ 4$	1 2 3 0 2 4 0 2 4	1 2 3 0 2 4 0 3 5	1 2 0 2 0 3
2		1 2 3 4 5 6 0 1 2 3 4 5 0 1 2 3 4 5	1 2 3 4 0 1 3 5 0 1 3 5	1 2 0 3 0 3
3			1 2 3 4 0 1 3 5 0 1 3 5	1 2 0 3 0 3
4				1 2 0 3 0 3

where omitting again the indices i,j,

$$(11.25) \qquad \frac{\partial A}{\partial s_m} \bigg] = \sum_{q=1}^{c} \frac{\partial A}{\partial v_q} \frac{dv_q}{\partial s_m} + \frac{\partial A}{\partial w_q} \bigg] \frac{dw_q}{\partial s_m} \quad .$$

From (11.19) we have

$$(11.26) \qquad \frac{\partial A}{\partial v_q} = w_q - w_{q+1}, \qquad \frac{\partial A}{\partial w_q} = v_q - v_{q-1} ,$$

whereas the quantities $\dfrac{dv_q}{ds_m}$, $\dfrac{dw_q}{ds_m}$, each of which is 0 or 1, can be determined from Table I for each (i,j). For instance, for $i = 1$, $j = 4$, we have

$\dfrac{dv_q}{ds_m}^{14}$	$q=1$	2	3	$\dfrac{dw_q}{ds_m}^{14}$	$q=1$	2	3
$m=1$	0	0	0	$m=1$	0	0	0
2	0	1	0	2	0	0	0
3	0	0	0	3	0	1	0
4	0	0	1	4	0	0	0
5	0	0	0	5	0	0	1

(11.27)

No rows need to be provided for $m = 0$ in these tabulations, because, since $s_0 = 0$, no differentiation of Y with respect to s_0 is needed. Using (11.27) in (11.25), and also using (11.26) with the v_q and w_q translated into s_r with the help of Table I, we find

$$\frac{\partial A_{14}}{\partial s_1} = 0, \qquad \frac{\partial A_{14}}{\partial s_2} = s_3 - s \qquad \frac{\partial A_{14}}{\partial s_3} = s_2,$$

(11.28)

$$\frac{\partial A_4}{\partial s_4} = s_3 - 1 \qquad \frac{\partial A_{14}}{\partial s_5} = s_4 - s_2.$$

Similar calculations for other (i,j)-pairs lead to similar results, which we put together, in Table II, in a statement of the coefficients of s_r in $\frac{\partial Y}{\partial s_m}$, derived from the information in Table I on the basis of (11.24), (11.25) and (11.26). The reader is invited to examine how the placement of the various entries in Table II is determined from Table I.

11.4 Most Economical Schedules

If a schedule is found that minimizes accumulation delay in a given structure, this does not necessarily mean that the most economical schedule has been found. There may be another structure containing a more economical schedule. However, because different structures may involve different hauling costs as soon as they differ in the number of trains on each stretch, this question can only be considered if we introduce further assumptions as to how hauling cost depends on the structure of the schedule. For instance, we may use the very simplest assumption incorporated in Figure 5 that hauling cost on each stretch is a given linear function of train length, which is independent of the timing of the train in relation to that of other trains on the same or other stretches.

If this function is expressed in dollars per train, it will still be necessary also to place a dollar value on the car-day, and on the day's delay to freight, before a balancing of haulage cost and accumulation delay is

Table II. Coefficients of s_r in $\frac{\partial Y}{\partial s_m}$

	r = 1	2	3	4	5
m=1	$2x_{23}+2x_{24}+2x_{34}$	$-x_{23}$	$-x_{24}-x_{34}$		
2	$-x_{23}$	$2x_{12}+2x_{13}+2x_{23}$	$x_{14}+x_{13},\ -x_{23}$	$-x_{12}-x_{13}$	
3	$-x_{24}-x_{34}$	$x_{14}+x_{13},\ -x_{23}$	$2x_{23}+2x_{24}+2x_{25}+2x_{34}+x_{13},+2x_{24},$	$-x_{23}$	$-x_{24}-x_{34}$
4	.	$-x_{12}-x_{13}$	$-x_{23}$	$2x_{12}+2x_{13}+2x_{23}$	$x_{14}-x_{23}$
5		$-x_{14}$	$-x_{24}-x_{34}$	$x_{14}-x_{23}$	$2x_{23}+2x_{24}+2x_{34}$

possible. All these things being given, the problem of finding a most economical schedule is one of trial and error, where one examines alternative structures, determines the haulage cost for each, the minimum accumulation delay for each, and chooses that structure for which the sum of accumulation delay cost and haulage cost is as small as possible. While again the particular circumstances of the problem are likely to permit short-cuts that reduce the number of comparisons between alternative structures that needs to be made, we will not press the search for such short-cuts here.

11. Scheduling on a Network with Circuits

In the cases considered so far, there is only one way in which a car can go from a terminal i to a terminal j, where $j > i$. This is done by joining successively a train from i to $i + 1$, a train from $i + 1$ to $i + 2$, and so on, and finally a train from $j - 1$ to j. This feature of our model made it possible to introduce the concept of referred time and thereby to achieve a simplification of the analysis.

In the present section we shall consider a number of extensions of the analysis which are best discussed and understood as scheduling problems on networks that contain circuits, even if at first glance this does not always seem to be the case in a literal sense. We shall only give an exploratory discussion of a few such extensions, without attempting a formal analysis. The concept of referred time is likely to be less useful in most of these cases. In principle, one would need to distinguish at a terminal j as many different concepts of "time referred to i" as there are ways in which a car could travel from i to j. We shall therefore use only "natural" time in our discussion. We shall continue to assume that

classification and travel times are independent of the flows of cars being classified or travelling, and that limitations of track capacity do not rule out any of the schedules that are otherwise considered.

It is also unrealistic, on more complicated networks, to ignore the problem of assignment of empty cars to next loading points. This problem is analyzed in more detail in Chapter 12. Here we shall assume that besides the flows of loaded cars prescribed by the program, which we shall here denote by $\bar{x}_{ij}$, most economical flows $\bar{\bar{x}}_{ij}$ of empty cars have already been determined. The scheduling problem thus is that of most economical scheduling of trains to accommodate the flows

$$x_{ij} = \bar{x}_{ij} + \bar{\bar{x}}_{ij}$$

of all cars, loaded and empty. This assumption ignores possible "interaction" between the empty car assignment problem and the scheduling problem. It also ignores possible differences in classification or travel times as between loaded and empty cars.

The analysis of Section 11.4 is still approximately valid for a network consisting of one double-track line, of which each track is travelled in one direction only. Perhaps the weakest spot in the assumptions made there is the assumed even flow of car originations from loading or discharging. The schedule itself indicates that cars are delivered for these operations in bunches. In order that releases of cars from these operations can be treated as an even flow, one would have to assume that there is either enough natural variability in loading or discharging times, or enough congestion in these operations, to convert a bunched input to these activities into an evenly flowing output. In the second case, one would have to recognize congestion

delays at loading or discharging tracks as an additional cost of the bunching of cars in trains.

Further complications arise if the line considered has side branches. In the network of Figure 14, with two-way traffic on each stretch there will be an advantage, at terminals A and B, in scheduling train arrivals and departures in both directions of the "main line" in such a way that,
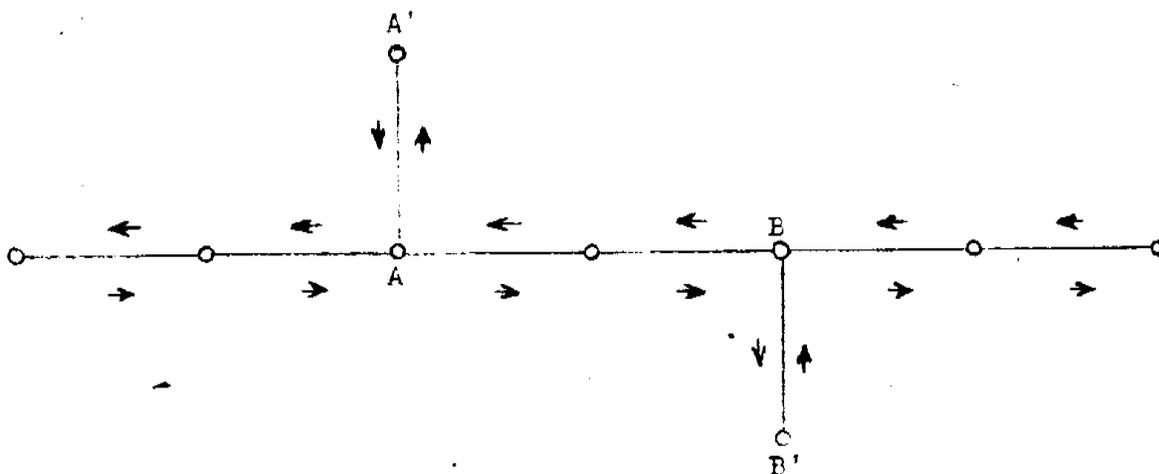

Figure 14.   Main Line with Branches

with suitably coordinated arrivals and departures of trains on the side branches, cars from A' or B' for either direction, and cars for A' and B' from either direction can make connections with little accumulation delay at A or B.   That this is essentially due to the presence of circuits in the network comes out more clearly if, in Figure 15, we draw the same network with a separate line for each direction of flow.
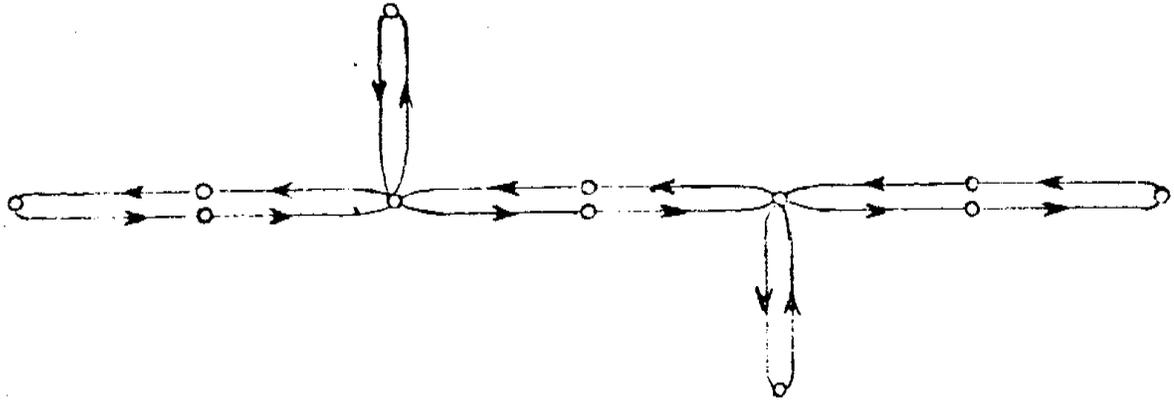
Figure 1. Network of Figure 14 Drawn with One-Way Lines

Circuits also arise. even on a single line with one-way traffic only, if the structure of the schedule includes maintrackers, that is, trains which by-pass certain intermediate yards. The network of Figure 16 illustrates such a case. Diagrams in natural time of the type of Figure 7 can
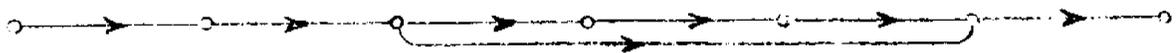

Figure 16. Single-Line. One-Way Network with Maintrackers

be used to analyze scheduling problems on networks such as that in Figure 16, even with two-way traffic and with maintrackers between several pairs of terminals. The limitations of two-dimensional diagrams begin to show up in

networks with branches such as that of Figure 14. and are fully apparent
with regard to networks such as that in Figure 17. in which circuits occur
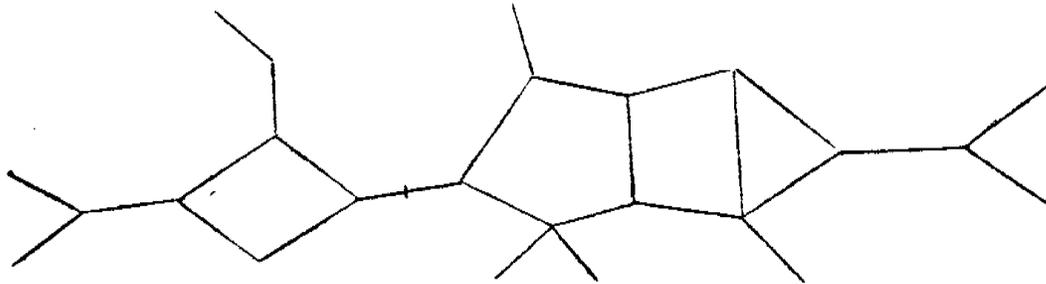even if one disregards direction of flow.

Figure 17. Network Containing Circuits

The method for determining a most economical schedule of given structure
under constant rates of origination, which was described in preceding sections
of this Chapter, can probably be extended without essential difficulties to
arbitrary networks containing any number of circuits. However, the more
complicated the network is, the more the essential part of the scheduling
problem shifts to that of determining the structure, rather than the timing,
of the schedule. The reason for this is that, in rather complicated networks,
often a slight change in the timing of some given train will cause some
traffic flow x_{ij} to miss a connection previously made or permit it to
make a connection previously missed, in either case changing the structure
of the schedule. Thus n networks as complicated as the United States
railroad system, or even a major railroad company network, the first and
foremost problem is that of choosing between a finite but very large number

of alternative schedule structures. Problems of this type are classified
in mathematics as "combinatorial" problems. While basic mathematical tools
to deal with combinatorial problems have been developed by many mathematicians,
their application in computational procedures for solving complicated practical
problems is still in its first beginnings.[1]

In conclusion we wish to offer a few intuitive remarks as to ways of
approaching the problem of choice of schedule structure on networks with
many circuits. It would seem natural to split this problem into two parts.
First, one would find a good schedule on what may be called the "core" of the
network. In the core we include all routes that are part of one circuit or
another. If this were to form a disconnected set of routes (as in the network
of Figure 17, where the routes belonging to circuits form two separate networks)
we add the minimum number of connecting routes necessary to make the core into
one connected network. Figure 18 shows the core so derived from the network
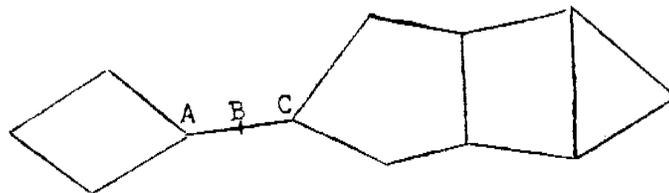of Figure 17. The route A B C has been added to reconnect the core.

Figure 18. Core of the Network in Figure 17

If we remove the core from a network, we are left with a number of
isolated branchings which we shall call trees (in accordance with established
mathematical terminology). The six trees of the network in Figure 17 are

[1] For a survey of computational aspects of combinatorial problems see
Tompkins [1952].

shown in Figure 19. The point where a tree feeds into the core will be called
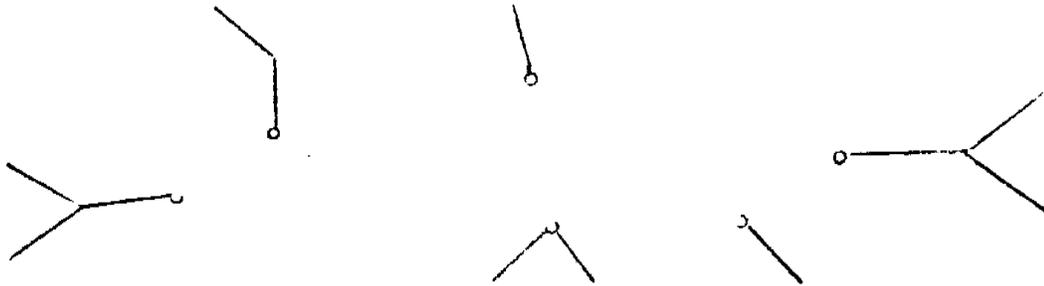its feeding point , shown by a circle in Figure 19.

Figure 19. Trees of the Network in Figure 17

Now a first approximation to the scheduling problem on a network with
circuits could be obtained by treating programmed traffic flows to and from
all terminals of a tree (other than the feeding point) as if they were pro-
grammed to and from a single terminal. The network of Figure 17 is thereby
simplified to that of Figure 20, in which the ramifications of the trees are
ignored. The underlying idea is that the scheduling on a tree is not encum-
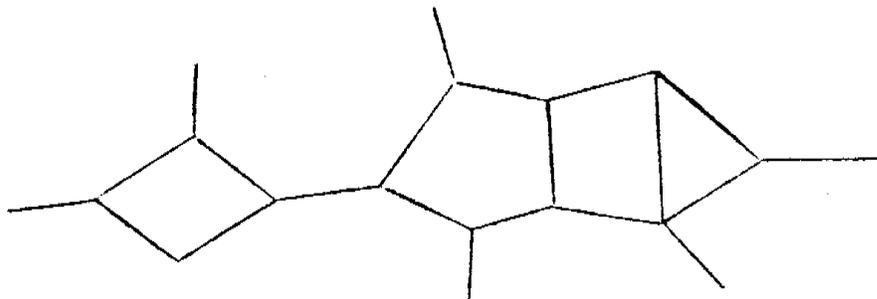bered by circuit problems, and can therefore perhaps be adjusted to whatever

Figure 20. Simplification of the Network of Figure 17

schedule is found to be best on the simplified network. Implicit in this
is the assumption that speedy turnaround of engines at the dead ends of the
trees is a problem of secondary importance, that can be neglected in a first
approximation.

Even after this simplification, the scheduling problem is bound to be
complicated if several circuits are present. A further simplification may
be achieved by first concentrating on routes with heavy traffic, and there-
after filling in the schedule on the more sparsely travelled routes. This
procedure will be even more natural if track capacity on the heavily travelled
routes is a limiting factor. Of course any of the simplifications suggested
has a cost in preventing one from finding the theoretically very best schedule
-- which however may be too costly to compute anyway.

One other thought may be advanced very tentatively as a suggestion for
further research. It is conceivable that a solution of the scheduling problem
could be made easier by the introduction of fictitious prices on the cars,
which vary with the location of the cars in such a way that they increase as
the car comes nearer to its destination. As cars accumulate at a certain
terminal on the way. a point in time will be reached where the gain in
"locational" value from running a train to the next terminal will be enough
to make up for the cost of that train.

Another aspect of the scheduling problem has not been mentioned at all
yet. This is random fluctuations, in travel times, in classification times,
and in traffic flows. Safety margins to allow for such fluctuations will be
needed in any realistic schedule. We have not attempted to introduce this
consideration in the present exploratory discussion of the scheduling problem.

The Short-Haul Routing of Empty Boxcars

12.1 Purposes of the Study

12.1.1 The Demands of a Transportation Program on the Freight-car Fleet

Throughout our study of railway operations a great deal of emphasis has been laid on the time required to carry out a shipment. As was pointed out in Chapter 5 the importance of time arises not only through its effect on the inventories customers need to carry and on railroad operating costs, but also through its effect on the amount of rolling stock the industry needs in order to carry out a given amount of business. Thus if we had estimates for some future year of the number of car miles and car loadings, and if we knew, from studies such as that of the Federal Coordinator of Transportation, discussed in Chapter 6, how long it took to load a car, to haul it so many miles, etc., we would have a very good start on estimating the amount of rolling stock that would be needed. An important part of this problem however remains to be discussed. This is the relation between the set of traffic flows from each place to each other place and the number of empty car miles required to sustain this set of flows.

Hultgren has shown[1] that loaded-car-miles and empty-car-miles rise and fall together, but not in the same ratio. The swings of empty-car-miles seem to be proportionately less than those of loaded-car-miles. One could estimate the level of empty-car-miles corresponding to a given set of traffic flows by using ratios like those in Hultgren's study. There are, however, serious shortcomings to such an approach and in addition several reasons for investigating a more direct one.

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[1] Hultgren (1948, pp. 111-120).

To give an idea of the magnitudes involved: in 1951 loaded-car-miles for Class I railways were 20.6 billion and empty car-miles were 10.6 billion[2]. The ratio of loaded- to total car-miles fluctuates between 60 and 70 percent.[3] The predominant movement of traffic in the United States is eastward. This arises from the fact that most manufacturing takes place in the East and most farming in the Midwest and West; so we have bulky raw materials moving east and compact manufactured goods moving west. The result is that empty freight cars for Western loading must be supplied from Eastern terminals, and a considerable amount of empty-car mileage is brought about in the process. Now if one year's transportation program is very much like another year's we should expect the empty-car mileage to be about the same for the two years; and if the two years were proportionately the same, that is, if the A to B traffic changed in the same ratio as the C to D traffic, etc., then we should expect that empty-car mileage would also vary in the same proportion. It is quite clear though that changes in the program which are not proportional can give rise to very radical changes in empty mileage, even though the total loaded car-miles remain nearly the same as before. If we desire, therefore, to make any statements as to the adequacy of freight car supply for these more unusual changes, we must develop a more direct method of calculating the empty mileage associated with the given program.

12.1.2 A Means of Evaluating the Existing Car Service Procedures

Except for "system cars" (cars on their owners tracks) within the confines of a single railroad, the movement of empty freight cars is governed by three things. First is the set of so-called Car Service Rules agreed upon by the various railroad companies and administered by the Car Service Division of

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2/ I.C.C., Statistics of Railways in the United States 1951.
3/ Hultgren [1948, p. 118, Chart 52].

the Association of American Railroads. Second are the Per Diem Agreements.
"Car Service Rules and Per Diem Agreements are established for the purpose of
providing an improved movement of cars interchanged between railroads, to mini-
mize the movement of empty cars, to coordinate equipment so that an improved car
supply can be had by shippers, to facilitate general movement of loaded cars,
and to establish uniform charges for cars on lines of other than the owners,
commonly known as Per Diem Charges".[4] The third controlling factor consists
of special directives issued by the Car Service Division. These represent
a cooperative attempt on the part of the different railroads to predict the
time place, and extent of extraordinary needs for loading, and to take
measures to supply these needs by directing that empty cars be moved in
certain ways, often contrary to those specified by the Car Service Rules.
These directives can be interpreted as a recognition of the fact that the
Car Service Rules and Per Diem Agreements do not by themselves lead to the
most efficient use of the freight car fleet. While the application of these
rules can become quite involved, the main idea they express with regard to
the movement of empties (that is, cars for which no load is available) is that
these cars should be moved in the direction of the home road. In a brief
space it is difficult to avoid doing the rules an injustice for they are not
quite this rigid,[5] but it is nevertheless true that car ownership is given
a more important role than considerations of the needs for and availabilities
of empty cars at various locations.

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[4]   Freight Traffic Red Book 1950, p. 951.
[5]   "If empty ... cars ... may be moved locally in an opposite direction
from the home road ... if to be loaded for delivery on or movement via the
home road". -- Rule 2(c), Code of Car Service Rules. Freight Traffic Red Book
1950, p. 952.

In the present chapter we shall examine these needs and availabilities at different times and we shall attempt to discover how empty cars should be routed if the only consideration were that empty car-miles be minimized. It should be emphasized at this point that, while for simplicity car-miles are the only cost we consider in this routing, many others, such as the costs of congestion, might very legitimately be brought in.

Once these hypothetical short-haul routings were known, it would in principle be possible to compare them with the actual routings brought about by the three agencies mentioned earlier, and in this way facilitate an evalue-tion of existing car service procedures. Because, however, our study is a rather crude one and because information on actual empty-car flows is sparse, we must be satisfied for the present with a somewhat more modest goal, as follows.

12.1.3 Stability of the Short-Haul Routing Patterns

We shall compare the short-haul routing patterns that are derived for empty cars for several different years and for the four quarters of one year. If the pattern is found to be a very unstable one we should consider this as evidence of difficulties that the car service procedures must overcome. In this case we should not expect a set of rules as simple as the Car Service Rules to provide efficient direction; judgment, organized flows of information, and special directives would carry most of the burden. If on the other hand a high degree of stability is found, then a simple set of rules that suffices might be found.

12.2   The Data

The data published by the Interstate Commerce Commission on commodity movements by rail are, generally speaking, of two sorts. For brevity we shall refer to these two as program data and flow data. Each relates to a given period of time. The program data tell us for instance, how much corn traffic originated in Illinois in 1949, and how much terminated in the same period. The flow data tell us how much corn moved from Illinois origins to Pennsylvania terminals.

While it is clear that program data can be obtained from flow data, the reverse is not the case unless some assumption is made which gives us a rule to use in assigning the supplies of, say, corn in one state to the demands for corn in other states. If we wished to carry out this process, market prices would provide one such rule. Namely, Illinois will ship to Pennsylvania if the price of corn in Illinois plus the cost of transporting corn to Pennsylvania is less than the price of corn in Pennsylvania. That is to say, those shipments will be made which are profitable.[5]

Notice however that the flow pattern which would result from such a calculation is a purely hypothetical one; it might or might not agree with the actual flow data, depending on the validity of the assumptions made. As far as carload commodities are concerned both program and flow data are available, in various degrees of detail. The program data are the result of complete tabulations of traffic by individual railroads, and are published by the I.C.C. for several degrees of classification. The flow data consist of estimates derived from a one percent carload waybill sampling procedure of the Commission. The classifications are years, states, and commodities in detail.

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[5]   In Fox [1953] this approach is applied, with the additional complication that the program itself depends on prices. Demand and supply curves are given for each location and flows, program, and prices are solved for simultaneously.

The one commodity however with which we are particularly concerned here appears in none of the published programs or flow data. This "commodity" is the empty freight car, and any information on its movements must be inferred from data on the movements of commodities proper.

We have seen above that the program data for any particular commodity could be used to set up a hypothetical flow pattern for that commodity. This is the procedure we shall follow in determining the flows for empty cars, with the added complication that the program data must also be estimated before we can start. These latter data will be derived from the program data for all other commodities.

12.3 Surpluses and Deficits of Empty Boxcars

If the number of empty freight cars that originates in a region during a given period is greater than the number that terminates there, the region will be called a freight-car surplus region. Conversely a deficit region will be one where the number of empties terminating is the greater. Finding the program for a given year then is simply a matter f determining the levels of the year's surpluses and deficits for all the regions. If all freight cars were equally useful for all purposes this determination would be much easier than it actually is. We must however take account of the fact that an empty gondola car is of no use to a shipper who wants to load oranges. Surplus and deficit figures relating simply to freight cars would not be very meaningful, for later, in the process of deriving flows, we might unwittingly assign one region's surplus, consisting mostly of boxcars, to the filling of another region's deficit, consisting mostly of flat cars. To reduce these errors due to the summing up of different kinds of things, we shall confine our whole discussion to the movement of boxcars alone. Precisely the same

procedure could be applied to each of the other types of freight car. By
thus confining our study to a relatively more homogeneous group we reduce
the seriousness of these objections, and by concentrating on boxcars we still
have in our study a sizable fraction of the whole freight car population.
The total cars in active service in 1950 and the breakdown by types of car
are shown in Table I where it can be seen that boxcars constitute more than
a third of the total stock of rolling equipment.

Table I.   Freight Cars On or About Dec. 31, 1950
(Class I Line-Haul Railways Plus Private Lines)

	Number	Percent
Boxcars	717,424	36.16
Flatcars	65,195	3.28
Stock Cars	47,971	2.42
Gondolas and Hoppers	866,489	43.67
Tank Cars	149,330	7.53
Refrigerator Cars	127,210	6.41
Other Freight-Carrying Cars	10,094	.53
Total	1,984,214	100.00

Source: Interstate Commerce Commission, Statistics of Railways
in the United States, 1950, Table 24, p. 29.

While we can be a little more comfortable with the assumption that every
boxcar is like every other than with the assumption that all cars are alike,
it should be realized that even this restricted simplification is far from
true. If by some standard it is thought that too much distortion is intro-
duced into the final estimates of empty car flows by this assumption of homogeneity,

the class of cars examined could be still further reduced. As we shall see

below however, such a procedure would very quickly lead to diminishing returns,

quite apart from the fact that computations would increase if the same number

of cars were to be included in the study. The reason is that the estimates

of the program for the type of car being studied quickly deteriorate if this

car is not one of the main classifications for which related information is

published by the I.C.C.

We shall now describe the method used to derive from the program data

for all commodities the estimated boxcar program of surpluses and deficits

for the different states in the U.S. for a period of a year.

Unfortunately, for our purposes, the unit of the published origination

and destination figures[7] is not the carload, but the ton. Thus it is necessary

to convert these figures to carloads through the use of statistics of average

tons per car for the various commodities. While the latter are available for

each of the 500 commodities in the I.C.C. classification, we shall for compu-

tational ease use only the ones corresponding to the coarse six-commodity

classification: Products of Agriculture, Animals and Products, Products of

Mines, Products of Forests, Manufactures and Miscellaneous, and Forwarder

Traffic.[8] Table II lists these tons-per-car figures by year and by commodity

group.

The first important thing to notice in the table is the absence of any

mention of the state concerned. These are U.S. averages; where the true

figures differ between states we have a source of error in our conversion

from tons to cars. The second feature to notice is the wide difference

[7]   Table N-A, I.C.C. Statistics of Railways in the United States, 1940-1952.
[8]   "The term 'forwarder traffic' means freight traffic consigned by or to
a forwarder, i.e., a company, firm, or individual recognized as engaged in
the business of collecting and accumulating less-than-carload shipments into
consolidated carloads without an ownership interest in the property so handled".
I.C.C., Statistics of Railways in the United States, 1940. p. 40.

Table II. Tons per Car of Carload Revenue Freight Originated by Commodity Group, 1940-1950.2/

Year	Products of Agriculture	Animals and Products	Products of Mines	Products of Forests	Manufactures and Miscellaneous	Forwarder Traffic
1940	27.0	12.9	33.8	31.6	27.8	12.1
1941	28.4	13.4	34.2	31.7	28.3	12.7
1942	30.7	14.5	34.4	33.0	30.7	17.4
1943	34.1	15.4	34.8	34.5	32.1	15.1
1944	32.5	15.3	35.2	34.5	31.4	12.7
1945	33.4	15.0	35.3	34.3	30.9	12.2
1946	32.6	14.2	35.6	34.5	29.7	17.8
1947	34.7	14.3	35.9	34.1	30.5	17.5
1948	34.3	14.5	36.3	34.3	30.8	17.2
1949	34.4	14.2	36.9	34.0	30.1	15.9
1950	35.3	14.5	37.3	33.8	30.1	14.3

2/ I.C.C., Statistics of Railways in the United States, 1950. Table 49, p. 41.

in the averages between the several commodities. This difference is somewhat
reassuring with respect to the source of error just mentioned. By treating
heavy commodities such as Products of Mines separately from light commodities
such as Animals and Products we are insuring to some extent against the chance
that a tons per car figure for a mining state exceeds the U.S. average for
the same complex of commodities while at the same time a similar figure for
a cattle-raising state falls below the U.S. Average. Using these average
load figures then, the following conversion from tons to cars can be made for
each state and each commodity:

$$\frac{\text{Tons of Commodity Group X Terminated} - \text{Tons of Commodity Group X Originated}}{\text{Tons of Commodity Group X Per Car}} = \frac{\text{Car Surplus Due To Commodity X}}{}$$

A negative surplus is taken to be a deficit.

Thus we have for each state six car surplus estimates. one for each
commodity group, from which we must now derive estimates of boxcar surpluses
(or flatcar surpluses. etc., if we so desired).

Through the early years of our study no data on the relative uses of
boxcars and other cars were published. In 1949 however the I.C.C. began
presenting in their Carload Waybill Analyses some very valuable information
on the distribution of car types in the haulage of each commodity in the
Commission's detailed classification.

From this source we learn for example that of the 36.965 freight cars
in the sample which were used in the carload transportation of Products of
Agriculture in 1952, 27,352 were boxcars, 7,127 refrigerator cars, 63 stock
cars. 791 gondolas, 797 hoppers, 1 flat, 617 special cars, and 37 tank cars.[10]

10/   I.C.C. Carload Waybill Statistics 1952. Statement 3354. File 40-C-8,
       p. 1.

These statistics, like all others in the Waybill Analyses, are based on a
one percent sample of the audited waybills of Class I railroads, and are
consequently subject to sampling errors. Furthermore the errors in an
estimate, as the caveat in the introduction to each of these reports states,
will tend to be larger when the number of cars (i.e. number of waybills) to
which it corresponds is smaller. This need not worry us too much here how-
ever for our commodity groups are very broad and the sample, while from a
percentage point of view small, is in absolute size large indeed. It is
well known that the absolute size of a random sample controls the accuracy
of estimates derived from it much more decisively than does the ratio
of the sample size to the whole population. Table III lists for the years
1949, 1950 and 1951, the ratios of boxcars used to freight cars used in the
transportation of each of our six commodity groups. It can be noticed first
that the variation in the ratio for any one commodity group from year to year
is very small. This is fortunate for we have no information about the ratios
for earlier years, 1940-1948. We shall use the 1949 ratios for each commodity
as estimates for these earlier years. On the other hand the variation of

Year	Products of Agriculture	Animals and Products	Products of Mines	Products of Forests	Manufacture and Misc.	Forwarder Traffic
1949	0.730	0.081	0.064	0.526	0.578	0.962
1950	0.72	0.094	0.062	0.523	0.585	0.954
1951	0.739	0.083	0.055	0.558	0.558	0.964

Table IV. Ratios of Boxcars Used to Freight Cars Used,
by Commodity Group, 1949-1951[11]

11/   I.C.C., Carload Waybill Statistics, Statements 5058, 5159, 5256, File
40-C-8, (1949, 1950 1951).

these "boxcar per car" figures between commodity groups is seen to be very
substantial, as we should expect. The lowest figures are those for Animals
and Products and for Products of Mines, both of the major parts of which call
for special purpose cars -- stock and refrigerator cars in the first case and
gondolas and hoppers in the second.

These figures, like the tons per car figures, are derived from data for
the U.S. as a whole, and therefore introduce errors into our calculations
whenever they differ from the corresponding (unknown) figures for the individual
states.

We can now translate the freight car surpluses into boxcar surpluses by
multiplying by the appropriate boxcars per car figure. The final formula is:

$$\frac{\text{Tons of Commodity Group X Terminated} - \text{Tons of Commodity Group X Originated}}{\text{Tons of Commodity Group X per Car}} \quad \times \quad \text{Boxcars per Car for Commodity Group X} \quad = \quad \text{Estimated Boxcar Surplus Due to Commodity Group X}$$

These six surpluses (some of which may of course be negative) can now be summed
to give the final boxcar surplus for the year and state being considered. The
results of these calculations for the eleven years in our study are printed
on the maps in Figures 3 through 13. For simplicity the states have been
aggregated into the twelve regions described in Table IV and Figure 1.

The derivation of the program of boxcar surpluses and deficits from the
programs for all other commodities is now complete. The next step is to build
up from this program data a hypothetical system of empty boxcar flows which
brings about the least amount of empty boxcar mileage possible. Before we
can go on to this routing problem however, a short discussion must be devoted
to the assumed distances between regions.

12.4 A Simplified Rail Network

Table IV describes the twelve region division of the U.S., and lists for each region a representative point. This grouping of states has been done by weighing the following factors: geographical proximity, similarity of economic status and product, pattern of railway facilities, and the possibility of selecting a single center for the area which would dominate other possible centers as a focal point for shipping. The purpose of picking representative points is to establish an origin from which to measure rail distances. The cities selected do not necessarily represent important gateways or rail centers; these would often be less satisfactory for present purposes than the points chosen. The criteria of selection are admittedly rather vague, but they include some of the following considerations. Since we are interested in the distances to and from ultimate origins and destinations of traffic, large centers of population are not necessarily the points to choose. One would expect this to be the case especially in the farm states. Wheat is not grown in large cities. Secondly, unless there is very good reason to think that originations and destinations are concentrated along the edge of a region, a point in such a place should be avoided, in order not to favor flows to the region bordering this edge. This consideration excludes St. Louis in Region VIII. One should attempt in setting up the regional classification to avoid bipolar regions such as Region XI which includes Los Angeles and San Francisco. Since the statewise character of our data preclude this, we picked an intermediate representative point. If we had for a region complete information about where empties arose or where they were needed, and in each case the precise number, then it might be possible to specify our representative point more closely in mathematical terms. Since in any case such information is not

available it is perhaps best to think of the point roughly as a traffic center of gravity for the region.

Table V gives the assumed representative distances between regions. The italicized entries in the table are short-line rail distances between representative points; the other entries are rail distances between representative points which differ only slightly from short-line ones; they are derived from the italicized entries by finding the shortest route over these links. The reason for not using short-line distances between all the points is that we have a somewhat simpler rail network as a result. As far as our final routing calculations are concerned this small change has no effect. The conceptual rail network corresponding to the distances of Table V is shown in Figure 2.

Table IV   Regions and Representative Points

Region	States Included	Representative Point
I	Maine New Hampshire Vermont Massachusetts Rhode Island Connecticut	Boston, Massachusetts
II	New York New Jersey Canada	New York, New York
III	Ohio Pennsylvania Michigan West Virginia	Cleveland, Ohio
IV	Maryland Delaware Virginia District of Columbia Florida Georgia North Carolina South Carolina	Columbia, South Carolina
V	Indiana Illinois Wisconsin	Chicago, Illinois
VI	Alabama Arkansas Kentucky Mississippi Tennessee Louisiana	Jackson, Mississippi
VII	Minnesota North Dakota South Dakota	Minneapolis, Minnesota
VIII	Colorado Iowa Kansas Nebraska Missouri	Kansas City, Missouri

Table IV continued

Region	States Included	Representative Point
IX	New Mexico Texas Oklahoma	Fort Worth. Texas
X	Wyoming Utah Idaho Montana	Ogden, Utah
XI	Arizona Nevada California	Monterey. California
XII	Oregon Washington	Portland. Oregon

Region	I (Boston)	II (New York)	III (Cleveland)	IV (Columbia)	V (Chicago)	VI (Jackson)	VII (Minneapolis)	VIII (Kansas City)	IX (Fort Worth)	X (Ogden)	XI (Monterey)	XII (Portland)
I (Boston)	0											
II (New York)	229	0										
III (Cleveland)	678	711	0									
IV (Columbia)	930	701	846	0								
V (Chicago)	1018	911	340	867	0							
VI (Jackson)	1465	1224	965	648	738	0						
VII (Minneapolis)	1424	1347	746	1273	406	1144	0					
VIII (Kansas City)	1469	1362	791	1111	421	592	420	0				
IX (Fort Worth)	1905	1675	1244	1110	961	442	1000	210	0			
X (Ogden)	2496	2386	1318	2231	1443	1801	1530	1170	1339	0		
XI (Monterey)	3298	3191	2620	2785	2236	2117	2132	1673	1671	802	0	
XII (Portland)	3217	3110	2339	3066	2159	2649	1793	2018	2207	848	953	0

Source: Italicized figures from Rand-McNally Commercial Atlas and Marketing Guide, 84th Edition. 1953; others derived from italicized figures by finding shortest route.

Table V. Rail Distances Between Regions.
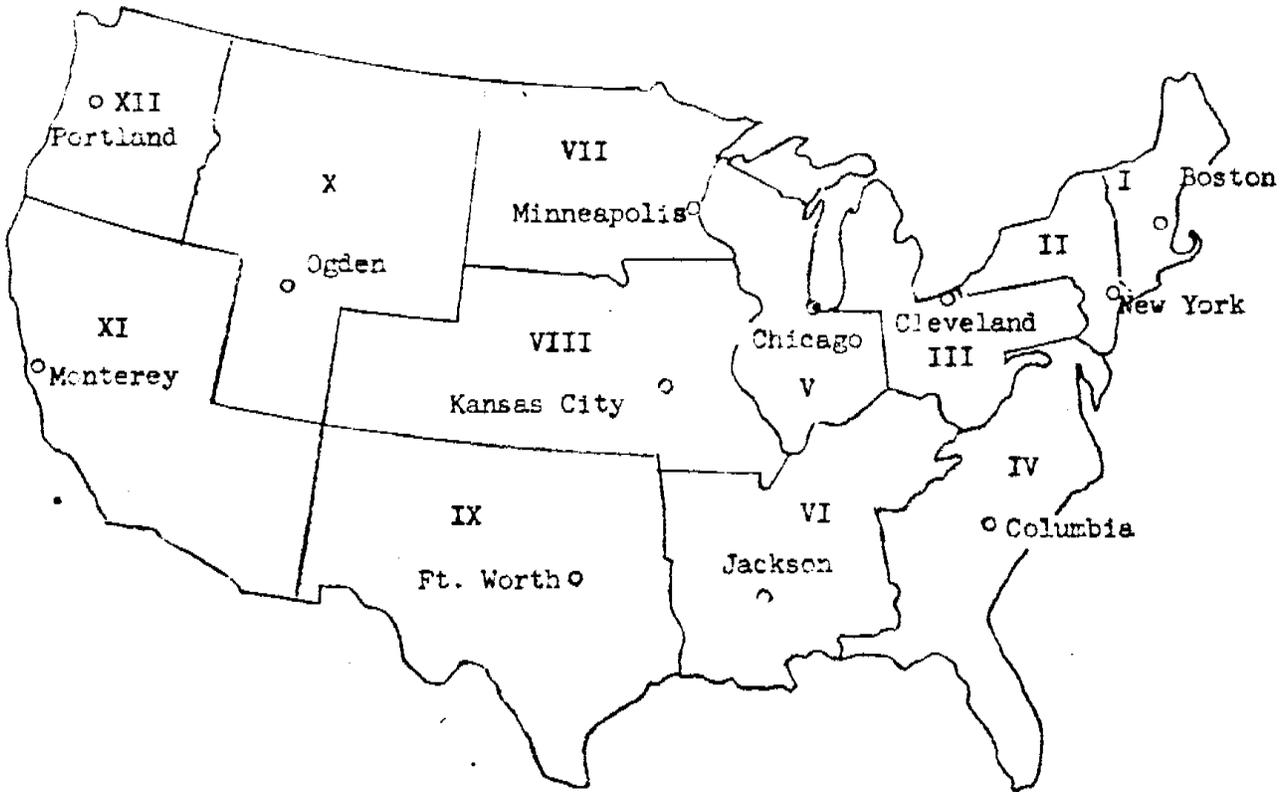
Figure 1. Regions and Representative Points
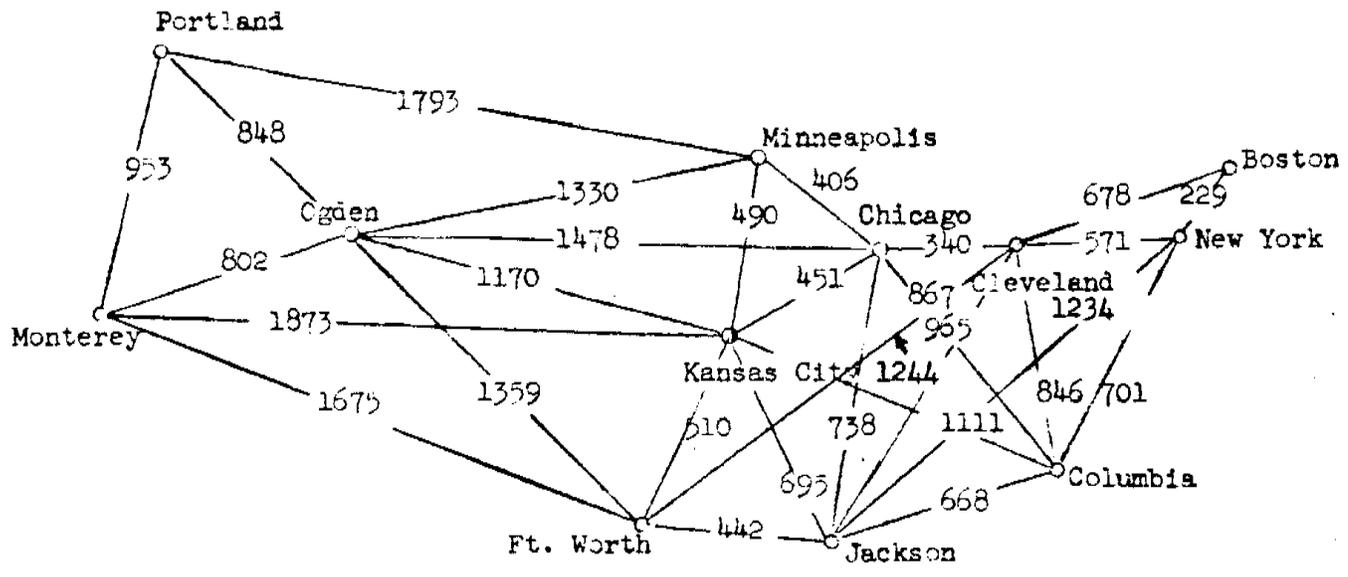
Figure 2. A Simplified Rail Network
with Distances in Miles

12.9 Finding the Short-Haul Routing Patterns

Once the surpluses and deficits are known, the short-haul routings can be computed by the so-called "simplex" method of Dantzig.[12] We shall not describe the method beyond stating the condition which is necessary and sufficient for the solution. This condition was first formulated by Koopmans[13] in relation to the problem of finding the most economical routing of empty ships. It is quite similar to those conditions developed in Chapter 4 for the efficient use of a highway network. Here it is somewhat simpler because we have disregarded congestion. If in the present problem we were concerned with the routing of all traffic, this simplification would have less justification.

Our condition is that there must exist a set of fictitious prices or "potentials"[14], one for each of our representative points, which has the following two characteristics. (1) The difference in potentials between any two points must not exceed the distance between these points, and (2) if the boxcar flow between any two points is positive then the potential at the receiving point must exceed the potential at the sending point by an amount just equal to the distance between the two points. Of course, distance is used here as a substitute for cost of empty movement, on the simplifying assumption that the latter is proportional to the former. Dantzig's method consists essentially in showing how to proceed in a systematic manner from an arbitrary initial routing pattern for which such a set of potentials does not exist to routing pattern for which one does.

[12] Dantzig [1951e, pp. 359-374]. For a description of the development of this problem, see our introduction, p. 2.
[13] Koopmans [1947]. See also Koopmans and Reiter [1951].
[14] A term suggested by an analogous problem in the flow of electricity through a network of conductors.

12.6 Results

Figures 3 through 13 show the routing patterns for the years 1940-1950 and Figures 14 through 17 for the four quarters of 1949. Surpluses for each region are shown as the boxed figures alongside each representative point, and the magnitudes of the various flows (in numbers of cars) are shown alongside each segment of the pattern.

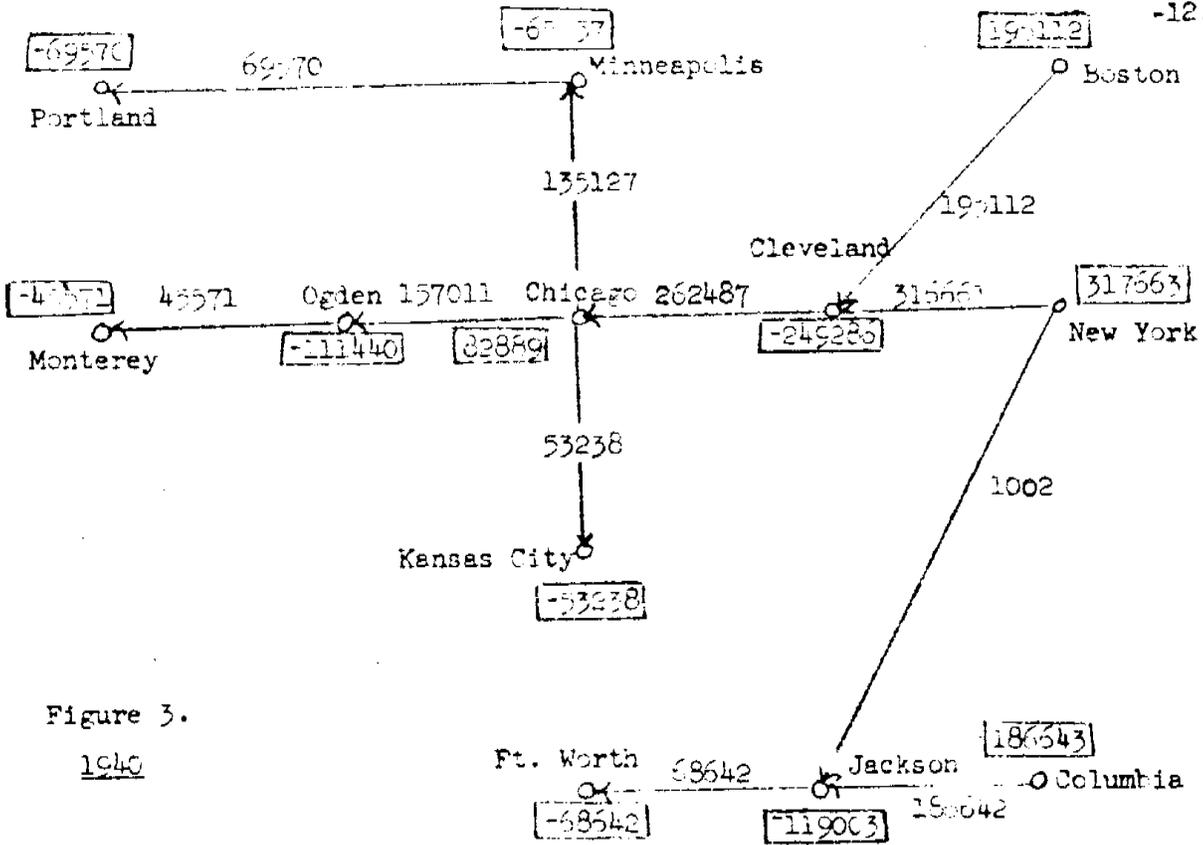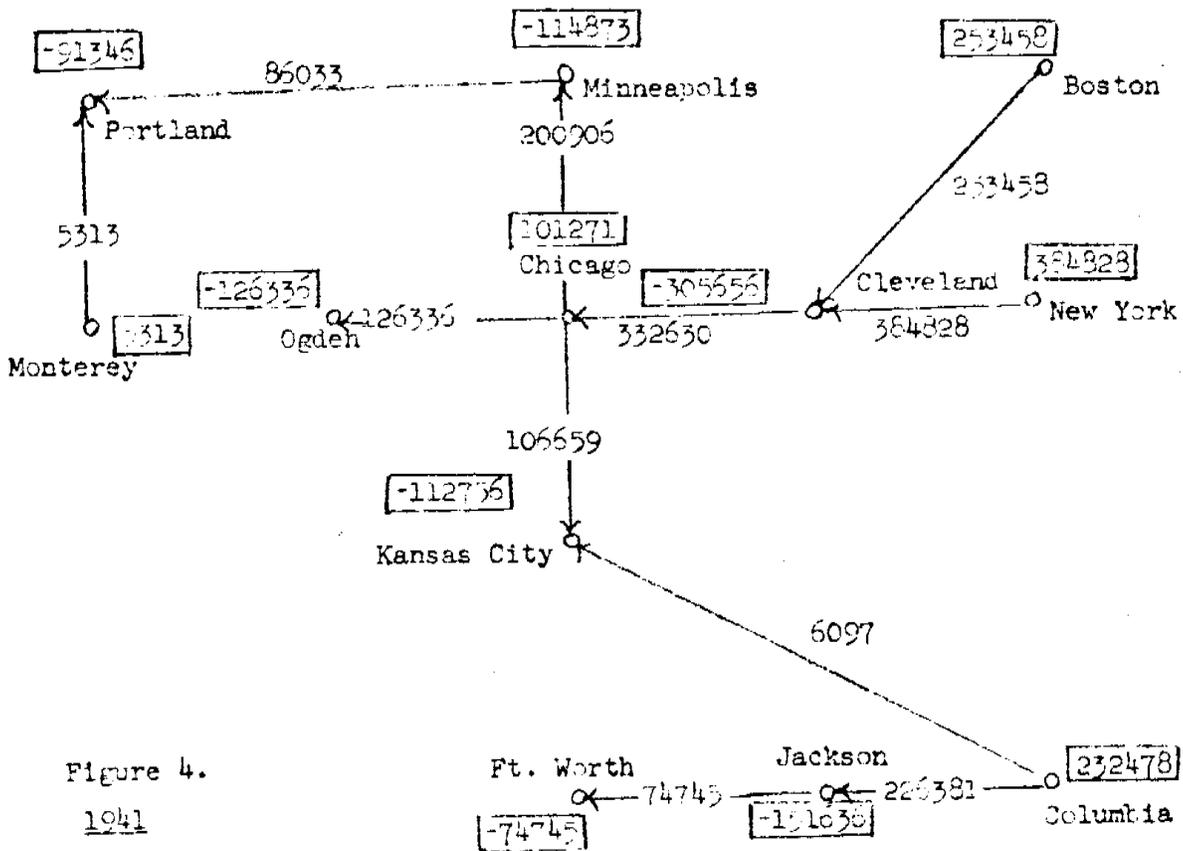Figures 3-17. Short-Haul Routing Patterns of Empty Boxcars for the Programs of Each of the Years 1940-195C and Each of the Quarters of 1949.

Figure 3.

1940

Figure 4.

1941
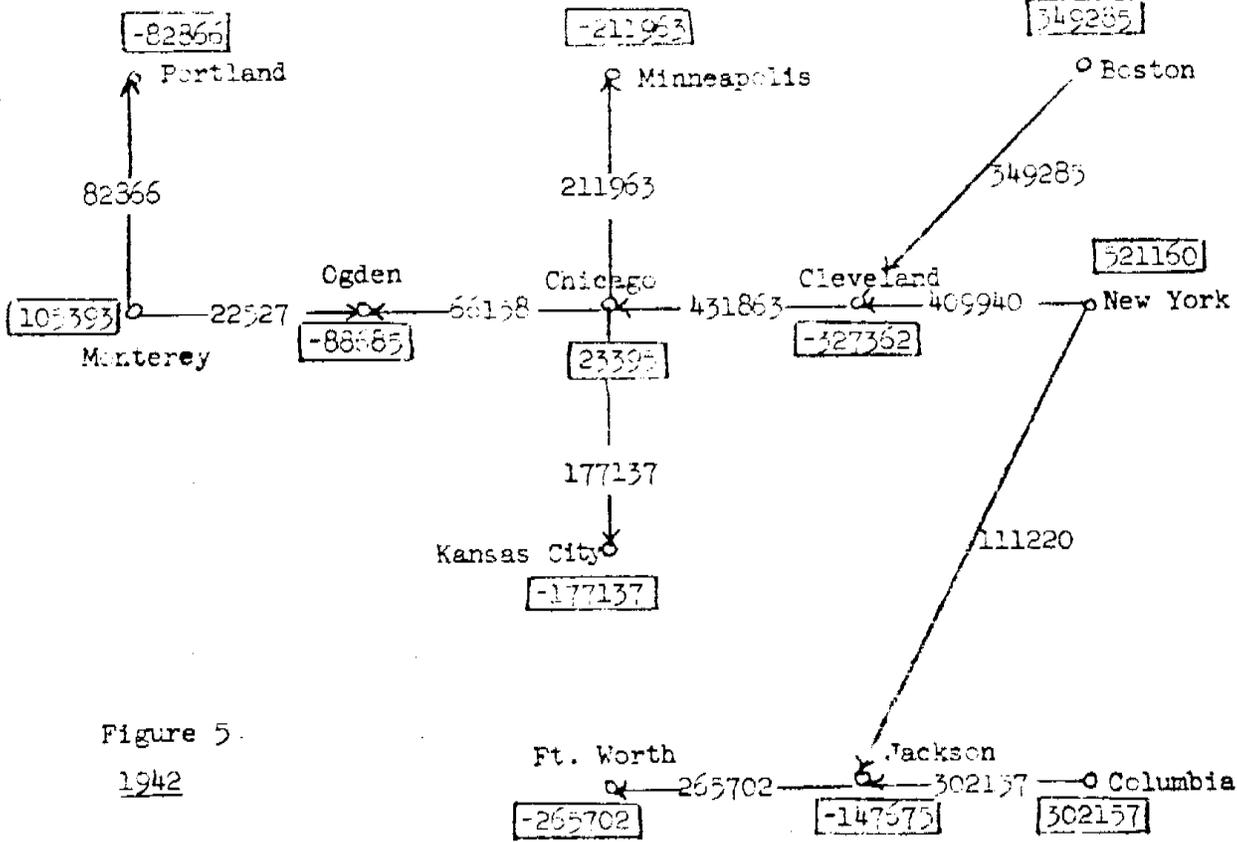
Figure 5
1942

Figure 6
1943

Portland
-70547

70547

173361   -147593   -43008   -123745   702270
Monterey   102814   Ogden   44779   Chicago   495749   Cleveland   304071   New York

-234913
Minneapolis

234913

172059

-172059
Kansas City

391423   Boston

398199

Jackson
Ft. Worth   418247   215921   215921   Columbia
-418247   -195573

Figure 7.
1944

Portland
-26479

26479

-84910   -144573
Monterey   144   Ogden

13767

-232592
Minneapolis

232592

241

Chicago

277430
-277430
Kansas City

319157   Boston

319157

Cleveland   531805
509781   505002   New York
-314378

26803

Jackson
Ft. Worth   247604   347277   Columbia
-361521   -126475   347277
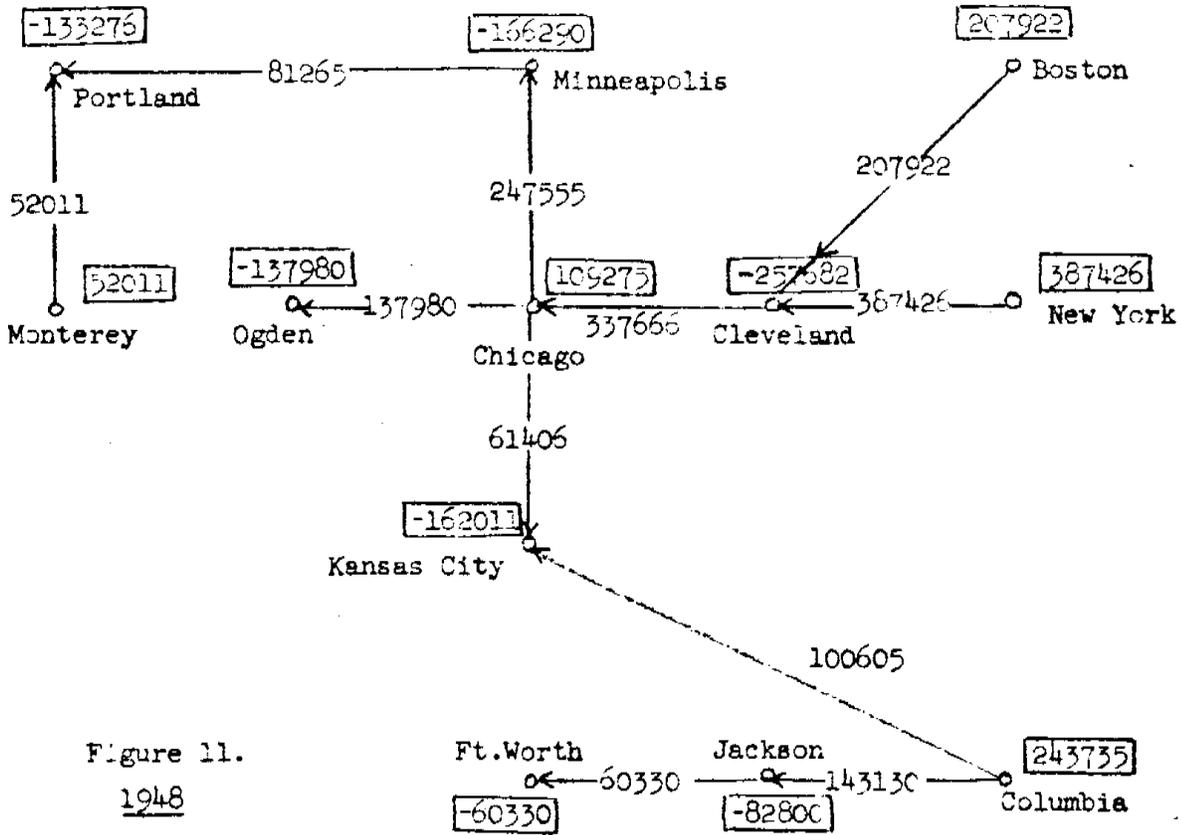
Figure 8.
1945

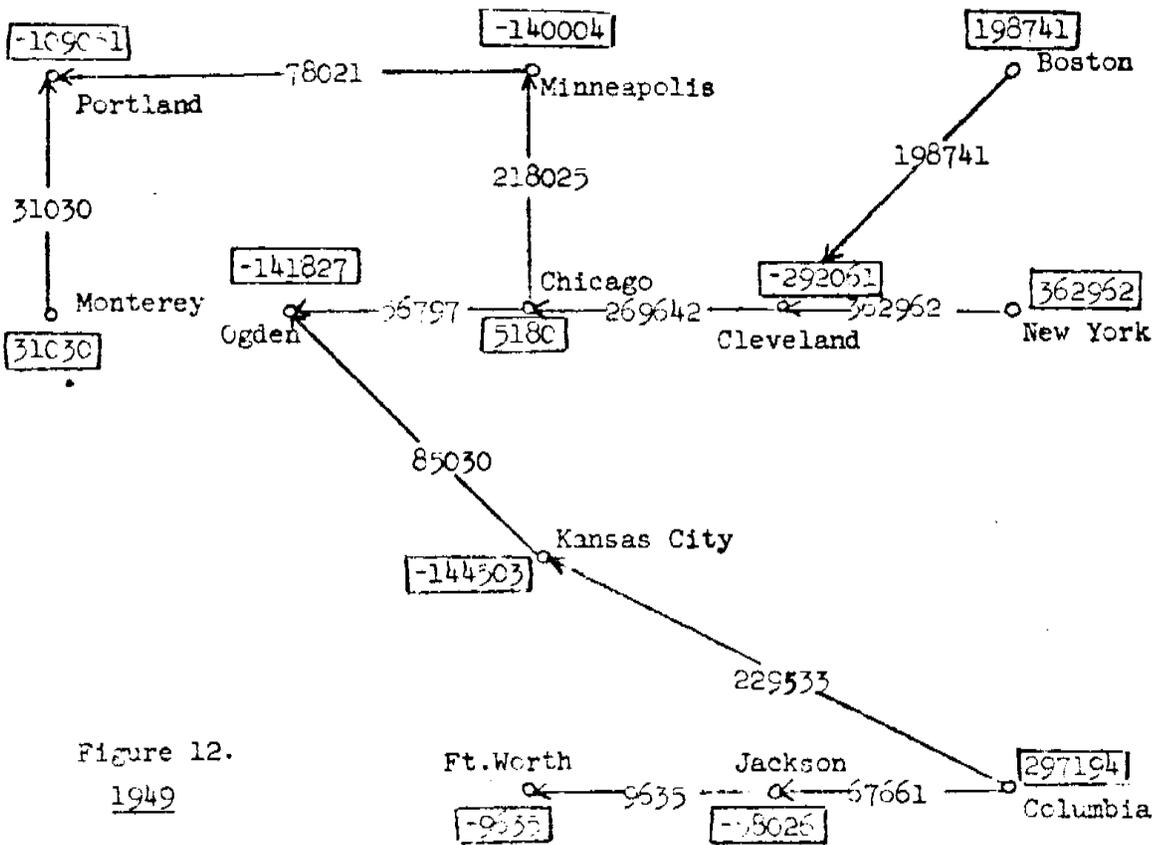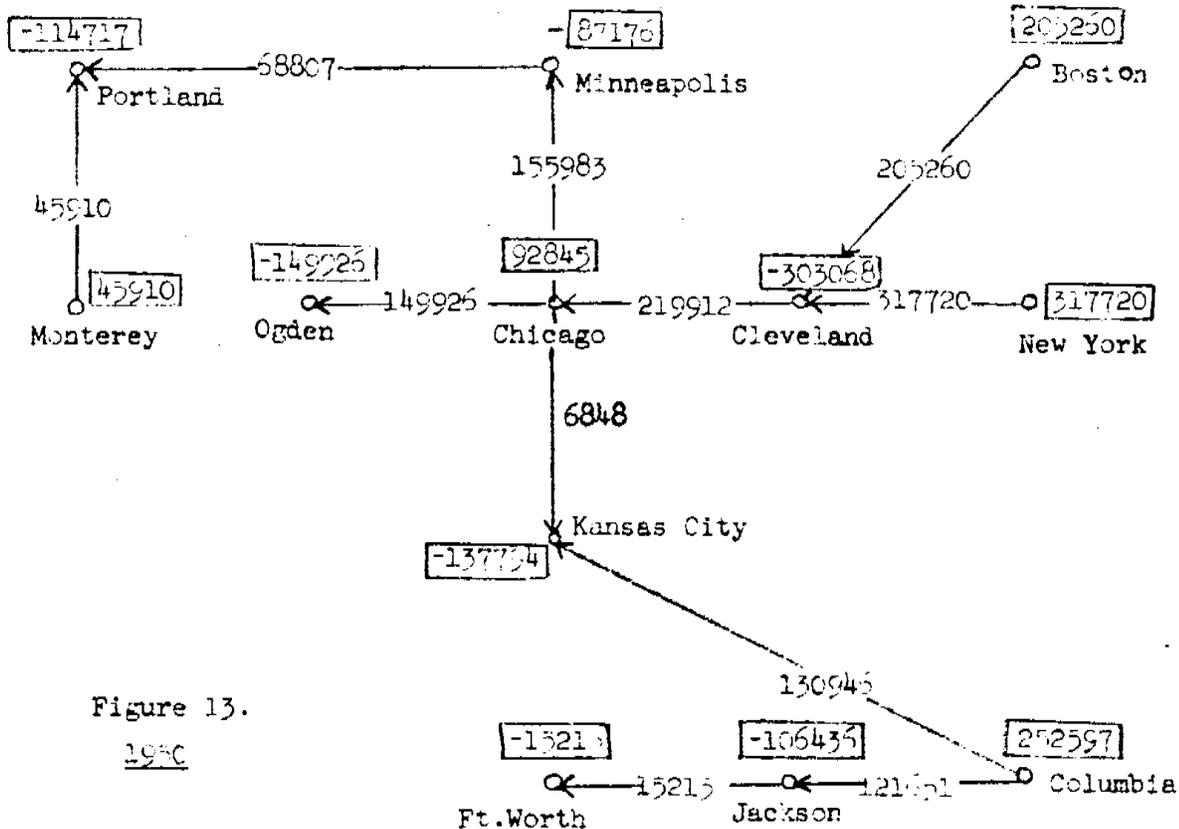Figure 9.
1946
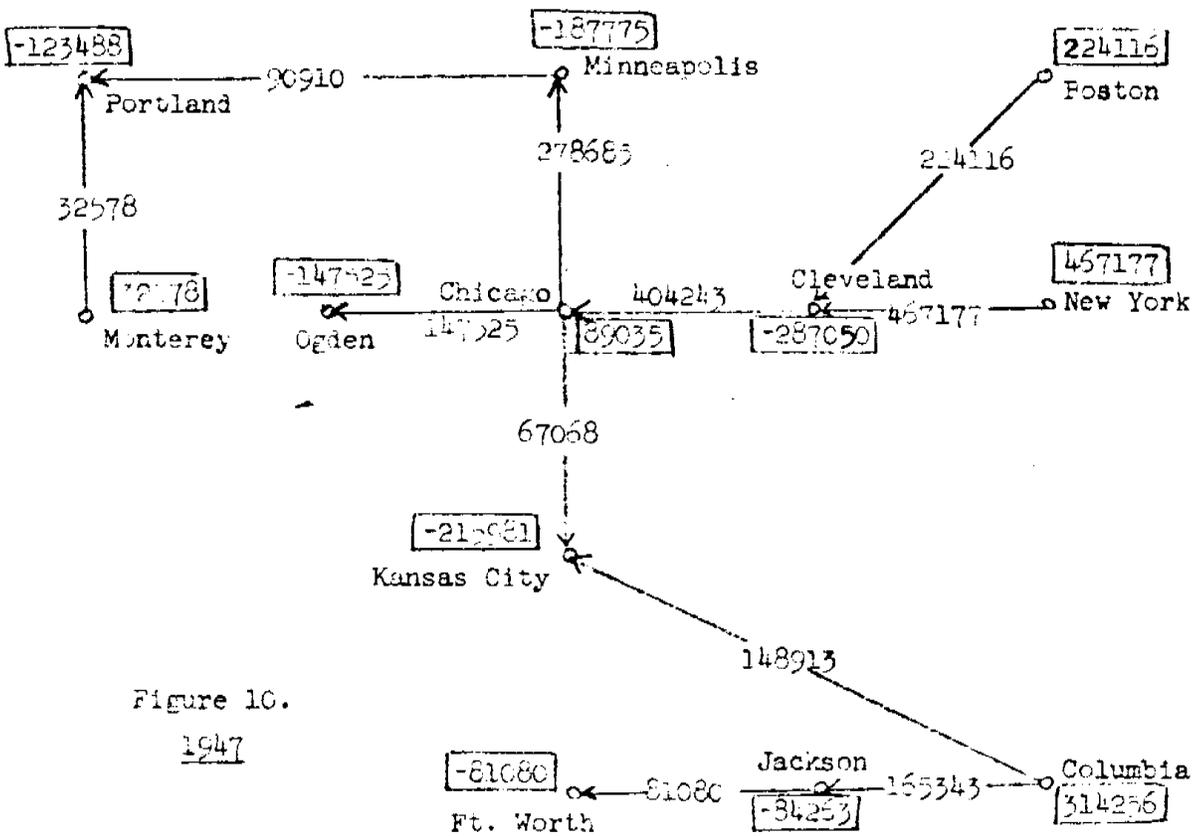
Figure 10.
1947

Figure 11.
1948
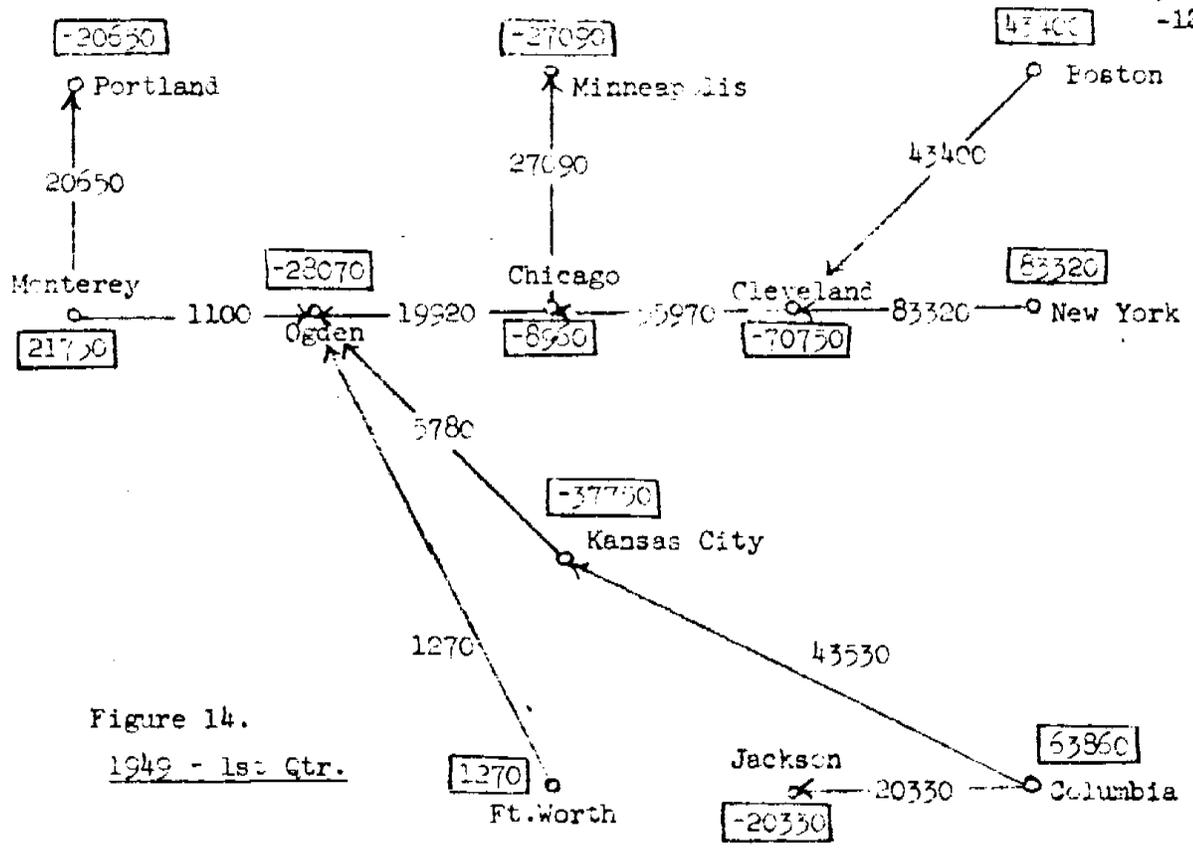

Figure 12.
1949

Figure 13.
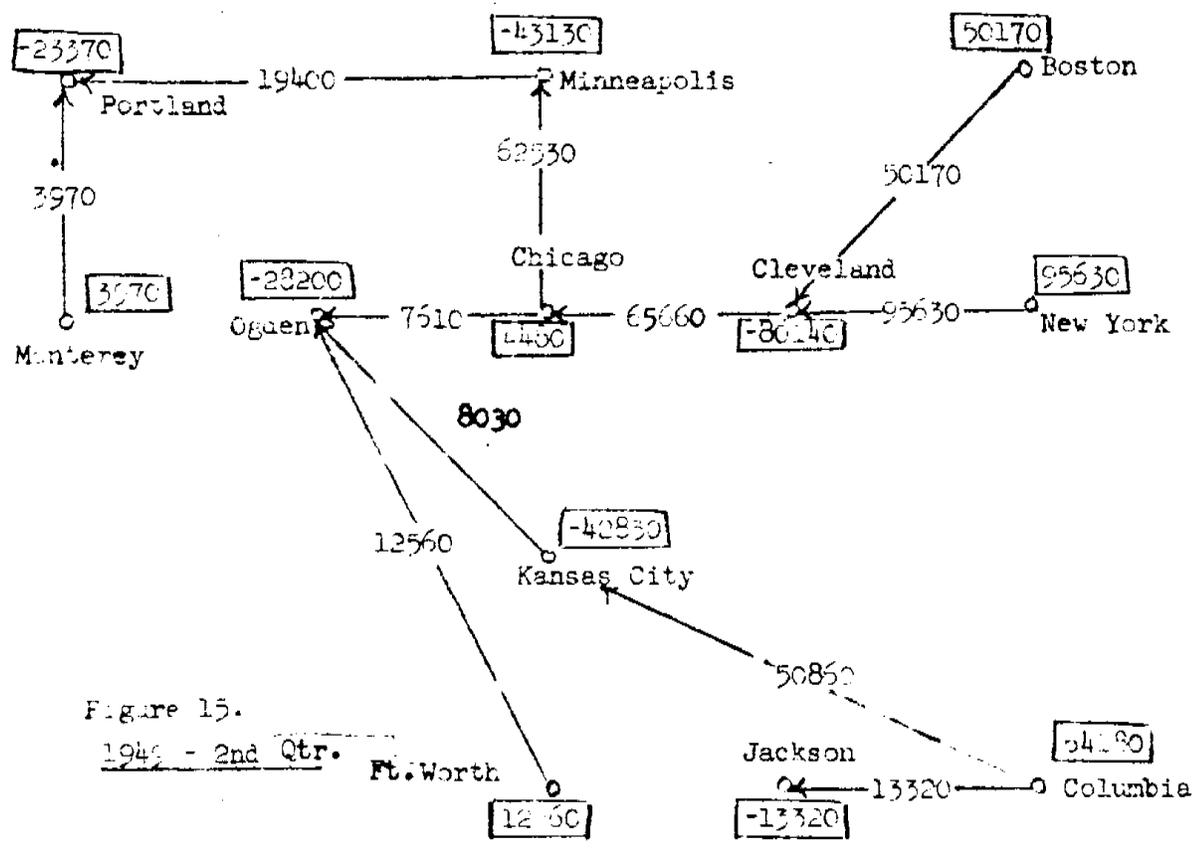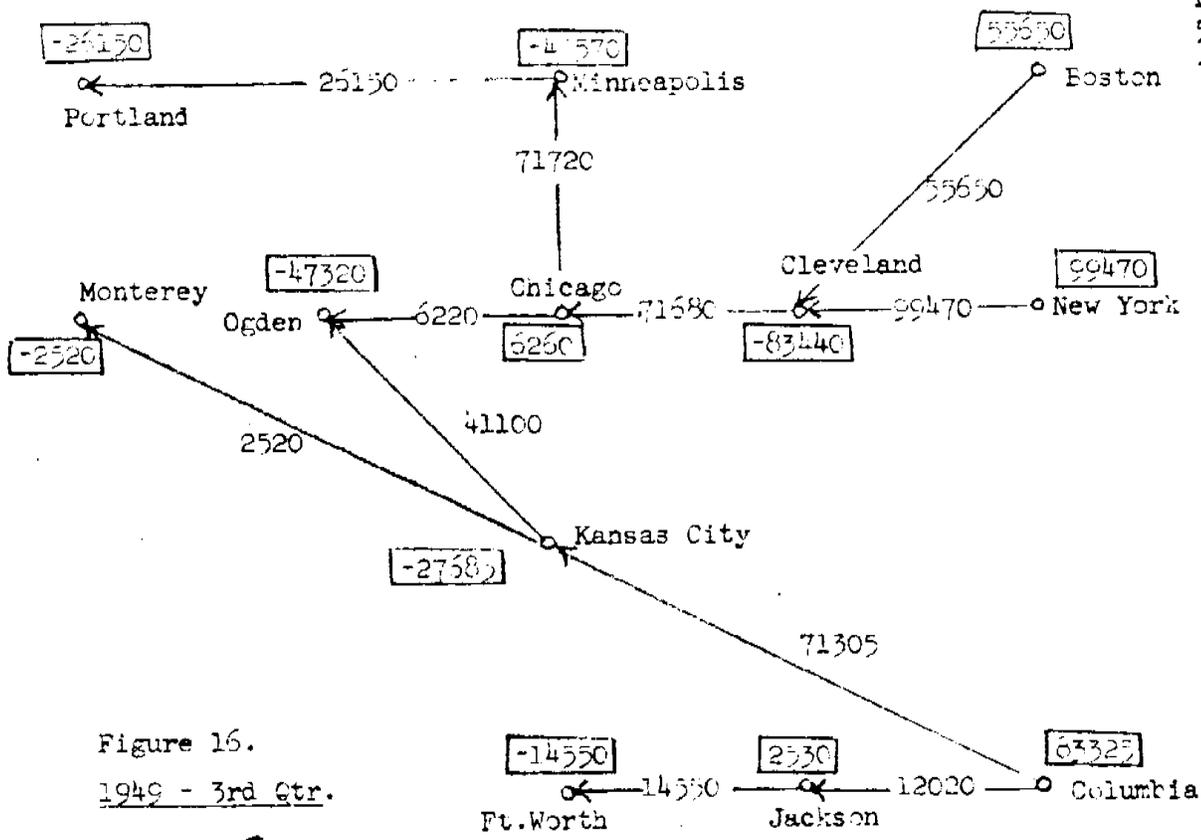1950

Figure 14.

1949 - 1st Qtr.

Figure 15.

1949 - 2nd Qtr.

Figure 16.
1949 - 3rd Qtr.

Figure 17.
1949 - 4th Qtr.

1940
(D)

1941
(A)

1942
(C)

1943
(B)

1944
(C)

1945
(B)

1946
(A)

1947
(A)

1948
(A)

1949
(E)

1950
(A)

1949
1st Qtr.
(F)

1949
2nd Qtr.
(G)

1949
3rd Qtr.
(H)
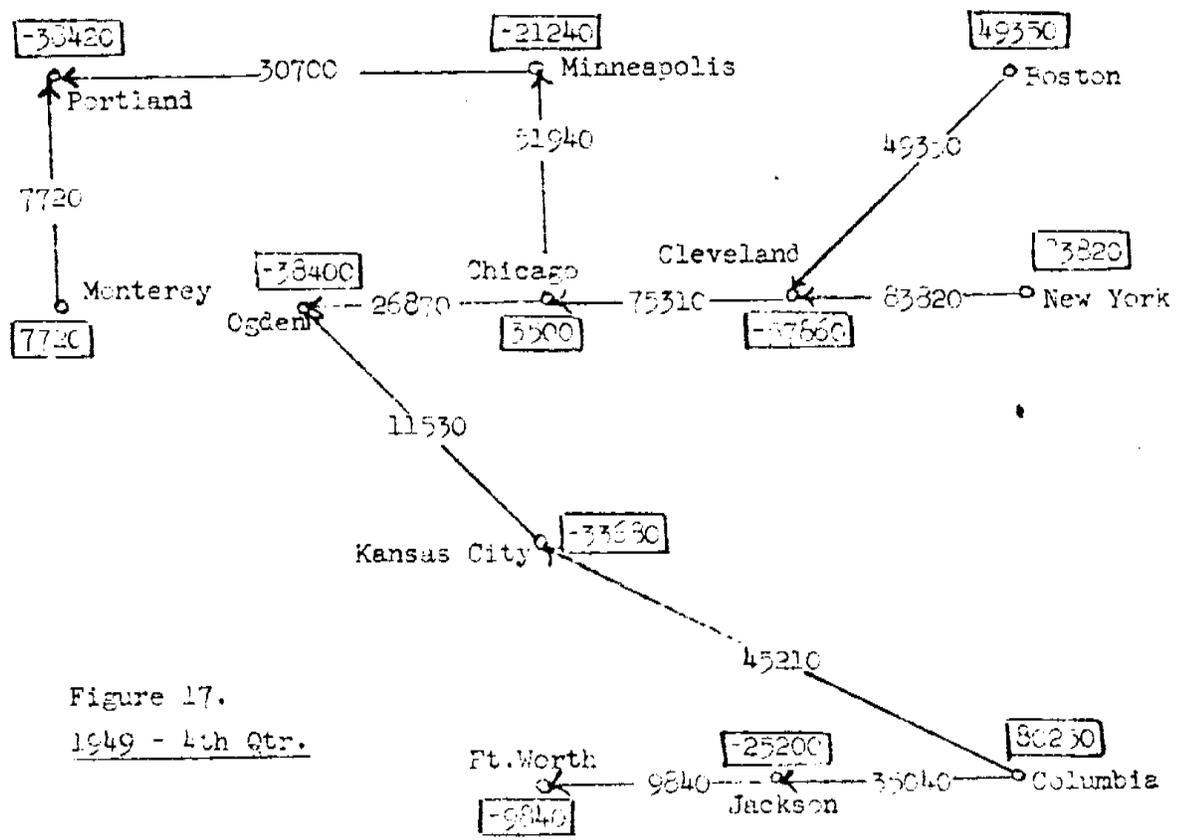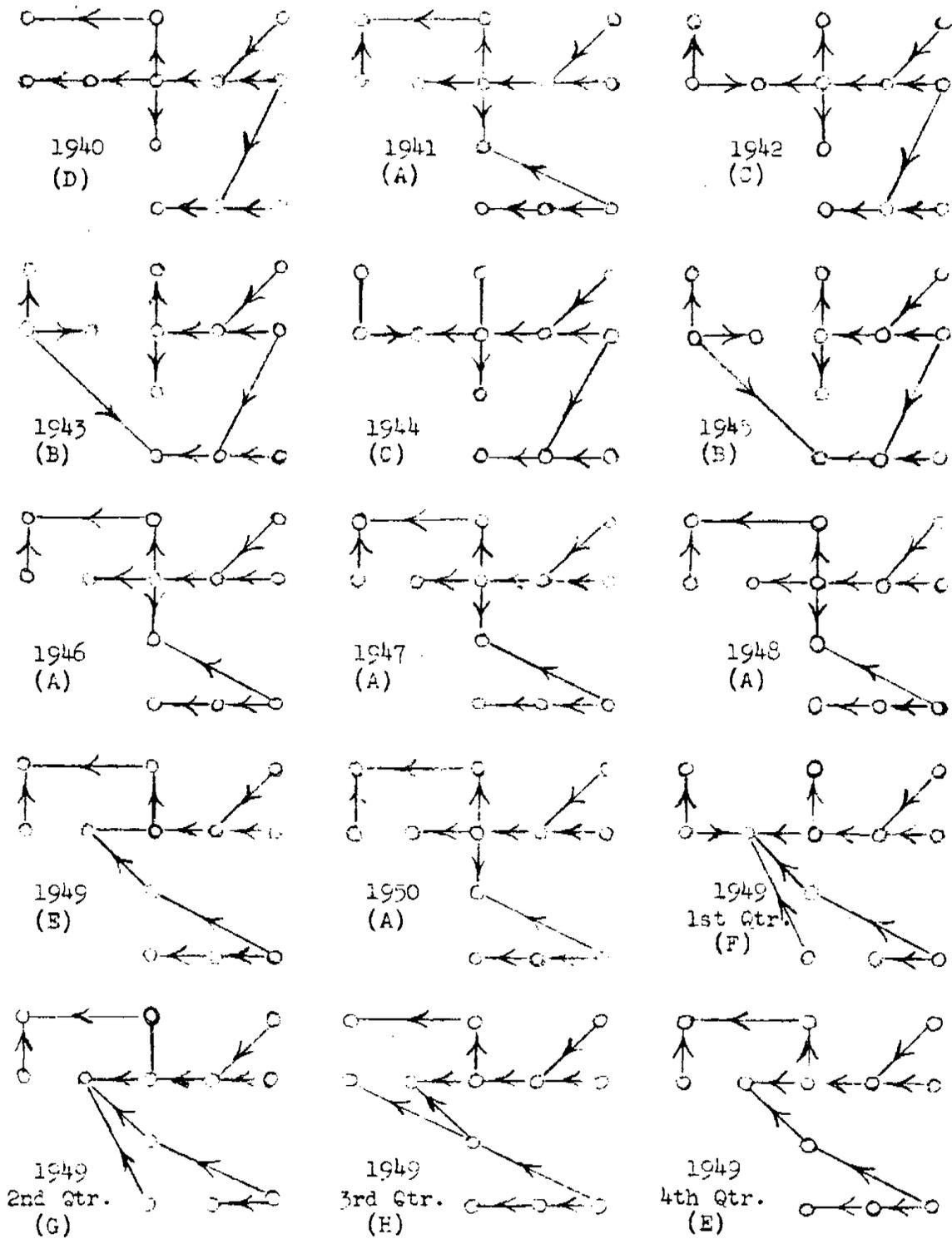
1949
4th Qtr.
(E)

Figure 19.   Short-Haul Routing Patterns.

12.7 Conclusions With Respect to Stability

In Figure 19 we have reproduced all of the resulting short-haul routing patterns on a single page where they can be seen at a glance for easy comparison. Since a judgment as to whether one pattern is very different from another is to some extent a subjective matter we shall leave part of the final decision concerning stability to the reader. Notice however that in the eleven years only five different yearly patterns arise: 1941, 1946, 1947, 1948, and 1950 are of one type, let us say, type A. 1943 and 1945 of type B. 1942 and 1944 of type C. 1940 alone of type D, and 1949 alone of type E. While the fourth quarter of 1949 is of the common type E, the other quarters of 1949 are unlike each other and unlike any of the yearly patterns. While the peacetime years taken by themselves, and the wartime years taken by themselves do seem to show a fair amount of stability, the quarterly patterns for 1949 seem to indicate that seasonal instability warrants a closer inspection of other years than we have given them. One encouraging aspect of our results is this. The stability found to exist seems sufficient to justify the hope that incremental-cost freight rates that take into account a shipment's contribution to empty-car movement can someday be put into effect. Short-haul empty-car routing patterns like the ones we have examined in this chapter are a prerequisite to the administration of such a rate structure. The computations for such purposes would have to be a good deal more detailed than the simple one we have carried out here, but it is clear that the complexities of the problem are not so great as to carry it beyond the reach of existing techniques.

12.5* (Appendix) Formal Description of the Surplus-Deficit Computation Procedure

In this section the exact procedure used in deriving the surplus-deficit estimates is described. It differs from the verbal account in 12.3 only insofar as some adjustments of the initial data are concerned.

Since only Class I line-haul railroads reported, there was a discrepancy between tons originated and tons terminated for the whole U.S., within each commodity group. If one had summed originations and terminations for any commodity group over each state and Canada, one should have found that total originations equaled total terminations within each commodity group. The actual discrepancy between the two was undoubtedly caused by error and by "leakage" of freight tonnage on to the Class II and Class III railroads (i.e., railroads which do not gross over $1,000,000 a year). Since no data were available for the true originations and terminations, it was decided to adjust the figures initially so that tons originated would equal tons terminated. The initial adjustment was carried out as follows: Let $\bar{O}_i^M$ and $\bar{T}_i^M$ equal the number of tons in commodity group M reported originated and terminated in state i, respectively. Then for each M

$$(12.1) \qquad \sum_i \bar{O}_i^M - \sum_i \bar{T}_i^M = C^M$$

where C^M equals the discrepancy between the two. To adjust $\bar{O}_i^M$ and $\bar{T}_i^M$ to find O_i^M and T_i^M, respectively, we divide C^M evenly and apportion each half among the $\bar{O}_i^M$ and $\bar{T}_i^M$ according to their percentage of $\sum_i \bar{O}_i^M$. That is,

$$(12.2) \qquad O_i^M = \bar{O}_i^M - \frac{C^M}{2} \cdot \frac{\bar{O}_i^M}{\sum_k \bar{O}_k^M} = \bar{O}_i^M \cdot \frac{\sum_i \bar{O}_i^M + \sum_k \bar{T}_k^M}{2 \sum_k \bar{O}_k^M} ;$$

$$(12.3) \qquad T_i^M = \bar{T}_i^M + \frac{C^M}{2} \cdot \frac{\bar{T}_i^M}{\sum_k \bar{T}_k^M} = \bar{T}_i^M \frac{\sum_k \bar{C}_k^M + \sum_k \bar{T}_k^M}{2 \sum_k \bar{T}_k^M} \quad .$$

Clearly,
$$\sum_i C_i^M = \sum_i T_i^M .$$

The transformation of the adjusted tons originated and tons terminated was carried out as follows: Let α^M equal the proportion of boxcars to all cars used in shipping commodities in group M for a given year. Let β^M equal the reciprocal of the average number of tons per car for commodities in group M and for a given year. Then

$$(12.4) \qquad \sum_M \alpha^M \beta^M (T_i^M - C_i^M)$$

gives us an estimate for surplus or deficit box cars attributable to each state i.

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