Studies on the Painlevé Equations (*).

I. - Sixth Painlevé Equation P_{VI}.

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Summary. – *In this series of papers, we study birational canonical transformations of the Painlevé system $\mathscr{H}$, that is, the Hamiltonian system associated with the Painlevé differential equations. We consider also τ-function related to $\mathscr{H}$ and particular solutions of $\mathscr{H}$. The present article concerns the sixth Painlevé equation. By giving the explicit forms of the canonical transformations of $\mathscr{H}$ associated with the affine transformations of the space of parameters of $\mathscr{H}$, we obtain the non-linear representation: $\mathsf{G} \to \mathsf{G}_*$, of the affine Weyl group of the exceptional root system of the type F_4. A canonical transformation of G_* can extend to the correspondence of the τ-functions related to $\mathscr{H}$. We show the certain sequence of τ-functions satisfies the equation of the Toda lattice. Solutions of $\mathscr{H}$, which can be written by the use of the hypergeometric functions, are studied in details.*

0. – Introduction.

Let $E(a, b, c)$ be the set of solutions of the hypergeometric differential equation

$$t(1-t)\frac{d^2 f}{dt^2} + \big(c-(a+b+1)t\big)\frac{df}{dt} - abf = 0\,. \tag{0.1}$$

If $f = f(t)$ is in $E(a, b, c)$ the function $f^- = f^-(t)$ defined by

$$f^- = \left[t\frac{d}{dt} + c - 1\right]f \tag{0.2}$$

is contained in $E(a, b, c-1)$. The linear map

$$L^-(c)\colon f \to f^-$$

from the two dimensional vector space $E(a, b, c)$ to the other $E(a, b, c-1)$ is an isomorphism. In fact, put

$$f^+ = \left[(1-t)\frac{d}{dt} + c - a - b\right]f\,, \tag{0.3}$$

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which defines the linear map from $E(a, b, c)$ to $E(a, b, c+1)$:

$$L^+(c): f \to f^+ .$$

We see that $(L^+(c-1)L^-(c))(f)$ is a constant multiple of f. In particular, if f is the hypergeometric series:

$$f = F(a, b, c; t) = \sum_{n=0}^{\infty} \frac{[a]_n[b]_n}{[c]_n n!} t^n ,$$

then we have

$$f^- = (c-1)F(a, b, c-1; t) , \tag{0.4}$$

$$f^+ = \frac{(c-a)(c-b)}{c} F(a, b, c+1; t) . \tag{0,5}$$

Here we assume that none of c, $c-a$, $c-b$ is integer and use the notation:

$$[a]_n = a(a+1) \dots (a+n-1) .$$

The relations (0.2)-(0.4) and (0.3)-(0.5) are known as the contiguity relations for the hypergeometric series of Gauss.

The main purpose of this series of papers is to obtain such relations for the set of solutions of the Painlevé equations. In the following of this series of papers, we will refer to each of the six Painlevé equations as P_J ($J =$ I, II, ..., VI). A solution of P_J is called a *Painlevé transcendental function.* Consider the Painlevé equation $\mathsf{P} = \mathsf{P}_J$, depending on a parameter $\boldsymbol{v}$: we will write the equation as $\mathsf{P}(\boldsymbol{v})$ and ℓ a transformation of a space V of parameters $\boldsymbol{v}$. A map of the form

$$\pi \colon S(\boldsymbol{v}) \to S(\ell(\boldsymbol{v})) , \quad \bar{q} = \pi(q)$$

is called a *contiguity relation associated with* ℓ, if $\bar{q}$ is *rational* in q and its derivatives with rational function coefficients of the independent variable t. Using this terminology, we say (0.2) is a contiguity relation of the hypergeometric differential equation associated with the parallel transformation of the parameter:

$$c \mapsto c-1 .$$

Now we give the table of the six Painlevé equations:

P_{I} $$\frac{d^2q}{dt^2} = 6q^2 + t$$

P_{II} $$\frac{d^2q}{dt^2} = 2q^3 + tq + \alpha$$

$\mathsf{P}_{\mathrm{III}}$ $$\frac{d^2q}{dt^2}=\frac{1}{q}\left(\frac{dq}{dt}\right)^2-\frac{1}{t}\frac{dq}{dt}+\frac{1}{t}(\alpha q^2+\beta)+\gamma q^3+\frac{\delta}{q}$$

P_{IV} $$\frac{d^2q}{dt^2}=\frac{1}{2q}\left(\frac{dq}{dt}\right)^2+\frac{3}{2}q^3+4tq^2+2(t^2-\alpha)q+\frac{\beta}{q}$$

P_{V} $$\frac{d^2q}{dt^2}=\left(\frac{1}{2q}+\frac{1}{q-1}\right)\left(\frac{dq}{dt}\right)^2-\frac{1}{t}\frac{dq}{dt}+\frac{(q-1)^2}{t^2}\left(\alpha q+\frac{\beta}{q}\right)+\frac{\gamma}{t}q+\delta\frac{q(q+1)}{q-1}$$

P_{VI} $$\frac{d^2q}{dt^2}=\frac{1}{2}\left(\frac{1}{q}+\frac{1}{q-1}+\frac{1}{q-t}\right)\left(\frac{dq}{dt}\right)^2-\left(\frac{1}{t}+\frac{1}{t-1}+\frac{1}{q-t}\right)\frac{dq}{dt}+$$
$$+\frac{q(q-1)(q-t)}{t^2(t-1)^2}\left[\alpha+\beta\frac{t}{q^2}+\gamma\frac{t-1}{(q-1)^2}+\delta\frac{t(t-1)}{(q-t)^2}\right].$$

Here α, β, γ and δ denote complex constants. We assume throughout the series of papers that $\delta\neq 0$ for P_{V} and $\gamma\delta\neq 0$ for $\mathsf{P}_{\mathrm{III}}$. We concern mainly the studies on the contiguity relations of the Painlevé equations, therefore the first equation P_{I} is not considered in the following. It contains no parameter.

The Painlevé equations P_J ($J=\mathrm{I}, \ldots, \mathrm{VI}$) are characterized as nonlinear ordinary differential equations of the second order without any movable critical point. They can be written in the Hamiltonian system:

$$\frac{dq}{dt}=\frac{\partial H}{\partial p}, \qquad \frac{dp}{dt}=-\frac{\partial H}{\partial q}, \tag{0.6}$$

with the Hamiltonian $H(t; q, p)$, rational in t and polynomial in (q, p) ([7], [8]). The Hamiltonian H_J associated with P_J is given by the following table;

H_{I} $$\frac{1}{2}p^2-2q^3-tq$$

H_{II} $$\frac{1}{2}p^2-\left(q^2+\frac{t}{2}\right)p-\left(\alpha+\frac{1}{2}\right)q$$

H_{III} $$\frac{1}{t}[q^2p^2-\{2\eta_\infty tq^2+(2\theta_0+1)q-2\eta_0 t\}p+2\eta_\infty(\theta_0+\theta_\infty)tq]$$

H_{IV} $$2qp^2-\{q^2+2tq+2\theta_0\}p+\theta_\infty q$$

H_{V} $$\frac{1}{t}[(q-1)^2p^2-\{\varkappa_0(q-1)^2+\theta q(q-1)-\eta tq\}p+\varkappa(q-1)]$$

H_{VI} $$\frac{1}{t(t-1)}[q(q-1)(q-t)p^2-$$
$$-\{\varkappa_0(q-1)(q-t)+\varkappa_1 q(q-t)+(\theta-1)q(q-1)\}p+\varkappa(q-t)].$$

Here the constants in H_J are connected to $\alpha, \beta, \gamma, \delta$ of the equations P_J as follows:

$$H_{\mathrm{III}}: \alpha = -4\eta_\infty\theta_\infty\,, \quad \beta = 4\eta_0(\theta_0+1)\,, \quad \gamma = 4\eta_\infty^2\,, \quad \delta = -4\eta_0^2\,,$$
$$H_{\mathrm{IV}}: \alpha = -\theta_0 + 2\theta_\infty + 1\,, \quad \beta = -2\theta_3^2\,,$$
$$H_{\mathrm{V}}: \alpha = \tfrac{1}{2}\varkappa_\infty^2\,, \quad \beta = -\tfrac{1}{2}\varkappa_0^2\,, \quad \gamma = -\eta(\theta+1)\,, \quad \delta = -\tfrac{1}{2}\eta^2\,,$$
$$\varkappa = \tfrac{1}{4}(\varkappa_0+\theta)^2 - \tfrac{1}{4}\varkappa_\infty^2\,,$$
$$H_{\mathrm{VI}}: \alpha = \tfrac{1}{2}\varkappa_\infty^2\,, \quad \beta = -\tfrac{1}{2}\varkappa_0^2\,, \quad \gamma = \tfrac{1}{2}\varkappa_1^2\,, \quad \delta = \tfrac{1}{2}(1-\theta^2)\,,$$
$$\varkappa = \tfrac{1}{4}(\varkappa_0+\varkappa_1+\theta-1)^2 - \tfrac{1}{4}\varkappa_\infty^2\,.$$

By the assumption, $\eta \neq 0$ for H_{V} and $\eta_\Delta \neq 0$ ($\Delta = 0, \infty$) for H_{III}. The Hamiltonian H_J has been introduced by the use of the theory of the isomonodromic deformation of a linear ordinary differential equation; see [2], [7], [8].

The Hamiltonian structure associated with the Painlevé equation P_J is represented by

$$\mathscr{H}_J = (q, p, H_J, t)\,. \tag{0.7}$$

We denote by $\boldsymbol{v}$ the set of parameters contained in the Hamiltonian H_J and by V a space of all parameters. When we consider the Hamiltonian system (0.6) at an arbitrarily fixed value $\boldsymbol{v}$ of parameters, the Hamiltonian structure (0.7) is written as

$$\mathscr{H}(\boldsymbol{v}) = \big(q(\boldsymbol{v}), p(\boldsymbol{v}), H(\boldsymbol{v}), t\big)\,.$$

Here $H(\boldsymbol{v}) = H_J(t; q, p; \boldsymbol{v})$ is the Hamiltonian given above. We call $\mathscr{H}(\boldsymbol{v})$ the *Painlevé system at* $\boldsymbol{v}$. The totality of $\mathscr{H}(\boldsymbol{v})$:

$$\mathscr{H} = \bigcup_{\boldsymbol{v}\in V} \mathscr{H}(\boldsymbol{v})$$

is the Painlevé system associated with P_J. In this series of papers, we will study mainly the dependence of $\mathscr{H} = \mathscr{H}_J$ on V.

A geometrical interpretation of the Hamiltonian structure $\mathscr{H}(\boldsymbol{v})$ at $\boldsymbol{v}$ has been studied in [4]. We constructed the fiber space with the foliated structure associated with $\mathscr{H}(\boldsymbol{v})$. The Painlevé system $\mathscr{H}$ itself can be regarded as a fiber space with the base space V: a fiber of this fibration is $\mathscr{H}(\boldsymbol{v})$ provided with the foliation. We do not discuss in what follows a geometrical structure of the Painlevé system, although this point of view will yield some interesting and important problems to be examined.

We shall see that for each J the space $\mathsf{V} = \mathsf{V}_J$ of parameters of $H = H_J$ is a complex affine space, whose dimension N_J is:

$$N_{\mathrm{II}} = 1\,, \quad N_{\mathrm{III}} = N_{\mathrm{IV}} = 2\,, \quad N_{\mathrm{V}} = 3\,, \quad N_{\mathrm{VI}} = 4\,.$$

For example, it seems the third equation $\mathsf{P}_{\mathrm{III}}$ depends on the four parameters $\alpha, \beta, \gamma, \delta$. On the other hand, by replacing q by λq and t by μt, we can put, without loss of generality,

$$\gamma = 4\,, \qquad \delta = -4\,,$$

λ, μ being constants.

Let $(q(\boldsymbol{v}; t), p(\boldsymbol{v}; t))$ be a solution of the Hamiltonian system (0.6) with the Hamiltonian $H(\boldsymbol{v}) = H(t; q, p; \boldsymbol{v})$. We call it simply a *solution of the Painlevé system* $\mathscr{H}(\boldsymbol{v})$ and write it as $(q(\boldsymbol{v}), p(\boldsymbol{v}))$. Consider the 2-form:

$$\Omega = dp \wedge dq - dH \wedge dt\,,$$

called the *fundamental form* attached to the Hamiltonian structure (0.7). We denote by $\Omega_{\boldsymbol{v}}$ the restriction of Ω on the Painlevé system $\mathscr{H}(\boldsymbol{v})$ at $\boldsymbol{v}$. A transformation of $\mathscr{H}$

$$\pi\colon \mathscr{H} \to \mathscr{H}\,,$$

is said *canonical* if Ω remains invariant under π:

$$\pi^* \Omega = \Omega\,.$$

Denote by $\pi_{\boldsymbol{v}}$ the restriction of π on the fiber $\mathscr{H}(\boldsymbol{v})$. For $\boldsymbol{v}$ of V, we have $\boldsymbol{v}'$ such that

$$\pi_{\boldsymbol{v}}\colon \mathscr{H}(\boldsymbol{v}) \to \mathscr{H}(\boldsymbol{v}')\,,$$

$$\pi_{\boldsymbol{v}}^*(\Omega_{\boldsymbol{v}'}) = \Omega_{\boldsymbol{v}}\,.$$

The transformation of V:

$$\ell\colon \boldsymbol{v} \mapsto \boldsymbol{v}'$$

is thus induced from the canonical transformation π. Let

$$q' = Q(t; q, p)\,, \qquad p' = P(t; q, p) \tag{0.8}$$

$$t' = \varphi(t) \tag{0.9}$$

$$H' = \varrho(t) H + \Phi(t; q, p) \tag{0.10}$$

be an representation of $\pi_{\boldsymbol{v}}$, where we put:

$$\mathscr{H}(\boldsymbol{v}) = (q, p, H, t)\,, \qquad \mathscr{H}(\boldsymbol{v}') = (q', p', H', t')\,.$$

The canonical transformation π is said to be *rational*, if for any $\boldsymbol{v}$, the functions Q, P, φ, ϱ and Φ are rational with respect to the canonical variables. By the definition, we have the following conditions:

$$\frac{\partial P}{\partial p}\frac{\partial Q}{\partial q}-\frac{\partial P}{\partial q}\frac{\partial Q}{\partial p}=1\,, \tag{0.11}$$

$$\frac{\partial P}{\partial q}\frac{\partial Q}{\partial t}-\frac{\partial P}{\partial t}\frac{\partial Q}{\partial q}-\frac{\partial \Phi}{\partial q}\frac{d\varphi}{dt}=0\,, \tag{0.12_1}$$

$$\frac{\partial P}{\partial p}\frac{\partial Q}{\partial t}-\frac{\partial P}{\partial t}\frac{\partial Q}{\partial p}-\frac{\partial \Phi}{\partial p}\frac{d\varphi}{dt}=0\,, \tag{0.12_2}$$

$$\varrho\frac{d\varphi}{dt}=1\,. \tag{0.13}$$

When (0.8)-(0.9) define a birational map from (q, p, t) to (q', p', t'), we call π a *birational canonical transformation* of $\mathscr{H}$. A rational canonical transformation of the form

$$q'=q\,,\quad p'=p\,,\quad t'=t\,,\quad H'=H+\Phi(t)$$

is said *trivial*. We do not distinguish such a transformation from the identity map, since the Hamiltonian system (0.6) is unchanged. We consider a canonical transformation modulo trivial one.

Let π be a birational canonical transformation, represented by (0.8)-(0.10). We say $\mathscr{H}$ is *stable* with respect to π, if, for any $\boldsymbol{v}$ of V, $\pi_{\boldsymbol{v}}(\mathscr{H}(\boldsymbol{v}))=\mathscr{H}(\boldsymbol{v})$. The transformation π is said to be of the *first kind*, if $t'=t$ in (0.9). Two birational canonical transformations π_i $(i=1,2)$ are said to be equivalent, if $\pi_2\circ\pi_1^{-1}=\pi$ is of the first kind and $\mathscr{H}$ is stable with respect to π. We will identify π_1 and π_2 if they are equivalent each other. The main subfect of this series of papers is to investigate a birational canonical transformation of $\mathscr{H}$ which induce an affine transformation of V. Let ℓ be an affine transformation of V. If for $i=1,2$ and for any $\boldsymbol{v}$

$$\pi_{i,\boldsymbol{v}}\colon \mathscr{H}(\boldsymbol{v})\to\mathscr{H}(\ell(\boldsymbol{v}))\,,$$

then $\mathscr{H}$ is stable with respect to $\pi=\pi_2\circ\pi_1^{-1}$. Moreover if π is of the first kind π_1 is equivalent to π_2. As for this equivalence relation we propose the following conjecture:

CONJECTURE 0.1. – *Suppose that $\mathscr{H}$ is stable with respect to a birational canonical transformation π. If π is of the first kind, then π is the identity transformation of $\mathscr{H}$.*

Here we identify a trivial transformation with the identity, as was remarked above.

Assuming that the assertion of the conjecture is established, we obtain for an affine transformation ℓ of V the unique birational canonical transformation $\pi=$

$= \{\pi_{v};\, v \in \mathsf{V}\}$, if it does exist. We have

$$\pi_{v}(\mathscr{H}(v)) = \mathscr{H}(\ell(v))$$

for any v. π is called a *representation* of ℓ on the Painlevé system and written as

$$\pi = \ell_{*}\,. \tag{0.14}$$

Given a birational canonical transformation π, we denote by V^{π} the set of v such that $\pi(\mathscr{H}(v)) = \mathscr{H}(v)$. In the case when π is of the first kind, Conjecture 0.1 means that $\mathsf{V}^{\pi} = \mathsf{V}$ if and only if π is the identity transformation. Furthermore, we make the following conjecture.

CONJECTURE 0.2. – *If π is of the first kind and has a non-empty set V^{π}, then V^{π} is a proper analytic subset of V.*

These conjectures are not verified yet. In any case, ℓ_{*} can be determined from ℓ, if it exists, uniquely up to a stable transformation of the first kind.

Let G be a subgroup of the group $\mathscr{A}(\mathsf{V})$ of affine motions in V, such that for any element g of G there exists a birational canonical transformation π which induces g:

$$\pi\colon \mathscr{H}(v) \to \mathscr{H}(g(v))\,.$$

We denote by $\mathscr{G}$ the set of such π's and by $\mathscr{G}_0$ the subset of $\mathscr{G}$ consisting of transformations π_0 of the first kind which keep $\mathscr{H}$ stable. Noting $\mathscr{G}_0$ is a normal subgroup of $\mathscr{G}$, we write the quotient group $\mathscr{G}/\mathscr{G}_0$ as G_{*}. The assertion of Conjecture 0.1 implies $\mathsf{G} = \mathsf{G}_{*}$. The homomorphism

$$\mathsf{G} \ni g \to g_{*} \in \mathsf{G}_{*}$$

is called a *nonlinear representation of* G *on the Painlevé system.* In other words, the image G_{*} of G is the family of the contiguity relations g_{*} of Painlevé transcendental functions. We will associate the group $\mathsf{G} = \mathsf{G}_J$ with each $\mathscr{H}_J$ ($J = \mathrm{II}, \ldots, \mathrm{VI}$) and give an explicit realization of the representation of G. The presentation of G_J being somewhat complicated, we will do it later for each G_J. The group G contains the affine Weyl $\tilde{W}$ group as a subgroup. To describe the group $\tilde{W}$, we have to introduce the notion of the τ-function of the Painlevé system.

Let $(q(t; v), p(t; v))$ be a solution of the Painlevé system $\mathscr{H}(v)$ at v. We define the τ-function $\tau(v; t)$ related to $\mathscr{H}(v)$ by:

$$\frac{d}{dt}\log\,\tau(t; v) = H(t; q(t; v), p(t; v); v)\,, \tag{0.15}$$

with ambiguity of a multiplicative constant (see [6]).

On the other hand, JIMBO and MIWA have defined in [2] τ-functions by using the theory of the isomonodromic deformation of linear ordinary differential equation. They coincide with (0.15) as for the Painlevé systems. A birational canonical transformation $\pi = g_*$ leads to the correspondence of τ-functions:

$$\tau(t;\, \boldsymbol{v}) \to \tau\big(t;\, g(\boldsymbol{v})\big)$$

in a natural way. We denote it also by $\pi = g_*$. We will make no distinction between two τ-functions τ_i $(i = 1, 2)$ such that

$$\frac{d}{dt}\log \tau_1 - \frac{d}{dt}\log \tau_2$$

is rational in t. They are mutually connected through a trivial canonical transformation. This identification will be in discard when we consider *rational* solutions or *classical* solutions of the Painlevé system.

Let G be the affine subgroup with the representation $\mathsf{G} \to \mathsf{G}_*$ on the Painlevé system $\mathscr{H}$. We say that the τ-function $\tau(\boldsymbol{v}) = \tau(t;\, \boldsymbol{v})$ *remains invariant* under the birational canonical transformation g_*, if the logarithmic derivative of the function

$$g_*\big(\tau(\boldsymbol{v})\big)/\tau(\boldsymbol{v})$$

is a rational function of t. Here we adopt the identification of τ-functions. We denote by W the subgroup of G such that $\tau(\boldsymbol{v})$ remains invariant under the representation w_* of any w of W. It will be shown for each J that W is a realization of the Weyl group $W(\mathsf{R})$ of the root system R. The type of each $\mathsf{R} = \mathsf{R}_J$ $(J = \mathrm{II}, \ldots, \mathrm{VI})$ is given as follows:

$$\mathsf{R}_{\mathrm{II}}\colon\ A_1$$

$$\mathsf{R}_{\mathrm{III}}\colon\ B_2$$

$$\mathsf{R}_{\mathrm{IV}}\colon\ A_2$$

$$\mathsf{R}_{\mathrm{V}}\colon\ A_3$$

$$\mathsf{R}_{\mathrm{VI}}\colon\ D_4\,.$$

Throughout this series of papers we use the notation used in [1] concerning the theory of root systems.

Moreover we will construct for each J the birational canonical transformation ℓ_* corresponding to the parallel transformation ℓ of V. The group $W = W(\mathsf{R}_J)$ and ℓ generate $\tilde{W}$, which is isomorphic to the affine Weyl group $W_a(\mathsf{R}_J)$. We will obtain the representation:

$$\tilde{W} \to \tilde{W}_*$$

on the Painlevé system. For g of $\tilde{W}$, the birational canonical transformation g_* is of the first kind.

Let $\tau(\boldsymbol{v}) = \tau(t;\boldsymbol{v})$ be the τ-function related to a solution $(q(\boldsymbol{v}), p(\boldsymbol{v})) = (q(t;\boldsymbol{v}), p(t;\boldsymbol{v}))$ of the Painlevé system $\mathscr{H}(\boldsymbol{v})$ at $\boldsymbol{v}$. For ℓ of the group G we define the set of τ-functions

$$\mathfrak{T}(\ell) = \{\tau_m;\ m \in \mathbb{Z}\} \tag{0.16}$$

by $\tau_0 = \tau(\boldsymbol{v})$, $\tau_m = (\ell_*)^m \tau(\boldsymbol{v})$. If ℓ is of infinite order, we call (0.16) a *τ-sequence defined by ℓ.* We will show for each $\mathscr{H}_J$ that there exists a parallel transformation ℓ such that the τ-sequence (0.16) satisfies the equation:

$$\delta^2 \log \tau_m = c_m \frac{\tau_{m-1}\tau_{m+1}}{\tau_m^2}, \tag{0.17}$$

c_m being a non-zero constant. Here δ is a derivation: we will see

$$\begin{aligned} \delta &= \frac{d}{dt} && \text{for } H_{\mathrm{II}},\ H_{\mathrm{IV}}, \\ \delta &= t\frac{d}{dt} && \text{for } H_{\mathrm{III}},\ H_{\mathrm{V}}, \\ \delta &= t(1-t)\frac{d}{dt} && \text{for } H_{\mathrm{VI}}. \end{aligned}$$

The constraint (0.17) for (0.16) is the well-known *Toda equation* for τ-functions. We can put in (0.17) $c_m = 1$ by choosing suitably normalization constants for τ_m. We will verify (0.17) without the help of the theory of the isomonodromic deformation of linear equations: compare with [2].

Let $\tau = \tau(v)$ be a τ-function related to the Painlevé system $\mathscr{H}(\boldsymbol{v})$ at $\boldsymbol{v}$. We have the family of τ-functions:

$$\mathfrak{T}(\mathsf{G};\boldsymbol{v}) = \{\tau_g;\ \tau_g^{\natural} = g_*\tau \text{ for } g \in \mathsf{G}\}.$$

where g_* denotes the representation of g on the Painlevé system. The Painlevé transcendental functions can be represented in terms of functions in $\mathfrak{T}(\mathsf{G};\boldsymbol{v})$. For example, we will show that there exist the τ-functions τ_1, τ_2 such that

$$q(\boldsymbol{v}) = \mathrm{const}\cdot\delta\log\frac{\tau_2}{\tau_1}, \tag{0.18}$$

$q(\boldsymbol{v})$ being a solution of the Painlevé equation. We note that the expressions (0.17)-(0.18) are written in consideration of the identification of τ-functions mentioned above: see (0.20).

It is known that, for particular values of the parameter $\boldsymbol{v}$, the Painlevé equation P_J possesses special solutions expressible in terms of the *classical transcendental functions*, that is, Gauss' hypergeometric functions, Bessel functions and so on. We will readjust these facts and obtain some results on special solutions of the Painlevé system, by taking the affine Weyl group $W_a(\mathbf{R})$ into consideration. To describe the results we use the reflection group $\tilde{W}$ of the affine space V, isomorphic to $W_a(\mathbf{R})$. Let $\pi = g_*$ be the representation of g of $\tilde{W}$ on the Painlevé system H. We will see that if $\boldsymbol{v}$ is contained in the subset V^π of V, the Painlevé system $\mathscr{H}(\boldsymbol{v})$ possesses a solution represented by classical transcendental functions. V^π is a wall of some Weyl chamber associated with $W_a(\mathbf{R})$. The list of classical transcendental functions which appear as special solutions of $\mathscr{H} = \mathscr{H}_J$ is the following:

H_{II}	Airy functions
H_{III}	Bessel functions
H_{IV}	Hermite-Weber functions
H_{V}	Confluent hypergeometric functions
H_{VI}	Hypergeometric functions .

Some rational solutions of the Painlevé systems will be studied.

The present article is the first part of the studies on the Painlevé systems. We study in the following the sixth Painlevé equation P_{VI}. The next part of the series of papers will be devoted to the theory of the fifth one P_{V}. The other three equations P_{II}, $\mathsf{P}_{\mathrm{III}}$, P_{IV} are relatively known and studied in many articles. We shall investigate also these equations in the forthcoming papers, by means of the method of birational canonical transformations.

Some results given in this series of papers have been announced in [5].

In § 1, we will firstly define the auxiliary Hamiltonian function $h = h(t)$ associated with the sixth Painlevé equation $\mathsf{P} = \mathsf{P}_{\mathrm{VI}}$. We will see that $h = h(t)$ satisfies the nonlinear ordinary differential equation $\mathsf{E} = \mathsf{E}_{\mathrm{VI}}$:

$$\frac{dh}{dt}\left[t(1-t)\frac{d^2h}{st^2}\right]^2 + \left[\frac{dh}{dt}\left\{2h-(2t-1)\frac{dh}{dt}\right\} + b_1b_2b_3b_4\right]^2 - \prod_{k=1}^{4}\left(\frac{dh}{dt} + b_k^2\right) = 0 \, .$$

Here the constants b_k $(k = 1, \ldots, 4)$ are defined by

$$b_1 = \tfrac{1}{2}(\varkappa_0 + \varkappa_1)\,, \quad b_2 = \tfrac{1}{2}(\varkappa_0 - \varkappa_1)\,, \quad b_3 = \tfrac{1}{2}(\theta - 1 + \varkappa_\infty)\,, \quad b_4 = \tfrac{1}{2}(\theta - 1 - \varkappa_\infty)\,.$$

We can regard $\boldsymbol{b} = (b_1, b_2, b_3, b_4)$ as a parameter of the Painlevé system $\mathscr{H} = \mathscr{H}_{\mathrm{VI}}$. We shall prove that *there is the one-to-one correspondence from a solution* (q, p) *of the Painlevé system* $\mathscr{H}$ *to a general solution* h *of* E. The nonlinear representation of the Weyl group W can be deduced from this fact.

Let ℓ be the parallel transformation:

$$\ell\colon (b_1, b_2, b_3, b_4) \to (b_1, b_2, b_3+1, b_4)\,. \tag{0.19}$$

We will construct the birational canonical transformation ℓ_* related to ℓ: see Proposition 1.6. Let G^0 be the group generated by the Weyl group W and ℓ, realized as the subgroup of affine motions on the space of parameters $\boldsymbol{b}$. G^0 contains the affine Weyl group $\tilde{W}=W_a(\mathsf{R})$ of the root system of the type D_4. The representation g_* of any g of G^0 is of the first kind. One of the main purposes of the present article is to obtain the explicit form of the birational canonical transformation g_*. We will do it in §§ 1.2. Proofs of the results stated in § 1 are given in § 2.

It is known that the Hamiltonian associated with the sixth Painlevé equation $\mathsf{P}=\mathsf{P}_{\mathrm{VI}}$ is invariant under some rational transformations *except permutations of constants*. For example, replacing in P, t by $1/t$ and q by $1/q$, we obtain equation:

$$\frac{d^2q}{dt^2}=\frac{1}{2}\left(\frac{1}{q}+\frac{1}{q-1}+\frac{1}{q-t}\right)\left(\frac{dq}{dt}\right)^2-\left(\frac{1}{t}+\frac{1}{t-1}+\frac{1}{q-t}\right)\frac{dq}{dt}+$$
$$+\frac{q(q-1)(q-t)}{t^2(t-1)^2}\left[-\beta-\alpha\frac{t}{q^2}+\gamma\frac{t-1}{(q-1)^2}+\delta\frac{t(t-1)}{(q-t)^2}\right].$$

This replacement extends to the canonical transformation:

$$(q, p, H, t) \to \left(\frac{1}{q}, \frac{1}{2}(\varkappa_0+\varkappa_1+\theta-1+\varkappa_\infty)q-q^2p, -\frac{1}{t^2}H, \frac{1}{t}\right)$$

and yields in the Hamiltonian the permutation of constants:

$$\varkappa_0 \to \varkappa_\infty\,.$$

We have the representation of the group G^1 of permutations of the four constants $\varkappa_0, \varkappa_1, \theta, \varkappa_\infty$, on the Painlevé system $\mathscr{H}$ (see [4], [10]). Let $\mathsf{G}=\mathsf{G}_{\mathrm{VI}}$ be tne group generated by G^0 and G^1, then we obtain the representation of G

$$\mathsf{G} \to \mathsf{G}_*$$

on $\mathscr{H}$. We define the affine space $\mathsf{V}=\mathsf{V}_{\mathrm{VI}}$ as the totality of vectors $\boldsymbol{v}=(v_1, v_2, v_3, v_4)$ such that

$$v_1=\theta-1\,,\quad v_2=\varkappa_0\,,\quad v_3=\varkappa_1\,,\quad v_4=\varkappa_\infty\,,$$

and regard it as the space of parameters of $\mathscr{H}$. We shall realize G as the subgroup of affine motions in V and see it is isomorphic to the affine Weyl group of the exceptional root system of the type F_4: The determination of G will be done in § 3; see Theorem 1.

The section 4 concerns the studies on the τ-functions of the Painlevé system $\mathscr{H}$. We show in Proposition 4.2 and in Theorem 2 that the τ-sequence defined by the parallel transformation (0.19) satisfies the Toda equation (0.17). Moreover a solution $q(t;\boldsymbol{v})$ of the sixth Painlevé equation $\mathsf{P}(\boldsymbol{v})$ is written in the form

$$v_4\big(q(t;\boldsymbol{v})-t\big)=\tau_1^{-1}\delta\tau_1-\tau_2^{-1}\delta\tau_2\,, \tag{0.20}$$

where τ_1, τ_2 are τ-functions of the family $\mathfrak{T}(\mathsf{G};\boldsymbol{v})$ and $\delta=t(t-1)(d/dt)$.

For certain values of the parameters $\boldsymbol{v}$, the Painlevé system $\mathscr{H}(\boldsymbol{v})$ at $\boldsymbol{v}$ possesses solutions such that in the expression (0.20) the τ-functions τ_1, τ_2 are represented in terms of hypergeometric functions (see [2], [4]). We say such solutions to be *classical*. Classical solutions of $\mathscr{H}(\boldsymbol{v})$ are the subject of the final section, § 5, where we will see that they appear in walls of a Weyl chamber of the affine Weyl group $\tilde{W}$ of the root system of the type D_4. The studies of the last section will lead us to a new view-point in the theory of hypergeometric functions through the theory of the Painlevé system. We will give some examples of τ-sequences whose τ-functions are classical and examine them in details.

1. – Sixth Painlevé equation.

1.1. *Auxiliary Hamiltonian function.*

In the present article, we study mainly the sixth Painlevé equation $\mathsf{P}_{\rm VI}$:

$$\frac{d^2q}{dt^2}=\frac{1}{2}\left(\frac{1}{q}+\frac{1}{q-1}+\frac{1}{q-t}\right)\left(\frac{dq}{dt}\right)^2-\left(\frac{1}{t}+\frac{1}{t-1}+\frac{1}{q-t}\right)\frac{dq}{dt}+\\
+\frac{q(q-1)(q-t)}{t^2(t-1)^2}\left[\alpha+\beta\frac{t}{q^2}+\gamma\frac{t-1}{(q-1)^2}+\delta\frac{t(t-1)}{(q-t)^2}\right].$$

The Hamiltonian $H_{\rm VI}$ associated with it is the following:

$$\frac{1}{t(t-1)}[q(q-1)(q-t)p^2-\\
-\{\varkappa_0(q-1)(q-t)+\varkappa_1 q(q-t)+(\theta-1)q(q-1)\}p+\varkappa(q-t)]\,,$$

where $\varkappa_\varDelta$ $(\varDelta=0,1)$, $\varkappa$, θ denote the constants such that

$$\varkappa_0=\sqrt{-2\beta}\,,\quad \varkappa_1=\sqrt{2\gamma}\,,\quad \theta=\sqrt{1-2\delta}\,,$$
$$\varkappa=\frac{1}{4}(\varkappa_0+\varkappa_1+\theta-1)^2-\frac{1}{4}\varkappa_\infty^2\,,\quad \varkappa_\infty=\sqrt{2\alpha}\,.$$

Let $\boldsymbol{e}_j$ $(j=1,\ldots,4)$ be the canonical basis of the four dimensional complex vector space $\mathbf{C}^4$ with a symmetric bilinear form $(\boldsymbol{b}|\boldsymbol{b}')$; we have by definition $(\boldsymbol{e}_i|\boldsymbol{e}_j)=$

$= (\boldsymbol{e}_j|\boldsymbol{e}_i) = \delta_{ij}$, δ_{ij} being the Kronecher's delta. We associate the constants of the Hamiltonian H_{VI} with a vector

$$\boldsymbol{b} = \sum_{j=1}^{4} b_j \boldsymbol{e}_j \tag{1.1}$$

in the following manner:

$$\varkappa_0 = b_1 + b_2 , \quad \varkappa_1 = b_1 - b_2 , \quad \theta = b_3 + b_4 + 1 , \quad \varkappa_\infty = b_3 - b_4 . \tag{1.2}$$

We consider the space $\mathbf{C}^4$ as the parameter space of the Painlevé system:

$$\mathscr{H}_{\mathrm{VI}} = (q, p, H_{\mathrm{VI}}, t) , \tag{1.3}$$

associated with P_{VI}, through (1.1)-(1.2). In the following of this paper, the vector (1.1) will be written simply as $\boldsymbol{b} = (b_1, b_2, b_3, b_4)$. Denote by $\sigma_k[\boldsymbol{b}]$ the k-th fundamental symmetric polynomial ($k = 1, \dots, 4$) of b_1, b_2, b_3, b_4 and by $\sigma'_s[\boldsymbol{b}]$ ($s = 1, 2, 3$) the s-th one with respect to b_1, b_3, b_4.

A Hamiltonian function $H_{\mathrm{VI}}(t)$ related to $\mathscr{H}_{\mathrm{VI}}$ is defined by

$$H_{\mathrm{VI}}(t) = H_{\mathrm{VI}}(t; q(t), p(t)) , \tag{1.4}$$

where $(q(t), p(t))$ is a solution of the Hamiltonian system

$$\frac{dq}{dt} = \frac{\partial H}{\partial p} , \quad \frac{dp}{dt} = -\frac{\partial H}{\partial q} \tag{1.5}$$

with the Hamiltonian $H_{\mathrm{VI}} = H_{\mathrm{VI}}(t; q, p)$. We call $H_{\mathrm{VI}}(t)$ simply a Hamiltonian function of $\mathscr{H}_{\mathrm{VI}}$. For the purpose of simplification of presentation, we omit in what follows the subscript from P_{VI}, $\mathscr{H}_{\mathrm{VI}}$, H_{VI} and so on, unless there is a risk of confusion. We introduce the auxiliary Hamiltonian function:

$$h(t) = t(t-1)H(t) + \sigma'_2[\boldsymbol{b}]t - \tfrac{1}{2}\sigma_2[\boldsymbol{b}] , \tag{1.6}$$

which plays an important role in our studies. In fact we obtain the following propositions.

PROPOSITION 1.1. – *The function $h = h(t)$ satisfies the nonlinear ordinary differential equation:*

$$\mathsf{E}_{\mathrm{VI}} \qquad \frac{dh}{dt}\left[t(1-t)\frac{d^2h}{dt^2}\right]^2 + \left[\frac{dh}{dt}\left\{2h - (2t-1)\frac{dh}{dt}\right\} + \sigma_4[\boldsymbol{b}]\right]^2 = \prod_{k=1}^{4}\left(\frac{dh}{dt} + b_k^2\right).$$

PROPOSITION 1.2. – *There is the one-to-one correspondence from a general solution h of $\mathsf{E} = \mathsf{E}_{\mathrm{VI}}$ to that (q, p) of the Painlevé system $\mathscr{H}$.*

This correspondence is denoted by

$$\Gamma(h) = (q, p) . \tag{1.7}$$

h is expressed as the polynomial in (q, p), by the definition (1.6). On the other hand, it is shown that q and p are rational with respect to h and its derivatives dh/dt, d^2h/dt^2. So we say (1.7) defines a birational correspondence. We will prove these propositions in the next section.

REMARK 1.1. – The equation **E** has the one-parameter family of *singular* solutions:

$$h = \lambda t + \mu . \tag{1.8}$$

Here (λ, μ) is on the elliptic curve:

$$\{\lambda(\lambda + 2\mu) + \sigma_4[\boldsymbol{b}]\}^2 = \prod_{j=1}^{4} (\lambda + b_k^2) . \tag{1.9}$$

In general, the function h is not written in the form (1.8) for (q, p), since there is no algebraic first integral for the Painlevé system. This fact has been known as the *transcendency* of the Painlevé equation ([10]).

1.2. *Invariance of the differential equation E.*

For a point $\boldsymbol{b} = (b_1, b_2, b_3, b_4)$ of $\mathbf{C}^4$, consider the following four linear transformations w_j $(j = 1, \ldots, 4)$:

$$\begin{aligned}
&w_1\colon (b_1, b_2, b_3, b_4) \to (b_2, b_1, b_3, b_3)\\
&w_2\colon (b_1, b_2, b_3, b_4) \to (b_1, b_3, b_2, b_4)\\
&w_3\colon (b_1, b_2, b_3, b_4) \to (b_1, b_2, b_4, b_3)\\
&w_4\colon (b_1, b_2, b_3, b_4) \to (b_1, b_2, -b_3, -b_4) .
\end{aligned}$$

If we put

$$\boldsymbol{a}_1 = \boldsymbol{e}_1 - \boldsymbol{e}_2 , \quad \boldsymbol{a}_2 = \boldsymbol{e}_2 - \boldsymbol{e}_3 , \quad \boldsymbol{a}_3 = \boldsymbol{e}_3 - \boldsymbol{e}_4 , \quad \boldsymbol{a}_4 = \boldsymbol{e}_3 + \boldsymbol{e}_4 ,$$

then for each j, w_j is a reflection in $\mathbf{C}^4$ with respect to $\boldsymbol{a}_j$, that is,

$$w_j(\boldsymbol{v}) = \boldsymbol{v} - 2\frac{(\boldsymbol{v}|\boldsymbol{a}_j)}{(\boldsymbol{a}_j|\boldsymbol{a}_j)}\boldsymbol{a}_j .$$

Let W be the group generated by $w_1, \ldots, w_4$. W is a subgroup of the complex orthogonal group $O_4(\mathbf{C})$ and moreover we have the

PROPOSITION 1.3 ([1]). – *W is isomorphic to the Weyl group $W(\mathsf{R})$ of simple root system R of the type D_4.*

In order to simplify notation, we will write $W(\mathsf{R})$ as $W(D_4)$.

We regard $\mathbf{C}^4$ also as the space of parameters of the nonlinear differential equation $\mathsf{E} = \mathsf{E}_{\mathrm{VI}}$. When considering the Painlevé system $\mathscr{H}[\boldsymbol{b}]$ at $\boldsymbol{b}$, we denote E by $\mathsf{E}[\boldsymbol{b}]$ and by $(q[\boldsymbol{b}], p[\boldsymbol{b}])$ a solution of $\mathscr{H}[\boldsymbol{b}]$ with the auxiliary Hamiltonian function $h[\boldsymbol{b}]$. It is easy to see the

PROPOSITION 1.4. – *For any w of W, we have*

$$\mathsf{E}[\boldsymbol{b}] = \mathsf{E}[w(\boldsymbol{b})] .$$

In fact, the coefficients of the equation E are the fundamental symmetric polynomials of $b_1^2, b_2^2, b_3^2, b_4^2$ and $\sigma_4(\boldsymbol{b})$, that is, the *invariant polynomials of the Weyl group $W(D_4)$*. For a solution $h = h[\boldsymbol{b}]$ of $\mathsf{E}[\boldsymbol{b}]$,

$$h_w = h[w(\boldsymbol{b})]$$

satisfies $\mathsf{E}[w(\boldsymbol{b})]$ and vice versa. Definitively, by putting

$$h = h_w , \tag{1.10}$$

we obtain the relation between $(q, p) = (q[\boldsymbol{b}], p[\boldsymbol{b}])$ and $(q_w, p_w) = (q[w(\boldsymbol{b}), p[w(\boldsymbol{b})])$ by means of the correspondence (1.7). In fact, we show the

PROPOSITION 1.5. – *There exists the birational canonical transformation of the Painlevé system:*

$$w_*: (q, p, H[\boldsymbol{b}], t) \to (q_w, p_w, H[w(\boldsymbol{b})], t) .$$

By the definition w_* is the representation of w and its explicit form will given in the proof of this proposition: see the section 2.2. Let W_* be the group generated by $(w_j)_*$ $(j = 1, \ldots, 4)$. W_* is homomorphic to the Weyl group $W = W(D_4)$. In particular, we have from the Proposition 1.5 the expressions:

$$\begin{aligned} q_w &= R(w; q, p) , & p_w &= S(w; q, p) , \\ q &= R(w^{-1}; q_w, p_w) , & p &= S(w^{-1}; q_w, p_w) , \end{aligned}$$

R, S denoting rational functions. Moreover, we can construct the representation of the affine Weyl group $W_a(\mathsf{R})$ associated with the root system of the type D_4 on the Painlevé system. We write $W_a(\mathsf{R})$ as $W_a(D_4)$ in the following of this paper.

1.3. *Realization of the parallel transformation.*

Let $h = h[\boldsymbol{b}]$ be an auxiliary Hamiltonian function and $(q, p) = \Gamma(h)$ a solution of the Painlevé system defined by the correspondence (1.7). We will prove the following proposition:

PROPOSITION 1.6. – *There exists the birational canonical transformation*:

$$\ell_*: (q, p, H[\boldsymbol{b}], t) \to (q_+, p_+, H[\boldsymbol{b}^+], t) , \tag{1.11}$$

where

$$\boldsymbol{b}^+ = \boldsymbol{b} + \boldsymbol{e}_3 . \tag{1.12}$$

If we denote by ℓ_j $(j = 1, \ldots, 4)$ the parallel transformation:

$$\boldsymbol{b} \to \boldsymbol{b} + \boldsymbol{e}_j ,$$

then ℓ_* is a representation of $\ell = \ell_3$. In order to prove the proposition, we introduce the other auxiliary function $h^+ = h^+[b]$ defined by

$$h^+ = h - q(q-1)p + (b_1 + b_4)q - \tfrac{1}{2}(b_1 + b_2 + b_4) . \tag{1.13}$$

We will verify the following two propositions.

PROPOSITION 1.7. – *h^+, dh^+/dt, d^2h^+/dt^2 are polynomials in (q, p) and rational in t. Conversely, q and p are written as rational functions of h^+, dh^+/dt, d^2h^+/dt^2 and t.*

PROPOSITION 1.8. – *h^+ satisfies the nonlinear differential equation* $\mathsf{E}[\boldsymbol{b}^+]$.

Proposition 1.6 is an immediate consequence of these propositions. In fact, note firstly that, by Proposition 1.8, we can put

$$h^+ = h[\boldsymbol{b}^+] = h[\ell(\boldsymbol{b})]$$

and then obtain by (1.6) and (1.13) the following:

$$H[\ell(\boldsymbol{b})] = H[\boldsymbol{b}] - \frac{1}{t(t-1)}\, Y , \tag{1.14}$$

where

$$Y = q(q-1)p - (b_1 + b_4)(q - t) . \tag{1.15}$$

If we regard h^+, dh^+/dt, d^2h^+/dt^2 as polynomials in (q^+, p^+) by means of the correspondence (1.7), then (q, p) can be written as rational functions of (q^+, p^+) and t:

$$q = Q(t; q^+, p^+)\,, \qquad p = P(t; q^+, p^+)\,, \tag{1.16}$$

by the second assertion of Proposition 1.7. Oppositely, we write (q^+, p^+) as rational function of h^+ and its derivatives by applying again Proposition 1.2. Then we deduce again from Proposition 1.7 the expression:

$$q^+ = Q_+(t; q, p)\,, \qquad p^+ = P_+(t; q, p)\,. \tag{1.17}$$

Consequently, the birational transformation (1.11) is given by (1.14), (1.16) and (1.17). We will see that it is a canonical transformation from $\mathscr{H}[\boldsymbol{b}]$ to $\mathscr{H}[\boldsymbol{b}^+] = \mathscr{H}[\ell(\boldsymbol{b})]$, by the use of the explicit forms of (1.16) and (1.17), given in the section 2.3.

REMARK 1.2. – We obtain from ℓ_* the birational canonical transformations $(\ell_j)_*$ $(j = 1, \dots, 4)$, by combining ℓ_* with W_* obtained in Proposition 1.5 as the representation of the Weyl group $W = W(D_4)$ on the Painlevé system. We have the representation

$$\mathsf{G}^0 \to \mathsf{G}^0_*$$

of the group generated by W and ℓ_j $(j = 1, \dots, 4)$: cf. the section 2.4.

2. – Realization of the affine Weyl group $W_a(D_4)$.

2.1. *Proof of Propositions* 1.1 *and* 1.2.

First of all we make an attempt to obtain a differential equation satisfied by the auxiliary Hamiltonian function $h = h(t)$. By the definition we have

$$\begin{aligned} h = q(q-1)(q-t)p^2 - \{b_1(2q-1)(q-t) - b_2(q-t) + (b_3+b_4)q(q-1)\}p\, + \\ + (b_1+b_3)(b_1+b_4)q - b_1^2 t - \tfrac{1}{2}\sigma_2[\boldsymbol{b}]\,. \end{aligned} \tag{2.1}$$

It follows from (1.6) that

$$\frac{dh}{dt} = -\,q(q-1)p^2 + \{b_1(2q-1) - b_2\}p - b_1^2\,, \tag{2.2}$$

since for the Hamiltonian function,

$$\frac{d}{dt}H(t) = \frac{\partial}{\partial t}H(t; q, p)\Big|_{(q,p)=(q(t),p(t))}.$$

We obtain from (2.1) and (2.2):

$$h - t\frac{dh}{dt} = q\left(-\frac{dh}{dt} + \sigma_2'[\boldsymbol{b}]\right) - (b_3 + b_4)q(q-1)p - \frac{1}{2}\sigma_2[\boldsymbol{b}]\,, \tag{2.3}$$

and then,

$$\begin{aligned} t(t-1)\frac{d^2h}{dt^2} = 2q\left(\sigma_1'[\boldsymbol{b}]\frac{dh}{dt} - \sigma_3'[\boldsymbol{b}]\right) - 2q(q-1)p\left(\frac{dh}{dt} - b_3b_4\right) - \\ - \sigma_1[\boldsymbol{b}]\frac{dh}{dt} + \sigma_3[\boldsymbol{b}]\,, \end{aligned} \tag{2.4}$$

by differentiating (2.3) and using the Hamiltonian system (1.5). It follows from (2.3) and (2.4) that

$$q = \frac{1}{2A}\left[(b_3 + b_4)B + \left(\frac{dh}{dt} - b_3b_4\right)C\right] \tag{2.5}$$

$$q(q-1)p = \frac{1}{2A}\left[-\left(\frac{dh}{dt} - \sigma_2'[\boldsymbol{b}]\right)B + \left(\sigma_1'[\boldsymbol{b}]\frac{dh}{dt} - \sigma_3'[\boldsymbol{b}]\right)C\right] \tag{2.6}$$

where

$$A = \det\begin{pmatrix} -\dfrac{dh}{dt} + \sigma_2'[\boldsymbol{b}] & -b_3 - b_4 \\ \sigma_1'[\boldsymbol{b}]\dfrac{dh}{dt} - \sigma_3'[\boldsymbol{b}] & -\dfrac{dh}{dt} + b_3b_4 \end{pmatrix} = \left(\frac{dh}{dt} + b_3^2\right)\left(\frac{dh}{dt} + b_4^2\right),$$

$$B = t(t-1)\frac{d^2h}{dt^2} + \sigma_1[\boldsymbol{b}]\frac{dh}{dt} - \sigma_3[\boldsymbol{b}]\,, \tag{2.7}$$

$$C = 2\left(t\frac{dh}{dt} - h\right) - \sigma_2[\boldsymbol{b}]\,. \tag{2.7$'$}$$

Rewriting (2.2) in the following form:

$$q(q-1)\left(\frac{dh}{dt} + b_1^2\right) = -(q(q-1)p)^2 + \{b_1(2q-1) - b_2\}q(q-1)p$$

and substituting (2.5), (2.6), we arrive at the differential equation $\mathsf{E} = \mathsf{E}_{\mathrm{VI}}$. The Proposition 1.1 is thus established. Given a solution $(q, p) = (q(t), p(t))$ of the Painlevé system, we have a solution $h = h(t)$ of the nonlinear differential equation E. Conversely, for a solution h of E, we define (q, p) by (2.5), (2.6). It can be verified by computation that (q, p) thus obtained is a solution of the Painlevé system, provided that h is not a singular solution, that is, $d^2h/dt^2 \neq 0$. This fact proves the Proposition 1.2; we do not enter into details of computation.

REMARK 2.1. – It may occur that $h = h(t)$ is a singular solution of E of the form (1.8). In this case, $q = q(t)$, a solution of the Painlevé equation P, satisfies also an algebraic differential equation of the first order, as it is easily seen by virtue of the Hamiltonian system. We will say such $q = q(t)$ is *semi-transcendental*. On the other hand, it is known that a Painlevé transcendental function is in general *transcendental*: it does not satisfy an algebraic differential equation of the first order except for some special value of the parameters. We will study such case in the section 5 and obtain semi-transcendental solutions of the Painlevé system.

2.2. *Weyl group* $W(D_4)$.

Let $W = W(D_4)$ be the Weyl group of the simple root system of the type D_4, and consider the realization of W given in the bebinning of the section 1.2. We obtain now the explicit form of the rational transformation

$$(q, p) \to (q_w, p_w) \tag{2.8}$$

for w of W, assuming that the auxiliary Hamiltonian function h related to (q, p) is not a singular solution of E. This transformation will be given by the relations:

$$\Gamma(h) = (q, p)\,, \qquad \Gamma(h_w) = (q_w, p_w)$$

and by (1.10). Since

$$\frac{dh}{dt} = \frac{dh_w}{dt}\,, \qquad \frac{d^2h}{dt^2} = \frac{d^2h_w}{dt^2}\,, \tag{2.9}$$

we have from (2.3), (2.4) the following relations:

$$F[\boldsymbol{b}]\begin{pmatrix} q \\ q(q-1)p \end{pmatrix} - G[\boldsymbol{b}] = F[w(\boldsymbol{b})]\begin{pmatrix} q_w \\ q_w(q_w-1)p_w \end{pmatrix} - G[w(\boldsymbol{b})] \tag{2.10}$$

where

$$F[\boldsymbol{b}] = \begin{pmatrix} -\dfrac{dh}{dt} + \sigma_2'[\boldsymbol{b}] & -b_3 - b_4 \\ \sigma_1'[\boldsymbol{b}]\dfrac{dh}{dt} - \sigma_3'[\boldsymbol{b}] & -\dfrac{dh}{dt} + b_3b_4 \end{pmatrix},$$

$$G[\boldsymbol{b}] = \begin{pmatrix} \dfrac{1}{2}\sigma_2[\boldsymbol{b}] \\ \sigma_1[\boldsymbol{b}]\dfrac{dh}{dt} - \sigma_3[\boldsymbol{b}] \end{pmatrix}.$$

Remark that the elements of $F[\boldsymbol{b}]$ and $G[\boldsymbol{b}]$ are polynomials in (q_w, p_w) as well as in (q, p) by means of (2.2). For example, if $w = w_1$, we obtain:

$$\frac{dh}{dt} = -q(q-1)p^2 + \\ + \{b_1(2q-1) - b_2\}p - b_1^2 = -q_1(q_1-1)p_1^2 + \{b_2(2q_1-1) - b_1\}p_1 - b_2^2 .$$

where $(q_1, p_1) = (q_w, p_w)$. The relations (2.10) give us the explicit form of (2.8), since

$$\det F[\boldsymbol{b}] = A = \left(\frac{dh}{dt} + b_3^2\right)\left(\frac{dh}{dt} + b_4^2\right)$$

is not zero by the assumption. Moreover, there exist polynomials $c_1(w; \boldsymbol{b})$, $c_2(w; \boldsymbol{b})$ of $b_1, \ldots, b_4$ such that

$$H[\boldsymbol{b}] - H[w(\boldsymbol{b})] = \frac{1}{t} c_1(w; \boldsymbol{b}) + \frac{1}{t-1} c_2(w; \boldsymbol{b}) . \tag{2.11}$$

In fact the Hamiltonian $H[\boldsymbol{b}]$ is connected to $h[\boldsymbol{b}]$ as follows:

$$H[\boldsymbol{b}] = \frac{1}{t(t-1)} h[\boldsymbol{b}] - \frac{\sigma_2[\boldsymbol{b}]}{2t} + \frac{\sigma_2[\boldsymbol{b}] - 2\sigma_2'[\boldsymbol{b}]}{2(t-1)} .$$

We have from (2.8), (2.10) the transformation:

$$w_*: \mathscr{H}[\boldsymbol{b}] \to \mathscr{H}[w(\boldsymbol{b})]$$

which can be easily seen to be canonical by the use of (0.11)-(0.13). Hence Proposition 1.5 is completely verified.

EXAMPLE 2.1. – Put $w = w_1 w_2 w_1$, that is, $w(\boldsymbol{b}) = (b_3, b_2, b_1, b_4)$. We obtain from (2.10)

$$\begin{pmatrix} q_w \\ q_w(q_w-1)p_w \end{pmatrix} = \frac{1}{dh/dt + b_1^2} \begin{pmatrix} \dfrac{dh}{dt} + b_1^2 & b_3 - b_1 \\ 0 & \dfrac{dh}{dt} + b_3^2 \end{pmatrix} \begin{pmatrix} q \\ q(q-1)p \end{pmatrix},$$

since $\sigma_s'[\boldsymbol{b}] = \sigma_s'[w(\boldsymbol{b})]$ and $G[\boldsymbol{b}] = G[w(\boldsymbol{b})]$, where

$$\frac{dh}{dt} = -(q-1)p^2 + \{b_1(2q-1) - b_2\}p - b_1^2 = \\ = -q_w(q_w-1)p_w^2 + \{b_3(2q_w-1) - b_2\}p_w - b_3^2 .$$

2.3. *Auxiliary function* h^+.

In this paragraph we prove Propositions 1.6, 1.7 and 1.8. Let $h = h[\boldsymbol{b}]$ be an auxiliary function and (q, p) the solution of the Painlevé system such that $\Gamma(h) = (q, p)$. By differentiating the both sides of (1.13) with respect to t and by using the system of differential equations:

$$(2.12)_1 \qquad t(t-1)\frac{dq}{dt} = 2q(q-1)(q-t)p - \\ - b_1(2q-1)(q-t) + b_2(q-t) - (b_3+b_4)q(q-1)\,,$$

$$(2.12)_2 \qquad t(t-1)\frac{dp}{dt} = \big(q(q-1) + (q-1)(q-t) + q(q-t)\big)p^2 + \\ + \{b_1(4q-2t-1) - b_2 + (b_3+b_4)(2q-1)\}p - (b_1+b_3)(b_1+b_4)\,,$$

we obtain first by (2.2):

$$(2.13) \quad t(t-1)\left(\frac{dh^+}{dt} + b_4^2\right) + \\ + (q-t)\left\{h^+ + (b_3 - b_4 + 1)X + \frac{1}{2}b_4^2(2t-1) - \frac{1}{2}b_1 b_2\right\} = 0\,.$$

Here we put

$$(2.14) \qquad X = q(q-1)p - (b_1+b_4)q + \tfrac{1}{2}(b_1+b_2+b_4)\,.$$

Moreover it follows from (2.12), (2.14) that

$$t(t-1)\left(\frac{dq}{dt} + b_3 + b_4\right) = -(b_3-b_4)(q-t)^2 + \big(2X - b_3(2t-1)\big)(q-t)\,,$$

$$(q-1)\frac{dX}{dt} = -(q-t)\left(\frac{dh^+}{dt} + b_4^2\right) - 2b_4X - b_1b_2 - \\ - \left[h^+ + (b_3 - b_4 + 1)X + \frac{1}{2}b_4^2(2t-1) - \frac{1}{2}b_1b_2\right].$$

Taking these equations into consideration, we arrive at the following expression:

$$(2.15) \qquad 2\left(\frac{dh^+}{dt} + (b_3+1)^2\right)X = \\ = t(t-1)\frac{d^2h^+}{dt^2} + (b_3+1)\left[(2t-1)\frac{dh^+}{dt} - 2h^+\right] + b_1b_2b_4\,.$$

Hence q and X are written as rational functions of h^+, dh^+/dt, d^2h^+/dt^2 and t by (2.13), (2.15), and so is the function p. The proof of the Proposition 1.7 is completed.

In order to obtain the differential equation satisfied by h^+, we eliminate (q, X) from (2.13), (2.15) and

$$h^+ = h - X\,. \tag{2.16}$$

We deduce firstly from (2.13), (2.16):

$$q(t-1)\left\{t\frac{dh^+}{dt} - h^+ - (b_3^+ + b_4^+)X + \frac{1}{2}(b_4^+)^2 - \frac{1}{2}b_1^+b_2^+\right\} + \\ + (q-t)\left\{X^2 - b_4^+X - \frac{1}{4}(b_1^+ + b_2^+)^2 + \frac{1}{4}(b_4^+)^2\right\} = 0\,, \tag{2.17}$$

where we write $\boldsymbol{b}^+ = (b_1^+, b_2^+, b_3^+, b_4^+)$. Note that by the definition $b_k^+ = b_k$ $(k \neq 3)$ and $b_3^+ = b_3 + 1$. It follows from (2.13), (2.17) that

$$4\left(\frac{dh^+}{dt} + (b_3^+)^2\right)X^2 + 4\left\{b_3^+\left(2h^+ - (2t-1)\frac{dh^+}{dt}\right) - b_1^+b_2^+b_4^+\right\}X + \left(2h^+ - (2t-1)\frac{dh^+}{dt}\right)^2 = \\ = \left(\frac{dh^+}{dt}\right)^2 + \left((b_1^+)^2 + (b_2^+)^2 + (b_4^+)^2\right)\frac{dh^+}{dt} + (b_1^+b_2^+)^2 + (b_2^+b_4^+)^2 + (b_1^+b_4^+)^2\,,$$

from which we obtain the differential equation

$$\frac{dh^+}{dt}\left\{t(t-1)\frac{d^2h^+}{dt^2}\right\}^2 + \left[\frac{dh^+}{dt}\left\{2h^+ - (2t-1)\frac{dh^+}{dt}\right\} + \sigma_4[\boldsymbol{b}^+]\right]^2 = \prod_{k=1}^{4}\left(\frac{dh^+}{dt} + (b_k^+)^2\right).$$

The proof of Proposition 1.8 is completed.

As we have discussed at the end of the last section, (2.13), (2.15) and (1.14) define the canonical transformation:

$$\ell_*: \mathscr{H}[\boldsymbol{b}] \to \mathscr{H}[\boldsymbol{b}^+]\,.$$

In fact, an expression of the form (1.16) is given by (2.13) and (2.15), if we regard $h^+ = h[\boldsymbol{b}^+]$ as polynomial in (q^+, p^+) through (2.1). Moreover we obtain the explicit form of (1.17) by applying (2.5) and (2.6) to the function h^+ and then by considering h^+ as function of (q, p). We do not enter into details of computation.

REMARK 2.2. – The canonical transformation ℓ_* is determined under the assumption that none of the auxiliary functions h, h^+ is linear in t. However, the formula (2.16) stands also for a singular solution $h = h(t)$ of the nonlinear differential equation E, unless h^+ is a linear function of t at the same time. In fact, as it has been shown in the proof of Proposition 1.8, the function h^+ defined by (2.16) satisfies the equation $\mathsf{E}[\boldsymbol{b}^+]$, provided that $dh^+/dt + (b_3^+)^2 \not\equiv 0$. Moreover if $d^2h^+/dt^2 \not\equiv 0$, then we have the correspondence $\Gamma(h^+) = (q^+, p^+)$ by Proposition 1.2.

REMARK 2.3. – Denote (1.15) by $Y[b]$: we have $H[\boldsymbol{b}^+] = H[\boldsymbol{b}] - (1/t(t-1))\,Y[\boldsymbol{b}]$. We obtain from (2.15):

$$Y[\boldsymbol{b}] = \frac{\bar{B}^+ + (b_3+1)\,C^+}{2(dh^+/dt + (b_3+1)^2)} \tag{2.18}$$

where

$$\bar{B}^+ = t(t-1)\frac{d^2h^+}{dt^2} - \sigma_1[\boldsymbol{b}^+]\frac{dh^+}{dt} + \sigma_3[\boldsymbol{b}^+]\,, \qquad C^+ = 2\left(t\frac{dh^+}{dt} - h^+\right) - \sigma_2[\boldsymbol{b}^+]\,.$$

On the other hand, it follows from (2.5) and (2.6) that

$$Y[\boldsymbol{b}] = \frac{-B + b_3 C}{2(dh/dt + b_2^2)}\,, \tag{2.19}$$

B, C being given by (2.7), (2.7)$'$. We will use (2.18) and (2.19) in the section 4.2.

2.4. *Parallel transformation and the affine Weyl group.*

Let ℓ_j be the parallel transformation of $\mathbb{C}^4$:

$$\boldsymbol{b} \to \boldsymbol{b} + \boldsymbol{e}_j \qquad (j = 1, \ldots, 4)$$

and $W = W(D_4)$ the Weyl group considered above. We denote by G^0 the group generated by W and $\ell_1, \ldots, \ell_4$: G^0 is a subgroup of the group $\mathscr{A}(\mathbb{C}^4)$ of affine motions. We have constructed in the previous sections the representation of G^0:

$$\mathsf{G}^0 \to \mathsf{G}^0_*$$

on the Painlevé system associated with the sixth Painlevé equation.

Consider the element w_0 of G such that

$$w_0(\boldsymbol{b}) = (b_1, b_2, -b_4 - 1, -b_3 - 1) \tag{2.20}$$

and let $\tilde{W}$ be the group generated by W and w_0. We will show that the representation of $\tilde{W}$ is given in a brief manner, although the realization G^0_* is a little complicated. To determine the representation of w_0 on the Painlevé system, we remark first that:

PROPOSITION 2.1 ([1]). – *$\tilde{W}$ is isomorphic to the affine Weyl group $W_a(D_4)$ associated with the root system of the type D_4.*

The explicit form of the representation $\pi = (w_0)_*$ of w_0 will be obtained by the use of the relation:

$$w_0 = w_4 w_3 \ell_3 w_3 \ell_3\,.$$

On the other hand, remarking that (2.20) is equivalent to the transformation:

$$\theta \to -\theta \tag{2.21}$$

of the constants of the Hamiltonian, we can construct the birational canonical transformation π by a straightforward way. In fact, consider the canonical transformation

$$(q, p, H, t) \to (q, \bar{p}, \bar{H}, t) \tag{2.22}$$

such that

$$\bar{p} = p - \frac{\theta}{q-t}, \qquad \bar{H} = H - \frac{\theta}{q-t} + \theta\left(\frac{\varkappa_0 - 1}{t} + \frac{\varkappa_1 - 1}{t-1}\right).$$

Then we have

$$\begin{aligned}\bar{H} = \frac{1}{t(t-1)}\,[q(q-1)(q-t)\bar{p}^2 -& \\ - \{\varkappa_0(q-1)(q-t) + \varkappa_1 q(q-t) - (\theta\, N\, 1)q(q-1)\}p + \bar{\varkappa}(q-t)]&\,, \\ \bar{\varkappa} = \tfrac{1}{4}(\varkappa_0 + \varkappa_1 - \theta - 1)^2 - \tfrac{1}{4}\varkappa_\infty^2&\,,\end{aligned}$$

which shows

$$(q, \bar{p}, \bar{H}, t) = \mathscr{H}[w_0(\boldsymbol{b})]\,.$$

It follows that

PROPOSITION 2.2. – *The transformation* (2.22) *defines* $\pi = (w_0)_*$.

We have thus the representation of the affine Weyl group $W_a(D_4)$ on the Painlevé system. The highest root of $\tilde{W}$ is the vector

$$\tilde{\boldsymbol{a}} = \boldsymbol{e}_1 + \boldsymbol{e}_2$$

and the reflection with respect to $-\tilde{\boldsymbol{a}}$ is of the form:

$$w[\boldsymbol{b}] = (-b_2 - 1, -b_1 - 1, b_3, b_4)\,. \tag{2.23}$$

The canonical transformation $\tilde{w}_*$ is obtained from π by the use of the relation

$$\tilde{w} = w' w_0 w'\,, \qquad w'[\boldsymbol{b}] = (b_3, b_4, b_1, b_2)\,.$$

For g of G^0, g_* is a birational canonical transformation of the first kind. In the following section we will consider a birational canonical transformation of the Painlevé system of more general type.

3. – Transformation group of the Painlevé system.

3.1. *Symmetry of the Painlevé equation.*

It is known ([10]) that, if we replace q by $1-q$ and t by $1-t$, the Painlevé equation is transformed into the following equation:

$$\frac{d^2q}{dt^2}=\frac{1}{2}\left(\frac{1}{q}+\frac{1}{q-1}+\frac{1}{q-t}\right)\left(\frac{dq}{dt}\right)^2-\left(\frac{1}{t}+\frac{1}{t-1}+\frac{1}{q-t}\right)\frac{dq}{dt}+ \\ +\frac{q(q-1)(q-t)}{t^2(t-1)^2}\left[\alpha-\gamma\frac{t}{q^2}-\beta\frac{t-1}{(q-1)^2}+\delta\frac{t(t-1)}{(q-t)^2}\right].$$

This change of variables extends to the canonical transformation of the Painlevé system. In fact, if we put

$$q_1=1-q\,,\quad p_1=-p\,,\quad t_1=1-t\,,\quad H_1=-H \tag{3.1}$$

and then rewrite (q_1, p_1, H_1, t_1) as (q, p, H, t), the Hamiltonian remains invariant except the change of the constants:

$$x^1\colon \varkappa_0\to\varkappa_1\,,\qquad \varkappa_1\to\varkappa_0\,.$$

For the sake of simplification of presentation, we denote the canonical transformation (3.1) and the succeeding replacement by

$$x^1_*\colon (q, p, H, t)\to(1-q, -p, -H, 1-t)\,.$$

Moreover, consider canonical transformations of the form:

$$x^2_*\colon (q, p, H, t)\to\left(\frac{1}{q}, \varepsilon q-q^2p, -\frac{1}{t^2}H, \frac{1}{t}\right),\qquad \varepsilon=\frac{1}{2}(\varkappa_0+\varkappa_1+\theta-1+\varkappa_\infty)\,,$$
$$x^3_*\colon (q, p, H, t)\to\left(\frac{t-q}{t-1}, -(t-1)p, (t-1)^2H+(t-1)(q-1)p, \frac{1}{t-1}\right),$$

where we use the abbreviated form of notation. They are connected to the changes of constants:

$$x^2\colon \varkappa_0\to\varkappa_\infty\,,\qquad \varkappa_\infty\to\varkappa_0\,,$$
$$x^3\colon \varkappa_0\to\theta\,,\qquad \theta\to\varkappa_0\,.$$

We have the

PROPOSITION 3.1 ([4]). – *The Hamiltonian H is invariant for each j $(j=1,2,3)$ under the transformation x^j_* except the permutation x^j of the constants $\varkappa_\Delta$, θ.*

Let G^1 be the group generated by x^j $(j = 1, 2, 3)$. This consists of the permutations of the finite set $\{\varkappa_0, \varkappa_1, \varkappa_\infty, \theta\}$ and then is isomorphic to the symmetric group $\mathfrak{S}_4$. On the other hand, the canonical transformations x^j_* generate a group G^1_* isomorphic to G^1; so we obtain a representation of G^1 on the Painlevé system.

REMARK 3.1. – The permutations x^j induce affine transformations of $\mathbf{C}^4$ as follows:

$$x^1\colon (b_1, b_2, b_3, b_4) \to (b_1, -b_2, b_3, b_4)$$

$$x^2\colon (b_1, b_2, b_3, b_4) \to$$
$$\to \big(\tfrac{1}{2}(b_1 - b_2 + b_3 - b_4),\ \tfrac{1}{2}(-b_1 + b_2 + b_3 - b_4),\ \tfrac{1}{2}(b_1 + b_2 + b_3 + b_4),\ \tfrac{1}{2}(-b_1 - b_2 + b_3 + b_4)\big),$$

$$x^3\colon (b_1, b_2, b_3, b_4) \to \big(\tfrac{1}{2}(b_1 - b_2 + b_3 + b_4 + 1),\ \tfrac{1}{2}(-b_1 + b_2 + b_3 + b_4 + 1),\ \tfrac{1}{2}(b_1 + b_2 + b_3 - b_4 - 1),\ \tfrac{1}{2}(b_1 + b_2 - b_3 + b_4 - 1)\big).$$

We denote also by G^1 the group generated by these affine transformations of $\mathbf{C}^4$.

3.2. *Affine space V of parameters.*

Let G be the subgroup of $\mathscr{A}(\mathbf{C}^4)$ generated by the two subgroups G^0 and G^1, considered above. Note that G is generated by w_j $(j = 1, \ldots, 4)$, ℓ_3 and x^j $(j = 1, 2, 3)$. We will prove the

PROPOSITION 3.2. – G *is isomorphic to the affine Weyl group of the simple root system of the type* F_4: $W_a(F_4)$.

To prove this proposition, we introduce firstly the space V of the parameters of the Painlevé system. It is a four dimensional vector space with canonical basis $\boldsymbol{f}_k$ $(k = 1, \ldots, 4)$ such that a vector of V of the form

$$\boldsymbol{v} = \sum_{k=1}^{4} v_k \boldsymbol{f}_k \tag{3.2}$$

is related to the constants of the Hamiltonian in the following manner:

$$\varkappa_0 = v_2, \quad \varkappa_1 = v_3, \quad \varkappa_\infty = v_4, \quad \theta = v_1 + 1.$$

It follows from (1.1) that

$$v_1 = b_3 + b_4, \quad v_2 = b_1 + b_2, \quad v_3 = b_1 - b_2, \quad v_4 = b_3 - b_4. \tag{3.3}$$

The group G can be regarded in a natural way as a subgroup of the group $\mathscr{A}(\mathsf{V})$ of affine motions of V. Let φ be the linear map from V to $\mathbf{C}^4$ defined by (3.3). For the sake of simplification of notation, we denote also by g the element $\varphi^* g$ of $\mathscr{A}(\mathsf{V})$ and by G the subgroup $\varphi^*\mathsf{G}$. It is convenient to adopt V as the space of parameters of the Painlevé system, so that we write the Painlevé system at $\boldsymbol{v}$ of V as $\mathscr{H}(\boldsymbol{v})$, the Hamiltonian as $H(\boldsymbol{v})$, a solution of $\mathscr{H}(\boldsymbol{v})$ as $(q(\boldsymbol{v}), p(\boldsymbol{v}))$ and so on. We may write $\mathscr{H}(\boldsymbol{v})$ as $\mathscr{H}[\boldsymbol{b}]$, if necessary, where $\boldsymbol{v}$ and $\boldsymbol{b}$ are mutually connected through the isomorphism φ. For each g of G, there exists the birational canonical transformation

$$g_*\colon \mathscr{H}(\boldsymbol{v}) \to \mathscr{H}(g(\boldsymbol{v})) ,$$

whose explicit form can be given by the use of the canonica ltransformations, $(w_j)_*$ $(j=1, \dots, 4)$, l_*, x^j_* $(j=1, 2, 3)$. We obtain the representation of G

$$\mathsf{G} \to \mathsf{G}_*$$

on the Painlevé system. If we write (3.2) simply as $\boldsymbol{v}=(v_1, v_2, v_3, v_4)$, then the elements of G, w_j, w_0, ℓ_3 and xi are realized as follows:

$$w_1\colon \boldsymbol{v} \to (v_1, v_2, -v_3, v_4) ,$$

$$w_2\colon \boldsymbol{v} \to \left(\tfrac{1}{2}(v_1+v_2-v_3-v_4),\ \tfrac{1}{2}(v_1+v_2+v_3+v_4),\ \tfrac{1}{2}(-v_1+v_2+v_3-v_4),\ \tfrac{1}{2}(-v_1+v_2-v_3+v_4)\right) ,$$

$$w_3\colon \boldsymbol{v} \to (v_1, v_2, v_3, -v_4) ,$$

$$w_4\colon \boldsymbol{v} \to (-v_1, v_2, v_3, v_4) ,$$

$$w_0\colon \boldsymbol{v} \to (-v_1-2, v_2, v_3, v_4) ,$$

$$\ell_3\colon \boldsymbol{v} \to (v_1+1, v_2, v_3, v_4+1) ,$$

$$x^1\colon \boldsymbol{v} \to (v_1, v_3, v_2, v_4) ,$$

$$x^2\colon \boldsymbol{v} \to (v_1, v_4, v_3, v_2) ,$$

$$x^3\colon \boldsymbol{v} \to (v_2-1, v_1+1, v_3, v_4) .$$

We denote by $(\boldsymbol{v}|\boldsymbol{v}')$ the symmetric bilinear form of V such that $(\boldsymbol{f}_k|\boldsymbol{f}_{k'})=(\boldsymbol{f}_{k'}|\boldsymbol{f}_k)=\delta_{kk'}$.

3.3. *Verification of Proposition* 3.2.

Consider the following elements of G:

$$s_1 = x^1 , \quad s_2 = x^1x^2x^1 , \quad s_3 = w_3 ,$$

$$s_4 = w_1w_2w_3w_2w_1 , \quad s_0 = x^3w_0x^1w_1x^1 .$$

Viewing them as elements of $\mathscr{A}(\mathsf{V})$, we have:

$$s_1\colon \boldsymbol{v} \to (v_1, v_3, v_2, v_4)\,,$$

$$s_2\colon \boldsymbol{v} \to (v_1, v_2, v_4, v_3)\,,$$

$$s_3\colon \boldsymbol{v} \to (v_1, v_2, v_3, -v_4)\,,$$

$$s_4\colon v \to \big(\tfrac{1}{2}(v_1+v_2+v_3+v_4),\ \tfrac{1}{2}(v_1+v_2-v_3-v_4),\ \tfrac{1}{2}(v_1-v_2+v_3-v_4),\ \tfrac{1}{2}(v_1-v_2-v_3+v_4)\big)\,,$$

$$s_0\colon \boldsymbol{v} \to (-v_2-1, -v_1-1, v_3, vl)\,.$$

We denote by G_s the subgroup of G generated by s_j $(j=0,1,\ldots,4)$. Put

$$\boldsymbol{a}_1=\boldsymbol{f}_2-\boldsymbol{f}_3\,,\quad \boldsymbol{a}_2=\boldsymbol{f}_3-\boldsymbol{f}_4\,,\quad \boldsymbol{a}_3=\boldsymbol{f}_4\,,$$
$$\boldsymbol{a}_4=\tfrac{1}{2}(\boldsymbol{f}_1-\boldsymbol{f}_2-\boldsymbol{f}_3-\boldsymbol{f}_4)\,,\quad \boldsymbol{a}_0=-\boldsymbol{f}_1-\boldsymbol{f}_2\,.$$

Then $\boldsymbol{a}_j$ $(j=1,\ldots,4)$ compose the set of fundamental roots of the exceptional root system of the type F_4 and each s_j is a reflection of V with respect to the hyperplane $(\boldsymbol{a}_j|\boldsymbol{v})=0$. $\boldsymbol{a}_0$ is the minus of the highest root and s_0 is a reflection with respect to $(\boldsymbol{a}_0|\boldsymbol{v})=1$: see [1]. Therefore s_j $(j=1,\ldots,4,0)$ generate the affine Weyl group $W_a(F_4)$ of the root system of the type F_4 and G_s is isomorphic to $W_a(F_4)$. The Coxeter graph of G_s is of the form:

$$\underset{s_0}{\circ}\!\!-\!\!\!-\!\!\!-\!\!\underset{s_1}{\circ}\!\!-\!\!\!-\!\!\!-\!\!\underset{s_2}{\circ}\!\!\overset{4}{-\!\!\!-\!\!\!-}\!\!\underset{s_3}{\circ}\!\!-\!\!\!-\!\!\!-\!\!\underset{s_4}{\circ}\,,$$

that is, we have the relations

$$s_j^0=1\qquad (j=1,\ldots,4,0)\,,$$
$$(s_0s_1)^3=(s_1s_2)^3=(s_3s_4)^3=1\,,$$
$$(s_2s_4)^4=1$$
$$(s_is_j)^2=1\qquad(\text{otherwise})\,.$$

To prove the Proposition 3.2, it is enough to show s_j $(j=1,\ldots,4,0)$ generate G. Recall that G is generated by the two subgroups G^0 and G^1, and G^0 is generated by the Weyl group W and the parallel transformation ℓ_3. We claim: *G^1 is a subgroup of G_s.* In fact, w_3 and x^1 are contained in G_s by the definition of s_j. Moreover w_1 and x^2 are in G_s, since $x^2=s_1s_2s_1$, $w_1=s_2s_3s_2$. On the other hand, putting $g_1=x^1w_1x^1$, we see $x^3=g_1s_0g_1$; note

$$g_1\colon \boldsymbol{v} \to (v_1, -v_2, v_3, v_4)\,.$$

We will show next that G^0 *is a subgroup of* G_s. Put $g_2 = s_3 s_2 s_3$, $g_3 = w_2 p_1 w_2$. It is easy to see:

$$g_2 \colon \boldsymbol{v} \to (v_1, v_2, -v_4, -v_3)\,, \qquad g_3 \colon \boldsymbol{v} \to (-v_1, v_2, v_3, v_4)\,.$$

Then W is a subgroup of G_s, since $w_2 = g_2 s_4 g_2$, $w_4 = g_3 w_3 g_3$. Finally putting

$$g_0 = s_3 s_4 s_3 s_2 s_3 s_4 s_3 = s_4 g_2 s_4\,,$$

we obtain the expression

$$\ell_3 = w_3 w_4 x^2 g_0 x^3 w_0 x^2 w_3\,,$$

which is verified by the use of

$$g_0 \colon \boldsymbol{v} \to (v_2, v_1, v_3, v_4)\,.$$

Definitively ℓ_3 is in G_s, since $w_0 = x^3 s_0 g_1$. The proof of the Proposition 3.2 is thus completed.

3.4. *Conclusion.*

Getting together the discussion given above, we arrive at the following theorem:

THEOREM 1. – *Let* G *be the realization of the affine Weyl group of the root system of the type* F_4 *as the reflection group of the four dimensional affine space* V. *Then there exists the representation*

$$\varrho \colon \mathsf{G} \to \mathsf{G}_* \tag{3.4}$$

on the Painlevé system $\mathscr{H}$ *associated with the sixth Painlevé equation, such that, for* g *of* G, $g_* = \varrho(g)$ *is a birational canonical transformation of* $\mathscr{H}$.

REMARK 3.2. – (3.4) is not an isomorphism. In fact, the Hamiltonian of the Painlevé system is invariant under $(s_3)_* \colon \mathscr{H}(\boldsymbol{v}) = \big(s_3(\boldsymbol{v})\big)$. We will use this fact in the following section in order to establish some expressions of a Painlevé transcendental function by means of τ-functions.

EXAMPLE 3.1. – If we put

$$u = \int_{\infty}^{q} \frac{dq}{\sqrt{q(q-1)(q-t)}}\,, \tag{3.5}$$

then the Painlevé equation $\mathsf{P} = \mathsf{P}_{\mathrm{VI}}$ is transformed into the equation:

$$t(1-t)\frac{d^2u}{dt^2} + (1-2t)\frac{du}{dt} - \frac{1}{4}u = \frac{1}{2t(1-t)}\frac{\partial}{\partial u}\psi(u;t)\,,$$

where

$$\psi(u;t) = \varkappa_\infty^2\,\wp(u;t) + \varkappa_0^2\frac{t}{\wp(u;t)} + \varkappa_1^2\frac{1-t}{\wp(u;t)-1} + \theta^2\frac{t(t-1)}{\wp(u;t)-t}\,,$$

$q = \wp(u;t)$ denoting the inverse function of (3.5). It follows that, if $\varkappa_0 = \varkappa_1 = \varkappa_\infty = \theta = 0$, then a general solution of P is of the form

$$q(t) = \wp\big(c_1\omega_1(t) + c_2\omega_2(t);\,t\big)\,, \tag{3.6}$$

where $\omega_i(t)$ $(i = 1, 2)$ are linearly independent solution of the hypergeometric differential equation

$$t(1-t)\frac{d^2u}{dt^2} + (1-2t)\frac{du}{dt} - \frac{1}{4}u = 0\,.$$

The function (3.6) with two parameters c_1, c_2 is called the *solution of E. Picard.* This occurs at the point $\boldsymbol{v}^0 = (-1, 0, 0, 0)$ of the affine space V.

Let $\mathcal{O}(\boldsymbol{v}^0; \mathsf{G})$ be the orbit of $\boldsymbol{v}^0$ by G. Then we have the

PROPOSITION 3.3. – *The Painlevé system $\mathscr{H}(\boldsymbol{v})$ at $\boldsymbol{v}$ of $\mathcal{O}(\boldsymbol{v}^0; G)$ is integrable by quadrature with elliptic functions, provided that a birational canonical transformation does exist for g, where $\boldsymbol{v} = g(\boldsymbol{v}^0)$.*

For instance, put $g = w_1 w_2 w_1$. Then a solution of the Painlevé system at $g(\boldsymbol{v}^0) = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2})$ is given by the formulas given in the example 2.1 with (3.6).

REMARK 3.3. – $\boldsymbol{v}^0$ is characterized by the equations:

$$\boldsymbol{v}^0 = s_0(\boldsymbol{v}^0) = s_1(\boldsymbol{v}^0) = s_2(\boldsymbol{v}^0) = s_3(\boldsymbol{v}^0)\,,$$

therefore the isotropy subgroup $\mathsf{G}(\boldsymbol{v}^0)$ of G at $\boldsymbol{v}^0$ is generated by s_j $(j = 1, 2, 3, 0)$. The Coxeter graph of $\mathsf{G}(\boldsymbol{v}^0)$ is

$$\underset{s_0}{\circ}\!\!-\!\!\!-\!\!\!-\!\!\underset{s_1}{\circ}\!\!-\!\!\!-\!\!\!-\!\!\underset{s_2}{\circ}\!\!\overset{4}{-\!\!\!-\!\!\!-}\!\!\underset{s_3}{\circ}\,,$$

that is, $\mathsf{G}(\boldsymbol{v}^0)$ is isomorphic to the Weyl group $W(B_4)$ of the root system of the type B_4.

4. – τ-function of the Painlevé system.

4.1. *τ-sequence and Toda equation.*

Let $\mathscr{H}(\boldsymbol{v})$ be the Painlevé system at $\boldsymbol{v}$ of V, $h(\boldsymbol{v})$ an auxiliary function and $(q(\boldsymbol{v}), p(\boldsymbol{v}))$ a solution of $H(\boldsymbol{v})$ such that $\Gamma(h(\boldsymbol{v})) = (q(\boldsymbol{v}), p(\boldsymbol{v}))$. A τ-function $\tau(\boldsymbol{v})$ of $\mathscr{H}(\boldsymbol{v})$ related to the Hamiltonian $H(\boldsymbol{v}) = H(t; q, p; v)$ is defined by

$$H(t; q(\boldsymbol{v}), p(\boldsymbol{v}); \boldsymbol{v}) = \frac{d}{dt}\log \tau(\boldsymbol{v})\,. \tag{4.1}$$

We have by (1.6)

$$h(\boldsymbol{v}) = t(t-1)\frac{d}{dt}\log \tau(\boldsymbol{v}) + \sigma_2'[\boldsymbol{b}]t - \frac{1}{2}\sigma_2[\boldsymbol{b}] \tag{4.2}$$

where $\boldsymbol{v}$ and $\boldsymbol{b}$ are mutually connected by the correspondence (3.3): we write $\boldsymbol{b} = \varphi(\boldsymbol{v})$. The τ-function $\tau(\boldsymbol{v})$ is holomorphic on the universal covering Riemann surface of $\mathbb{C}\setminus\{0, 1\}$; see [6]. For any g of G, we have constructed the birational canonical transformation:

$$g_*\colon \mathscr{H}(\boldsymbol{v}) \to \mathscr{H}(g(\boldsymbol{v}))\,,$$

which induces in a natural way the correspondence from the τ-function $\tau(\boldsymbol{v})$ to the other $\tau(g(\boldsymbol{v}))$. Disregarding ambiguity about multiplicative constants, we denote it also by g_*. As we have mentioned in the introduction, the two τ-function τ_1, τ_2 are identified, if the logarithmic derivative of the quotient τ_1/τ_2 is a rational function of t. By adopting this identification, we obtain from the preceding section the following proposition:

PROPOSITION 4.1. – *The τ-function is invariant under the group W_* of the canonical transformations.*

W_* is a realization of the Weyl group $W(D_4)$.

DEFINITION 4.1. – A *τ-sequence defined by g* is by definition a sequence of τ-functions, written as

$$\mathfrak{T}(g) = \{\tau_m;\ m \in Z\}\,, \tag{4.3}$$

such that for any integer m,

$$g_*\tau_{m-1} = \tau_m\,,$$

g_* being the representation of g of G on the Painlevé system.

One of the most important examples of the τ-sequence is related to the parallel transformation $\ell = \ell_3$, studied in the proposition 1.6. It is defined by $\ell[\boldsymbol{b}] = (b_1, b_2, b_3 + 1, b_4)$ or $\ell(\boldsymbol{v}) = \varphi^* \ell(\boldsymbol{v}) = (v_1 + 1, v_2, v_3, v_4 + 1)$. For an arbitrary fixed point $\boldsymbol{v}$ of V, we put for an integer m,

$$\boldsymbol{v}_m = \ell^m(\boldsymbol{v}) , \quad \boldsymbol{v}_0 = \boldsymbol{v} , \quad \boldsymbol{b}_m = \ell^m[\boldsymbol{b}] , \quad \boldsymbol{b}_0 = \boldsymbol{b} \quad (\boldsymbol{b} = \varphi(\boldsymbol{v})) .$$

So, starting from $\mathscr{H}_0 = \mathscr{H}(\boldsymbol{v})$, we have the sequence of the Painlevé systems

$$\mathscr{H}_m = \ell_*^m \mathscr{H}_0 = \mathscr{H}(\boldsymbol{v}_m)$$

and that of the functions (q_m, p_m), h_m such that $\Gamma(h_m) = (q_m, p_m)$. Here ℓ_* denotes the birational canonical transformation (1.11). Let τ_m the τ-function of $\mathscr{H}_m$ related to (q_m, p_m). We will prove the following proposition:

PROPOSITION 4.2. – *The τ-sequence $\mathfrak{T}(\ell) = \{\tau_m;\ m \in \mathbb{Z}\}$ is subject to the constraint:*

$$\frac{d}{dt}\, t(t-1) \frac{d}{dt} \log \tau_m + (b_1 + b_3 + m)(b_3 + b_4 + m) = c(m) \frac{\tau_{m-1}\tau_{m+1}}{\tau_m^2} , \tag{4.4}$$

$c(m)$ being a nonzero constant.

REMARK 4.1. – (4.4) is equivalent to:

$$\delta^2 \log \tau_m = c(m) \frac{\tau_{m-1}\tau_{m+1}}{\tau_m^2} , \tag{4.5}$$

where

$$\delta = t(t-1) \frac{d}{dt} . \tag{4.6}$$

In fact, put

$$\bar{\tau}_m = \tau_m \{t(t-1)\}^{c_m} , \qquad c_m = \tfrac{1}{2}(b_1 + b_3 + m)(b_3 + b_4 + m) , \tag{4.7}$$

therefore $\bar{\tau}_m$ satisfy (4.5). The substitution (4.7) is corresponding to a trivial canonical transformation of the form

$$H_m \to H_m - c_m \left(\frac{1}{t} + \frac{1}{t-1} \right) .$$

We can put $c(m) = 1$, by taking suitably multiplicative constants of the τ-functions. The equation (4.5) is known as the Toda equation.

The preceding proposition establishes the following theorem:

THEOREM 2. – *The τ-sequence $\mathfrak{T}(\ell)$ of the Painlevé system satisfies the Toda equation.*

REMARK 4.2. – We can deduce from Proposition 4.1 the results similar to Theorem 2 concerning the τ-sequence $\mathfrak{T}(\ell_j)$ $(j = 1, \dots, 4)$.

4.2. *Proof of Proposition* 4.2.

Remark first $\ell^m[\boldsymbol{b}] = (b_1, b_2, b_3 + m, b_4)$. Putting

$$X_m = q_m(q_m - 1)p_m - (b_1 + b_4)q_m + \tfrac{1}{2}(b_1 + b_2 + b_4)\,,$$
$$Y_m = q_m(q_m - 1)p_m - (b_1 + b_4)(q_m - t)\,,$$

we have by (1.14) and (2.16)

$$H_{m+1} = H_m - \frac{1}{t(t-1)}Y_m \tag{4.8}$$

$$h_{m+1} = h_m - X_m\,. \tag{4.9}$$

Moreover, we deduce from (2.18), (2.19):

$$Y_m = \frac{\bar{B}_{m+1} + (b_3 + m + 1)C_{m+1}}{2\big(dh_{m+1}/dt + (b_3 + m + 1)^2\big)} = \frac{-B_m + (b_3 + m)C_m}{2\big(dh_m/dt + (b_3 + m)^2\big)}\,, \tag{4.10}$$

where

$$B_m = t(t-1)\frac{d^2h_m}{dt^2} + \sigma_1[\boldsymbol{b}_m]\frac{dh_m}{dt} - \sigma_3[\boldsymbol{b}_m]\,,$$
$$\bar{B}_m = t(t-1)\frac{d^2h_m}{dt^2} - \sigma_1[\boldsymbol{b}_m]\frac{dh_m}{dt} + \sigma_3[\boldsymbol{b}_m]\,,$$
$$C_m = 2\left(t\frac{dh_m}{dt} - h_m\right) - \sigma_2[\boldsymbol{b}_m]\,;$$

see Remark 2.3. If follows from (4.10) that

$$Y_{m-1} - Y_m = \frac{t(t-1)(d^2h_m/dt^2)}{dh_m/dt + (b_3 + m)^2} = t(t-1)\frac{d}{dt}\log\left(\frac{dh_m}{dt} + (b_3 + m)^2\right). \tag{4.11}$$

On the other hand, since, by (4.8) and by the definition of the τ-function,

$$Y_m = t(t-1)\frac{d}{dt}\log\frac{\tau_m}{\tau_{m+1}}\,, \tag{4.12}$$

we obtain from (4.11):

$$c(m)\frac{\tau_{m-1}\tau_{m+1}}{\tau_m^2} = \frac{dh_m}{dt} + (b_3 + m)^2\,, \tag{4.13}$$

$c(m)$ being a non-zero constant. Taking (4.2) into consideration, we see the right hand side of (4.13) equals

$$\frac{d}{dt}t(t-1)\frac{d}{dt}\log\tau_m + (b_1+b_3+m)(b_3+b_4+m)\,,$$

which establishes the proposition.

4.3. *τ-function and Painlevé transcendental functions.*

Let $(q, p) = (q(\boldsymbol{v}), p(\boldsymbol{v}))$ be a solution of the Painlevé system $\mathscr{H}(\boldsymbol{v})$ at $\boldsymbol{v}$. We have from (4.12):

$$q(q-1)p-(b_1+b_4)(q-t) = t(t-1)\frac{d}{dt}\log\frac{\tau(\boldsymbol{v})}{\tau(\ell_3(\boldsymbol{v}))}\,. \tag{4.14}$$

As we have mentioned in Remark 3.2, the Painlevé system $\mathscr{H}(\boldsymbol{v})$ is invariant under the canonical transformation $(s_3)_*$: in particular $(q, p) = \big(q(s_3(\boldsymbol{v})), p(s_3(\boldsymbol{v}))\big)$. It follows from (4.14) that

$$\begin{aligned} q(q-1)p-(vb_1+b_3)(q-t) = t(t-1)\frac{d}{dt}\log\frac{\tau(s_3(\boldsymbol{v}))}{\tau(\ell_3\circ s_3(\boldsymbol{v}))} = \\ = t(t-1)\frac{d}{dt}\log\frac{\tau(\boldsymbol{v})}{\tau(\ell_4(\boldsymbol{v}))}\,, \end{aligned} \tag{4.15}$$

where $\ell_4 = s_3\ell_3 s_3$. Therefore we have from (4.14) and (4.15):

$$(b_3-b_4)(q-t) = t(t-1)\frac{d}{dt}\log\frac{\tau(\ell_4(\boldsymbol{v}))}{\tau(\ell_3(\boldsymbol{v}))}\,.$$

We arrive at the following proposition:

PROPOSITION 4.3. – *A solution (q, p) of $\mathscr{H}(\boldsymbol{v})$ is represented in terms of τ-functions as* (4.14) *and*

$$v_4(q-t) = t(t-1)\frac{d}{dt}\log\frac{\tau(\ell_4(\boldsymbol{v}))}{\tau(\ell_3(\boldsymbol{v}))}\,. \tag{4.16}$$

4.4. *Particular solutions and τ-functions.*

Consider the transformation g_3 of G such that $g_3(\boldsymbol{v}) = (-v_1, v_2, v_3, v_4)$ and put $g_4 = g_3 s_4 g_3$ (see the section 3.3). Let $\mathsf{V}(g_4)$ be the hyperplane of V defined by:

$$\boldsymbol{v} = g_4(\boldsymbol{v})\,. \tag{4.17}$$

It is easy to see (4.17) is equivalent to each of the following three expressions:

$$v_1+v_2+v_3+v_4 = 0\,,\qquad \varkappa_0+\varkappa_1+\theta-1+\varkappa_\infty = 0\,,\qquad b_1+b_3 = 0\,.$$

In this case, the Painlevé system $\mathscr{H}(\boldsymbol{v})$ is possessed of a family of special solutions of the form:

$$(4.18)\quad t(t-1)\frac{dq}{dt}=-\varkappa_0(q-1)(q-t)-\varkappa_1 q(q-t)-(\theta-1)q(q-1)\,,\qquad p\equiv 0\,;$$

cf. [3], [4]. The Hamiltonian function and the auxiliary function related to such a solution are:

$$(4.19)\qquad H(t;\boldsymbol{v})\equiv 0\,,\qquad h(t;\boldsymbol{v})=b_1b_3t-\tfrac{1}{2}(b_1b_3+b_2b_4)\,.$$

Put

$$a=b_1+b_4=\tfrac{1}{2}(v_1+v_2+v_3-v_4)=\varkappa_0+\varkappa_1+\theta-1\,.$$

If $a=0$, then (4.18) is reduced to the linear equation

$$t(t-1)\frac{dq}{dt}=\{(\varkappa_0+\varkappa_1)t-\varkappa_1\}q-\varkappa_0 t\,.$$

So we assume $a\neq 0$. It follows from (4.14) that

$$(4.20)\qquad a(q-t)=t(t-1)\frac{d}{dt}\log\tau\big(\ell_3(\boldsymbol{v})\big)\,,$$

where $\tau(\boldsymbol{v})$ is reduced to a constant by (4.19); we can put $\tau_0=\tau(\boldsymbol{v})=1$ without loss of generality. Remark that (4.14) is valid even if $h_0=h(\boldsymbol{v})$ is a linear function in t; see Remark 2.2. Inserting (4.20) into (4.18), we see the function $\tau_1=\tau\big(\ell_3(\boldsymbol{v})\big)$ satisfies the following hypergeometric differential equation:

$$(4.21)\qquad t(1-t)\frac{d^2\tau_1}{dt^2}+\big(c-(a+b+1)t\big)\frac{d\tau_1}{dt}-ab\tau_1=0\,,$$

where, by (4.17),

$$b=v_1+1=b_3+b_4+1=\theta\,,\qquad c=v_1+v_2+1=b_2+b_4+1=\varkappa_0+\theta\,.$$

Starting from $\tau_0=1$, and the hypergeometric function:

$$\tau_1=\oint u^{b-1}(1-u)^{c-b-1}(1-tu)^{-a}\,du\,,$$

we obtain successively the τ-functions τ_m for $m\geqq 2$ by the use of the Toda equation (4.4). For example, τ_2 is a constant multiple of:

$$\big(c-1-(a+b-1)t\big)\tau_1\frac{d\tau_1}{dt}-t(t-1)\left(\frac{d\tau_1}{dt}\right)^2-b(a-1)\tau_1^2\,.$$

The auxiliary function h_m defined by (4.2) is no longer a singular solution of the differential equation $\mathsf{E}_m = \mathsf{E}[\ell_3^m(\boldsymbol{b})]$ for $m \geqq 1$. Hence, the solution $(q_m, p_m) = \Gamma(h_m)$ of the Painlevé system $\mathscr{H}_m = \mathscr{H}(\ell_3^m(\boldsymbol{v}))$ is well-defined and written as rational function of the hypergeometric function τ_1 and its first derivative. We obtain the *semi-sequence* of τ-functions:

$$\mathfrak{T}_+(\ell) = \{\tau_m;\ m \geqq 0\}\,. \tag{4.22}$$

REMARK 4.3. – It is known that, if

$$\tau_0 = 1\,, \qquad \delta^2 \log \tau_m = \frac{\tau_{m-1}\tau_{m+1}}{\tau_m^2} \qquad (m \geqq 1)\,,$$

then τ_m $(m \geqq 2)$ are given by:

$$\tau_m = \begin{vmatrix} \tau_1, & \delta\tau_1, & \dots, & \delta^{m-1}\tau_1 \\ \delta\tau_1, & \delta^2\tau_1, & \dots, & \delta^m\tau_1 \\ \dots & \dots & \dots & \dots \\ \delta^{m-1}\tau_1, & \delta^m\tau_1, & \dots, & \delta^{2m-2}\tau_1 \end{vmatrix}\,, \tag{4.23}$$

with an arbitrary function τ_1. This fact might be remarked for the first time by G. DARBOUX: see *Leçon sur la théorie générale des surfaces*, vol. II. If we define for (4.22) the functions $\bar{\tau}_m$ by (4.7) and normalize their multiplicative constants as $c(m) = 1$ in (4.5), then $\bar{\tau}_m$ $(m \geqq 2)$ are written in the form (4.23).

5. – Classical solutions.

5.1. *Weyl chamber of $W_a(D_4)$.*

In this section, we study a solution of the Painlevé system $\mathscr{H}$ which is written by the use of elementary functions or classical transcendental functions: hypergeometric function, Bessel function and so on. We call such a solution a *classical solution* of $\mathscr{H}$. We adopt the notation in the section 1 and consider a vector $\boldsymbol{b} = (b_1, b_2, b_3, b_4)$ as a parameter of $\mathscr{H}$. Let $\tilde{W}$ be the realization of the affine Weyl group $W_a(D_4)$ of the type D_4, for which we have constructed in the section 2 the representation

$$\tilde{W} \to \tilde{W}_*$$

on the Painlevé system. We denote by $\mathfrak{C}$ a Weyl chamber of $\tilde{W}$ in the space $\mathbf{C}^4$ of parameters of H and by $\partial\mathfrak{C}$ the set of walls of $\mathfrak{C}$. For a generic point of $\boldsymbol{b}$ of $\mathfrak{C}$, the Painlevé system $\mathscr{H}[\boldsymbol{b}]$ has no classical solution. This fact is an immediate consequence of the *irreducibility* of the Painlevé transcendental functions: cf. ([10]). On the other hand, we have the following theorem:

THEOREM 3. – *If* $\boldsymbol{b}$ *is contained in* $\partial\mathfrak{C}$, *then* $\mathscr{H}[\boldsymbol{b}]$ *has a classical solution.*

We can assume that $\partial\mathfrak{C}$ is defined by the following five hyperplane:

$$b_1 - b_2 = 0\,, \tag{5.1}$$

$$b_2 - b_3 = 0\,, \tag{5.2}$$

$$b_3 - b_4 = 0\,, \tag{5.3}$$

$$b_3 + b_4 = 0\,, \tag{5.4}$$

$$b_1 + b_2 - 1 = 0\,. \tag{5.5}$$

In fact, for another $\mathfrak{C}'$, there exists w of $\tilde{W}$ which transforms $\mathfrak{C}'$ onto $\mathfrak{C}$. Applying the representation w_* to the Painlevé system $\mathscr{H}[\boldsymbol{b}']$ at a point b' of $\partial\mathfrak{C}'$, we can verify the theorem for $\mathfrak{C}'$, even if it happens that the auxiliary function $h[\boldsymbol{b}']$ degenerates into a singular solution of $\mathsf{E}[\boldsymbol{b}']$. We will study in details cases of degeneration for some examples.

REMARK 5.1. – The Weyl chamber $\mathfrak{C}$ defined by (5.1)-(5.5) is a simplex with the vertices: O (the origin), $P_1(\boldsymbol{e}_1)$, $P_2(\frac{1}{2}(\boldsymbol{e}_1 + \boldsymbol{e}_2))$, $P_3(\frac{1}{2}(\boldsymbol{e}_1 + \boldsymbol{e}_2 + \boldsymbol{e}_3 - \boldsymbol{e}_4))$, $P_4(\frac{1}{2}(\boldsymbol{e}_1 + \boldsymbol{e}_2 + \boldsymbol{e}_3 + \boldsymbol{e}_4))$. Here $\boldsymbol{e}_j$ $(j = 1, \ldots, 4)$ are the canonical basis of $\mathbf{C}^4$ with respect to the symmetric bilinear form $(\boldsymbol{b}|\boldsymbol{b}')$; see the section 1.2. Each P_j is of the form

$$\frac{1}{n_j}\varpi_j\,,$$

where ϖ_j denote the weight vectors of the Weyl group $W = W(D_4)$ and $(n_1, n_2, n_3, n_4) = (1, 2, 1, 1)$: cf. [1].

5.2. *Proof of Theorem* 3.

(i) *Case* (5.1). – If $\boldsymbol{b}$ is on the hyperplane (5.1), then $\varkappa_1 = 0$. It is easy to see $\mathscr{H}[\boldsymbol{b}]$ has a family of solutions of the form:

$$q \equiv 1\,, \quad t(t-1)\frac{dp}{dt} = (t-1)p^2 + (\{-(t-1)\varkappa_0 + \theta - 1\}p - \varkappa\,,$$

which is a *singular solution* of the Painlevé equation $\mathsf{P} = \mathsf{P}_{\mathrm{VI}}$.

(ii) *Case* (5.3). – Apply the canonical transformation x_*^2, introduced in the section 3.1, to the Painlevé system $\mathscr{H}[\boldsymbol{b}]$ at $\boldsymbol{b}$. By putting

$$x_*^2\,[\boldsymbol{b}] = \mathscr{H}[x^2(\boldsymbol{b})] = (q, p, H, t)\,,$$

we have a solution of $\mathscr{H}[x^2(\boldsymbol{b})]$ of the form:

$$q \equiv 0\,, \quad t(t-1)\frac{dp}{dt} = -tp^2 - (t\varkappa_1 + \theta - 1)p - \varkappa\,. \tag{5.6}$$

It corresponds to a singular solution $q \equiv \infty$ of P: note $\varkappa_\infty = 0$.

(iii) *Case* (5.2). – Apply again the transformation x^2_* to $\mathscr{H}[\boldsymbol{b}]$ and put $x^2_*\mathscr{H}[\boldsymbol{b}] = (q, p, H, t)$. The transformation x^2 of V induces the alternation of the constants $\varkappa_0$ and $\varkappa_\infty$. We have the particular solutions:

$$\frac{dq}{dt} = -\varkappa_\infty(q-1)(q-t) - \varkappa_1 q(q-t) - (\theta-1)q(q-1)\,, \quad p \equiv 0\,, \tag{5.7}$$

since (5.2) implies:

$$\varkappa_0 = \varkappa_1 + \theta - 1 + \varkappa_\infty\,.$$

The Riccati equation (5.7) is solved by use of the Gauss hypergeometric differential equation; see the section 4.4. We obtain from (5.7) a family of classical solutions of $\mathscr{H}[\boldsymbol{b}]$.

(iv) *Case* (5.5). – We have $\varkappa_0 = 1$ from (5.5). This case is reduced to (5.4) by the transformation x^3_*, since x^3 replaces $\varkappa_0$ by θ, and so (5.5) by (5.4).

(v) *Case* (5.4). – Let $\boldsymbol{b} = (b_1, b_2, b_3, b_4)$ be a point on the hyperplane (5.4) and $\mathscr{H}[\boldsymbol{b}] = (q, p, H, t)$ be the Painlevé system at $\boldsymbol{b}$. We prove the following proposition, which establishes the theorem.

PROPOSITION 5.1. – *$\mathscr{H}[\boldsymbol{b}]$ has a classical solution of the form*

$$q = -\frac{Z^2 - b_4 Z}{2b_4 Z + (b_1 + b_4)(b_2 + b_4)}\,, \tag{5.8}$$

$$X_0 = -Z - \tfrac{1}{2}(b_1 + b_2 + b_4)\,, \tag{5.8$'$}$$

where

$$X_0 = q(q-1)p - (b_1 + b_3)q + \tfrac{1}{2}(b_1 + b_2 + b_3)$$

and Z is a solution of the equation:

$$t(t-1)\frac{dZ}{dt} = -Z^2 + \big(1 - b_1 - b_2 - 2(b_3 - 1)t\big(Z - (t + b_2 + b_4)(b_1 + b_4)t\,. \tag{5.9}$$

5.3. *Proof of Proposition* 5.1.

Consider the hyperplane:

$$b_3 + b_4 + 1 = 0\,. \tag{5.10}$$

It is easy to see that for a point $\tilde{\boldsymbol{b}}$ of (5.10), $\mathscr{H}[\tilde{\boldsymbol{b}}]$ has a solution of the form:

$$(5.11)\quad \tilde{q}-t\equiv 0\,,\qquad t(t-1)\frac{d\tilde{p}}{dt}=-t(t-1)\tilde{p}^2+\{\tilde{\varkappa}_0(t-1)+t\tilde{\varkappa}_1-(2t-1)\}\tilde{p}-\tilde{\varkappa}\,.$$

If $\boldsymbol{b}$ is on (5.4), $\tilde{\boldsymbol{b}}=\ell^{-1}(\boldsymbol{b})$ is on (5.10), where $\ell=\ell_3$. In this case the auxiliary Hamiltonian function $h^-=h^-(t)$ related to (5.11) is:

$$(5.12)\qquad h^-=t(t-1)\tilde{p}(t)-(b_3^2-b_3+b_1)t+\tfrac{1}{2}(b_3^2-b_3+b_1+b_2-b_1b_2)\,.$$

Moreover for a solution $\tilde{p}=\tilde{p}(t)$ of (5.11) with $\tilde{\boldsymbol{b}}=\ell^{-1}(\boldsymbol{b})$, the function

$$(5.13)\qquad Z(t)=t(t-1)\tilde{p}-(b_1+b_4)t$$

is a solution of (5.9). Since the auxiliary function $h=h(t)$ of $H[\boldsymbol{b}]$ is connected to (5.12) as:

$$h=h^- - Z-\tfrac{1}{2}(b_1+b_2+b_4)\,,$$

we have

$$(5.14)\qquad h=-b_3^2t+\tfrac{1}{2}(b_3^2-b_1b_2)\,,$$

for which the expressions (2.5), (2.6) are not defined. In fact, writing for $j=1,\dots,4$

$$A_j=\frac{dh}{dt}+b_j^2\,,$$

we obtain from (5.13)

$$A_3\equiv A_4\equiv 0\,.$$

On the other hand, there does exist a solution $(\tilde{q},\tilde{p})$ of $\mathscr{H}[\tilde{\boldsymbol{b}}]=\mathscr{H}[\ell^{-1}(\boldsymbol{b})]$ which is not classical. Such solutions constitute a family with two-parameters, from which we obtain the special solution (5.11) by taking the limit: $\tilde{q}-t\to 0$. The birational canonical transformation ℓ_*^{-1} defined by (2.13), (2.15) can be applied to $(\tilde{q},\tilde{p})$ except for (5.11). Put for such $(\tilde{q},\tilde{p})$

$$\tilde{X}=\tilde{q}(\tilde{q}-1)\tilde{p}-(b_1+b_4)\tilde{q}+\tfrac{1}{2}(b_1+b_2+b_4)\,.$$

The auxiliary Hamiltonian function h of $\mathscr{H}[\boldsymbol{b}]$ is given by:

$$h=h^- -\tilde{X}$$

and is no longer of the form (5.14). It represents $(\tilde{q}, \tilde{p})$ by the formulae (2.13), (2.15) with $\tilde{\boldsymbol{b}} = \ell^{-1}(\boldsymbol{b})$. We obtain from (2.5), (2.6)

$$q = \frac{C}{2A_3}, \qquad q(q-1)p = \frac{1}{2A_3}[-B + b_1 C],$$

where, by (5.4)

$$B = t(t-1)\frac{d^2h}{dt^2} + (b_1 + b_2)A_3, \qquad C = 2\left(t\frac{dh}{dt} - h\right) - b_1 b_2 + b_3^2.$$

Moreover, we can deduce from (2.15):

$$2A_3\tilde{X} = t(t-1)\frac{d^2h}{dt^2} + 2b_3\left(t\frac{dh}{ht} - h\right) - b_3\frac{dh}{dt} - b_1 b_2 b_3 = B - (b_1 + b_2 + b_3)A_3 + b_3 C.$$

It follows that:

(5.15) $$q(q-1)p - (b_1 + b_3)q = -\tilde{X} - \tfrac{1}{2}(b_1 + b_2 + b_3).$$

Since by (5.13)

$$\tilde{X}|_{\tilde{q}=t} = Z + \tfrac{1}{2}(b_1 + b_2 + b_4),$$

we deduce (5.8)′ from (5.15) by putting $\tilde{q} = t$. To verify (5.8), consider (2.13) and (2.17), which are in this case written as:

$$t(t-1)A_3 + (\tilde{q} - t)\left[h + 2b_3\tilde{X} + \frac{1}{2}b_2^2(2t-1) - \frac{1}{2}b_1 b_2\right] = 0,$$

$$\tilde{q}(t-1)\left\{t\frac{dh}{dt} - h + \frac{1}{2}(b_{21}^2 b_3 - b)\right\} + (\tilde{q} - t)\left\{\tilde{X}^2 + b_3\tilde{X} - \frac{1}{4}(b_1 + b_2)^2 + \frac{1}{4}b_2^2\right\} = 0.$$

By eliminating $\tilde{q} - t$ from these relations, we have

$$\frac{C}{2A_3} = \frac{\tilde{X}^2 + b_3\tilde{X} - \frac{1}{4}(b_1 + b_2)^2 + \frac{1}{4}b_4^2}{h + 2b_3\tilde{X} + \frac{1}{2}b_3^2(2t-1) - \frac{1}{2}b_1 b_2}\cdot\frac{t}{\tilde{q}},$$

which reduces to (5.8) after the limiting: $\tilde{q} \to t$. Since (5.8) and (5.8)′ defines the canonical transformation from $\mathscr{H}[\ell^{-1}(\boldsymbol{b})]$ to $\mathscr{H}[\boldsymbol{b}]$, they give a solution of the Painlevé system $\mathscr{H}[\boldsymbol{b}]$. The proof of Proposition 5.1 is thus completed.

5.4. *Examples of τ-functions.*

Consider again the hyperplane (4.17); we have determined the semi-sequence of τ-functions $\mathfrak{T}^+(\ell) = \{\tau_m;\ m \geqq 0\}$ with

$$\tau_0 \equiv 1, \qquad \tau_1 = F(b_1 + b_4, 1 + b_3 + b_4, 1 + b_2 + b_4; t).$$

In what follows, we will obtain τ-functions τ_m also for $m < 0$. To do so, we have to compute the canonical transformation ℓ_*^{-1} from $\mathscr{H}(\boldsymbol{v})$ to $\mathscr{H}(\ell^{-1}(\boldsymbol{v}))$ by a similar manner to Proposition 5.1, since for (4.18)

$$A_1 \equiv A_3 \equiv 0 .$$

Put $\mathscr{H}(\boldsymbol{v}) = (q, p, H, t)$, $\mathscr{H}(\ell^{-1}(\boldsymbol{v})) = (q^-, p^-, H^-, t)$ for $\boldsymbol{v}$ of $V(g_4)$, and moreover

$$X^- = q^-(q^- - 1)p^- - (b_1 + b_4)q^- + \tfrac{1}{2}(b_1 + b_2 + b_4) .$$

By assuming $b_3 + b_4 \neq 0$, $p \not\equiv 0$, we deduce from (2.2), (2.3), (2.4) and (2.15) that

$$X^- = (b_3 + b_4)q - q(q-1)\mu - \frac{1}{2}(b_2 + b_3 + b_4) + 2b_3(b_3 + b_4)\frac{q(q-1)p}{A_3}, \tag{5.16}$$

$$\frac{q(q-1)p}{A_3} = \frac{q(q-1)}{-q(q-1)p + b_1(2q-1) - b_2} .$$

Now we put $p = 0$. It follows from (5.16) that:

$$X^- = -(b_3 + b_4)Z_0 - \frac{1}{2}(b_2 + b_3 + b_4) , \qquad Z_0 = \frac{(b_2 + b_3)q}{b_1(2q-1) - b_2}, \tag{5.17}$$

from which we have, by using (2.13),

$$\frac{t - q^-}{t-1} = \frac{t}{t + Z_0} . \tag{5.18}$$

PROPOSITION 5.2. – *A function q^- given by* (5.18) *is a solution of the Painlevé equation $\mathsf{P}(\ell^{-1}(\boldsymbol{v}))$ at $\ell^{-1}(\boldsymbol{v})$.*

PROOF. – Since q is a solution of the Riccati equation (4.18), Z_0 satisfies

$$t(t-1)\frac{dZ_0}{dt} = (b_3 + b_4)Z_0^2 + (2b_3 t + b_2 + b_4)Z_0 + (b_2 + b_3)t . \tag{5.19}$$

It follows from (5.18) that

$$t(t-1)\frac{dq^-}{dt} = \varkappa_0(q^- - 1)(q^- - t) + \varkappa_1 q^-(q^- - t) + \theta q^-(q^- - 1) , \tag{5.20}$$

which shows q^- is a solution of $\mathsf{P}(\ell^{-1}(\boldsymbol{v}))$.

REMARK 5.2. – Define the canonical transformation

$$(q, p, H, t) \to (q, \tilde{p}, \tilde{H}, t) \tag{5.21}$$

by:

$$\tilde{p} = p - \frac{\varkappa_0}{q} - \frac{\varkappa_1}{q-1} - \frac{\theta}{q-t}, \qquad \tilde{H} = H - \frac{\theta}{q-t} - \frac{\varkappa_0+\theta}{t} - \frac{\varkappa_1+\theta}{t-1}.$$

We have

$$\tilde{H} = \frac{1}{t(t-1)}[q(q-1)(q-t)\tilde{p}^2 + \\ + \{\varkappa_0(q-1)(q-t) + \varkappa_1 q(q-t) + (\theta+1)q(q-1)\}\tilde{p} + \tilde{\varkappa}(q-t)],$$

$$\tilde{\varkappa} = \tfrac{1}{4}(\varkappa_0 + \varkappa_1 + \theta + 1)^2 - \tfrac{1}{4}\varkappa_\infty^2,$$

Applying (5.21) to $\mathscr{H}[\ell^{-1}(\boldsymbol{v})]$, we see immediately (5.20) gives a family of classical solutions of $\mathsf{P}(\ell^{-1}(\boldsymbol{v}))$.

It is not difficult to determine the τ-function $\tau_{-1} = \tau(\ell^{-1}(\boldsymbol{v}))$. In fact, taking (4.14) into consideration we obtain from (5.17)

$$-(b_3+b_4)Z_0 + t(b_1+b_4) - (b_2+b_4) = t(t-1)\frac{d}{dt}\log\tau_{-1},$$

where we put $\tau_0 = 1$. It follows from (5.19) that $\tau = \tau_{-1}$ satisfies:

$$t(1-t)\frac{d^2\tau}{dt^2} + [c' - (a'+b'+1)t]\frac{d\tau}{pt} - a'b'\tau = 0,$$

where

$$a' = -b_1 - b_4, \quad b' = -b_3 - b_4 + 1, \quad c' = 1 - b_2 - b_4.$$

We can thus determine τ-functions τ_m also for $m < 0$ by means of (4.4). We arrive at the following proposition.

PROPOSITION 5.3. – *If we write a point* $\boldsymbol{b}$ *on the hyperplane* $b_1 + b_3 = 0$ *in the form:*

(5.22) $\boldsymbol{b} = \left(\tfrac{1}{2}(a-b+1),\ c - \tfrac{1}{2}(a+b+1),\ -\tfrac{1}{2}(a-b+1),\ \tfrac{1}{2}(a+b-1)\right),$

then the Painlevé system $\mathscr{H}[\boldsymbol{b}]$ *at* $\boldsymbol{b}$ *has particular solutions defined by the Riccati equation*

$$t(t-1)\frac{dq}{dt} = -aq^2 + \{(a-b+1)t + c - 1\}q - (c-b)t, \quad p \equiv 0.$$

Starting from such a solution (q, p), *we have the* τ*-sequence at* $\boldsymbol{b}$:

$$\mathfrak{T}(\ell) = \{\tau_m;\ m \in Z\}$$

such that

$$\tau_0 = 1, \quad \tau_1 = F(a, b, c; t), \quad \tau_{-1} = F(-a, 2-b, 2-c; t).$$

If we define $\bar{\tau}_m$ by (4.7), they satisfy

$$\delta^2 \log \bar{\tau}_m = c(m) \frac{\bar{\tau}_{m-1}\,\bar{\tau}_{m+1}}{\bar{\tau}_m^2}$$

for $m \geqq 1$ and for $-m \geqq 1$ separately.

REMARK 5.3. – By normalizing multiplicative constants of $\bar{\tau}_m$ as $c(m) = 1$, we obtain the expression (4.23) also for $-m \geqq 2$.

REMARK 5.4. – Put for an integer n

$$F_n = F(a, b, c + n; t) .$$

By assuming none of c, $c - a$ and $c - b$ is integer, we have

$$\left(t\frac{d}{dt} + c + n - 1\right) F_n = (c + n - 1) F_{n-1} , \tag{5.24}$$

$$\left((1-t)\frac{d}{dt} + c + n - a - b\right) F_n = \frac{(c + n - a)(c + n - b)}{c + n} F_{n+1} , \tag{5.25}$$

which are known as the contiguity relations of Gauss (confer (0.2), (0.3)). It is known ([9]) that the function

$$G_n = \{t(1 - t)\}^{c_n} a_n F_n$$

satisfies the Toda equation:

$$\delta^2 \log G_n = \frac{G_{n-1}G_{n+1}}{G_n^2} , \qquad \delta = t(1-t)\frac{d}{dt} ,$$

where a_n is some constant and

$$2c_n = (c + n - 1)^2 - (a + b - 1)(c + n - 1) + ab .$$

We see from Proposition 5.3 the Function F_n is a τ-function at $\ell_2^n \ell_3[\boldsymbol{b}]$ for a point $\boldsymbol{b}$ of $\mathbf{C}^n$ of the form (5.22). It follows that (5.24), (5.25) can be obtained from the birational canonical transformation $(\ell_2)_*$ from $\mathscr{H}_n$ to $\mathscr{H}_{n+1}$, or to $\mathscr{H}_{n-1}$, where $\mathscr{H}_n$ denotes the Painlevé system $\mathscr{H}[\ell_2^n \ell_3[\boldsymbol{b}]]$.

5.5. *Rational solutions.*

Recall a hypergeometric function is reduced to a polynomial (Jacobi polynomial, Gegenbauer polynomial and so on) for a special value of the parameters a, b, c. Hence the Painlevé system has a rational solution at a point $\boldsymbol{b}$ of the form (5.22). We see it occurs certainly at the intersection of walls of the Weyl chamber. We give an example of rational solutions of the Painlevé system.

PROPOSITION 5. – *The Painlevé system $\mathscr{H}(\boldsymbol{v}_m)$ at*

$$\boldsymbol{v}_m = (-3 - m, 0, 1, -m)$$

has the rational solution:

$$(q_m, p_m) = \left(\frac{m+1}{t+m}, \frac{t+m}{t+m+1}\right), \tag{5.26}$$

m being non-negative integers.

PROOF. – It is easy to see the Painlevé system $\mathscr{H}(\boldsymbol{v})$ at

$$\boldsymbol{v} = (-3, 0, 1, 0)$$

possesses a solution of the form

$$(q, p) = \left(\frac{1}{t}, -\frac{t}{1+t}\right), \tag{5.27}$$

from which we have the Hamiltonian functions:

$$H(t) = \frac{1}{1+t} - \frac{2}{t}, \qquad h(t) = -\frac{1}{4}t - 1 + \frac{2}{1+t}$$

and then the τ-function

$$\tau_0 = \tau(\boldsymbol{v}) = t^{-2}(1+t) .$$

On the other hand, we obtain from (5.27)

$$Y_0 \equiv q(q-1)p - (b_1 + b_4)(q - t) = \frac{1}{1+t} - \frac{2}{t},$$

hence we can put

$$\tau_1 = \tau(\ell_3(\boldsymbol{v})) = 1 .$$

It follows from (4.4) with $c(m) = (m+1)(m+2)$ that

$$\tau_m = \tau(\boldsymbol{v}_m) = t^{-m-2}(t + m + 1) \tag{5.28}$$

where $\boldsymbol{v}_m = \ell_{-3}^{-m}(\boldsymbol{v})$. Consequently we have (5.26) and

$$Y_m = -t + \frac{(m+1)(m+2)}{t+m+1} - \frac{m(m+1)}{t+m}$$

by means of (5.28) and (4.12), which proves the proposition.

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