Study of Deuteron Stripping Reaction by Coupled Channel Theory. II

——Properties of Interaction Kernel and Method of Numerical Solution——

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The basic equations of the coupled channel method for (d, p) reactions derived in a previous paper, I [T. Ohmura, B. Imanishi, M. Ichimura and M. Kawai, Prog. Theor. Phys. 41 (1969), 391], are transformed into forms which are convenient for practical calculations in which the interactoin kernels are symmetric with respect to channels. The calculated S-matrix is then also shown to be symmetric, despite the presence of the imaginary part of the optical potentials. An explicit form of the interaction kernel is given and some numerical examples are presented. The properties of the kernel is discussed in detail. The method of numerical solution of the basic equations is described in detail and examples of the calculated cross section are presented. In the calculations the non-orthogonality term in the interaction kernel is neglected. Rather detailed discussions are given on the Pauli principle between the participating nucleons. It is shown that the effect of the Pauli principle is expressed in a certain approximation by a multiplication of the interaction kernel by the square root of the spectroscopic factor.

§ 1. Introduction

The purpose of the present series of papers is to study the deuteron stripping reaction in terms of the method of coupled channels in order to investigate the theoretical back ground of distorted wave Born approximation (DWBA) for the nuclear rearrangement collision which has been studied only inadequately compared with the nuclear inelastic scattering, in contrast to the great success of DWBA in the actual analysis of the experiment.

In a previous paper[1] (hereafter referred to as I) we have derived a set of close coupling equations for the deuteron stripping reaction by explicitly using a variational principle for the rearrangement collision. The generalized Kato identity[2] was also given for the error in the calculated S-matrix by applying

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the theory of finite variations. The coupled equations are considerably different from those for the inelastic scattering because of the non-orthogonality of the initial and final channels. Discussions were given there on various aspects of the basic equations, namely, 1) the effects of non-orthogonality term in the interaction kernel on the S-matrix elements, 2) relation of the present method to DWBA and to the method of projection operators, 3) the non-uniqueness in defining the wave functions of relative motion in the deuteron and proton channels which gives rise to an ambiguity in the optical potentials and so on.

In the present paper we transform the basic equations into a form which is convenient for practical calculations. We discuss the nature of the equations, especially the interaction kernels, in detail and describe the method of numerical solution of the equations. Some examples of numerical calculations are shown. In application to the actual (d, p) process one encounters the problem of taking into account the Pauli principle between the captured neutron and the target neutrons in the same orbit. We discuss this point in detail.

In § 2 the basic equation is brought into a form in which the interaction kernels are explicitly symmetric with respect to channels, which were not so in the original form of the equations given in I. One can then show that the S-matrix in the present approximation is also symmetric. In § 3 the interaction kernels are given by expressions which are convenient for practical calculations. Some examples of the calculated kernels are presented and the nature of the kernels is discussed in some detail. The kernel is decomposed into the interaction term and the non-orthogonality term. The interaction term which is used in the calculation in § 6 is illustrated separately and is shown to be well behaved compared to the non-orthogonality term which is rapidly oscillating and has a very long range. In § 4 the Pauli principle between the captured and target neutrons is discussed in detail. In § 5 a method of numerical solution of the basic equations is described. A cutoff procedure is introduced to avoid the difficulty encountered in the numerical calculations caused by the singularity of the centrifugal potential at the origin. In § 6 examples of calculated cross sections are presented and the results are briefly discussed for (d, p) reactions on ^{40}Ca, ^{16}O and ^{12}C in which the non-orthogonality terms are neglected from the interaction kernel. A summary is given in § 7.

§ 2. Transformation of the basic equations and the symmetry properties of the interaction kernels and the S-matrix

First, let us review the notation and the basic equations given in I.[1] Neglecting the spin of nucleons and the D-state of the deuteron, the L-th partial wave of the total wave funtcion, Ψ_{L0}, with the z-component of the angular momentum zero has the form

$$\Psi_{L0} = \Phi_{00}^A(\xi) \phi_d(r_{np}) Y_{L0}(\widehat{R}) u_L(R) / R$$

$$+ \sum_{l=|L-l_n|}^{L+l_n} [\Phi_{l_n}^B(\xi, r_n), Y_l(\hat{r})]_{L0} v_l^{(L)}(r)/r + \psi_{L0}, \qquad (2 \cdot 1)$$

where $\Phi_{00}^A(\xi)$, $\phi_d(r_{np})$ and $\Phi_{l_n}^B(\xi, r_n)$ are the internal wave functions of the target nucleus A, the deuteron and the residual nucleus B respectively. We assume $\Phi_{l_n}^B(\xi, r_n)$ to be of the form

$$\Phi_{l_n}^B(\xi, r_n) = \Phi_{00}^A(\xi) \phi_{l_n m_n}(r_n).$$

The wave function of relative motion in the deuteron channel is given by $(u_L(R)/R) Y_{L0}(\hat{R})$ and that of the proton channel by $(v_l^{(L)}(r)/r) Y_{lm}(\hat{r})$. The notation $[\Phi_{l_n}^B, Y_l]_{L0}$ is defined by

$$[\Phi_{l_n}^B, Y_l]_{LM} = \sum_{m_n m} (l_n l m_n m | LM) \Phi_{l_n m_n} Y_{lm}.$$

Fig. 1. Coordinate system for the reaction $A(d,p)B$.

The function ψ_{L0} vanishes asymptotically in the deuteron and proton channels and has only outgoing waves in the asymptotic region of all the other open channels. The coordinate system used in Eq. (2·1) is shown in Fig. 1. Taking Ψ_{L0} with ψ_{L0} put equal to zero as the trial function of a variational principle, one gets, according to I, the basic equations for the radial wave functions u_L and $v_l^{(L)}$,
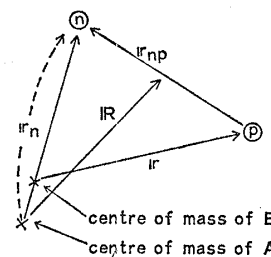
$$\frac{-a_d}{R} \left\{ \frac{d^2}{dR^2} + k_d^2 - \frac{L(L+1)}{R^2} \right\} u_L(R) + U_{dA}(R) \frac{u_L(R)}{R}$$

$$+ \int \Phi_{00}^{A*} \phi_d^* Y_{L0}^*(\hat{R}) \left[-a_p (\Delta_r|_n + k_p^2) + V_{pA}(r_p') + V_{np}(r_{np}) \right]$$

$$\times \sum_l [\Phi_{l_n}^B(\xi, r_n), Y_l(\hat{r})]_{L0} \frac{v_l^{(L)}(r)}{r} d\xi dr_{np} d\hat{R} = 0 \qquad (2 \cdot 2)$$

and

$$-\frac{a_p}{r} \left\{ \frac{d^2}{dr^2} + k_p^2 - \frac{l(l+1)}{r^2} \right\} v_l^{(L)}(r) + U_{pB}(r) \frac{v_l^{(L)}(r)}{r}$$

$$+ \int [\Phi_{l_n}^{B*}(\xi, r_n), Y_l^*(\hat{r})]_{L0} \left[-a_p (\Delta_r|_n + k_p^2) + V_{pA}(r_p') + V_{np}(r_{np}) \right]$$

$$\times \Phi_{00}^A(\xi) \phi_d(r_{np}) Y_{L0}(\hat{R}) \frac{u_L(R)}{R} d\xi dr_n d\hat{r} = 0, \qquad (2 \cdot 3)$$

where $a_p = \hbar^2/2\mu_p$, $a_d = \hbar^2/2\mu_d$ and $\Delta_r|_n$ stands for the Laplacian with respect to r with r_n kept constant when r and r_n are taken to be independent variables. The potentials $U_{dA}(R)$ and $U_{pB}(r)$ are suitably chosen complex phenomenological potentials in the d- and p-channels respectively and the potentials $V_{pA}(r_p')$ and $V_{np}(r_{np})$ represents the interaction of the outgoing proton with the core A and the captured neutron n respectively.

In the case of a (d, p) reaction, the set of boundary condition imposed on

$u_L(R)$ and $v_l^{(L)}(r)$ is that they should vanish at the origin of the respective arguments and have the asymptotic forms

$$u_L(R) \sim \zeta_L^d F_L(k_d R) + \zeta_L^d T_{dL,dL}^{(L)} H_L^{(+)}(k_d R) \qquad (2\cdot 4a)$$

and

$$v_l^{(L)}(r) \sim \zeta_l^p T_{pl,dL}^{(L)} H_l^{(+)}(k_p r), \qquad (2\cdot 4b)$$

respectively, where F_L and $H_L^{(+)}$ are the regular and outgoing wave Coulomb functions defined in Eq. (2·34) of I and $\zeta_L^d = i^L \exp(i\sigma_L^d)/\sqrt{v_d}$ and $\zeta_l^p = i^l \exp(i\sigma_l^p)/\sqrt{v_p}$ where the σ's are the Coulomb phase shifts and the v's the speeds of the particles at the infinite distances. Equations (2·4) define the reaction amplitudes $T_{dL,dL}^{(L)}$ and $T_{pl,dL}^{(L)}$ the values of which are obtained by solving Eqs. (2·2) and (2·3).

More generally the reaction amplitude $T_{\beta\alpha}^{(L)}$ for the total angular momentum L is defined by the asymptotic form of the radial wave function $u_{\beta\alpha}^{(L)}$ as

$$u_{\beta\alpha}^{(L)}(r_\beta) \sim \zeta_\beta (\delta_{\beta\alpha} F_\beta + T_{\beta\alpha}^{(L)} H_\beta^{(+)}), \qquad (r_\beta \to \infty) \qquad (2\cdot 5)$$

where α and β stand for (d, L), (p, l), etc., and $u_{\beta\alpha}^{(L)}(r_\beta)$ is the radial wave function in the β-channel of the solution with the incoming wave only in the α-channel. The S-matrix element corresponding to $T_{\beta\alpha}^{(L)}$ is given by

$$S_{\beta\alpha}^{(L)} = e^{2i\sigma_\beta} (\delta_{\beta\alpha} + 2i T_{\beta\alpha}^{(L)}). \qquad (2\cdot 6)$$

Now, Eqs. (2·2) and (2·3) are not explicitly symmetric with respect to channels. Furthermore, the integration kernels contain differential operators operating on the unknown functions $u_L(R)$ and $v_l^{(L)}(r)$. These features are very inconvenient in practical calculations. Fortunately, however, the derivatives of unknown functions can be eliminated from the integral term by the procedure described below.

Let us first consider the quantities $I_{dp}(\mathbf{R})$ and $I_{pd}(r)$ defined, respectively, by

$$I_{dp}(\mathbf{R}) = \int \Phi_{00}^{A*}(\xi) \phi_d^*(r_{np}) \nabla_r^2 |_n \Phi_{l_n m_n}^B(\xi, r_n) \chi_p(r) d\xi dr_{np}, \qquad (2\cdot 7)$$

where

$$\chi_p(r) = Y_{lm}(\hat{r}) r^{-1} v_l^{(L)}(r), \qquad (2\cdot 8)$$

and

$$I_{pd}(r) = \int \Phi_{l_n 0}^{B*}(\xi, r_n) \nabla_r^2 |_n \Phi_{00}^A(\xi) \phi_d(r_{np}) \chi_d(\mathbf{R}) d\xi dr_n, \qquad (2\cdot 9)$$

where

$$\chi_d(\mathbf{R}) = Y_{L0}(\hat{\mathbf{R}}) R^{-1} u_L(R). \qquad (2\cdot 10)$$

Let us change the integration variables into r and $\mathbf{R}$ using the relations

$$r_n = s\mathbf{R} + tr, \qquad r_{np} = s'\mathbf{R} + t'r, \qquad (2\cdot 11)$$

where s, t, s' and t' are given by

$$s = \frac{2(A+1)}{A+2}, \qquad t = -\frac{A+1}{A+2}, \qquad s' = \frac{2A}{A+2}, \qquad t' = -\frac{2(A+1)}{A+2}, \quad (2\cdot12)$$

A being the mass of the target nucleus. In the following we omit the subscripts denoting the magnetic quantum numbers to simplify notations. The integral in Eq. (2·7) can then be transformed as

$$I_{dp}(\boldsymbol{R}) = (-t')^3 \int \phi_{l_n}(s\boldsymbol{R}+t\boldsymbol{r}) \phi_d^*(s'\boldsymbol{R}+t'\boldsymbol{r}) \nabla_r^2|_n \chi_p(\boldsymbol{r}) d\boldsymbol{r}$$

$$= (-t')^3 \int \phi_{l_n}\phi_d^* (\nabla_r^2|_R - 2ts^{-1}\nabla_r|_R \cdot \nabla_R|_r + t^2 s^{-2}\nabla_R^2|_r) \chi_p(\boldsymbol{r}) d\boldsymbol{r}$$

$$= (-t')^3 \int \chi_p(\boldsymbol{r}) \nabla_r^2|_R [\phi_{l_n}(s\boldsymbol{R}+t\boldsymbol{r}) \phi_d^*(s'\boldsymbol{R}+t'\boldsymbol{r})] d\boldsymbol{r}, \qquad (2\cdot13)$$

where in the step leading to the last line Green's theorem is used together with the fact $\phi_{l_n}\phi_d^*$ tends to zero as r goes to infinity.

A similar transformation can be carried out also for I_{pd}. First, one writes it in the form

$$I_{pd}(\boldsymbol{r}) = s^3 \int \phi_{l_n}^*(s\boldsymbol{R}+t\boldsymbol{r}) (\nabla_r^2|_R - 2ts^{-1}\nabla_r|_R \cdot \nabla_R|_r + t^2 s^{-2}\nabla_R^2|_r) \phi_d(s'\boldsymbol{R}+t'\boldsymbol{r}) \chi_d(\boldsymbol{R}) d\boldsymbol{R}.$$
$$(2\cdot14)$$

By means of Green's theorem, one can rewrite Eq. (2·14) by integration by part as follows. Using an abbreviated notation $f(\boldsymbol{r},\boldsymbol{R}) = \phi_d(s'\boldsymbol{R}+t'\boldsymbol{r})\chi_d(\boldsymbol{R})$, one has for the second term of the integrand in Eq. (2·14),

$$-2ts^{-1} \int \phi_{l_n}^*(s\boldsymbol{R}+t\boldsymbol{r}) \nabla_r|_R \cdot \nabla_R|_r f(\boldsymbol{r},\boldsymbol{R}) d\boldsymbol{R}$$

$$= 2ts^{-1} \int [\nabla_R|_r \phi_{l_n}^*(s\boldsymbol{R}+t\boldsymbol{r})] \nabla_r|_R f(\boldsymbol{r},\boldsymbol{R}) d\boldsymbol{R}$$

$$= 2 \int [\nabla_r|_R \phi_{l_n}^*(s\boldsymbol{R}+t\boldsymbol{r})] \nabla_r|_R f(\boldsymbol{r},\boldsymbol{R}) d\boldsymbol{R}, \qquad (2\cdot15)$$

since the surface integral at infinity vanishes on account of the factor ϕ_{l_n} and the factor ϕ_d in $f(\boldsymbol{r},\boldsymbol{R})$. Similarly, the third term of the integrand in Eq. (2·14) can be rewritten as

$$t^2 s^{-2} \int \phi_{l_n}^*(s\boldsymbol{R}+t\boldsymbol{r}) \nabla_R^2|_r f(\boldsymbol{r},\boldsymbol{R}) d\boldsymbol{R} = \int f(\boldsymbol{r},\boldsymbol{R}) \nabla_r^2|_R \phi_{l_n}(s\boldsymbol{R}+t\boldsymbol{r}) d\boldsymbol{R}. \quad (2\cdot16)$$

Hence, from Eqs. (2·14), (2·15) and (2·16) one has

$$I_{pd}(\boldsymbol{r}) = s^3 \int [\phi_{l_n}^* \nabla_r^2|_R f + 2(\nabla_r|_R \phi_{l_n}^*) \nabla_r|_R f + f \nabla_r^2|_R \phi_{l_n}^*] d\boldsymbol{R}$$

$$= s^3 \int \chi_d(\boldsymbol{R}) \nabla_r^2|_R [\phi_{l_n}^*(s\boldsymbol{R}+t\boldsymbol{r}) \phi_d(s'\boldsymbol{R}+t'\boldsymbol{r})] d\boldsymbol{R}, \qquad (2\cdot17)$$

since the factor $\chi_d(\boldsymbol{R})$ in $f(\boldsymbol{r},\boldsymbol{R})$ is independent of $\boldsymbol{r}$,

Now, using Eqs. (2·13) and (2·17) one can rewrite Eqs. (2·2) and (2·3) as

$$-a_d \left\{ \frac{d^2}{dR^2} + k_d{}^2 - \frac{L(L+1)}{R^2} \right\} u_L(R) + U_{dA}(R) u_L(R)$$

$$+ \sum_l \int_0^\infty K_{dL,pl}^{(L)}(R, r) v_l^{(L)}(r) dr = 0 \qquad (2·18)$$

and

$$-a_p \left\{ \frac{d^2}{dr^2} + k_p{}^2 - \frac{l(l+1)}{r^2} \right\} v_l^{(L)}(r) + U_{pB}(r) v_l^{(L)}(r)$$

$$+ \int_0^\infty K_{pl,dL}^{(L)}(r, R) u_L(R) dR = 0, \qquad (2·19)$$

respectively. Here the kernels $K_{dL,pl}^{(L)}(R, r)$ and $K_{pl,dL}^{(L)}(r, R)$ are given by

$$K_{dL,pl}^{(L)}(R, r) = \frac{8(A+1)^3}{(A+2)^3} \sum_m (ll_n - mm|L0) rR \int Y_{L0}^*(\widehat{\boldsymbol{R}}) Y_{l-m}(\widehat{\boldsymbol{r}})$$

$$\times \{ [-a_p(\nabla_r{}^2|_R + k_p{}^2) + U_{pB}] + [V_{np} + U_{pA} - U_{pB}] \}$$

$$\times [\phi_d{}^*(s'\boldsymbol{R} + t'\boldsymbol{r}) \phi_{l_n m}(s\boldsymbol{R} + t\boldsymbol{r})] d\widehat{\boldsymbol{r}} d\widehat{\boldsymbol{R}} \qquad (2·20)$$

and

$$K_{pl,dL}^{(L)}(r, R) = K_{dL,pl}^{(L)}(R, r), \qquad (2·21)$$

where

$$U_{pA} = \int |\Phi_{00}^A|^2 V_{pA} d\xi . \qquad (2·22)$$

Equations (2·18) and (2·19) constitute the basic equations of our theory. Equation (2·21) manifestly expresses the symmetry of the interaction kernel. One sees from Eq. (2·20) that the kernel consists of two parts, one coming from the usual $V_{np} + U_{pA} - U_{pB}$ term and the other due to the non-orthogonality of the initial and final channel wave functions. As in I, we call these terms the interaction term and the non-orthogonality term respectively. In the actual calculation we make further assumption that U_{pA} can be replaced by U_{pB} in Eqs. (2·20) and (2·21), which corresponds to the assumption in ordinary DWBA that $(U_{pA} - U_{pB})$ can be neglected in the residual interaction.[*] The kernel is symmetric also in this approximation. The parameters of the complex "distorting" potentials U_{dA} and U_{pB} should be determined so that the calculated cross sections agree with the experiment.

Now, let us examine the symmetry property of the S-matrix calculated by the present method. Because of the imaginary part of the distorting potentials the flux is not conserved and the time reversal invariance does not hold. Nevertheless one can show that the S-matrix is symmetric if one writes Eqs. (2·18)

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[*] Validity of this assumption is being examined by K. Kubo (private communication).

and $(2 \cdot 19)$ in an abbreviated form

$$(\boldsymbol{E} - \boldsymbol{H}) \psi^{(L)} = 0 , \tag{2.23}$$

using $f \times f$ dimensional matrices $\boldsymbol{E}$ and $\boldsymbol{H}$ and a vector $\psi^{(L)}$ whose components are f radial wave functions $u_L, v_l^{(L)}, \cdots$, where f is the number of channels coupled in Eqs. $(2 \cdot 18)$ and $(2 \cdot 19)$. The "energy matrix" $\boldsymbol{E}$ is a diagonal matrix of the form

$$\boldsymbol{E} = \begin{bmatrix} a_d k_d^2 & 0 & \cdots \\ 0 & a_p k_p^2 & \\ \vdots & & \end{bmatrix} . \tag{2.24}$$

Explicit form of the "Hamiltonian matrix" $\boldsymbol{H}$ will not be given here but should be clear from Eqs. $(2 \cdot 18)$, $(2 \cdot 19)$ and the definitions of $\psi^{(L)}$ and $\boldsymbol{E}$ given above. The matrix $\boldsymbol{H}$ contains a diagonal matrix of the differential operators

$$\boldsymbol{T} = \begin{pmatrix} a_d \dfrac{d^2}{dR^2} & 0 & \cdots \\ 0 & a_p \dfrac{d^2}{dr^2} & \\ \vdots & & \end{pmatrix} , \tag{2.25}$$

and a matrix of integral operators which contain the potential term and the interaction kernels. The Hamiltonian matrix is not real but is symmetric in the sense that

$$\boldsymbol{H}^T = \boldsymbol{H} , \tag{2.26}$$

where $\boldsymbol{H}^T$ is the transpose of $\boldsymbol{H}$ with respect to channel indices.

According to Bilhorn et al.[3],[4] Eq. $(2 \cdot 26)$ is sufficient for the symmetry of the calculated S-matrix. To show this, let us define two solutions of Eq. $(2 \cdot 23)$, $\psi_{dL}^{(L)}$ and $\psi_{pl}^{(L)}$ which have the asymptotic forms of the incoming wave only in the dL- and pl-channels, respectively, plus outgoing waves in all channels. The asymptotic forms of the component functions of $\psi_{dL}^{(L)}$ and $\psi_{pl}^{(L)}$ are given by Eq. $(2 \cdot 5)$ with $\alpha = dL$ and $\alpha = pl$, respectively. Then, the identity

$$\int d\tau \{ \psi_{pl}^{(L)T} (\boldsymbol{H} - \boldsymbol{E}) \psi_{dL}^{(L)} - \psi_{dL}^{(L)T} (\boldsymbol{H} - \boldsymbol{E}) \psi_{pl}^{(L)} \} = 0 \tag{2.27}$$

holds where the ψ^T's are the transpose of the ψ's. Using Green's theorem to convert the volume integral containing the matrix $\boldsymbol{T}$, which are the only remaining integral on the left-hand side of Eq. $(2 \cdot 27)$, and using Eqs. $(2 \cdot 4)$ and $(2 \cdot 6)$ one gets, after a little manipulation,

$$(-1)^l S_{pl,dL}^{(L)} = (-1)^L S_{dL,pl}^{(L)} , \tag{2.28}$$

which is the symmetry of the S-matrix which we set out to prove.

§ 3. The Interaction kernel

In this section we derive an expression for the interaction kernel, $K_{dL,pl}^{(L)}(R, r)$, which is useful for practical calculations, and discuss some of the properties of $K_{dL,pl}^{(L)}$ which are revealed by the numerical calculation using that expression.

To evaluate the integral on the right-hand side of Eq. (2·20) with the approximation, $U_{pA} = U_{pB}$, we first transform the Laplacian in the integrand as

$$\nabla_r^2|_R (\phi_d^* \phi_{l_n m}) = (t^2 \nabla_n^2|_{np} + 2tt' \nabla_n|_{np} \cdot \nabla_{np}|_n + t'^2 \nabla_{np}^2|_n) \phi_d^*(r_{np}) \phi_{l_n m}(r_n). \tag{3·1}$$

We then obtain

$$K_{dL,pl}^{(L)}(R, r) = \left\{ \frac{2(A+1)}{A+2} \right\}^3 \sum_m (ll_n - mm|L0) rR \int Y_{L0}^*(\widehat{R}) Y_{l-m}(\hat{r})$$

$$\times \{F(r, R) - 2a_p tt' \nabla_n|_{np} \cdot \nabla_{np}|_n\} \phi_d^*(r_{np}) \phi_{l_n m}(r_n) d\hat{r} d\widehat{R}, \tag{3·2}$$

where

$$F(r, R) = a_p a_n^{-1} t^2 (E_n - V_{nA}) + a_p a_{np}^{-1} t'^2 (E_d - V_{np}) + U_{pB} + V_{np} - a_p k_p^2. \tag{3·3}$$

If one puts

$$\phi_{l_n m_n}(r_n) = w_{l_n}(r_n) r_n^{l_n} Y_{l_n m_n}(\hat{r}_n), \tag{3·4}$$

one can carry out the differentiation on the right-hand side of Eq. (3·2) as[5]

$$(\nabla_{np} \phi_d^*(r_{np})) \cdot (\nabla_n \phi_{l_n m_n}(r_n)) = (\bar{r}_{np} \cdot \bar{r}_n) \phi_d^{*\prime} w_{l_n}' r_n^{l_n} Y_{l_n}(\hat{r}_n)$$

$$+ (4\pi l_n (2l_n+1)/3)^{1/2} w_{l_n} \phi_d^{*\prime} r_{np}^{-1} [r_n^{l_n-1} Y_{l_n-1}(\hat{r}_n), r_{np} Y_1(\hat{r}_{np})]_{l_n m_n}, \tag{3·5}$$

where

$$\phi_d' = d\phi_d/dr_{np}, \quad w_{l_n}' = dw_{l_n}/dr_n, \quad \bar{r}_{np} = r_{np}/r_{np} \quad \text{and} \quad \bar{r}_n = r_n/r_n.$$

To carry out integration with respect to $\widehat{R}$ and $\hat{r}$ in Eq. (3·2), let us expand the quantity, $(F(r, R) - 2a_p tt' \nabla_n|_{np} \cdot \nabla_{np}|_n) \phi_d^*(r_{np}) w_{l_n}(r_n)$ in spherical harmonics of $\widehat{R}$ and $\hat{r}$. We first note that each factor of this quantity is scalar with respect to rotation of the coordinate space so that they are functions of the scalar, $\mu = r \cdot R/rR$ and so can be expanded in the series of the Legendre function, $P_L(\mu)$. Let us therefore put

$$F(r, R) w_{l_n}(r_n) \phi_d^*(r_{np}) - 2a_p tt' (\bar{r}_{np} \cdot \bar{r}_n) \phi_d^{*\prime}(r_{np}) w_{l_n}'(r_n)$$

$$= \sum_L [(2L+1)/2] j_L^{(l_n)}(r, R) P_L(\mu) \tag{3·6a}$$

and

$$-a_p tt' w_{l_n}(r_n) r_{np}^{-1} \phi_d^{*\prime}(r_{np}) = \sum_L [(2L+1)/2] q_L^{(l_n)}(r, R) P_L(\mu). \tag{3·6b}$$

$P_L(\mu)$ can then be expressed by $Y_L(\hat{r})$ and $Y_L(\widehat{R})$.

The coordinates r_n and r_{np} are related to r and R through Eqs. (2·11) as

$$r_n = (s^2 R^2 + t^2 r^2 + 2strR\mu)^{1/2}$$

and

$$r_{np} = (s'^2R^2 + t'^2r^2 + 2s't'rR\mu)^{1/2}.$$

One also has

$$(\vec{r}_{np} \cdot \vec{r}_n) = \{ss'R^2 + tt'r^2 + (s't + st')Rr\mu\}/r_n r_{np}. \tag{3.7}$$

Putting (3.5) through (3.7) into (3.2) and using the formulae[3]

$$r_n{}^l Y_{lm}(\hat{\vec{r}}_n) = \sum_{\lambda=0}^{l} \sqrt{\frac{4\pi(2l+1)!}{(2\lambda+1)!(2l-2\lambda+1)!}} (sR)^{l-\lambda}(tr)^\lambda [Y_{l-\lambda}(\hat{\vec{R}}), Y_\lambda(\hat{\vec{r}})]_{lm} \tag{3.8a}$$

and

$$r_{np} Y_{1m}(\hat{\vec{r}}_{np}) = \sum_{\lambda=0}^{1} \sqrt{4\pi}\, (s'R)^{1-\lambda}(t'r)^\lambda [Y_{1-\lambda}(\hat{\vec{R}}), Y_\lambda(\hat{\vec{r}})]_{1m}, \tag{3.8b}$$

one finally obtains the desired expression for $K_{dL,pl}^{(L)}(R, r)$:

$$K_{dL,pl}^{(L)}(R, r) = \frac{8(A+1)^3}{(A+2)^3} rR \sum_{\nu=0}^{\infty} \{ \sum_{\lambda=0}^{l_n} B[l_n lL; \lambda\nu] K[l_n; \lambda\nu; rR]$$

$$+ \sum_{\lambda=0}^{l_n-1} \sum_{\lambda'=0}^{1} D[l_n lL; \lambda'\lambda\nu] L[l_n; \lambda'\lambda\nu; rR] \}, \tag{3.9}$$

where

$$B[l_n lL; \lambda\nu] = (-)^\nu (2l_n+1)(2\nu+1)\sqrt{\frac{\pi(2l_n)!}{(2L+1)(2\lambda)!(2l_n-2\lambda)!}}$$

$$\times (l_n - \lambda\nu 00|L0)(\lambda\nu 00|l0) W(Lll_n-\lambda\lambda; l_n\nu), \tag{3.10a}$$

$$K[l_n; \lambda\nu; rR] = (sR)^{l_n-\lambda}(tr)^\lambda j_\nu{}^{(l_n)}(r, R), \tag{3.10b}$$

$$D[l_n lL; 0\lambda\nu] = (-1)^\lambda (2l_n+1)(2\nu+1)\sqrt{\frac{\pi(2l_n-1)(2l_n)!}{(2L+1)2(2\lambda)!(2l_n-2\lambda-2)!}}$$

$$\times (\lambda\nu 00|l0) \sum_k \sqrt{2k+1}\,(l_n-\lambda-1\nu 00|k0)(k100|L0) W(kll_n-\lambda-1\lambda; l_n-1\nu)$$

$$\times W(kl_n-1Ll_n; l1), \tag{3.10c}$$

$$D[l_n lL; 1\lambda\nu]$$

$$= (-1)^{\lambda+1}(2l_n+1)(2\nu+1)\sqrt{\frac{\pi(2l_n-1)(2l_n)!}{(2L+1)2(2\lambda)!(2l_n-2\lambda-2)!}}(l_n-\lambda-1\nu 00|L0)$$

$$\times \sum_k \sqrt{2k+1}\,(\lambda\nu 00|k0)(k100|l0) W(Lkl_n-\lambda-1\lambda; l_n-1\nu) W(l_n-1kl_n l; L1) \tag{3.10d}$$

and

$$L[l_n; \lambda'\lambda\nu; rR] = (sR)^{l_n-\lambda-1}(tr)^\lambda (s'R)^{1-\lambda'}(t'r)^{\lambda'} q_\nu{}^{(l_n)}(r, R). \tag{3.10e}$$

From Eqs. (3.10a), (3.10c) and (3.10d) it is seen that B and D vanish if $l_n + l + L = $ odd.

Let us now discuss some properties of the kernels. Let us first investigate the kernel for small r or small R. For this purpose let us consider an arbitrary,

well behaved function of r_n, $f(r_n)$. We first note that we can always write

$$r_n = x\sqrt{1 + h^2 + 2h\mu},$$

where $x = \max(|sR|, |tr|)$ and $h = \min(|sR|, |tr|)/x$. For a small value of h one can expand $f(r_n)$ in powers of $(h^2 + 2h\mu)$ as

$$f(r_n) = \sum_{m=0}^{\infty} \frac{a_m(x)}{m!} (h^2 + 2h\mu)^m,$$

where $a_m(x)$ is a function of x. Now, $(h^2 + 2h\mu)^m$ can be written as a polynomial of μ of degree m in which the coefficient of μ^m is $2^m h^m$. Since this polynomial of μ can be written as a linear combination of Legendre polynomials, $P_\nu(\mu)$, with $\nu \leq m$, $f(r_n)$ can be rewritten in the form

$$f(r_n) = \sum_{i=0}^{\infty} \left(\sum_{j=i}^{\infty} a_{ji}(x) h^j \right) P_i(\mu). \tag{3.11}$$

Similarly, an arbitrary, well-behaved function $g(r_{np})$ of r_{np} can be written in the form

$$g(r_{np}) = \sum_{i=0}^{\infty} \left(\sum_{j=i}^{\infty} b_{ji}(x') h'^j \right) P_i(\mu), \tag{3.12}$$

where $x' = \max(|s'R|, |t'r|)$ and $h' = \min(|s'R|, |t'r|)/x'$.

The product of $f(r_n)$ and $g(r_{np})$ can then be written in the form

$$f(r_n) g(r_{np}) = \sum_{k=0}^{\infty} \sum_{i=0}^{\infty} \sum_{j=k-i}^{\infty} c_{ij,k}(x, x') h^i h'^j P_k(\mu). \tag{3.13}$$

Equation (3.13) shows that the coefficient of $P_k(\mu)$ in the expansion of $f(r_n) g(r_{np})$ is r^k in the lowest order for sufficiently small r which satisfies $|tr| < |sR|$ and $|t'r| < |s'R|$ and R^k in the lowest order if $|sR| < |tr|$, $|s'R| < |t'r|$. Applying the above considerations to Eqs. (3.6a) and (3.6b) with Eq. (3.7), one sees that $j_\nu^{(l_n)}(r, R)$ and $q_\nu^{(l_n-1)}(r, R)$ are r^ν and R^ν in the lowest order according as $r \ll R$ or $R \ll r$. Equation (3.10b) then shows that $K[l_n; \lambda\nu; rR]$ is at least of the order or $r^{\lambda+\nu}$ for small r and $R^{l_n-\lambda+\nu}$ for small R. Because of the Clebsch-Gordan coefficients in $B[l_n lL; \lambda\nu]$, however, $l \geq \lambda + \nu$ and $L \geq l_n - \lambda + \nu$. Hence, the contribution of the first term in the curly brackets on the right-hand side of Eq. (3.9) gives r^{l+1} in the lowest possible order of r for small r and R^{L+1} for small R. Similarly, the second term on the right-hand side of Eq. (3.9) gives the same dependence on r and R for small r and R respectively. Hence, $K_{dL,pl}^{(L)}(R, r)$ is at least of the order of r^{l+1} for small r and R^{L+1} for small R.

In the limit $r \to \infty$ and/or $R \to \infty$, the kernel is expected to decrease exponentially since $\phi_{l_n m}$ and/or ϕ_d decrease in that limit. In fact, $\phi_{l_n}(s\boldsymbol{R} + t\boldsymbol{r})$ is large only when $r_n = |s\boldsymbol{R} + t\boldsymbol{r}|$ is within about the radius of the single particle orbit while $\phi_d(s'\boldsymbol{R} + t'\boldsymbol{r})$ is large only for $r_{np} = |s'\boldsymbol{R} + t'\boldsymbol{r}|$ within about the deuteron radius and they decay exponentially outside these ranges. Since, however, $|t/s| = 1/2$ and $|t'/s'| = (A+1)/A$ according to Eqs. (2.11), at least one of r_n

and r_{np} must be large in the limit of r and/or R tending to infinity.

The interaction kernel, $K_{dL,pl}^{(L)}(R,r)$, is a continuous function of R and r but its first derivative is in general discontinuous along the lines in r-R plane corresponding to $r_n = 0$ and $r_{np} = 0$. This is due to the cusps which occur when r_n and r_{np} are expanded in $P_L(\mu)$. The lines of the cusps in the r-R plane is given by the equations

$$R = |ts^{-1}|r = r/2 \quad \text{and} \quad R = |t's'^{-1}|r = (A+1)r/A . \tag{3.14}$$

However, the cusps are not prominent, in fact unrecognizable, in the cases which we discuss in the present paper because of the following reasons. First, U_{nA} is presumably nearly constant at the nuclear centre, $r_n = 0$. Then, the function $w_{l_n}(r_n)$, when expanded in a power series of r_n around the origin, has only even powers of r_n in the lower order terms. The cusps, therefore, are not prominent. On the other hand, Eq. (3.8) shows that $r_n^l Y_{lm}(\hat{r}_n)$ is a smooth function of r and R. If V_{np} is a function of r_{np}^2 alone it is obviously a smooth function of r and R since r_{np}^2 is a bilinear function of r and R. This is the case when V_{np} has a Gaussian form. Under these circumstances, therefore, $K_{dL,pl}^{(L)}(R,r)$ has practically no cusps and is a smooth function of r and R.

The imaginary part of $K_{dL,pl}^{(L)}$ arises from the imaginary part of U_{pB}. Since the latter is presumably peaked at the nuclear surface, $\mathrm{Im}(K_{dL,pl}^{(L)})$ is appreciable only for $r \simeq r_0 A^{1/3}$. The absolute value of $\mathrm{Im}(K_{dL,lp}^{(L)})$ is in general much smaller than $\mathrm{Re}(K_{dL,pl}^{(L)})$ as long as one assumes the optical potential U_{pB} to have an imaginary part which is much smaller than the real part.

Some of the examples of $K_{dL,pl}^{(L)}(R,r)$ with $U_{pA} = U_{pB}$ calculated by Eq. (3.9) are shown in Figs. 2, 3 and 4. The optical potentials were assumed to be of the form

$$U(r) = -V_0 f_{W-S}(r;r_0,a) - 4iW_0 a' \frac{d}{dr} f_{W-S}(r;r_0',a') + U_C ; \tag{3.15}$$

where

$$f_{W-S}(r;r_0,a) = \left[1 + \exp \frac{r - r_0 A^{1/3}}{a} \right]^{-1}, \tag{3.16}$$

and U_C is the Coulomb potential due to a uniformly charged sphere of radius $r_0 A^{1/3}$.

As is clear from these figures, the real part of the interaction kernel has a very long range and often changes sign as a function of r and R. The long range of the kernel is due to the fact that the deuteron wave function $\phi_d(r_{np})$ spreads over a long range of r_{np} because of the low binding energy of deuteron.

Figure 3 and the dotted lines in Fig. 4 show the contribution of V_{np} alone to $K_{dL,pl}^{(L)}(R,r)$. DWBA is equivalent to taking this as the interaction kernel and making a first order perturbation calculation. In zero range DWBA, the kernel is further approximated by a delta function, $D\delta(r_{np})\phi_{l_n}(r_n)$ where D is

the strength of the coupling. These figures show how the terms other than V_{np} give complicated features to the interaction kernel. We have called in I the part of $K_{dL,pl}^{(L)}(R,r)$ which is due to these terms the non-orthogonality term,
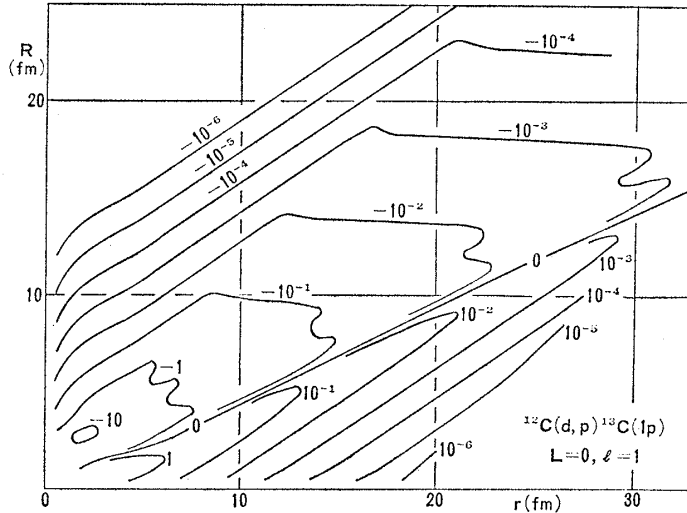

Fig. 2. The interaction kernel $K_{d0,p1}^{(0)}(R,r)$ in fm^{-2} MeV for the reaction $^{12}C(d,p)$ ^{13}C with $E_d^{(lab)}=11.8$ MeV, $Q=2.722$ MeV and $l_n=1$. The distorting and the binding potentials appearing in the kernel are as follows: $V_{np}=-72.15$ $\exp[-r^2_{np}/(1.484)^2]$, $V_{nA}=-33.92\,f_{W-S}(r_n;1.32,0.57)$ and the parameters used for U_{pB} listed in Table II. Since ^{12}C is not a closed shell nucleus in the sense of the LS-coupling, $K_{dL,pl}^{(L)}$ in Eq. (3·9) is multiplied by the factor $1/\sqrt{3}$ coming from c.f.p. (see §4).
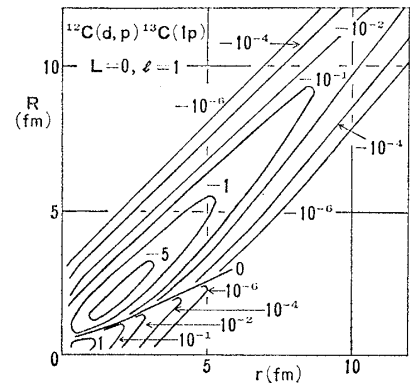
Fig. 3. The interaction kernel $K_{d0,p1}^{(0)}(R,r)$, coming from the V_{np} term in (2·20) only, in fm^{-2} MeV for the reaction $^{12}C(d,p)$ ^{13}C with $E_d^{(lab)}$ $=11.8$ MeV, $Q=2.722$ MeV and $l_n=1$. V_{np} is the same as given in Fig. 2.
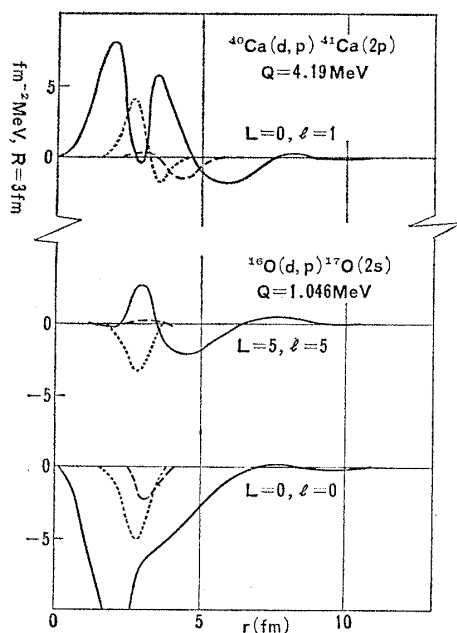

Fig. 4. The interaction kernel $K_{dL,pl}^{(L)}(R,r)$ in fm^{-2} MeV for $R=3$ fm for the reaction (i) $^{40}Ca(d,p)$ ^{41}Ca ($Q=4.19$ MeV and $l_n=1$) with $E_d^{(lab)}=11$ MeV, and $V_{nA}=-58.83\,f_{W-S}(r_n;1.20,0.65)$ and (ii) $^{16}O(d,p)^{17}O$ ($Q=1.046$ MeV and $l_n=0$) with $E_d^{(lab)}=11.8$ MeV and $V_{nA}=-51.25f_{W-S}(r_n;1.25,$ $0.58)$. V_{np} is the same as given in Fig. 2 for both cases and the parameters used for U_{pB} are listed in Table II. The real and imaginary parts of the kernel are represented by the full and dashed curve, respectively. The dotted lines shows the kernels coming up from the term V_{np} in (2·20) only.

since it is given rise to by the non-orthogonality of the deuteron and proton channels.

§ 4. Many valence nucleons and the Pauli principle

So far we have considered the (d, p) reaction in which the neutron is transferred into an empty orbit outside an inert even-even core of the target nucleus. In the general (d, p) reaction, however, the capturing orbit in the target nucleus is not necessarily empty but is already occupied by several neutrons. It then becomes necessary to consider the effect of the Pauli principle between the captured neutron and the target neutrons in the same orbit. In the ordinary DWBA calculations this effect is taken into account by the spectroscopic factor in the cross section formula.

In this section we investigate how the effect of the Pauli principle is taken into account in the coupled equations. We treat only the neutrons in the outer orbit as participating in the reaction and treat the rest of the target nucleus as an inert even-even core. We neglect the Pauli principle between the nucleons in the deuteron and those in the core. Hence, we still assume the stripping process as the sole reaction mechanism. We entirely suppress the coordinates of the nucleons in the core in the notations throughout the subsequent paragraphs in this section. We still neglect the intrinsic spin of nucleons.

Now, let us assume that the outer orbit of the target nucleus with the orbital angular momentum l_n is occupied by $N-1$ neutrons and gives rise to the spin I_A and its z-component M_A of the nucleus A. The anti-symmetrized wave function of A is written in the form

$$\Phi^A = \Phi^A (l_n^{N-1} I_A M_A | r_n^{(1)}, \cdots, r_n^{(N-1)}), \qquad (4\cdot1)$$

where $r_n^{(i)}$ is the coordinate of the i-th neutron. The captured neutron goes into the orbit l_n to form the residual nucleus B with spin I_B and its z-component M_B whose antisymmetrized wave function is written as

$$\Phi^B = \Phi^B (l_n^N I_B M_B | r_n^{(1)}, \cdots, r_n^{(N-1)}, r_n^{(N)}). \qquad (4\cdot2)$$

The Pauli principle between the captured neutron and the target neutrons in the l_n-orbit can be taken into account by anti-symmetrizing the trial wave function as

$$\Psi_{JM} = \frac{1}{\sqrt{N}} \mathcal{A} \{ \sum_L f_{L,JM}^{(d,N)} R^{(N)-1} u_L^{(J)} (R^{(N)}) \} + \sum_l f_{l,JM}^{(p)} r^{-1} v_l^{(J)} (r), \qquad (4\cdot3)$$

where

$$f_{L,JM}^{(d,N)} = [\Phi^A (l_n^{N-1} I_A | r_n^{(1)}, \cdots, r_n^{(N-1)}) \phi_d(r_{np}^{(N)}), Y_L(\hat{R}^{(N)})]_{JM}, \qquad (4\cdot4)$$

with

$$r_{np}^{(N)} = r_n^{(N)} - r_p' \qquad \text{and} \qquad R^{(N)} = (r_n^{(N)} + r_p')/2 \qquad (4\cdot5)$$

and

$$f_{l,JM}^{(p)} = [\Phi^B(l_n^{(N)}I_B|r_n^{(i)}), Y_l(\hat{r})]_{JM}, \tag{4·6}$$

and $\mathcal{A}$ is the anti-symmetrization operator with respect to the N-th neutron and other neutrons in the l_n-orbit. We assume here that the mass of the inert core is much larger than the sum of the masses of the outer neutrons so that the centre of mass of the inert core coincides with that of the target A. We therefore neglect the dependence of r_p' on the particle number, (N). Henceforth, we omit the suffices J and M to simplify notations and write $f_{L,JM}^{(d,N)}$, $f_{l,JM}^{(p)}$, $u_L^{(J)}$ and $v_l^{(J)}$ simply as $f_L^{(d,N)}$, $f_l^{(p)}$, u_L and v_l respectively.

One then gets

$$\int d\tau^{(d,N)} f_L^{(d,N)}(H-E)\Psi = 0 \tag{4·7}$$

and

$$\int d\tau^{(p)} f_l^{(p)}(H-E)\Psi = 0, \tag{4·8}$$

where

$$d\tau^{(d,N)} = d\hat{R}^{(N)} dr_{np}^{(N)} d\tau^{(A)} \tag{4·9a}$$

and

$$d\tau^{(p)} = d\hat{r} dr_n^{(N)} d\tau^{(A)}, \tag{4·9b}$$

where

$$d\tau^{(A)} = \prod_{i=1}^{N-1} dr_n^{(i)}. \tag{4·9c}$$

Let us introduce the notations A' and B' to indicate the outer nucleons in A and B, so that A consits of A' and C, the inert core, and B consists of B' and C. Let us assume that the Hamiltonian is of the form

$$H = K_p + U_{pC} + V_{pB'} + H_B, \tag{4·10a}$$

where

$$V_{pB'} = \sum_{i=1}^{N-1} V_{np}(r_{np}^{(i)}) + V_{np}(r_{np}^{(N)}), \tag{4·10b}$$

and

$$H_B = \sum_{i=1}^{N}(K_i + U_{iC}) + \sum_{i>j=1}^{N} V_{ij} + H_C, \tag{4·10c}$$

where K_p and K_i are the kinetic energies of the proton and the i-th neutron respectively, U_{pC} and U_{iC} are the interaction of the core with the outgoing proton and the i-th neutron respectively, $V_{np}(r_{np}^{(i)})$ is the interaction between the proton and the i-th neutron and V_{ij} is the interaction between the i-th and j-th neutrons.

Using Eq. (4·3) in Eqs. (4·7) and (4·8) one gets

$$\left[-a_d \left\{ \frac{d^2}{dR^2} + k_d{}^2 - \frac{L(L+1)}{R^2} \right\} + U_d{}^C(R) \right] u_L(R)$$

$$+ \sum_{L'} \left\{ U_{d,LL'}(R) u_{L'}(R) + \int U_{d,LL'}(R, R') u_{L'}(R') dR' \right\}$$

$$= -\sqrt{N} \sum_l \int_0^\infty \widetilde{K}_{Ll}^{(d)}(R, r) v_l(r) dr \qquad (4·11)$$

and

$$\left[-a_p \left\{ \frac{d^2}{dr^2} + k_p{}^2 - \frac{l(l+1)}{r^2} \right\} + U_p{}^C(r) \right] v_l(r) + \sum_{l'} U_{p,ll'}(r) v_{l'}(r)$$

$$= -\sqrt{N} \sum_L \int_0^\infty \widetilde{K}_{lL}^{(p)}(r, R) u_L(R) dR, \qquad (4·12)$$

where $\sqrt{N}$ is written out explicitly on the right-hand side of the above equations for the sake of later convenience. In Eq. (4·11) $U_d{}^C(R)$ represents the interaction between d and C and is given by

$$U_d{}^C(R^{(N)}) = \int \phi_d{}^*(r_{np}{}^{(N)}) \left[U_{pC}(r_p') + U_{nC}(r_n{}^{(N)}) \right] \phi_d(r_{np}{}^{(N)}) dr_{np}{}^{(N)}. \qquad (4·13)$$

In the same equation $U_{d,LL'}(R)$ represents the direct part of the interaction between d and A' and is given by

$$U_{d,LL'}(R^{(N)}) = \int d\tau^{(d,N)} f_L^{(d,N)*} (V_{pA'}^{(N)} + \sum_{i=1}^{N-1} V_{Ni}) f_{L'}^{(d,N)}, \qquad (4·14)$$

where $V_{pA'}^{(N)} = \sum_{i=1}^{N-1} V_{np}(r_{np}{}^{(i)})$. The non-local operator $U_{d,LL'}(R, R')$, which appears in the same equation is the exchange part of the same interaction as $U_{d,LL'}(R)$ and is given by

$$\int_0^\infty U_{d,LL'}(R^{(N)}, R') u_{L'}(R') dR'$$

$$= R^{(N)} \int d\tau^{(d,N)} f_L^{(d,N)*} (H-E) (\mathcal{A}-1) (f_{L'}^{(d,N)} R^{(N)-1} u_{L'}(R^{(N)})). \qquad (4·15)$$

In Eq. (4·12), $U_{p,ll'}(r)$ is the interaction between p and B' and is given by

$$U_{p,ll'}(r) = \int d\tau^{(p)} f_l^{(p)*} V_{pB'} f_{l'}^{(p)}. \qquad (4·16)$$

One sees from Eq. (4·14) through Eq. (4·16) that the interaction between two complex particles or between a complex particle and a nucleon is spherically non-symmetrical and non-local. The interaction kernels $\widetilde{K}_{lL}^{(d)}$ and $\widetilde{K}_{lL}^{(p)}$ are given by

$$\int \widetilde{K}_{Ll}^{(d)}(R^{(N)}, r) v_l(r) dr = R^{(N)} \int d\tau^{(d,N)} f_L^{(d,N)*} (H-E) f_l^{(p)} r^{-1} v_l(r) \qquad (4·17)$$

and

$$\int \widetilde{K}_{lL}^{(p)}(r, R^{(N)}) u_L(R^{(N)}) dR^{(N)} = r \int d\tau^{(p)} f_l^{(p)*} (H-E) f_L^{(d,N)} R^{(N)-1} u_L(R^{(N)}),$$

(4·18)

where use has been made of $[\mathcal{A}, H] = 0$ and $\mathcal{A}\Phi^B = N\Phi^B$. Explicit form of the interaction kernels can be obtained by the procedure described in § 2 and can be shown to be symmetric,

$$\widetilde{K}_{Ll}^{(d)}(R, r) = \widetilde{K}_{lL}^{(p)}(r, R).$$

Next, let us investigate the kernels in Eqs. (4·17) and (4·18) further. If one expands Φ^B using c.f.p., one has for $f_l^{(p)}$ the expression

$$f_l^{(p)} = \sum_{I_{A'}} \langle l_n^{N-1}(I_{A'}) l_n | \} l_n^N I_B \rangle f_{l,I_{A'}}^{(p,N)},$$

(4·19)

where

$$f_{l,I_{A'}}^{(p,N)} = [[\Phi^A(l_n^{N-1} I_{A'} | r_n^{(1)} \cdots r_n^{(N-1)}), \phi_{l_n}(r_n^{(N)})]_{I_B}, Y_l(\hat{r})]_{JM},$$

(4·20)

where $I_{A'}$ stands not only for the angular momentum but also for the quantum numbers necessary for specifying the state of the $N-1$ particle system. Using Eq. (4·19) in Eq. (4·17) one obtains

$$\widetilde{K}_{Ll}^{(d)}(R, r) = \langle l_n^{N-1}(I_A) l_n | \} l_n^N I_B \rangle \widetilde{K}_{Ll, I_A}^{(d),1}(R, r) + \sum_{I_{A'} \neq I_A} \langle l^{N-1}(I_{A'}) l_n | \} l_n^N I_B \rangle \widetilde{K}_{Ll, I_{A'}}^{(d),2}(R, r),$$

(4·21)

where

$$\int \widetilde{K}_{Ll, I_A}^{(d),1}(R^{(N)}, r) v_l(r) dr$$

$$= R^{(N)} \int d\tau^{(d,N)} f_L^{(d,N)*} (K_p + U_p^C + V_{pA'}^{(N)} - E_p + V_{np}(r_{np}^{(N)})) f_{l,I_A}^{(p,N)} r^{-1} v_l(r)$$

(4·22)

and

$$\int \widetilde{K}_{Ll, I_{A'}}^{(d),2}(R^{(N)}, r) v_l(r) dr$$

$$= R^{(N)} \int d\tau^{(d,N)} f_L^{(d,N)*} V_{pA'}^{(N)} f_{l,I_{A'}}^{(p,N)} r^{-1} v_l(r) \qquad \text{for } I_{A'} \neq I_A.$$

(4·23)

Comparing Eq. (4·22) with Eq. (2·20) one sees that $\widetilde{K}_{Ll, I_A}^{(d),1}$ is just a generalization of $K_{dL, pl}^{(L)}$ to the case where the target nucleus has a finite spin.

The kernel $K_{Ll, I_{A'}}^{(d),2}$, on the other hand, corresponds to the rearrangement process due to $V_{pA'}$ in which the target nucleus is brought into an excited state with the spin $I_{A'}$ while the transferred neutron plays a role of a spectator.

If the terms with $\widetilde{K}_{Ll, A'}^{(d),2}$ can be neglected in $\widetilde{K}_{Ll}^{(d)}$ the latter becomes just the kernel obtained in the previous sections, provided $U_p^C + V_{pA'}^{(N)}$ is approximated

by the optical potential U_{pB}. Then the Pauli principle is expressed by the factor,

$$\sqrt{N}\,(l_n{}^{N-1}(I_A)I_B|\}l_n{}^{N}I_B),\tag{4·24}$$

multiplying the interaction kernels. Such is the case, for instance, when $(l_n{}^{N-1}(I_A)I_B|\}l_n{}^{N}I_B)=1$. If $\widetilde{K}_{LL',I_{A'}}^{(d);2}$ cannot be neglected, the interaction kernels take on complicated expressions containing terms with $(l_n{}^{N-1}(I_{A'})I_B|\}l_n{}^{N}I_B)$.

§ 5. Method of numerical calculations

In this section we describe the method of numerical calculation of the scattering amplitude. We describe in § 5.1 the principle of the method and in § 5.2 the numerical method of solution. In § 5.3 the effect of the singularity of Green's function at the origin is discussed and a cutoff procedure to avoid the difficulty due to such a singularity is described.

5.1 *The principle of the method*

Let us use the notations α, $\beta \cdots$ introduced in § 2 for the channel suffices. The basic equations (2·18) and (2·19) then read

$$(E_\beta - H_\beta(r_\beta))u_{\beta\alpha}(r_\beta) = \sum_\gamma \int K_{\beta\gamma}(r_\beta, r_\gamma)u_{\gamma\alpha}(r_\gamma)dr_\gamma,\tag{5·1}$$

where $u_{\beta\alpha}$ is the radial wave function in the channel β when the incident wave is in the channel α and

$$H_\beta(r_\beta) = -a_\beta \frac{d^2}{dr_\beta{}^2} + a_\beta \frac{l_\beta(l_\beta+1)}{r_\beta{}^2} + U_\beta(r_\beta).\tag{5·2}$$

The asymptotic form of $u_{\alpha\beta}(r_\beta)$ for large r_β, which we denote by R_β, is given by Eq. (2·5) as

$$u_{\beta\alpha}(R_\beta) = \zeta_\beta(F_\beta(R_\beta)\delta_{\beta\alpha} + T_{\beta\alpha}H_\beta{}^{(+)}(R_\beta)),\tag{5·3}$$

which defines the scattering amplitude $T_{\beta\alpha}$.

Now the form of Eq. (5·1) is independent of α. Therefore, the wave functions corresponding to the different α's are actually the linearly independent solutions of the same set of equations satisfying different sets of boundary condition, Eq. (5·3). If f is the number of coupled channels there are f such linearly independent sets of solutions.

The principle of the present method is to solve Eq. (5·1) first under f appropriately chosen sets of boundary condition to get f linearly independent solutions $\bar{u}_{\beta i}(i=1\sim f)$. The choice of boundary condition is made so as to facilitate the subsequent numerical works. The wave functions u are then obtained from the $\bar{u}$'s by a linear transformation

$$u_{\beta\alpha} = \sum_i \bar{u}_{\beta i}y_{i\alpha},\tag{5·4}$$

where the transformation matrix $y = [y_{i\alpha}]$ is determined by the condition that

the u's satisfy boundary condition (5·3) which leads to

$$k_\beta \sum_i \bar{u}_{\beta i}(R_\beta) y_{i\alpha} = \rho_\beta [F_\beta(R_\beta) \delta_{\beta\alpha} + T_{\beta\alpha} H_\beta^{(+)}(R_\beta)] \tag{5·5}$$

and

$$k_\beta \sum_i \bar{u}'_{\beta i}(R_\beta) y_{i\alpha} = \rho_\beta [F_\beta'(R_\beta) \delta_{\beta\alpha} + T_{\beta\alpha} H_\beta^{(+)'}(R_\beta)], \tag{5·6}$$

where

$$\rho_\beta = (\mu_\beta k_\beta \hbar^{-1})^{1/2} i^{l_\beta} e^{i\sigma_\beta} \tag{5·7}$$

and $f_\beta' = df_\beta/dr_\beta$. Using the Wronskian relation

$$W[F_\beta, H_\beta^{(+)}] \equiv F_\beta H_\beta^{(+)'} - H_\beta^{(+)} F_\beta' = k_\beta \,,$$

one obtains from Eqs. (5·5) and (5·6)

$$\rho_\beta T_{\beta\alpha} = \sum_i W[F_\beta, \bar{u}_{\beta i}] y_{i\alpha} \tag{5·8}$$

and

$$\delta_{\beta\alpha} \rho_\beta = \sum_i W[H_\beta^{(+)}, \bar{u}_{\beta i}] y_{i\alpha} \tag{5·9}$$

for β running over all the coupled channels.

Equation (5·9) is a linear simultaneous equation for $y_{i\alpha}$ for a given α, from which $y = [y_{i\alpha}]$ is determined. The scattering amplitude $T_{\beta\alpha}$ is then determined from Eq. (5·8) using the $y_{i\alpha}$'s and the S-matrix element, $S_{\beta\alpha}$, is obtained from $T_{\beta\alpha}$ using Eq. (2·6).

5.2 *Numerical solution*

In the actual calculation we choose the following boundary condition for the u's:

$$\bar{u}_{\beta\alpha}(0) = 0 \tag{5·10}$$

and

$$\bar{u}_{\beta\alpha}(\varDelta r_\beta) = c_\beta \delta_{\beta\alpha} \tag{5·11}$$

for all α and β where c_β is a constant. The values of $\varDelta r_\beta$ and c_β are determined according to computational convenience.

For the numerical solution of the equation

$$(E_\beta - H_\beta(r_\beta)) \bar{u}_{\beta\alpha}(r_\beta) = \sum_\gamma \int K_{\beta\gamma}(r_\beta, r_\gamma) \bar{u}_{\gamma\alpha}(r_\gamma) dr_\gamma \,, \tag{5·12}$$

the (r, R) space is divided into mesh. The differential operator contained in H_β is approximated by an appropriate difference operator and the integral is approximated by a weighted sum using appropriate formulae of numerical differentiation and integration respectively. Equation (5·12) is then approximated

by a set of linear simultaneous equations

$$\sum_{j=1}^{N_\beta} A_\beta(i,j)\bar{u}_\beta(j) + \sum_\gamma \sum_{j=1}^{N_\gamma} C_{\beta\gamma}(i,j)\bar{u}_\gamma(j) = 0 , \tag{5·13}$$

where $\bar{u}_\beta(j)$ is the value of $\bar{u}_\beta(r_\beta)$ at the j-th mesh point, with $j=1$ corresponding to the origin, and N_β is the number of mesh points on the r_β-axis. Since the form of Eq. (5·12) is independent of α, the subscript α is dropped in Eq. (5·13). The coefficients $A_\beta(i,j)$ and $C_{\beta\gamma}(i,j)$ correspond to $(E_\beta - H_\beta)$ and $K_{\beta\gamma}$ in Eq. (5·12) respectively and their explicit forms depend on the choice of the difference operator and the integration formula which one adopts in the approximation as are shown in the Appendix.

Now, let us take $\varDelta r_\beta$ as the first mesh point from the origin and $c_\beta = j_{l_\beta}(k_\beta \varDelta r_\beta)$, where j_l is the spherical Bessel function. Equations (5·10) and (5·11) then become

$$\bar{u}_{\beta\alpha}(1) = 0 \tag{5·14}$$

and

$$\bar{u}_{\beta\alpha}(2) = \delta_{\beta\alpha} \cdot c_\beta , \tag{5·15}$$

respectively, and Eq. (5·13) can be written as

$$\sum_{j=3}^{N_\beta} A_\beta(i,j)\bar{u}_\beta(j) + \sum_\gamma \sum_{j=3}^{N_\gamma} C_{\beta\gamma}(i,j)\bar{u}_\gamma(j) = U_\beta(i), \quad (i=2 \sim (N_\beta-1)) \tag{5·16}$$

where

$$U_\beta(i) = -\sum_{j=1,2} \left\{ A_\beta(i,j)\bar{u}_\beta(j) + \sum_\gamma C_{\beta\gamma}(i,j)\bar{u}_\gamma(j) \right\}. \tag{5·17}$$

Equation (5·16) with Eq. (5·17), which gives $U_\beta(i)$ because of Eqs. (5·14) and (5·15), is a set of linear inhomogeneous simultaneous equations for $\bar{u}_\beta(j)$ $(3 \leq j \leq N_\beta)$ which may be rewritten in a matrix form as

$$A_\beta \bar{u}_\beta + \sum_\gamma C_{\beta\gamma} \bar{u}_\gamma = U_\beta , \tag{5·18}$$

where $(A_\beta)_{ij} = A_\beta(i+1,j+2)$, $(C_{\beta\gamma})_{ij} = C_{\beta\gamma}(i+1,j+2)$, $(\bar{u})_j = \bar{u}(j+2)$ and $(U_\beta)_i = U_\beta(i+1)$.

Now we notice that $K_{\beta\gamma}$ is zero if $\beta=\gamma$ and if both β and γ refer to the proton channel. Also, for a given total angular momentum J there is only one deuteron channel present. Hence, if one uses d, p, p', etc., in place of α, β, γ, etc., Eq. (5·18) can be written in the form

$$A_d \bar{u}_d + \sum_p C_{dp} \bar{u}_p = U_d \tag{5·19}$$

and

$$A_p \bar{u}_p + C_{pd} \bar{u}_d = U_p . \tag{5·20}$$

The equation for $\bar{u}_d$ is therefore

$$\left(A_d - \sum_p C_{dp} A_p^{-1} C_{pd}\right) \bar{u}_d = U_d - \sum_p C_{dp} A_p^{-1} \bar{u}_p . \qquad (5\cdot21)$$

Carrying out a matrix inversion $(A_d - \sum_p C_{dp} A_p^{-1} C_{pd})^{-1}$, one can get the required solution $\bar{u}_d$. Equation $(5\cdot20)$ then gives $\bar{u}_p$.

Now, there is a difficulty in carrying out the procedure described above in the actual numerical calculation because of the singularity of H_p at the origin due to the centrifugal potential $l_p(l_p+1)/r_p^2$. This singularity makes $(A_p)_{ij}$ extremely large near the origin, i.e. for small i, especially for large l_p. This gives rise to a difficulty in evaluating $(A_d - \sum C_{dp} A_p^{-1} C_{pd})^{-1}$ because of the presence of A_p^{-1}. The situation is analogous to the difficulty encountered in numerically evaluating the Green function $(E_p + i\varepsilon - H_p)^{-1}$ near the origin because of the singularity of the irregular solution for H_p at the origin.

The difficulty described above can be avoided by cutting off the region of small r_p from the domain of integration of K_{pd} which corresponds to cutting off small i and j from the summation containing $C_{pd}(i,j)$ and $C_{dp}(i,j)$ in the numerical calculation. We shall describe such a cutoff procedure in more detail in the next subsection.

5.3 *The cutoff procedure*

Let us consider the right-hand side of the basic equation $(5\cdot1)$.

$$F_{\beta\alpha}(r_\beta) = \sum_\gamma \int_0^\infty dr_\gamma K_{\beta\gamma}(r_\beta, r_\gamma) u_{\gamma\alpha}^{(+)}(r_\gamma) . \qquad (5\cdot22)$$

The radial wave function $u_{\gamma\alpha}^{(+)}(r_\gamma)$ is zero at the origin. The value of $u_{\gamma\alpha}^{(+)}(r_\gamma)$ for small values of r_γ depends presumably on the angular momentum l_γ of the channel such that the larger l_γ is, the smaller is the value of $u_{\gamma\alpha}^{(+)}(r_\gamma)$ for $r_\gamma \sim 0$ because of the higher centrifugal barrier present in the channel. Further, $K_{\beta\gamma}(r_\beta, r_\gamma)$ is zero if r_β and/or $r_\gamma = 0$ because of the factor $r_\beta r_\gamma$ as is seen in Eq. $(3\cdot9)$. One expects, therefore, that one can safely cut off from the region of integration in Eq. $(5\cdot22)$ the region near the origin, say $(0, \hat{r}_\gamma)$.

Now, $K_{\beta\gamma}(r_\beta, r_\gamma)$ is appreciable only within the region in which $|r_\beta - r_\gamma| < \sigma$ where σ is a constant. If the non-orthogonality term is neglected σ is essentially the range of V_{np}. The smallness of σ enables one to neglect $K_{\beta\gamma}(r_\beta, r_\gamma)$ for small values of r_β, say $r_\beta < \tilde{r}_{\beta\gamma}$, since for such values r_β the values r_γ in the region $|r_\beta - r_\gamma| < \sigma$ nearly always fall into the region $r_\gamma \leq \hat{r}_\gamma$. If one denotes the smallest of all $\tilde{r}_{\beta\gamma}$'s by $\tilde{r}_\beta$, $F_{\beta\alpha}(r_\beta)$ is then negligible for $r_\beta < \tilde{r}_\beta$.

The above considerations show that one can cut off small values of r_β and r_γ's in Eq. $(5\cdot22)$. For practical calculation it is convenient to take a symmetric cutoff in the sense that the cutoff radius for a channel is the same whether the coordinate corresponding to that channel appears as the first or the second argument of the kernel. Let us denote the cutoff radius by attaching a bar. Then,

our approximation consists in using for $F_{\beta\alpha}$ the approximate one, $\widehat{F}_{\beta\alpha}$, defined by

$$\widehat{F}_{\beta\alpha}(r_\beta) = \sum_\gamma \int_0^\infty dr_\gamma \widehat{K}_{\beta\gamma}(r_\beta, r_\gamma) u_{\gamma\alpha}^{(+)}(r_\gamma), \tag{5.23}$$

where

$$\widehat{K}_{\beta\gamma}(r_\beta, r_\gamma) = \theta(r_\beta - \bar{r}_\beta) K_{\beta\gamma}(r_\beta, r_\gamma) \theta(r_\gamma - \bar{r}_\gamma), \tag{5.24}$$

with

$$\theta(x) = \begin{cases} 0 & \text{if } x < 0, \\ 1 & \text{if } x \geq 0. \end{cases} \tag{5.25}$$

The actual values of the cutoff radii are empirically determined in the course of calculation. The ensuing error in the calculated S-matrix can be estimated by the method described later. For practical purpose it is convenient to take the cutoff radius as large as possible under the condition that the error in the S-matrix be smaller than certain limit.

That the kernel has a finite, small range is important for the validity of the assumption that u's are small in the neighbourhood of the origin. In fact, if the kernel had an infinite range, $F_{\beta\alpha}(r_\beta)$ could be large even for small values of r_γ since the integration extends over whole space of r_γ. Equation (5·1) then shows that $u_{\beta\alpha}$ operated on by the operator $(E_\beta - H_\beta)$ could be large. This may invalidate the assumption that $u_{\beta\alpha}$ be small for small r_β, except for very special cases.

Let us now estimate the error introduced by the cutoff procedure. The cutoff is equivalent to approximating the integration kernel $K_{\beta\gamma}$ by $\widehat{K}_{\beta\gamma}$ which is equal to the right-hand side of Eq. (5·24). Let us denote the approximate wave function corresponding to this approximation by $\hat{u}_{\gamma\alpha}$ and the S-matrix element by $\widehat{S}_{\beta\alpha}$. Then the error is given by

$$\varDelta S_{\beta\alpha} = S_{\beta\alpha} - \widehat{S}_{\beta\alpha}. \tag{5.26}$$

From Eq. (5·1) one has the expression for the form of the S-matrix elements,

$$S_{\beta\alpha} = S_\beta^{(0)} \delta_{\beta\alpha} - \frac{4i}{\hbar} (-)^{l_\beta+1} \langle f_\beta^{(-)} | \sum_\gamma K_{\beta\gamma} u_{\gamma\alpha}^{(+)} \rangle, \tag{5.27}$$

where $f_\beta^{(-)}$ is the regular solution of homogeneous equation corresponding to Eq. (5·1) with the asymptotic form of the incident plus incoming scattering wave. Hence, $\varDelta S_{\beta\alpha}$ is written as

$$\varDelta S_{\beta\alpha} = \frac{4i}{\hbar} (-)^{l_\beta+1} \langle f_\beta^{(-)} | (\sum_\gamma K_{\beta\gamma} u_{\gamma\alpha}^{(+)} - \sum_\gamma \widehat{K}_{\beta\gamma} \hat{u}_{\gamma\alpha}^{(+)}) \rangle. \tag{5.28}$$

Approximating $u_{\gamma\alpha}^{(+)}$ in the above equation by $\hat{u}_{\gamma\alpha}^{(+)}$, one obtains

Table I. The relative error of $S_{pl,dL}^{(L)}$ due to the cutoff procedure for $^{40}Ca(d,p)^{41}Ca(1f)$.

L	$\bar{R}_{dL}$(fm)	l	$\bar{r}_{pl}$ (fm)	$|S_{pl,dL}^{(L)}|$	$|\Delta S_{pl,dL}^{(L)}/S_{pl,dL}^{(L)}|$
4	1.4	1	0.6	0.120	9.1×10^{-3}
		3	1.2	0.107	6.0×10^{-5}
		5	1.8	0.0738	1.10×10^{-4}
		7	2.4	0.0286	3.3×10^{-6}
6	2.0	3	1.2	0.0838	4.0×10^{-3}
		5	1.8	0.0662	3.6×10^{-5}
		7	2.4	0.0297	6.9×10^{-6}
		9	3.0	0.00920	2.3×10^{-6}
8	3.8	5	2.6	0.0281	7.5×10^{-3}
		7	3.4	0.0156	2.01×10^{-4}
		9	4.4	0.00611	4.0×10^{-5}
		11	5.2	0.00208	2.7×10^{-5}
10	5.8	7	4.2	0.00652	1.93×10^{-3}
		9	5.4	0.00308	2.02×10^{-4}
		11	6.4	0.00118	7.8×10^{-5}
		13	7.6	0.000403	2.4×10^{-4}

$$\Delta S_{\beta\alpha} \simeq \frac{4i}{\hbar}(-)^{l_\beta+1}\langle f_\beta^{(-)}| \sum_\tau (K_{\beta\tau}-\hat{K}_{\beta\tau})\hat{u}_{\tau\alpha}^{(+)}\rangle . \tag{5.29}$$

Another estimate of $\Delta S_{\beta\alpha}$ is obtained by using

$$f_\beta^{(-)} = \hat{u}_{\beta\beta}^{(-)} - \sum_\tau G_\beta{}^* \hat{K}_{\beta\tau}^* \hat{u}_{\tau\alpha}^{(-)} ,$$

on the right-hand side of Eq. (5·28) where G_β is the outgoing Green function for channel β. Then Eq. (5·28) becomes

$$\frac{4i}{\hbar}(-)^{l_\beta+1}\sum_\tau \{\langle \hat{u}_{\beta\beta}^{(-)}| (K_{\beta\tau}u_{\tau\alpha}^{(+)}-\hat{K}_{\beta\tau}\hat{u}_{\tau\alpha}^{(+)})\rangle$$

$$-\langle G_\beta{}^*\hat{K}_{\beta\tau}^*\hat{u}_{\tau\alpha}^{(-)}| (\sum_{\tau'} K_{\beta\tau'}u_{\tau'\alpha}^{(+)} - \sum_{\tau'}\hat{K}_{\beta\tau'}\hat{u}_{\tau'\alpha}^{(+)})\rangle\}. \tag{5.30}$$

The second term in the curly bracket of Eq. (5·30) can be written as

$$\langle \hat{K}_{\beta\tau}^*\hat{u}_{\tau\alpha}^{(-)}| u_{\beta\alpha}^{(+)}-\hat{u}_{\beta\alpha}^{(+)}\rangle$$

because of the symmetry of Green's function G_β. Hence,

$$\Delta S_{\beta\alpha} = \frac{4i}{\hbar}(-)^{l_\beta+1}\sum_\tau \{\langle \hat{u}_{\beta\beta}^{(-)}| (K_{\beta\tau}u_{\tau\alpha}^{(+)}-\hat{K}_{\beta\tau}\hat{u}_{\tau\alpha}^{(+)})\rangle - \langle \hat{K}_{\beta\tau}^*\hat{u}_{\tau\alpha}^{(-)}| u_{\beta\alpha}^{(+)}-\hat{u}_{\beta\alpha}^{(+)}\rangle\}.$$

$$\tag{5.31}$$

If one approximates in this expression as $u_{\tau\alpha}^{(+)} \simeq \hat{u}_{\tau\alpha}^{(+)}$ and $u_{\beta\alpha}^{(+)} \simeq \hat{u}_{\beta\alpha}^{(+)}$ one obtains

$$\Delta S_{\beta\alpha} \simeq \frac{4i}{\hbar} (-)^{l_\beta+1} \langle \hat{u}_{\beta\beta}^{(-)} | \sum_{\gamma} (K_{\beta\gamma} - \hat{K}_{\beta\gamma}) \hat{u}_{\gamma\alpha}^{(+)} \rangle \,. \tag{5.32}$$

In Table I examples of cutoff and the estimate of the ensuing error using Eq. (5·32) are shown for the case of ^{40}Ca$(d, p)^{41}$Ca transition to the $1f$ state of ^{41}Ca with the Q-value 6.14 MeV at the incident energy 11.0 MeV (lab.). The values of parameters used in the calculation are listed in Table II. The calculated $|S_{pl, dL}^{(L)}|$ and the estimated relative error $|\Delta S_{pl, dL}^{(L)}/S_{pl, dL}^{(L)}|$ are shown in Table I for the deuteron angular momentum $L=4$, 6, 8 and 10 and the corresponding proton angular momentum l. It is seen that the relative error is the largest for the smallest l for each L and is of the order of 10^{-2} or less. For the other combinations of L and l the error is of the order of 10^{-4} or less. These results are sufficient to justify the cutoff procedure employed here. In actual calculations the cutoff radii are determined by the condition that the calculated S-matrix element is insensitive to the small variation of the radii in the neighbourhood of the chosen values and the estimated error is small.

§ 6. Numerical examples

In this section, we give some examples of the numerical results obtained for ^{40}Ca$(d, p)^{41}$Ca$(1f)$, ^{40}Ca$(d, p)^{41}$Ca$(2p)$, ^{16}O$(d, p)^{17}$O$(2s)$ and ^{12}C$(d, p)^{13}$C$(1p)$. For simplicity, we neglected the non-orthogonality term and $U_{pA} - U_{pB}$ and kept only V_{np} in the interaction kernel. The kernel used is, therefore, that which is

Table II. Parameters for potential used in the calculations (Energy is in MeV; length in fm).

nucleus	E(c.m.)	V_0	r_0	a	W_0	r_0'	a'	r_{0C}
				for deuteron				
^{12}C[a]	10.11	118.0	0.886	0.907	5.80	1.57	0.777	1.30
^{16}O[b]	10.49	118.0	0.934	0.792	5.95	1.58	0.777	1.30
^{40}Ca[d]	10.48	120.7	0.966	0.846	16.4	1.48	0.492	1.30
				for proton				
^{13}C[a]	12.83	45.0	1.32	0.57	11.0	1.32	0.345	1.30
^{17}O[c]	11.54	49.3	1.25	0.58	11.1	1.25	0.143	1.25
^{41}Ca[d]	14.67	51.7	1.20	0.65	11.0	1.25	0.470	1.25
^{41}Ca[d]	16.62	52.7	1.20	0.65	11.0	1.25	0.470	1.25
				for neutron				
^{12}C[a]	-4.95	33.93	1.32	0.57				
^{16}O	-3.27	51.25	1.25	0.58				
^{40}Ca[d]	-6.42	58.83	1.20	0.65				
^{40}Ca[d]	-8.37	60.29	1.20	0.65				

a) Reference 8).
b) Type I in reference 9).
c) Based on Table III of reference 10).
d) Reference 11).
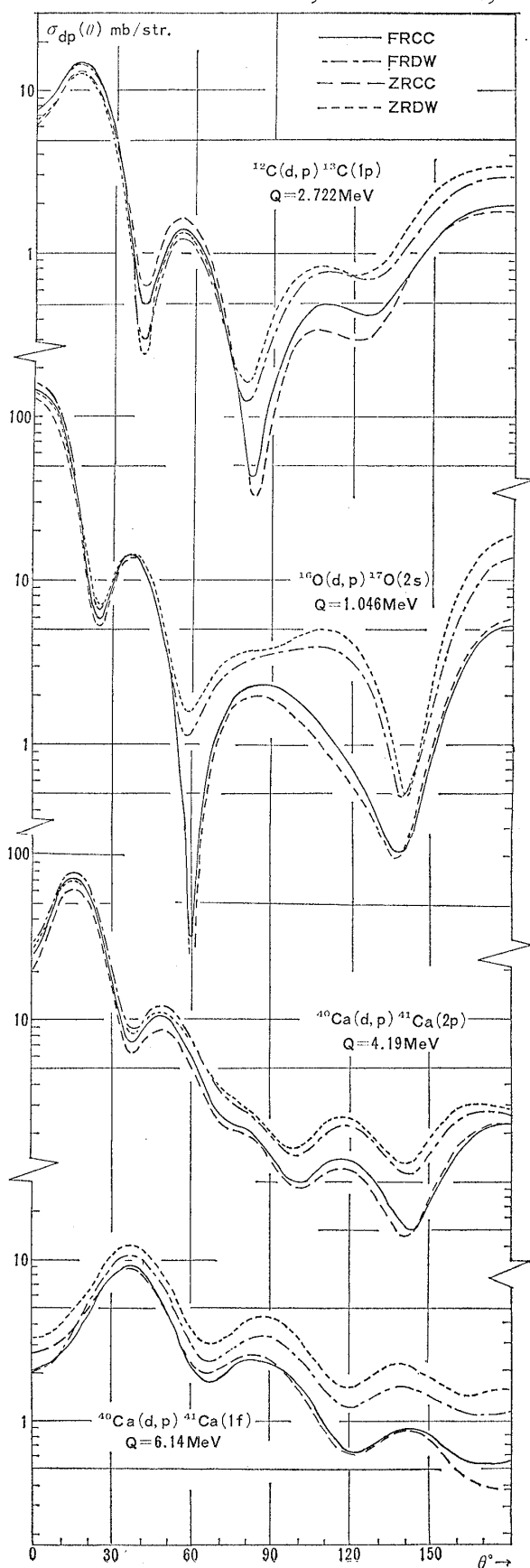
used in the ordinary DWBA calculation. The difference between the coupled channel (CC) calculation and the corresponding DWBA calculation is due entirely to the higher order terms in V_{np} in the CC calculation.

For V_{np}, a potential of a Gaussian form with a range of 1.484 fm and the depth of 72.15 MeV was used. Zero-range approximation with the strength $D = 124.7$ MeV fm$^{3/2}$ was also used for comparison in both DWBA and CC calculations.

The results of the calculation are shown in Fig. 5. The values of the parameters used in the calculation are listed in Table II. It is apparent from these figures that the difference between the results of the CC and DWBA calculations is much larger than that between the finite-range and zero-range DWBA calculations. In other words the zero-range approximation in the frame work of DWBA gives rise to an error which is less than the error inherent in DWBA itself caused by neglect of higher order terms in V_{np}.

The deviation of DWBA cross section from CC cross section is large in backward angles. This indicates that the effect of the strong

Fig. 5. Angular distributions of (d, p) reactions $^{12}C(d, p)^{13}C(1p)$, $^{16}O(d, p)^{17}O(2s)$, $^{40}Ca(d, p)^{41}Ca(2p)$ and $^{40}Ca(d, p)^{41}Ca(1f)$ calculated by the method of FRCC, FRDW, ZRCC and ZRDW. The incident energies in laboratory frame are 11.8 MeV for ^{12}C and ^{16}O and 11.0 MeV for ^{40}Ca. The parameters of the distorting potentials are listed in Table II,

coupling of channels is large for low angular momentum states of L affecting the lower partial waves of the outgoing particle p. These as well as other features of the CC calculations will be discussed in detail in a future publication.

§ 7. Summary and conclusion

The set of close coupling equations for the (d, p) reaction derived in I by the variational method is inconvenient for numerical calculations since the interaction kernel includes a differential operator acting on the unknown function. In the present paper the equations were rewritten in the forms in which the differential operator in the kernels is eliminated and the kernels are explicitly symmetric with respect to channels. The S-matrix was then shown to be symmetric despite the fact that the effective Hamiltonian is not hermitian because of the imaginary part in the optical potentials.

The explicit form of the kernel, $K_{dL,pl}^{(L)}(R, r)$, was given for the (d, p) reaction with a closed shell target nucleus. The closed shell was treated as an inert core and the intrinsic spin of nucleons was neglected. $K_{dL,pl}^{(L)}(R, r)$ was shown to have the following properties.

(1) $K_{dL,pl}^{(L)}(R, r) \propto R^{L+1}$ for small $R(\ll r)$ and $K_{dL,pl}^{(L)}(R, r) \propto r^{l+1}$ for small $r(\ll R)$,
(2) $K_{dL,pl}^{(L)}(R, r)$ has cusps on the lines $r_n = 0$ and $r_{np} = 0$ on the (R, r)-plane and
(3) $K_{dL,pl}^{(L)}(R, r)$ tends exponentially to zero for R and/or r tending to infinity.
The kernel consists of two parts: the interaction term containing $V_{np} + U_{pA} - U_{pB}$ and the non-orthogonality term which arises from the non-orthogonality of the d- and p-channels. The numerical examples given in the present paper showed that the interaction term behaves well while the non-orthogonality term has a very long range and sometimes oscillates rapidly which may bring serious trouble in the numerical calculation.

The discussion was extended to the case in which there are some valence neutrons in the orbit l_n into which the neutron is captured in the (d, p) stripping process. The Pauli principle between the target and captured neutrons was taken into account by anti-symmetrizing the trial wave function with respect to the neutrons. The interaction kernel was then shown to consist of two parts. The one part is just the interaction kernel discussed previously multiplied by the square root of the spectroscopic factor, $\sqrt{N}(l_n^{N-1}(I_A)I_B|\}l_n^{N}I_B)$. The other part of the kernel corresponds to the rearrangement process due to V_{pA} in which A is excited by p to a state A^* while the transferred neutron plays the role of a spectator. In the actual calculation on ^{12}C the latter part of the kernel was entirely neglected.

A method of numerical solution of the basic equation was described in detail and some difficulties encountered in the calculations were pointed out. It was shown that the difficulty may be circumvented by a cutoff procedure which consists in cutting off the values of the kernel for small values of the arguments.

The error introduced into the S-matrix by such a procedure was estimated and some numerical examples were given.

Results of some actual calculations were presented, by way of examples, for ^{40}Ca (d, p) ^{41}Ca with $l_n = 1$ and 3, ^{16}O (d, p) ^{17}O with $l_n = 0$ and for ^{12}C (d, p) ^{13}C with $l_n = 1$. Calculations were carried out by means of (a) the finite-range coupled channel method (FRCC), (b) the zero-range coupled channel method (ZRCC), (c) finite-range DWBA (FRDW) and (d) zero-range DWBA (ZRDW). It was found that the difference between DWBA and CC is larger than that between FRCC and ZRCC or FRDW and ZRDW. Features of the angular distributions in the forward angles in the four types of calculations were shown to be very close to one another. In the backward angles the values of the cross sections by different methods can be considerably different.

Acknowledgements

Appendix

Let us put Eq. (5·12) in the form

$$\left\{\frac{d^2}{dr_\beta^2} + P_\beta(r_\beta)\right\} \bar{u}_{\beta\alpha}(r_\beta) - \frac{1}{a_\beta} \sum_\gamma \int_0^\infty K_{\beta\gamma}(r_\beta, r_\gamma) \bar{u}_{\gamma\alpha}(r_\beta) dr_\gamma = 0, \qquad \text{(A·1)}$$

where

$$P_\beta(r_\beta) = k_\beta^2 - \frac{l_\beta(l_\beta + 1)}{r_\beta^2} - \frac{U_\beta(r_\beta)}{a_\beta}. \qquad \text{(A·2)}$$

For a sufficiently large value of r_β, say R_β, non-Coulomb part of $U_\beta(R_\beta)$ and $K_{\beta\gamma}(R_\beta, r_\gamma)$ for all γ are negligibly small. Let us divide the range $(0, R_\beta)$ of r_β into $N_\beta - 1$ equal parts of length h_β and denote the value of r_β and $f_\beta(r_\beta)$ which correspond to the i-th dividing point by $r_\beta(i)$ and $f_\beta(i)$ respectively.

Then, we first approximate the second derivative

$$\frac{d^2}{dr_\beta^2} \bar{u}_{\beta\alpha}(r_\beta) \qquad \text{by} \qquad \frac{d^2}{dr_\beta^2} \bar{u}_{\beta\alpha}(r_\beta) \simeq \frac{1}{h_\beta^2}\left(1 - \frac{1}{12}\delta_\beta^2\right)\delta_\beta^2 \bar{u}_{\beta\alpha}(i),$$

where δ_β is the central difference operator by which $\delta_\beta^2 \bar{u}_{\beta\alpha}(i)$ is given by

$$\delta_\beta^2 \bar{u}_{\beta\alpha}(i) = \bar{u}_{\beta\alpha}(i+1) - 2\bar{u}_{\beta\alpha}(i) + \bar{u}_{\beta\alpha}(i-1). \qquad \text{(A·3)}$$

Secondly, we approximate the integral, $\int K_{\beta\gamma}(r_\beta, r_\gamma) \bar{u}_{\gamma\alpha}(r_\gamma) dr_\gamma$ by a sum,

$$\int K_{\beta\gamma}(r_\beta, r_\gamma) \bar{u}_{\gamma\alpha}(r_\gamma) dr_\gamma \sim \sum_{j=1}^{N_\gamma} w_j h_\gamma K_{\beta\gamma}(i,j) \bar{u}_{\gamma\alpha}(j), \qquad (A\cdot4)$$

where w_j is the weight coefficient depending on the numerical integration formula we take. In actual calculations we used Newton-Cotes' formula of the forth order.

Approximating Eq. (A·1) by the procedure described above and then operating $h_\beta^2(1+1/12\cdot\delta_\beta^2)$ on both sides, one gets for $i=2, 3\cdots, N_\beta-1$ up to δ_β^6

$$\sum_{j=1}^{N_\beta} A_\beta(i,j) \bar{u}_{\beta\alpha}(j) + \sum_{\gamma} \sum_{j=1}^{N_\gamma} C_{\beta\gamma}(i,j) \bar{u}_{\gamma\alpha}(j) = 0, \qquad (A\cdot5)$$

where

$$A_\beta(i,j) = \begin{cases} -2 + \frac{5}{6} h_\beta^2 P_\beta(j) & \text{for } j=i, \\ 1 + \frac{1}{12} h_\beta^2 P_\beta(j) & \text{for } j=i+1,\ i-1, \\ 0 & \text{otherwise}, \end{cases}$$

and

$$C_{\beta\gamma}(i,j) = -\frac{h_\gamma h_\beta^2}{12 a_\beta} w_j \{K_{\beta\gamma}(i+1,j) + 10 K_{\beta\gamma}(i,j) + K_{\beta\gamma}(i-1,j)\}. \qquad (A\cdot6)$$

Equation (A·5) shows that $\bar{u}_{\beta\alpha}$ for all different α is the solution of the same set of equations,

$$\sum_{j=1}^{N_\beta} A_\beta(i,j) \bar{u}_\beta(j) + \sum_{\gamma} \sum_{j=1}^{N_\gamma} C_{\beta\gamma}(i,j) \bar{u}_\gamma(j) = 0. \qquad (A\cdot7)$$

One can straightforwardly generalize the above derivation to allow for the dependence of h_β on i such as

$$h_{\beta i} = \begin{cases} h_\beta^{(1)} & \text{for } 1 \leq i \leq N_\beta^{(1)} - 1, \\ h_\beta^{(2)} & \text{for } N_\beta^{(1)} \leq i \leq N_\beta^{(2)} - 1, \\ \vdots \\ h_\beta^{(n\beta)} & \text{for } N_\beta^{(n\beta-1)} \leq i \leq N_\beta - 1. \end{cases}$$

Such modification is useful because the behaviour of the integrand is much different for different parts of the integration range. In actual calculations we took $n_\beta = 3$ and made $h^{(i+1)} = 2h^{(i)}$.

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