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Study of laplace and related probability distributions and their applications

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Study of Laplace and Related Probability Distributions and Their Applications

by

Gokarna Raj Aryal

A dissertation submitted in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy
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Dedication

To My Parents

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Study of Laplace and Related Probability Distributions and Their Applications

Gokarna Raj Aryal

ABSTRACT

The aim of the present study is to investigate a probability distribution that can be derived from the Laplace probability distribution and can be used to model various real world problems. In the last few decades, there has been a growing interest in the construction of flexible parametric classes of probability distributions. Various forms of the skewed and kurtotic distributions have appeared in the literature for data analysis and modeling. In particular, various forms of the skew Laplace distribution have been introduced and applied in several areas including medical science, environmental science, communications, economics, engineering and finance, among others. In the present study we will investigate the skew Laplace distribution based on the definition of skewed distributions introduced by O'Hagan and extensively studied by Azzalini. A random variable X is said to have the skew-symmetric distribution if its probability density function is $f(x) = 2g(x)G(\lambda x)$, where g and G are the probability density function and the cumulative distribution function of a symmetric distribution around 0 respectively and λ is the skewness parameter. We will investigate the mathematical properties of this distribution and apply it to real applications. In particular, we will consider the exchange rate data for six different currencies namely, Australian Dollar, Canadian Dollar, European Euro, Japanese Yen, Switzerland Franc and United Kingdom Pound versus United States Dollar.

To describe a life phenomenon we will be mostly interested when the random variable is positive. Thus, we will consider the case when the skew Laplace pdf is truncated to the left at 0 and we will study its mathematical properties. Comparisons with other

life time distributions will be presented. In particular we will compare the truncated skew laplace (TSL) distribution with the two parameter Gamma probability distribution with simulated and real data with respect to its reliability behavior. We also study the hypoexponential pdf and compare it with the TSL distribution. Since the TSL pdf has increasing failure rate(IFR) we will investigate a possible application in system maintenance. In particular we study the problem related to the preventive maintenance.

Chapter 1

Introduction

The quality of the procedures used in a statistical analysis depends heavily on the assumed probability model or distributions. Because of this, considerable effort over the years has been expended in the development of large classes of standard distributions along with relevant statistical methodologies, designed to serve as models for a wide range of real world phenomena. However, there still remain many important problems where the real data does not follow any of the classical or standard models. Very few real world phenomenon that we need to statistically study are symmetrical. Thus the popular normal model would not be a useful model for studying every phenomenon. The normal model at a times is a poor description of observed phenomena. Skewed models, which exhibit varying degrees of asymmetry, are a necessary component of the modeler's tool kit. Genton, M. [8] mentions that actually an introduction of non-normal distributions can be traced back to the nineteenth century. Edgeworth [7] examined the problem of fitting asymmetrical distributions to asymmetrical frequency data.

The aim of the present study is to investigate a probability distribution that can be derived from the Laplace probability distribution and can be used to model various real world problems. In fact, we will develop two probability models namely the skew Laplace probability distribution and the truncated skew Laplace probability distribution and show that these models are better than the existing models to model some of the real world problems. Here is an outline of the study:

In chapter two we will study the development of the Laplace probability distribution

and its basic properties. We will make a comparisons of this model with the Gaussian distribution and the Cauchy distribution. Also we will present some representations of the Laplace distribution in terms of other well known distributions.

In chapter three we will study the statistical model called the skew Laplace probability distribution. With the term skew Laplace we mean a parametric class of probability distributions that extends the Laplace probability distribution by additional shape parameter that regulates the degree of skewness, allowing for a continuous variation from Laplace to non Laplace. We will study the mathematical properties of the subject model.

In chapter four we will present an application of the skew Laplace distribution in financial study. In fact, we will use the currency exchange data of six different currencies, namely, Australian Dollar, Canadian Dollar, European Euro, Japanese Yen, Switzerland Franc and the United Kingdom Pound with respect to the US Dollar.

In chapter five we will develop a probability distribution from the skew Laplace distribution presented in chapter two. In fact, we will truncate the skew Laplace distribution at zero on the left and we will call it the truncated skew Laplace probability distribution. We will present some of its mathematical properties.

In chapter six we will make a comparison of the truncated skew Laplace distribution with two existing models namely, two parameter gamma and the hypoexponential probability distributions.

In chapter seven we will seek an application of the truncated skew Laplace distribution in the maintenance system. We will develop a model that can be used to find the optimum time in order to minimize the cost over a finite time span.

In the last chapter we will present possible extension of the present study.

Chapter 2

The Laplace Probability Distribution

2.1 Introduction

The quality of the procedures used in a statistical analysis depends heavily on the assumed probability model or distributions. Because of this, considerable effort over the years has been expended in the development of large classes of standard distributions along with relevant statistical methodologies, designed to serve as models for a wide range of real world phenomena. However, there still remain many important problems where the real data does not follow any of the classical or standard models. The aim of the present study is to investigate a probability distribution that can be derived from the Laplace distribution and can be used on modeling and analyzing real world data.

In the 1923 issue of the *Journal of American Statistical Association* two papers entitled "First and Second Laws of Error" by E.B. Wilson and "The use of median in determining seasonal variation" by W.L. Crum were published. In the first paper E.B. Wilson states that both laws of error were originated by Laplace.

The first law proposed in 1774, states that the frequency of an error could be expressed as an exponential function of the numerical magnitude of the error, or, equivalently that the logarithm of the frequency of an error (regardless of the sign) is a linear function of the error.

The second law proposed in 1778, states that the frequency of the error is an exponential function of the square of the error, or equivalently that the logarithm of the frequency is a quadratic function of the error.

The second Laplace law is called the normal or Gaussian probability distribution. Since the first law consists the absolute value of the error it brings a considerable mathematical difficulties in manipulation. The reasons for the far greater attention being paid for the second law is the mathematical simplicity because it involves the variable x^2 if x is the error. The Laplace distribution is named after Pierre-Simon Laplace (1749-1827), who obtained the likelihood of the Laplace distribution is maximized when the location parameter is set to be the median. The Laplace distribution is also known as the law of the difference between two exponential random variables. Consequently, it is also known as *double exponential distribution*, as well as the *two tailed distribution*. It is also known as the *bilateral exponential law*.

2.2 Definitions and Basic Properties

The *classical Laplace probability distribution* is denoted by $L(\theta, \phi)$ and is defined by the probability density function, pdf,

$$f(x; \theta, \phi) = \frac{1}{2\phi} \exp\left(-\frac{|x - \theta|}{\phi}\right), \quad -\infty < x < \infty \quad (2.2.1)$$

where $\theta \in (-\infty, \infty)$ and $\phi > 0$ are location and scale parameters, respectively. This is the probability distribution whose likelihood is maximized when the location parameter is to be median. It is a symmetric distribution whose tails fall off less sharply than the Gaussian distribution but faster than the Cauchy distribution. Hence, it is our interest to compare the Laplace pdf with the Gaussian pdf and Cauchy pdf. The probability density functions of the Gaussian or Normal, $N(\mu, \sigma^2)$ and the Cauchy, $C(x_0, \Gamma)$ distributions are respectively given by

$$f(x; \mu, \sigma) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x - \mu)^2}{2\sigma^2}\right) \quad (2.2.2)$$

and

$$f(x; x_0, \Gamma) = \frac{1}{\pi} \frac{\Gamma}{\Gamma^2 + (x - x_0)^2} \quad (2.2.3)$$

where $\sigma > 0$, $-\infty < \mu < \infty$, $\Gamma > 0$ and $-\infty < x_0 < \infty$.

Figure 2.1 gives a graphical display of probability density functions of standard Cauchy $C(0, 1)$, standard Laplace $L(0, 1)$ and standard Normal $N(0, 1)$ pdf's.

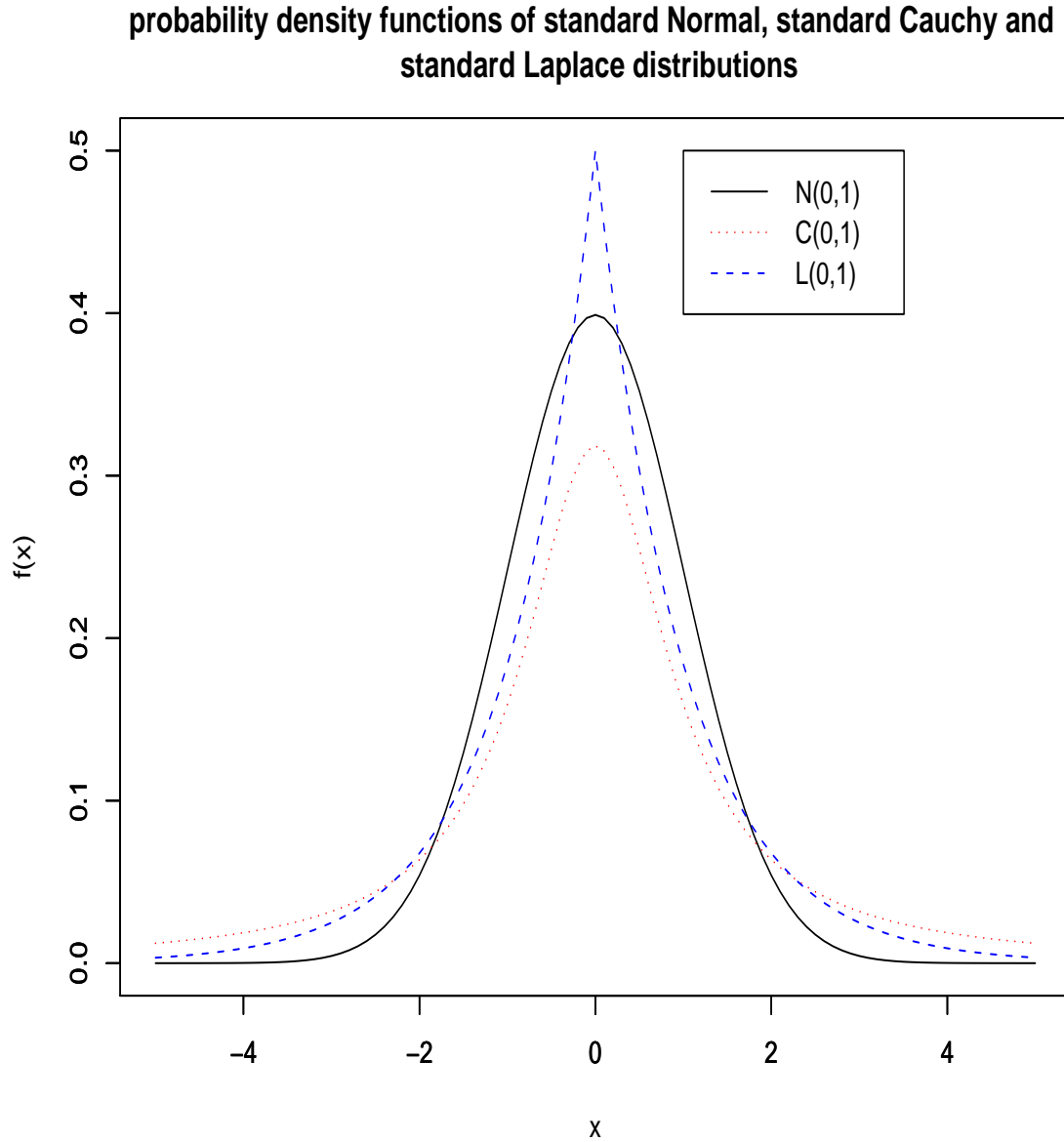


Figure 2.1: PDF of standard Laplace, Normal and Cauchy distributions

The Laplace pdf has a cusp, discontinuous first derivative, at $x = \theta$, the location parameter. Table 2.1 gives some of the basic and useful properties of Laplace, Gaussian and Cauchy pdf's and table 2.2 gives the comparison of the the estimates of sample mean, sample median, estimator of semi-interquartile range(S), which is an estimator for half-width at half maximum(HWHM) and the variance estimator (s^2) of these three pdf's.

The variance of the sample mean is simply the variance of the distribution divided by the sample size n . For large n the variance of the sample median m is given by $V(m) = 1/4nf^2$ where f is the functional value at the median.

By definition $S = \frac{1}{2}(Q_3 - Q_1)$ and $s^2 = \frac{1}{n-1} \sum_{i=1}^n (x_i - \bar{x})^2$.

Hence,

$$\begin{aligned} V(S) &= \frac{1}{4} (V(Q_1) + V(Q_3) - 2Cov(Q_1, Q_3)) \\ &= \frac{1}{64n} \left(\frac{3}{f_1^2} + \frac{3}{f_3^2} - \frac{2}{f_1 f_3} \right) \end{aligned}$$

and

$$V(s^2) = \frac{\mu_4 - \mu_2^2}{n} + \frac{2\mu_2^2}{n(n-1)}$$

where f_1 and f_3 are the functional values at the first quartile Q_1 and the third quartile Q_3 , respectively. Also μ_2 and μ_4 are the second and fourth central moments of the random variable.

Distribution	$E(X)$	$V(X)$	Sk(X)	Kur(X)	HWHM	Char. Function	Entropy
Normal	μ	σ^2	0	3	$\sigma\sqrt{2\ln 2}$	$\exp(\mu it - \frac{\sigma^2 t^2}{2})$	$\ln(\sigma\sqrt{2\pi e})$
Laplace	θ	$2\phi^2$	0	6	$\phi \ln 2$	$\frac{1}{1+\phi^2 t^2} \exp(\theta it)$	$1 + \ln(2\phi)$
Cauchy	<i>Und.</i>	∞	<i>Und.</i>	∞	Γ	$\exp(x_0 it - \Gamma t)$	$\ln(4\pi\Gamma)$

Table 2.1: Some properties of Normal, Laplace and Cauchy distributions

Distribution	$E(\bar{X})$	$V(\bar{X})$	E(m)	$V(m)$	$E(s^2)$	$V(s^2)$	E(S)	V(S)
Normal	μ	$\frac{\sigma^2}{n}$	μ	$\frac{\pi\sigma^2}{2n}$	σ^2	$\frac{2\sigma^4}{n-1}$	0.6745σ	$\frac{1}{16nf(Q_1)^2}$
Laplace	θ	$\frac{2\phi^2}{n}$	θ	$\frac{\phi^2}{n}$	$2\phi^2$	$\frac{20\phi^2}{n}\beta$	$\phi\ln 2$	$\frac{\phi^2}{n}$
Cauchy	Und.	∞	x_0	$\frac{\pi^2\Gamma^2}{4n}$	Γ	∞	∞	$\frac{\pi^2\Gamma^2}{4n}$

Table 2.2: Parameter estimators of Normal, Laplace and Cauchy distributions

Where $\beta = 1 + \frac{0.4}{n-1}$ if we include the second term in the expression of $V(s^2)$ and 1 otherwise and in the table Und. stands for undefined and m denotes the median.

2.3 Discriminating between the Normal and Laplace Distributions

Both the normal and Laplace pdf's can be used to analyze symmetric data. It is well known that the normal pdf is used to analyze symmetric data with short tails, whereas the Laplace pdf is used for data with long tails. Although, these two distributions may provide similar data fit for moderate sample sizes, however, it is still desirable to choose the correct or more nearly correct model, since the inferences often involve tail probabilities, and thus the pdf assumption is very important.

For a given data set, whether it follows one of the two given probability distribution functions, is a very well known and important problem. Discriminating between any two general probability distribution functions was studied by Cox [6].

Recently Kundu [17] consider different aspects of discriminating between the Normal and Laplace pdf's using the ratio of the maximized likelihoods (RML). Let X_1, X_2, \dots, X_n be a random sample from one of the two distributions. The likelihood functions, assuming that the data follow $N(\mu, \sigma^2)$ or $L(\theta, \phi)$, are

$$l_N(\mu, \sigma) = \prod_{i=1}^n f_N(X_i, \mu, \sigma)$$

and

$$l_L(\theta, \phi) = \prod_{i=1}^n f_L(X_i, \theta, \phi),$$

respectively. The logarithm of RML is defined by

$$T = \ln \left\{ \frac{l_N(\hat{\mu}, \hat{\sigma})}{l_L(\hat{\theta}, \hat{\phi})} \right\}.$$

Note that $(\hat{\mu}, \hat{\sigma})$ and $(\hat{\theta}, \hat{\phi})$ are the maximum likelihood estimators of (μ, σ) and (θ, ϕ) respectively based on a random sample X_1, X_2, \dots, X_n . Therefore, T can be written as

$$T = \frac{n}{2} \ln 2 - \frac{n}{2} \ln \pi + n \ln \hat{\phi} - n \ln \hat{\sigma} + \frac{n}{2}$$

where

$$\begin{aligned} \hat{\mu} &= \frac{1}{n} \sum_{i=1}^n X_i, & \hat{\sigma}^2 &= \frac{1}{n} \sum_{i=1}^n (X_i - \hat{\mu})^2, \\ \hat{\theta} &= \text{median}\{X_1, X_2, \dots, X_n\}, & \hat{\phi} &= \frac{1}{n} \sum_{i=1}^n |X_i - \hat{\theta}|. \end{aligned}$$

The discrimination procedure is to choose the normal pdf if the test statistic $T > 0$, otherwise choose the Laplace pdf as the preferred model. Note that if the null distribution is $N(\mu, \sigma^2)$, then the distribution of T is independent of μ and σ . Similarly, if the null distribution is $L(\theta, \phi)$, then the distribution of T is independent of θ and ϕ .

2.4 Representation and Characterizations

In this section we would like to present various representations of Laplace random variables in terms of the other well known random variables as presented by Kotz *et al.* [15]. These various form of the Laplace pdf will be useful to the present study. We shall derive the relations for standard classical Laplace Random variable whose probability density function is given by

$$f(x) = \frac{1}{2} \exp(-|x|), \quad -\infty < x < \infty$$

1. Let W be a standard exponential random variable (r.v.) with probability density

$$f_W(w) = \exp(-w), \quad w > 0$$

and Z be standard normal r.v. with probability density

$$f_Z(z) = \frac{1}{\sqrt{2\pi}} \exp(-z^2/2), \quad -\infty < z < \infty$$

then $X = \sqrt{2W}Z$ has standard classical Laplace pdf.

2. Let R be a Rayleigh r.v. with probability density given by

$$f_R(x) = x \exp(-x^2/2), \quad x > 0$$

and Z be standard normal r.v. with probability density

$$f_Z(z) = \frac{1}{\sqrt{2\pi}} \exp(-z^2/2), \quad -\infty < z < \infty$$

then $X = R.Z$ has standard classical Laplace pdf.

3. Let T be brittle fracture r.v. with probability density

$$f_T(x) = 2x^{-3} \exp(1/x^2) \quad x > 0$$

and Z be standard normal distribution r.v. with probability density

$$f_Z(z) = \frac{1}{\sqrt{2\pi}} \exp(-z^2/2), \quad -\infty < z < \infty$$

then $X = \sqrt{2}Z/T$ has standard classical Laplace pdf.

4. Let W_1 and W_2 be i.i.d standard exponential random variables then $X = W_1 - W_2$ has standard classical Laplace pdf.

5. Let Y_1 and Y_2 be i.i.d χ^2 r.v with two degrees of freedom i.e. having the probability density

$$f(x) = \frac{1}{2} \exp(-x/2)$$

then $X = (Y_1 - Y_2)/2$ has standard classical Laplace pdf.

6. Let W be a standard exponential r.v then $X = IW$, where I takes on values ± 1 with probabilities $1/2$, has standard classical Laplace pdf.

7. Let P_1 and P_2 are i.i.d. Pareto Type I random variables with density $f(x) = 1/x^2$, $x \geq 1$ then $X = \log(P_1/P_2)$ has standard classical Laplace pdf.

8. Let U_1 and U_2 be i.i.d. uniformly distributed on $[0, 1]$ then $X = \log(U_1/U_2)$ has standard classical Laplace pdf.

9. Let $U_i, i = 1, 2, 3, 4$ be i.i.d. standard normal variables then the determinant

$$X = \begin{vmatrix} U_1 & U_2 \\ U_3 & U_4 \end{vmatrix} = U_1U_4 - U_2U_3$$

has standard classical Laplace pdf.

10. Let $\{X_n, n \geq 1\}$ be a sequence of uncorrelated random variables then $X = \sum_{n=1}^{\infty} b_n X_n$ has a classical Laplace pdf, where,

$$b_n = \frac{\xi_n}{\sqrt{2}J_0(\xi_n)} \int_0^{\infty} x \exp(-x) J_0(\xi_n \exp(-\frac{x}{2})) dx$$

and

$$X_n = \frac{\sqrt{2}}{\xi_n J_0(\xi_n)} J_0(\xi_n \exp(-\frac{|x|}{2})),$$

where ξ_n is the n th root of J_1 . Here, J_0 and J_1 are the Bessel functions of the first kind of order 0 and 1, respectively. The Bessel function of the first kind of order i is

defined by

$$J_i(u) = u^i \sum_{k=0}^{\infty} \frac{(-1)^k u^{2k}}{2^{2k+\lambda} k! \Gamma(i+k+1)}.$$

This is also called the orthogonal representation of Laplace random variables.

2.5 Conclusions

In this chapter we have studied the development of Laplace probability distribution and its basic properties. More specifically we have derived analytical expressions for all important statistics and their corresponding estimates, as summarized in Table 2.1 and Table 2.2. In addition we have developed an analytical comparison with the famous and highly popular Gaussian probability distribution and Cauchy probability distribution. The reason for the subject comparison is the fact that the Laplace pdf is symmetric and whose tails fall off less sharply than the Gaussian pdf but faster than the Cauchy pdf.

We also provide a method when to choose the Laplace pdf over Gaussian pdf in analyzing and modeling a real world phenomenon. A list of various representations of the Laplace pdf in terms of some other well known and useful pdf's is also provided.

Chapter 3

On the Skew Laplace Probability Distribution

3.1 Introduction

Very few real world phenomenon that we need to statistically study are symmetrical. Thus the popular normal model would not be a useful pdf for studying every phenomenon. The normal model at times is a poor description of observed phenomena. Skewed models, which exhibit varying degrees of asymmetry, are a necessary component of the modeler's tool kit. Genton, M. [8] mentions that actually an introduction of non-normal distributions can be traced back to the nineteenth century. Edgeworth [7] examined the problem of fitting asymmetrical distributions to asymmetrical frequency data. Our interest in this study is about the skew Laplace pdf.

With the term skew Laplace (SL) we mean a parametric class of probability distributions that extends the Laplace pdf by an additional shape parameter that regulates the degree of skewness, allowing for a continuous variation from Laplace to non-Laplace. On the applied side, the skew Laplace pdf as a generalization of the Laplace law should be a natural choice in all practical situations in which there is some skewness present.

Several asymmetry forms of skewed Laplace pdf have appeared in the literature. One of the earliest studies is due to McGill [19] who considered the distributions with pdf given by

$$f(x) = \begin{cases} \frac{\phi_1}{2} \exp(-\phi_1 |x - \theta|), & \text{if } x \leq \theta, \\ \frac{\phi_2}{2} \exp(-\phi_2 |x - \theta|), & \text{if } x > \theta, \end{cases} \quad (3.1.1)$$

while Holla *et al.* in 1968 studied the distribution with pdf given by

$$f(x) = \begin{cases} p\phi \exp(-\phi |x - \theta|), & \text{if } x \leq \theta, \\ (1 - p)\phi \exp(-\phi |x - \theta|), & \text{if } x > \theta, \end{cases} \quad (3.1.2)$$

where $0 < p < 1$. Lingappaiah study (3.1.1) terming the distribution as two-piece double exponential. Poiraud-Casanova *et al.* [21] studied a skew Laplace distribution with p.d.f.

$$f(x) = \alpha(1 - \alpha) \begin{cases} \exp(-(1 - \alpha) |x - \theta|), & \text{if } x < \theta, \\ \exp(-\alpha |x - \theta|), & \text{if } x \geq \theta, \end{cases} \quad (3.1.3)$$

where $\theta \in (-\infty, \infty)$ and $\alpha \in (0, 1)$.

Another manner of introducing skewness into a symmetric distribution has been proposed by Fernandez *et al.* (1998). The idea is to convert a symmetric pdf into a skewed one by postulating inverse scale factors in the positive and negative orthants. Thus, a symmetric pdf f generates the following class of skewed distributions

$$f(x|\kappa) = \frac{2\kappa}{(1 + \kappa^2)} \begin{cases} f(\kappa x), & \text{if } x \geq 0, \\ f(\kappa^{-1}x) & \text{if } x < 0, \end{cases} \quad (3.1.4)$$

where $\kappa > 0$.

Therefore, if f is the standard classical Laplace pdf given by

$$f(x) = \frac{1}{2} \exp(-|x|), \quad -\infty < x < \infty$$

then, we have the pdf of the skew Laplace r.v. will be

$$f(x) = \frac{\kappa}{1 + \kappa^2} \begin{cases} \exp(-\kappa x), & \text{if } x \geq 0, \\ \exp(\kappa^{-1}x), & \text{if } x < 0, \end{cases}$$

The addition of location and scale parameters leads to a three parameter family of

pdf given by

$$f(x) = \frac{1}{\phi} \frac{\kappa}{1 + \kappa^2} \begin{cases} \exp\left(-\frac{\kappa}{\phi}(x - \theta)\right), & \text{if } x \geq \theta, \\ \exp\left(-\frac{1}{\phi\kappa}(x - \theta)\right), & \text{if } x < \theta, \end{cases} \quad (3.1.5)$$

where $\phi > 0$ and $\kappa > 0$. Note that for $\kappa = 1$ we obtain the pdf of symmetric Laplace pdf. This was introduced by Hinkly *et al.*(1977) and this distribution is termed as *asymmetric Laplace (AL)* pdf. An in depth study on the skew-Laplace distribution was reported by Kotz et al.[14]. They consider a three parameters skew-Laplace distribution with pdf given by

$$f(x; \alpha, \beta, \mu) = \begin{cases} \frac{\alpha\beta}{\alpha + \beta} \exp(-\alpha(\mu - x)), & \text{if } x \leq \mu, \\ \frac{\alpha\beta}{\alpha + \beta} \exp(-\beta(x - \mu)), & \text{if } x > \mu, \end{cases} \quad (3.1.6)$$

where μ is the mean and the parameters α and β describes the left and right-tail shapes, respectively. A value of α greater than β suggests that the left tails are thinner and thus, that there is less population to the left side of μ than to the right side; the opposite is of course true if β is greater than α . If $\alpha = \beta$, the distribution is the classical symmetric Laplace pdf.

In this chapter we will study in detail the skewed Laplace pdf using the idea introduced by O'Hagan and extensively studied by Azzalini [4].

The standard Laplace random variable has the probability density function and the cumulative distribution function, cdf, specified by

$$g(x) = \frac{1}{2\phi} \exp\left(-\frac{|x|}{\phi}\right) \quad (3.1.7)$$

and

$$G(x) = \begin{cases} \frac{1}{2} \exp\left(\frac{x}{\phi}\right), & \text{if } x \leq 0, \\ 1 - \frac{1}{2} \exp\left(-\frac{x}{\phi}\right), & \text{if } x \geq 0, \end{cases} \quad (3.1.8)$$

respectively, where $-\infty < x < \infty$ and $\phi > 0$. A random variable X is said to have the skew Laplace pdf with skewness parameter λ , denoted by $SL(\lambda)$, if its probability density function is given by

$$f(x) = 2g(x)G(\lambda x), \quad (3.1.9)$$

where $x \in \Re$ and $\lambda \in \Re$, the real line. The Laplace pdf given by (3.1.7)–(3.1.8) has been quite commonly used as an alternative to the normal pdf in robustness studies; see, for example, Andrews *et al.* [3] and Hoaglin *et al.* [10]. It has also attracted interesting applications in the modeling of detector relative efficiencies, extreme wind speeds, measurement errors, position errors in navigation, stock return, the Earth's magnetic field and wind shear data, among others. The main feature of the skew-Laplace pdf (3.1.9) is that a new parameter λ is introduced to control skewness and kurtosis. Thus, (3.1.9) allows for a greater degree of flexibility and we can expect this to be useful in many more practical situations.

It follows from (3.1.9) that the pdf $f(x)$ and the cdf $F(x)$ of X are respectively given by

$$f(x) = \begin{cases} \frac{1}{2\phi} \exp\left\{-\frac{(1+|\lambda|)|x|}{\phi}\right\}, & \text{if } \lambda x \leq 0, \\ \frac{1}{\phi} \exp\left(-\frac{|x|}{\phi}\right) \left\{1 - \frac{1}{2} \exp\left(-\frac{\lambda x}{\phi}\right)\right\}, & \text{if } \lambda x > 0 \end{cases} \quad (3.1.10)$$

and

$$F(x) = \begin{cases} \frac{1}{2} + \frac{\text{sign}(\lambda)}{2} \left[\frac{1}{1+|\lambda|} \exp\left\{-\frac{(1+|\lambda|)|x|}{\phi}\right\} - 1 \right], & \text{if } \lambda x \leq 0, \\ \frac{1}{2} + \text{sign}(\lambda) \left[\frac{1}{2} - \exp\left(-\frac{|x|}{\phi}\right) \Phi(\lambda) \right], & \text{if } \lambda x > 0, \end{cases} \quad (3.1.11)$$

where $\Phi(\lambda) = 1 - \frac{1}{2(1+|\lambda|)} \exp\left(-\frac{\lambda x}{\phi}\right)$.

Throughout the rest of our study, unless otherwise stated, we shall assume that $\lambda > 0$ since the corresponding results for $\lambda < 0$ can be obtained using the fact that $-X$ has a pdf given by $2g(x)G(-\lambda x)$.

Figure 3.1 illustrates the shape of the pdf (3.1.10) for various values of λ and $\phi = 1$.

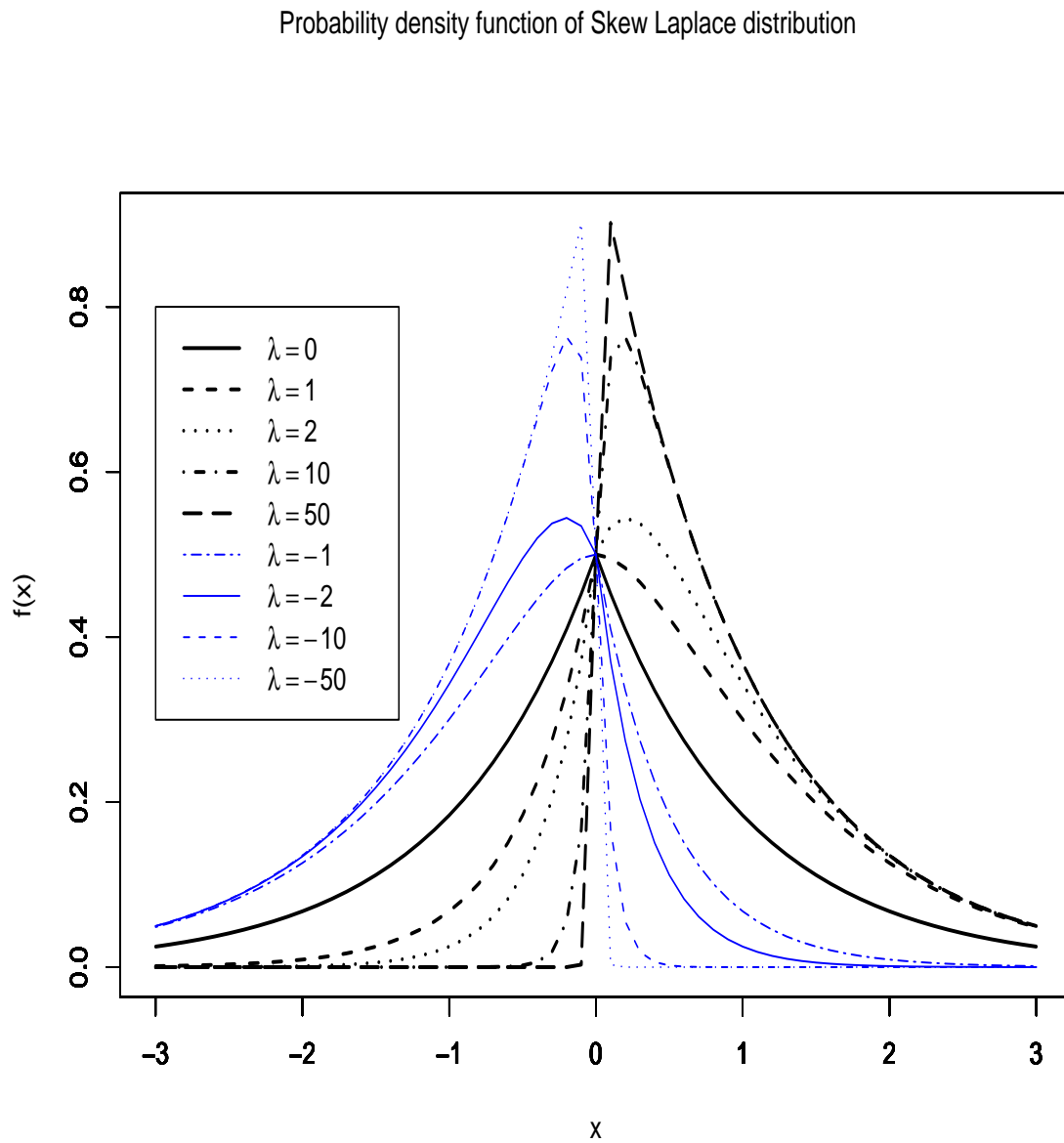


Figure 3.1: PDF of skew Laplace distribution for $\phi = 1$ and different values of λ

The following properties are very immediate from the definition.

Property 1. The pdf of the $SL(0)$ is identical to the pdf of the Laplace pdf.

Property 2. As $\lambda \rightarrow \infty$, $f(x; \lambda)$ tends to $2f(x)I_{x>0}$ which is the exponential distribution.

Property 3. If X has $SL(\lambda)$ then $-X$ has $SL(-\lambda)$.

One interesting situation where the skew Laplace random variable may occur is the following:

Proposition. Let Y and W be two independent $L(0, 1)$ random variables and Z is defined to be equal to Y conditionally on the event $\{\lambda Y > W\}$ then the resulting distribution Z will have skew-Laplace distribution.

Proof: We have

$$\begin{aligned}
 P(Z \leq z) &= P(Y \leq z | \lambda Y > W) \\
 &= \frac{P(Y \leq z, \lambda Y > W)}{P(\lambda Y > W)} \\
 &= \frac{1}{P(\lambda Y > W)} \int_{-\infty}^z \int_{-\infty}^{\lambda y} g(y)g(w)dw dy \\
 &= \frac{1}{P(\lambda Y > W)} \int_{-\infty}^z g(y)G(\lambda y)dy
 \end{aligned}$$

Note that $P(\lambda Y > W) = P(\lambda Y - W > 0) = 1/2$ as $\lambda Y - W$ has Laplace pdf with mean 0. Hence

$$P(Z \leq z) = 2 \int_{-\infty}^z g(y)G(\lambda y)dy$$

Differentiating the above expression with respect to z we obtain the skew Laplace pdf.

This proposition gives us a quite efficient method to generate random numbers from a skew Laplace pdf. It shows that in fact it is sufficient to generate Y and W from

L(0,1) and set

$$Z = \begin{cases} Y, & \text{if } \lambda Y > W, \\ -Y, & \text{if } \lambda Y \leq W, \end{cases}$$

Again, if we consider the Laplace distribution with location parameter being θ which we call the classical Laplace distribution (also known as first law of Laplace) denoted by $\mathcal{CL}(\theta, \phi)$ in this case the pdf and cdf of are respectively

$$g(x) = \frac{1}{2\phi} \exp\left(-\frac{|x - \theta|}{\phi}\right)$$

and

$$G(x) = \begin{cases} \frac{1}{2} \exp\left(-\frac{\theta - x}{\phi}\right), & \text{if } x \leq \theta, \\ 1 - \frac{1}{2} \exp\left(-\frac{x - \theta}{\phi}\right), & \text{if } x \geq \theta. \end{cases}$$

Hence in this case for $\lambda > 0$ the corresponding pdf and cdf of the skew-Laplace random variable are, respectively, given by

$$f(x) = \begin{cases} \frac{1}{2\phi} \exp\left\{-\frac{(1 + \lambda)(\theta - x)}{\phi}\right\}, & \text{if } x \leq \theta, \\ \frac{1}{\phi} \exp\left(-\frac{x - \theta}{\phi}\right) \left\{1 - \frac{1}{2} \exp\left(-\frac{\lambda(x - \theta)}{\phi}\right)\right\}, & \text{if } x > \theta \end{cases}$$

and

$$F(x) = \begin{cases} \frac{1}{2(1 + \lambda)} \exp\left(-\frac{(1 + \lambda)(\theta - x)}{\phi}\right), & \text{if } x \leq \theta, \\ 1 - \exp\left(-\frac{x - \theta}{\phi}\right) \left\{1 - \frac{1}{2} \exp\left(-\frac{\lambda(x - \theta)}{\phi}\right)\right\} & \text{if } x \geq \theta. \end{cases}$$

The skew Laplace pdf – in spite of its simplicity – appears not to have been studied in detail.

The only work that appears to give some details of this distribution is Gupta *et al.* [9] where the pdf of skew Laplace distribution is given by

$$f(x) = \frac{\exp(-|x|/\sigma)[1 + \text{sign}(\lambda x)(1 - \exp(-|\lambda x|/\sigma))]}{2\sigma} \quad x \in \mathfrak{R} \quad (3.1.12)$$

where $\lambda \in \mathfrak{R}$, the real line and $\sigma > 0$.

Also they give the expressions for the expectation, variance, skewness and the kurtosis. But these expressions are not entirely correct as pointed out by Aryal *et al.*[1]. In this study we will provide a comprehensive description of the mathematical properties of (3.1.10) and its applications. In particular, we shall derive the formulas for the k th moment, variance, skewness, kurtosis, moment generating function, characteristic function, cumulant generating function, the k th cumulant, mean deviation about the mean, mean deviation about the median, Rényi entropy, Shannon's entropy, cumulative residual entropy and the asymptotic distribution of the extreme order statistics. We shall also obtain the estimates of these analytical developments and perform a simulation study to illustrate the usefulness of the skew-Laplace distribution. Our calculations make use of the following special functions: the gamma function defined by

$$\Gamma(a) = \int_0^{\infty} t^{a-1} \exp(-t) dt;$$

the beta function defined by

$$B(a, b) = \int_0^1 t^{a-1}(1-t)^{b-1} dt;$$

and, the incomplete beta function defined by

$$B_x(a, b) = \int_0^x t^{a-1}(1-t)^{b-1} dt, \quad (3.1.13)$$

where $a > 0$, $b > 0$ and $0 < x < 1$.

We also use a result from analysis which says that a root of the transcendental equation

$$1 - x + wx^\beta = 0 \quad (3.1.14)$$

is given by

$$x = 1 + \sum_{j=1}^{\infty} \binom{\beta j}{j-1} \frac{w^j}{j}; \quad (3.1.15)$$

see, for example, page 348 in Pólya *et al.* [22].

3.2 Moments

The moments of a probability distributions is a collection of descriptive constants that can be used for measuring its properties. Using the definition of the gamma function, it is easy to show that the k th moment of a skew Laplace random variable X is given by

$$E(X^k) = \begin{cases} \phi^k \Gamma(k+1), & \text{if } k \text{ is even,} \\ \phi^k \Gamma(k+1) \left\{ 1 - \frac{1}{(1+\lambda)^{k+1}} \right\}, & \text{if } k \text{ is odd.} \end{cases} \quad (3.2.16)$$

Also we know that

$$\sum_{i=0}^k a_k = \begin{cases} a_0 + \sum_{i=1}^{\frac{k}{2}} a_{2i} + \sum_{i=1}^{\frac{k}{2}} a_{2i-1}, & \text{if } k \text{ is even,} \\ a_0 + \sum_{i=1}^{\frac{k-1}{2}} a_{2i} + \sum_{i=1}^{\frac{k+1}{2}} a_{2i-1}, & \text{if } k \text{ is odd.} \end{cases}$$

Using the Binomial expansion and (3.2.16), the k th central moment of X can be derived as

$$E \left\{ (X - \mu)^k \right\} = \begin{cases} \mu^k + \sum_{j=1}^{\frac{k}{2}} \binom{k}{2j} \mu^{k-2j} \phi^{2j} \Gamma(2j+1) \\ \quad - \sum_{j=1}^{\frac{k}{2}} \binom{k}{2j-1} \mu^{k-2j+1} \phi^{2j-1} \Gamma(2j) \Psi(\lambda) & \text{if } k \text{ is even} \\ -\mu^k - \sum_{j=1}^{\frac{k-1}{2}} \binom{k}{2j} \mu^{k-2j} \phi^{2j} \Gamma(2j+1) \\ \quad + \sum_{j=1}^{\frac{k+1}{2}} \binom{k}{2j-1} \mu^{k-2j+1} \phi^{2j-1} \Gamma(2j) \Psi(\lambda) & \text{if } k \text{ is odd.} \end{cases} \quad (3.2.17)$$

where $\mu = E(X)$ is the expectation of X and $\Psi(\lambda) = 1 - \frac{1}{(1+\lambda)^{2j}}$.

It follows from (3.2.16) and (3.2.17) that the expectation, variance, skewness and the kurtosis of X are derived to be

$$\text{Exp}(X) = \phi \left\{ 1 - \frac{1}{(1+\lambda)^2} \right\},$$

$$\text{Var}(X) = \frac{\phi^2 (2 + 8\lambda + 8\lambda^2 + 4\lambda^3 + \lambda^4)}{(1+\lambda)^4},$$

$$\text{Ske}(X) = \frac{2\lambda (6 + 15\lambda + 20\lambda^2 + 15\lambda^3 + 6\lambda^4 + \lambda^5)}{(2 + 8\lambda + 8\lambda^2 + 4\lambda^3 + \lambda^4)^{3/2}},$$

and

$$\text{Kur}(X) = \frac{3 (8 + 64\lambda + 176\lambda^2 + 272\lambda^3 + 276\lambda^4 + 192\lambda^5 + 88\lambda^6 + 24\lambda^7 + 3\lambda^8)}{(2 + 8\lambda + 8\lambda^2 + 4\lambda^3 + \lambda^4)^2}.$$

Note that these four expressions are valid only for $\lambda > 0$. The corresponding expressions given in Gupta *et al.* [9] are the same as the above, but they appear to claim the validity of the expressions for all $\lambda \in \Re$.

As pointed out by Aryal *et al.* [2] if $\lambda < 0$, one must replace λ by $-\lambda$ in each of the four expressions; in addition, the expressions for the expectation and the skewness must be multiplied by -1 .

Figure 3.2 illustrates the behavior of the above four analytical expressions for $\lambda = -10, \dots, 10$. Both the expectation and the skewness are increasing functions of λ with

$$\lim_{\lambda \rightarrow -\infty} E(X) = -\phi, \quad \lim_{\lambda \rightarrow -\infty} \text{Skewness}(X) = -2,$$

and

$$\lim_{\lambda \rightarrow \infty} E(X) = \phi, \quad \lim_{\lambda \rightarrow \infty} \text{Skewness}(X) = 2.$$

Note that the variance and the kurtosis are even functions of λ . The variance decreases from $2\phi^2$ to ϕ^2 as λ increases from 0 to ∞ which is a significant gain on introducing the new shape parameter λ in the model. The kurtosis decreases for $0 \leq \lambda \leq \lambda_0$ but then increases for all $\lambda > \lambda_0$, where λ_0 is the solution of the equation given by

$$4 + 6\lambda = 14\lambda^2 + 56\lambda^3 + 84\lambda^4 + 70\lambda^5 + 34\lambda^6 + 9\lambda^7 + \lambda^8.$$

Numerical calculations show that for $\lambda_0 \approx 0.356$. At $\lambda = 0$, $\lambda = \lambda_0$ and as $\lambda \rightarrow \infty$, the kurtosis takes the values 6, 5.810 (approx) and 9, respectively.

Behavior of Expectation, Variance, Skewness and Kurtosis

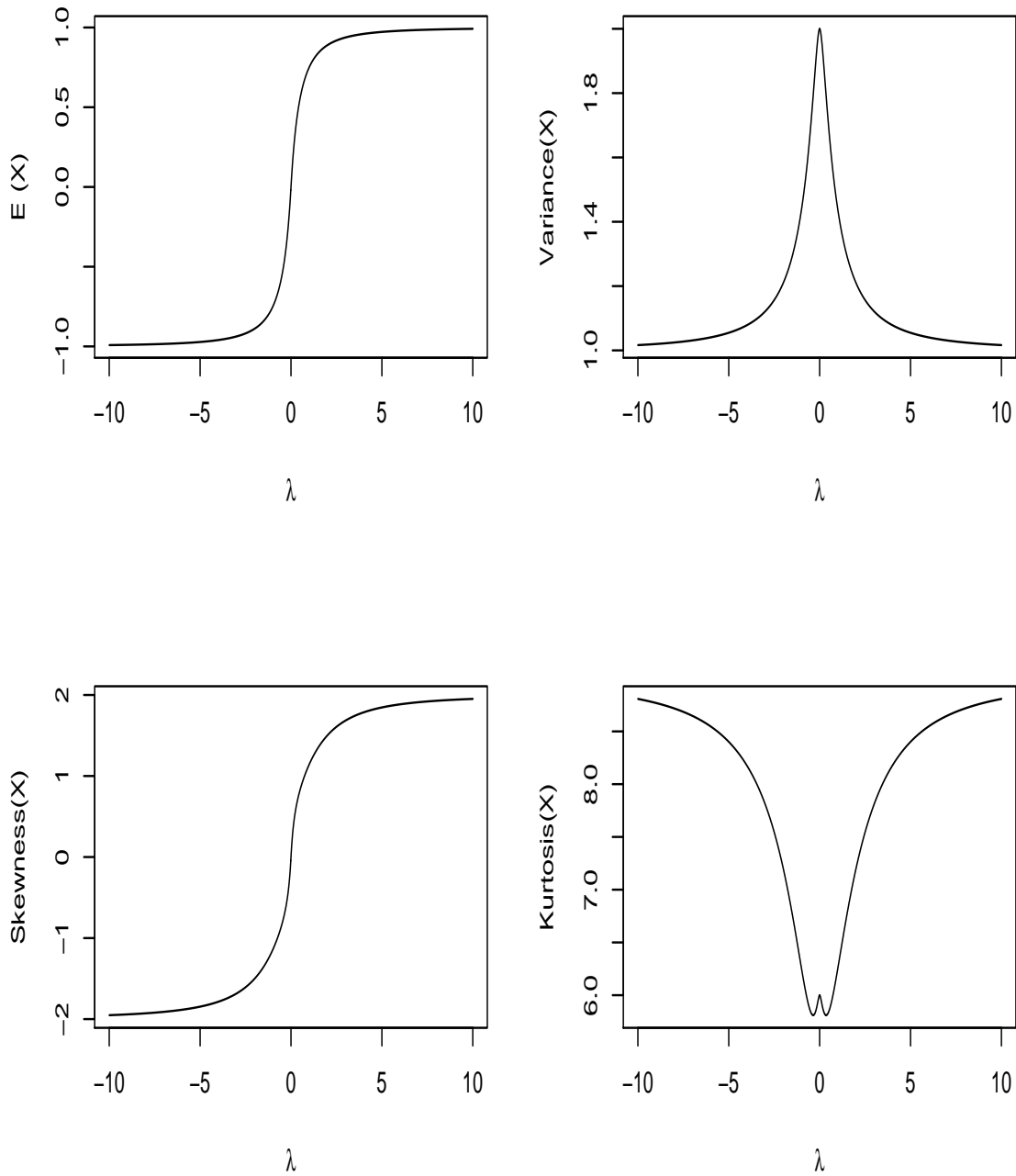


Figure 3.2: Behavior of expectation, variance, skewness and kurtosis of SL random variable as a function of λ for $\phi = 1$

We know that the skewness of a random variable X is defined by

$$\gamma = \frac{\text{third moment about the mean}}{(\text{s.d})^3}$$

Proposition: There is a one to one correspondence between γ , the skewness of a SL random variable and λ , the shape parameter of SL pdf.

Proof: We know that the skewness for a Skew-Laplace pdf is given by

$$\gamma = \frac{2\lambda(6 + 15\lambda + 20\lambda^2 + 15\lambda^3 + 6\lambda^4 + \lambda^5)}{(2 + 8\lambda + 8\lambda^2 + 4\lambda^3 + \lambda^4)^{3/2}},$$

which can be written as

$$\frac{\gamma}{2} = \frac{(\lambda + 1)^6 - 1}{[(\lambda + 1)^4 + 2(\lambda + 1)^2 - 1]^{3/2}} \quad (3.2.18)$$

and setting $(\lambda + 1)^2 = x$ we have

$$\gamma = \frac{2(x^3 - 1)}{(x^2 + 2x - 1)^{3/2}} \quad (3.2.19)$$

If we are given a value of γ we can get the corresponding value of x using simple calculations. It is clear that x is positive. Table 2.1 below gives some values of λ for a given values of γ when $\gamma > 0$.

γ	x	λ
0	1	0
0.5	1.4319	0.1966
1.0	3.0409	0.7438
1.5	8.9808	1.9968
1.75	20.9875	3.5812
1.9	56.9946	6.5495
1.99999	599996.9989	773.5947

Table 3.1: Values of the parameter λ for selected values of skewness γ

Note that if γ is greater than or equal to 2 we will have only imaginary roots. Also note that when we have $\lambda < 0$, then we know that the corresponding expression for skewness is obtained on replacing λ by $-\lambda$ and multiplying the whole expression by -1.

In this case we have

$$\gamma = -\frac{2(y^3 - 1)}{(y^2 + 2y - 1)^{3/2}} \quad (3.2.20)$$

where $y = (\lambda - 1)^2$. Again we can find the value of y and λ once we know the value of γ . Hence, knowing the value of skewness we can compute the corresponding unique value of λ .

Consider the case when the location parameter being θ and $\lambda > 0$ the k th moments are given by

$$E(X^k) = \begin{cases} \phi^k \exp\left(\frac{\theta}{\phi}\right) \Gamma\left(k+1; \frac{\theta}{\phi}\right) - \frac{\phi^k}{2\lambda_1^{k+1}} \left\{ \exp\left(\frac{\lambda_1\theta}{\phi}\right) \Gamma\left(k+1; \frac{\lambda_1\theta}{\phi}\right) \right. \\ \left. + \frac{\phi^k}{2\lambda_1^{k+1}} \left\{ \exp\left(-\frac{\lambda_1\theta}{\phi}\right) \Gamma\left(k+1; -\frac{\lambda_1\theta}{\phi}\right) \right\} \right\} & \text{if } k \text{ is even,} \\ \phi^k \exp\left(\frac{\theta}{\phi}\right) \Gamma\left(k+1; \frac{\theta}{\phi}\right) + \frac{\phi^k}{2\lambda_1^{k+1}} \left\{ \exp\left(\frac{\lambda_1\theta}{\phi}\right) \Gamma\left(k+1; \frac{\lambda_1\theta}{\phi}\right) \right. \\ \left. - \frac{\phi^k}{2\lambda_1^{k+1}} \left\{ \exp\left(-\frac{\lambda_1\theta}{\phi}\right) \Gamma\left(k+1; -\frac{\lambda_1\theta}{\phi}\right) \right\} \right\} & \text{if } k \text{ is odd.} \end{cases}$$

where $\lambda_1 = \lambda + 1$

3.3 MGF and Cumulants

The moment generating function, MGF, of a random variable X is defined by $M(t) = E(\exp(tX))$. When X has the pdf given (3.1.10), direct integration yields that

$$M(t) = \frac{t\phi}{(t\phi)^2 - (1 + \lambda)^2} + \frac{1}{1 - t\phi}$$

for $t < 1/\phi$. Thus, the characteristic function defined by $\psi(t) = E(\exp(itX))$ and the cumulant generating function defined by $K(t) = \log M(t)$ are of the form

$$\psi(t) = \frac{it\phi}{(it\phi)^2 - (1 + \lambda)^2} + \frac{1}{1 - it\phi}$$

and

$$K(t) = \log \left\{ \frac{t\phi}{(t\phi)^2 - (1 + \lambda)^2} + \frac{1}{1 - t\phi} \right\},$$

respectively, where $i = \sqrt{-1}$. By expanding the cumulant generating function as

$$K(t) = \sum_{k=1}^{\infty} a_k \frac{(t)^k}{k!},$$

one obtains the cumulants a_k given by

$$a_k = \begin{cases} (k-1)! \phi^k \left\{ 1 - \frac{1}{(1+\lambda)^{2k}} \right\}, & \text{if } k \text{ is odd,} \\ (k-1)! \phi^k \left\{ 1 + \frac{2}{(1+\lambda)^k} - \frac{1}{(1+\lambda)^{2k}} \right\}, & \text{if } k \text{ is even.} \end{cases}$$

One interesting characterization of a skew Laplace pdf is the following:

We have seen that the characteristic function of a SL random variable X is given by

$$\psi(t) = \frac{it\phi}{(it\phi)^2 - (1 + \lambda)^2} + \frac{1}{1 - it\phi}$$

It is clear that

$$\psi_X(t) + \psi_X(-t) = \frac{1}{1 - it\phi} + \frac{1}{1 + it\phi} = \frac{2}{1 - (it\phi)^2}$$

In fact we have the following proposition.

Proposition: Let Y be a $L(0, 1)$ random variable with probability density function $g(x)$ and X be $SL(\lambda)$ derived from Y , then the even moments of X are independent of λ and are the same as that of Y .

Proof: Let $\psi_X(t)$ be the characteristic function of X so that

$$\psi_X(t) = \int_{-\infty}^{\infty} \exp(itx)[2g(x)G(\lambda x)]dx.$$

Now,

$$\begin{aligned} \psi_X(-t) &= \int_{-\infty}^{\infty} \exp(-itx)[2g(x)G(\lambda x)]dx \\ &= \int_{\infty}^{-\infty} -\exp(itz)[2g(-z)G(-\lambda z)]dz \\ &= \int_{-\infty}^{\infty} \exp(itz)[2g(z)(1 - G(\lambda z))]dz \\ &= \int_{-\infty}^{\infty} \exp(itx)[2g(x)(1 - G(\lambda x))]dx. \end{aligned}$$

Note that the second from the last expression follows from the fact that g is symmetric about 0. Hence, we have

$$h(t) = \psi_X(t) + \psi_X(-t) = 2 \int_{-\infty}^{\infty} \exp(ity)g(y)dy = 2\psi_Y(t)$$

which is independent of λ .

Also we can show that if $(-1)^n h^{(2n)}(0)/2$ and $(-1)^n \psi_Y^{(2n)}(0)$ exist then they are the even order moments of X and Y , respectively, and they are the same.

3.4 Percentiles

A percentile is a measure of relative standing of an observation against all other observations. The p th percentile has at least $p\%$ of the values below that point and at least $(100 - p)\%$ of the data values above that point. To know the expression of percentile is very important to generate random numbers from a given distribution. The 100 p th percentile x_p is defined by $F(x_p) = p$, where F is given by (3.1.11). If $0 \leq p \leq F(0) = 1/\{2(1 + \lambda)\}$ then inverting $F(x_p) = p$, one gets the simple form

$$x_p = \frac{\phi}{1 + \lambda} \log \{2(1 + \lambda)p\}. \quad (3.4.21)$$

However, if $1/\{2(1 + \lambda)\} < p \leq 1$ then x_p is the solution of the transcendental equation

$$1 - \exp\left(-\frac{x_p}{\phi}\right) \left\{1 - \frac{1}{2(1 + \lambda)} \exp\left(-\frac{\lambda x_p}{\phi}\right)\right\} = p.$$

Substituting $y_p = (\exp(-x_p/\phi))/(1 - p)$, this equation can be reduced to

$$1 - y_p + \frac{(1 - p)^\lambda}{2(1 + \lambda)} y_p^{1 + \lambda} = 0,$$

which takes the form of (3.1.14). Thus, using (3.1.15), y_p is given by

$$y_p = 1 + \sum_{j=1}^{\infty} \binom{(1 + \lambda)j}{j - 1} \frac{(1 - p)^{\lambda j}}{j 2^j (1 + \lambda)^j}$$

and hence the solution for x_p is given by

$$x_p = -\phi \log \left\{1 - p + (1 - p) \sum_{j=1}^{\infty} \binom{(1 + \lambda)j}{j - 1} \frac{(1 - p)^{\lambda j}}{j 2^j (1 + \lambda)^j}\right\}. \quad (3.4.22)$$

3.5 Mean Deviation

The amount of scatter in a population is evidently measured to some extent by the totality of deviations from the mean or the median. These are known as the mean deviation about the mean and the mean deviation about the median. Mean deviation

is an important descriptive statistic that is not frequently encountered in mathematical statistics. This is essentially because while we consider the mean deviation the introduction of the absolute value makes analytical calculations using this statistic much more complicated. But still sometimes it is important to know the analytical expressions of these measures. The mean deviation about the mean and the median are defined by

$$\delta_1(X) = \int_{-\infty}^{\infty} |x - \mu| f(x) dx$$

and

$$\delta_2(X) = \int_{-\infty}^{\infty} |x - M| f(x) dx,$$

respectively, where $\mu = E(X)$ and M denotes the median. These measures can be calculated using the relationships that

$$\delta_1(X) = \int_{-\infty}^{\mu} (\mu - x) f(x) dx + \int_{\mu}^{\infty} (x - \mu) f(x) dx$$

and

$$\delta_2(X) = \int_{-\infty}^0 (M - x) f(x) dx + \int_0^M (M - x) f(x) dx + \int_M^{\infty} (x - M) f(x) dx,$$

where $M > 0$ because $F(0) = 1/\{2(1+\lambda)\} < 1/2$ for $\lambda > 0$. Simple calculations yield the following expressions:

$$\delta_1(X) = \phi \left[2 - \frac{1}{(1+\lambda)^2} \exp \left\{ -\frac{\lambda^2(2+\lambda)}{(1+\lambda)^2} \right\} \right] \exp \left\{ -\frac{\lambda(2+\lambda)}{(1+\lambda)^2} \right\}$$

and

$$\begin{aligned} \delta_2(X) = & M - \phi + 2\phi \exp \left(-\frac{M}{\phi} \right) + \frac{\phi}{(1+\lambda)^2} \left[1 - \exp \left\{ -\frac{M(1+\lambda)}{\phi} \right\} \right] \\ & + \frac{M}{2(1+\lambda)} \left[\exp \left\{ -\frac{M(1+\lambda)}{\phi} \right\} - \exp \{ -(1+\lambda) \} \right]. \end{aligned}$$

The corresponding expressions for $\lambda < 0$ are the same as above with λ replaced by $-\lambda$.

3.6 Entropy

An entropy of a random variable X is a measure of variation of the uncertainty. Rényi entropy is defined by

$$\mathcal{J}_R(\gamma) = \frac{1}{1-\gamma} \log \left\{ \int f^\gamma(x) dx \right\}, \quad (3.6.23)$$

where $\gamma > 0$ and $\gamma \neq 1$. See [24] for details. For the pdf (3.1.10), note that

$$\begin{aligned} \int f^\gamma(x) dx &= \frac{1}{(2\phi)^\gamma} \int_{-\infty}^0 \exp \left\{ \frac{\gamma(1+\lambda)x}{\phi} \right\} dx \\ &\quad + \frac{1}{\phi^\gamma} \int_0^\infty \exp \left(-\frac{\gamma x}{\phi} \right) \left\{ 1 - \frac{1}{2} \exp \left(-\frac{\lambda x}{\phi} \right) \right\}^\gamma dx. \end{aligned}$$

By substituting $y = (1/2) \exp(-\lambda x/\phi)$ and then using (3.1.13), one could express the above in terms of the incomplete beta function. It follows then that the Rényi entropy is given by

$$\mathcal{J}_R(\gamma) = \frac{1}{1-\gamma} \log \left\{ \frac{\lambda + \gamma(1+\lambda)2^{\gamma(1+1/\lambda)} B_{1/2} \left(\frac{\gamma}{\lambda}, \gamma+1 \right)}{\gamma \lambda 2^\gamma \phi^{\gamma-1} (1+\lambda)} \right\}. \quad (3.6.24)$$

Shannon's entropy defined by $E[-\log f(X)]$ is the limiting case of (3.6.23) for $\gamma \rightarrow 1$. In fact Shannon developed the concept of entropy to measure the uncertainty of a discrete random variable. Suppose X is a discrete random variable that obtains values from a finite set x_1, x_2, \dots, x_n , with probabilities p_1, p_2, \dots, p_n . We look for a measure of how much choice is involved in the selection of the event or how certain we are of the outcome. Shannon argued that such a measure $H(p_1, p_2, \dots, p_n)$ should obey the following properties

1. H should be continuous in p_i .
2. If all p_i are equal then H should be monotonically increasing in n .
3. If a choice is broken down into two successive choices, the original H should be

the weighted sum of the individual values of H . Shannon showed that the only H that satisfies these three assumptions is of the form

$$H = k \sum_{i=1}^n p_i \log p_i$$

and termed it the entropy of X .

It is well known that for any distribution limiting $\gamma \rightarrow 1$ in Rényi we get the shanon entropy. Hence from (3.6.24) and using L'Hospital's rule and taking the limits we have

$$\begin{aligned} E[-\log f(X)] &= 1 + \log(2\phi) + \frac{\lambda}{2(1+\lambda)^2} \\ &\quad + 2^{1/\lambda} \left\{ \frac{1}{1+\lambda} B_{1/2} \left(2 + \frac{1}{\lambda}, 0 \right) - B_{1/2} \left(1 + \frac{1}{\lambda}, 0 \right) \right\}. \end{aligned}$$

Rao *et al.* [23] introduced the *cumulative residual entropy* (CRE) defined by

$$\mathcal{E}(X) = - \int \Pr(|X| > x) \log \Pr(|X| > x) dx,$$

which is more general than Shannon's entropy as the definition is valid in the continuous and discrete domains. However, the extension of this notion of Shanon entropy to continuous probability distribution posses some challanges. The straightforward extension of the discrete case to continuous distribution F with pdf density f called differential entropy and is given by

$$H(F) = - \int f(x) \log f(x) dx.$$

However differential entropy has several drawbacks as pointed out in the paper by Rao *et al.* [23].

For the pdf (3.1.10), note that

$$\Pr(|X| > x) = \exp\left(-\frac{x}{\phi}\right)$$

as λ approaches ∞ . Thus, in this case we have

$$\log \Pr(|X| > x) = -\frac{x}{\phi}$$

for $x > 0$. Hence, the CRE takes the simple form, $\mathcal{E}(X) = \phi$.

3.7 Asymptotics

If X_1, \dots, X_n is a random sample from (3.1.10) and if $\bar{X} = (X_1 + \dots + X_n)/n$ denotes the sample mean then by using the Central Limit Theorem, $\sqrt{n}(\bar{X} - E(X))/\sqrt{\text{Var}(X)}$ approaches the standard normal distribution as $n \rightarrow \infty$. Sometimes one would be interested in the asymptotics of the extreme values, $M_n = \max(X_1, \dots, X_n)$ and $m_n = \min(X_1, \dots, X_n)$. For the cdf (3.1.11), it can be seen that

$$\lim_{t \rightarrow \infty} \frac{1 - F(t + \phi x)}{1 - F(t)} = \exp(-x)$$

and

$$\lim_{t \rightarrow \infty} \frac{F\left(-t - \frac{\phi}{1+\lambda}x\right)}{F(-t)} = \exp(-x).$$

Thus, it follows from Theorem 1.6.2 in Leadbetter *et al.* [15] that there must be norming constants $a_n > 0$, b_n , $c_n > 0$ and d_n such that

$$\Pr\{a_n(M_n - b_n) \leq x\} \rightarrow \exp\{-\exp(-x)\}$$

and

$$\Pr\{c_n(m_n - d_n) \leq x\} \rightarrow 1 - \exp\{-\exp(x)\}$$

as $n \rightarrow \infty$. The form of the norming constants can also be determined. For instance, using Corollary 1.6.3 in Leadbetter *et al.* [18], one can see that $a_n = 1/\phi$ and $b_n = \phi \log n$.

3.8 Estimation

Given a random sample X_1, X_2, \dots, X_n from (3.1.10), we wish to estimate the parameters stated above by using the method of moments. By equating the theoretical expressions for $E(X)$ and $E(X^2)$ with the corresponding sample estimates, one obtains the equations:

$$\phi \left\{ 1 - \frac{1}{(1 + \lambda)^2} \right\} = m_1 \quad (\text{if } \lambda > 0), \quad (3.8.25)$$

$$\phi \left\{ \frac{1}{(1 - \lambda)^2} - 1 \right\} = m_1 \quad (\text{if } \lambda < 0) \quad (3.8.26)$$

and

$$2\phi^2 = m_2, \quad (3.8.27)$$

where

$$m_1 = \frac{1}{n} \sum_{i=1}^n x_i$$

and

$$m_2 = \frac{1}{n} \sum_{i=1}^n x_i^2.$$

From (3.8.27), one obtain an estimate of the parameter ϕ , given by

$$\hat{\phi} = \sqrt{\frac{m_2}{2}}. \quad (3.8.28)$$

Substituting this into (3.8.25) and (3.8.26), one get the estimate of λ , namely

$$\hat{\lambda} = \left(1 - m_1 \sqrt{\frac{2}{m_2}} \right)^{-1/2} - 1 \quad (3.8.29)$$

and

$$\hat{\lambda} = 1 - \left(1 + m_1 \sqrt{\frac{2}{m_2}}\right)^{-1/2} \quad (3.8.30)$$

respectively. Note that $\hat{\lambda}$ in (3.8.29) is positive if and only if $m_1 > 0$ and $\hat{\lambda}$ in (3.8.30) is negative if and only if $m_1 < 0$. Thus, depending on whether $m_1 > 0$ or $m_1 < 0$, one would choose either (3.8.29) or (3.8.30) as the estimate of λ .

The estimation of the parameters by the method of Maximum likelihood, MLE, is described below. Let X_1, X_2, \dots, X_n be a random sample from the skew Laplace pdf. Then the likelihood function is given by

$$\begin{aligned} L(\phi, \lambda; x_1, x_2, \dots, x_n) &= \prod_{i=1}^j \left(\frac{1}{2\phi} \exp\left(\frac{(1+\lambda)x_i}{\phi}\right) \right) \\ &\quad \times \prod_{i=j+1}^n \left(\frac{1}{\phi} \exp\left(-\frac{x_i}{\phi}\right) \left[1 - \frac{1}{2} \exp\left(-\frac{\lambda x_i}{\phi}\right)\right] \right) \end{aligned}$$

where we assume that the first j observations take negative values and the rest assume the positive values. To estimate the parameters λ and ϕ we consider the log-likelihood and set equal to zero after differentiating with respect to λ and ϕ respectively and we get the following pair of equations

$$\sum_{i=1}^j x_i + \sum_{i=j+1}^n \frac{x_i \exp\left(-\frac{\lambda x_i}{\phi}\right)}{2 - \exp\left(-\frac{\lambda x_i}{\phi}\right)} = 0 \quad (3.8.31)$$

and

$$n + \frac{1}{\phi} \sum_{i=1}^j x_i - \frac{1}{\phi} \sum_{i=j+1}^n x_i = 0, \quad (3.8.32)$$

Solving this system of equations we obtain the MLE of ϕ ,

$$\hat{\phi} = \frac{1}{n} \left[\sum_{i=j+1}^n x_i - \sum_{i=1}^j x_i \right] \quad (3.8.33)$$

Now, to estimate λ one can simply substitute the value of $\hat{\phi}$ in (3.8.31). In fact if we suppose that all the observations in a random sample is coming from a random variable $X \sim SL(\phi, \lambda)$, with $\lambda > 0$ then the maximum likelihood method produces $\hat{\phi} = \bar{X}$ and $\hat{\lambda} = \infty$, that is, it is as though we are estimating the data set as coming from an exponential distribution.

3.9 Simulation Study

In this section we perform a simulation study to illustrate the flexibility of (3.1.10) over (3.1.7). An ideal technique for simulating from (3.1.10) is the inversion method.

(i) If $\lambda > 0$ then, using equations (3.4.21) and (3.4.22), one would simulate X by

$$X = \frac{\phi}{1 + \lambda} \log \{2(1 + \lambda)U\} \quad (3.9.34)$$

if $0 \leq U \leq 1/\{2(1 + \lambda)\}$ and by

$$X = -\phi \log \left\{ 1 - U + (1 - U) \sum_{j=1}^{\infty} \binom{(1 + \lambda)j}{j - 1} \frac{(1 - U)^{\lambda j}}{j 2^j (1 + \lambda)^j} \right\} \quad (3.9.35)$$

if $1/\{2(1 + \lambda)\} < U \leq 1$, Hence, if $\lambda > 0$ then, using equations one would simulate X by

$$X = \begin{cases} \frac{\phi}{1 + \lambda} \log \{2(1 + \lambda)U\}, & \text{if } 0 \leq U \leq 1/\{2(1 + \lambda)\} \\ -\phi \log \left\{ 1 - U + (1 - U) \sum_{j=1}^{\infty} \binom{(1 + \lambda)j}{j - 1} \frac{(1 - U)^{\lambda j}}{j 2^j (1 + \lambda)^j} \right\}, & \text{if } 1/\{2(1 + \lambda)\} < U \leq 1 \end{cases}$$

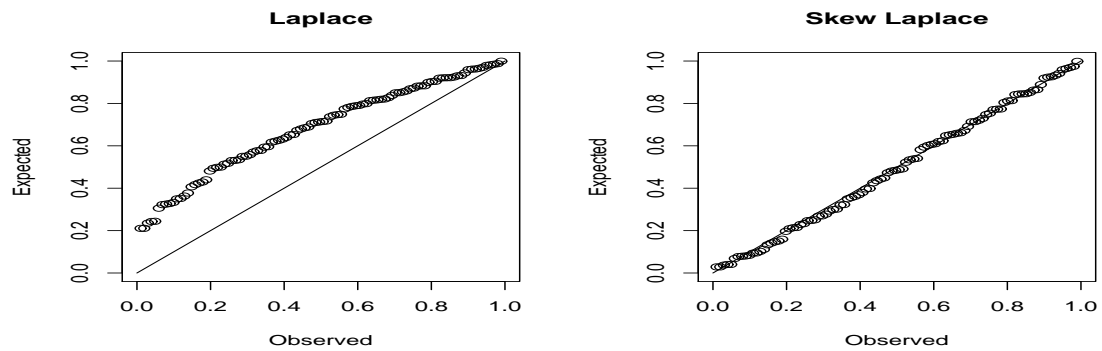
where $U \sim U(0, 1)$ is a uniform random number.

(ii) If $\lambda < 0$ then the value of $-X$ can be simulated by using the same equations with λ replaced by $-\lambda$. Using (3.9.34) and (3.9.35), we simulated two independent samples of size $n = 100$. The parameters were chosen as $(\phi, \lambda) = (1, 1)$ for one of the samples and as $(\phi, \lambda) = (1, -2)$ for the other. This means that one of the samples is

positively skewed while the other is negatively skewed. We fitted both these samples to the two models described by the standard Laplace pdf (equation (3.1.7)) and the skew Laplace pdf (equation (3.1.10)). We used the method of moments described by equations (3.8.28)–(3.8.30) to perform the fitting. All the necessary calculations were implemented by using the R language by Ihaka *et.al*, [11]. The P-P plots arising from these fits are shown in the following Figure 3.3.

It is evident that the skew-Laplace distribution provides a very significant improvement over standard Laplace pdf for both positively and negatively skewed data.

p-p plots for $\phi=1, \lambda=1$



pp plot for $\phi=1$ and $\lambda=-2$

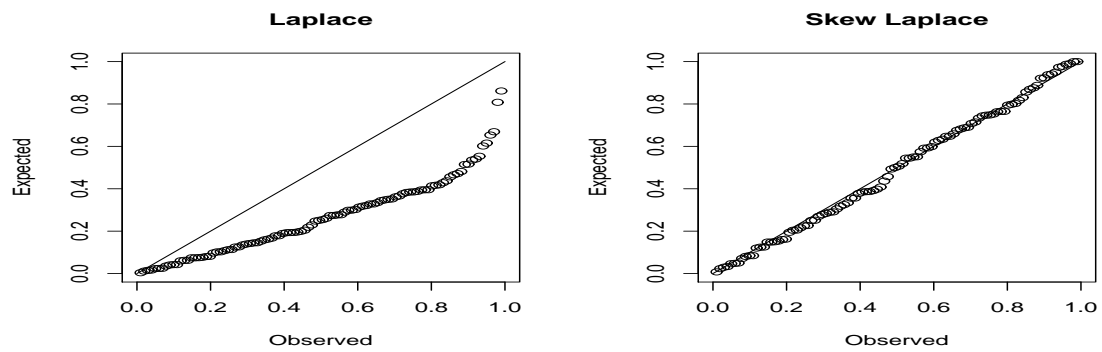


Figure 3.3: P-P plots of Laplace and skew Laplace pdf's for different values of λ

3.10 Conclusion

In this chapter we have completely developed the skew Laplace probability distribution. That is, its mathematical properties, analytical expressions for all important statistical characterization along with the estimations. Utilizing a precise numerical simulation we have illustrated the usefulness of our analytical developments with respect to positive and negative aspects of the skew Laplace pdf.

Finally, we have concluded that the subject pdf is better model for skewed type of data than the other popular models. It is easier to work with because of its analytical tractability and the one to one correspondence between the skewness γ and the shape parameter λ of the model.

Chapter 4

Application of Skew Laplace Probability Distribution

4.1 Introduction

Various versions of the Laplace and skew Laplace pdf have been applied in sciences, engineering and business studies. Recently Julia *et.al* [12] has applied the skew Laplace pdf in Gram-negative bacterial axenic cultures. In this study the cytometric side light scatter(SS) values in Gram-negative bacteria were fitted using the skew-Laplace pdf proposed by Kotz. *et.al* [15] given by

$$f(x; \alpha, \beta, \mu) = \begin{cases} \frac{\alpha\beta}{\alpha + \beta} \exp(-\alpha(\mu - x)), & \text{if } x \leq \mu, \\ \frac{\alpha\beta}{\alpha + \beta} \exp(-\beta(x - \mu)), & \text{if } x > \mu, \end{cases}$$

4.2 An Application of SL Distribution in Finance

In the present study we will present an application of the skew Laplace model presented in the previous chapter for modeling some financial data. Actually an area where the Laplace and related probability distributions can find most interesting and successful application is on modeling of financial data. Traditionally these type of data were modeled using the Gaussian pdf but because of long tails and asymmetry present in the data it is necessary to look for a probability distribution which can account for the skewness and kurtosis differing from Gaussian. Since the Laplace pdf can account for leptokurtic behavior it is the natural choice and moreover if skewness is present in the data then the skew Laplace pdf will take care of it. Hence the skew

Laplace pdf should be considered as the first choice for skewed and kurtotic data. Klein [14] studied yield interest rates on average daily 30 year Treasury bond from 1977 to 1990 and found that empirical distribution is too peaky and fat-tailed so the normal pdf won't be an appropriate model. Kozubowski *et al.* [16] suggested that an asymmetric Laplace model to be the appropriate model for interest rate data arguing that this model is easy and capable of capturing the peakedness, fat-tailedness, skewness and high kurtosis present in the data. Actually they fitted the model for the data set consisting of interest rates on 30 year Treasury bonds on the last working day of the month covering the period of February 1977 through December 1993. Kozubowski and Podgorski [16] fitted asymmetric Laplace (AL) model whose density is given by

$$f(x, \sigma, \kappa) = \frac{1}{\sigma} \frac{\kappa}{1 + \kappa^2} \exp\left(-\frac{\kappa}{\sigma}x^+ - \frac{1}{\kappa\sigma}x^-\right), \quad x \in \mathcal{R}, \sigma > 0, \kappa > 0$$

where $x^+ = \max\{x, 0\}$ & $x^- = \max\{-x, 0\}$ to fit the data set on currency exchange. Actually they fitted the model for German Deutschmark vs. U.S. Dollar and the Japanese Yen Vs. the U.S. Dollar. The observation were daily exchange rate from 01/01/1980 to 12/07/1990, approximately 2853 data points.

Here, we shall illustrate an application of the skew Laplace pdf (3.1.10) that we have studied in the previous chapter to the financial data. The data we consider are annual exchange rates for six different currencies as compared to the United States Dollar, namely, Australian Dollar, Canadian Dollar, European Euro, Japanese Yen, Switzerland Franc and United Kingdom Pound . The data were obtained from the web site <http://www.globalfindata.com/>. The standard change in the log(rate) from year t to year t+1 is used.

The skew Laplace pdf was fitted to each of these data sets by using the method of maximum likelihood. A quasi-Newton algorithm in R was used to solve the likelihood equations. Table 4.1 shows the range of the data and the descriptive statistics of the data. It includes number of observations(n), mean, standard deviation (SD), Skewness (SKEW) and kurtosis (KURT) of the data. Estimation of the parameters ϕ and λ

and the Kolmogorov- Smirnov D statistic considering the Normal, the Laplace and the SL models is given in Table 4.2 for the subject data.

Currency	Years of Data	n	$Mean$	SD	$SKEW$	$KURT$
Australian Dollar	1822-2003	182	3.626	1.648	-0.703	2.047
Canadian Dollar	1858-2003	146	1.106	0.176	2.882	16.12
European Euro	1950-2003	54	1.088	0.147	0.307	3.06
Japanese Yen	1862-2003	142	101.672	138.918	0.981	2.294
Switzerland Franc	1819-2003	185	4.105	1.269	-0.932	2.845
United Kingdom Pound	1800-2003	204	4.117	1.384	0.0264	5.369

Table 4.1: Descriptive statistics of the currency exchange data

$Currency$	$\hat{\phi}$	$\hat{\lambda}$	D_{Normal}	$D_{Laplace}$	$D_{SkewLaplace}$
Australian Dollar	0.0431	0.0123	0.3414	0.2983	0.1823
Canadian Dollar	0.3846	0.0182	0.4125	0.3846	0.1254
European Euro	0.5556	0.0157	0.2310	0.1765	0.1471
Japanese Yen	0.0651	0.0131	0.3140	0.2971	0.2246
Switzerland Franc	0.0312	$-1.627e - 07$	0.3168	0.2772	0.1957
United Kingdom Pound	0.0245	0.0039	0.3114	0.2709	0.1478

Table 4.2: Estimated parameters and Kolmogorov-Smirnov D-statistic of currency exchange data

The figures (4.1-4.6) show how well the this financial data fits the skew Laplace pdf for the subject data sets. It is evident that the fits are good. The Kolmogorov-Smirnov D-statistic on fitting the SL pdf is compared with the Normal and Laplace pdf's for each data. Table 4.2 shows that for each data the skew Laplace pdf fits better than the the Gaussian and the Laplace pdf.

We also fitted the data considering the Box-cox transformation. This is a transformation defined by

$$f(x) = \begin{cases} \frac{x^\eta - 1}{\eta} & \text{if } \eta \neq 0 \\ \log x & \text{if } \eta = 0. \end{cases}$$

Using the Box-Cox transformation we have a significant improvement over the log transformation.

Table 4.3 shows the values of the transformation parameter η . Also the table includes the estimated parameters of the skew Laplace pdf for transformed data and the Kolmogorov-Smirnov D-statistic for the SL pdf, the Laplace pdf and the Normal pdf.

<i>Currency</i>	$\hat{\eta}$	$\hat{\phi}$	$\hat{\lambda}$	D_{Normal}	$D_{Laplace}$	$D_{SkewLaplace}$
Australian Dollar	1.36824	0.053	0.0137	0.3133	0.2486	0.1326
Canadian Dollar	-5.00	0.357	0.0162	0.4040	0.3675	0.0461
European Euro	0.5	0.555	0.0181	0.2330	0.1765	0.1471
Japanese Yen	-0.1727	0.0630	0.0098	0.3349	0.3043	0.2536
Switzerland Franc	2	0.0530	0.0094	0.3085	0.2446	0.1413
United Kingdom Pound	1	0.0369	0.0069	0.2950	0.2414	0.1281

Table 4.3: Kolmogorov-Smirnov D-statistic for currency exchange data using Box-Cox transformations

In each of the figures (4.1-4.6) we can see that the actual data is picky and skewed so it is obvious that neither the usual Gaussian nor the Laplace pdf would fit the data. Hence, we choose SL pdf to fit the data using both log transformation and Box-Cox transformation. In fact, if we carefully observe the figures the data with Box-Cox transformation support the SL pdf better than the log transformation.

Percentage Change in Exchange Rates of Australian Dollar to US Dollar

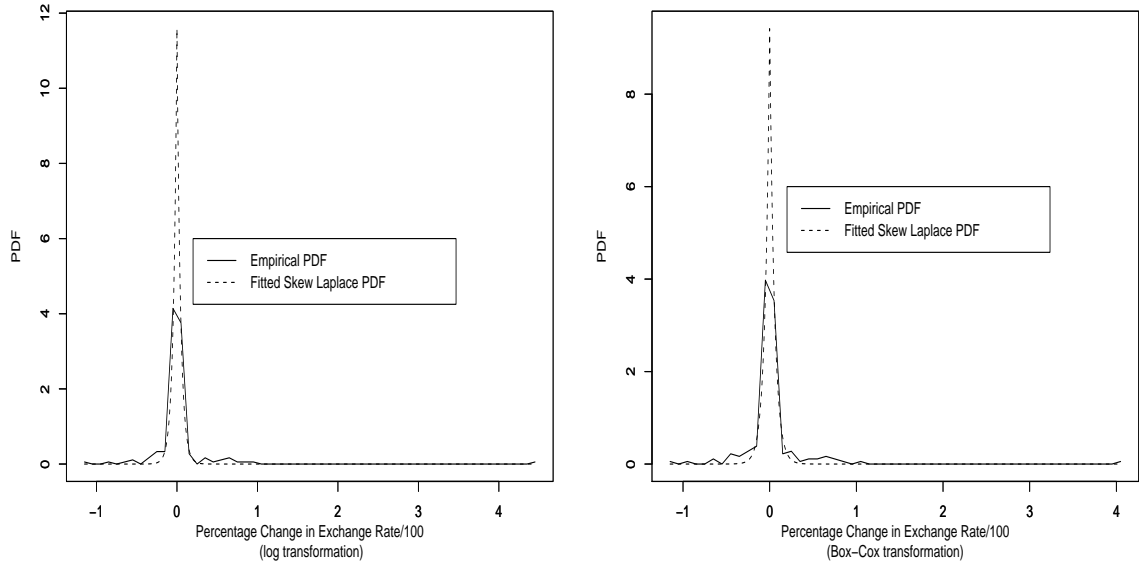


Figure 4.1: Fitting SL model for Australian Dollar exchange rate data

Percentage Change in Exchange Rates of Canadian Dollar to US Dollar

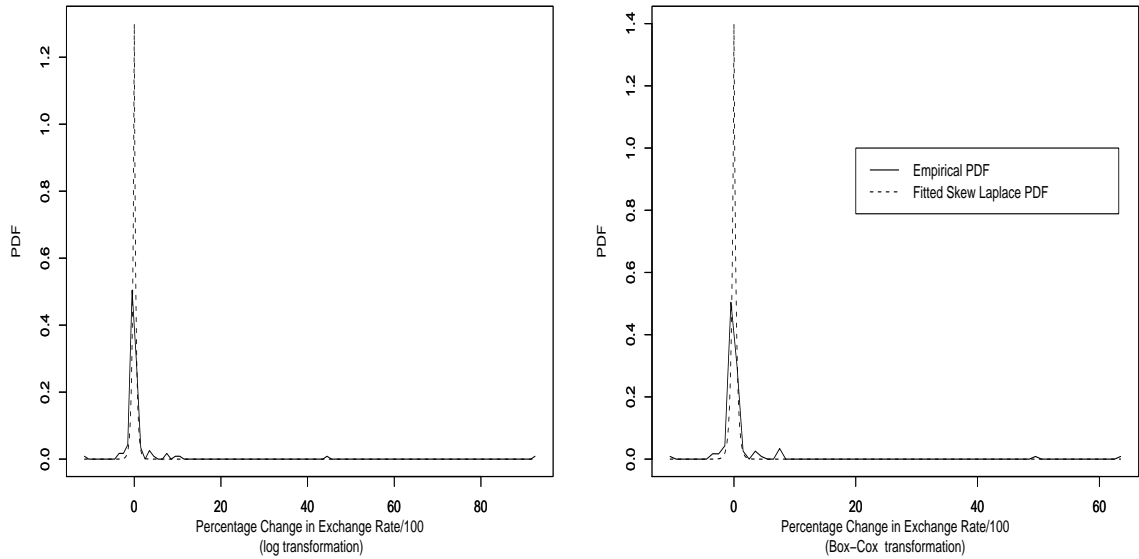


Figure 4.2: Fitting SL model for Canadian Dollar exchange rate data

Percentage Change in Exchange Rates of European Euro to US Dollar

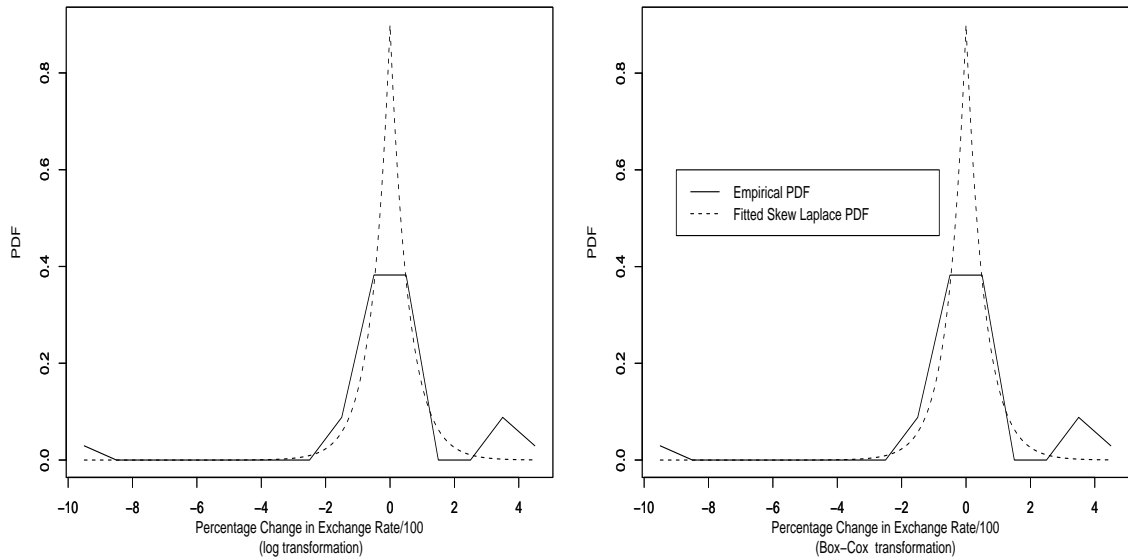


Figure 4.3: Fitting SL model for European Euro exchange rate data

Percentage Change in Exchange Rates of Japanese Yen to US Dollar

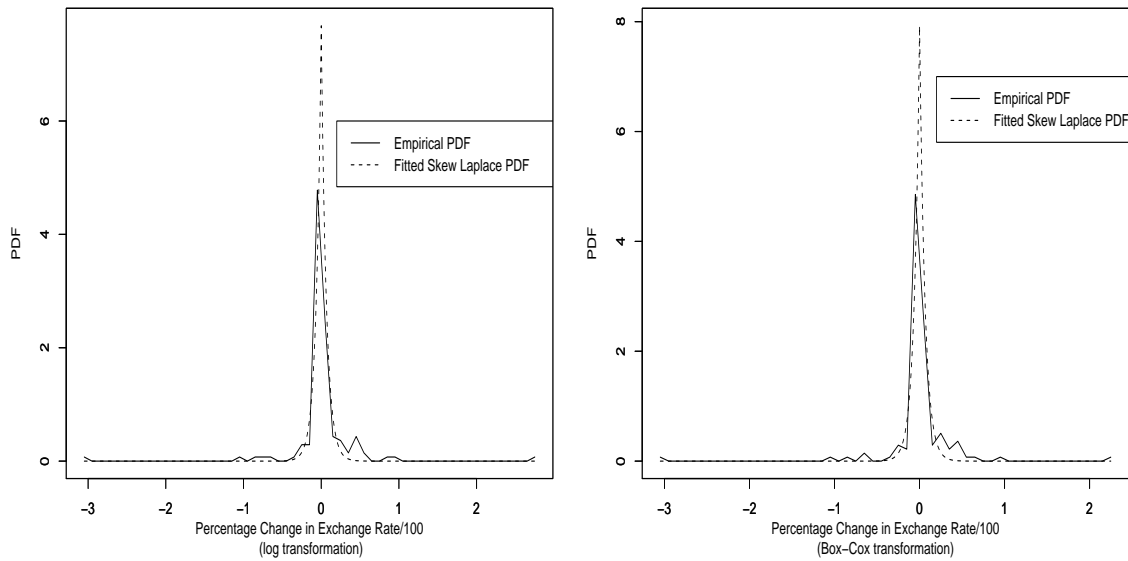


Figure 4.4: Fitting SL model for Japanese Yen exchange data

Percentage Change in Exchange Rates of Switzerland Franc to US Dollar

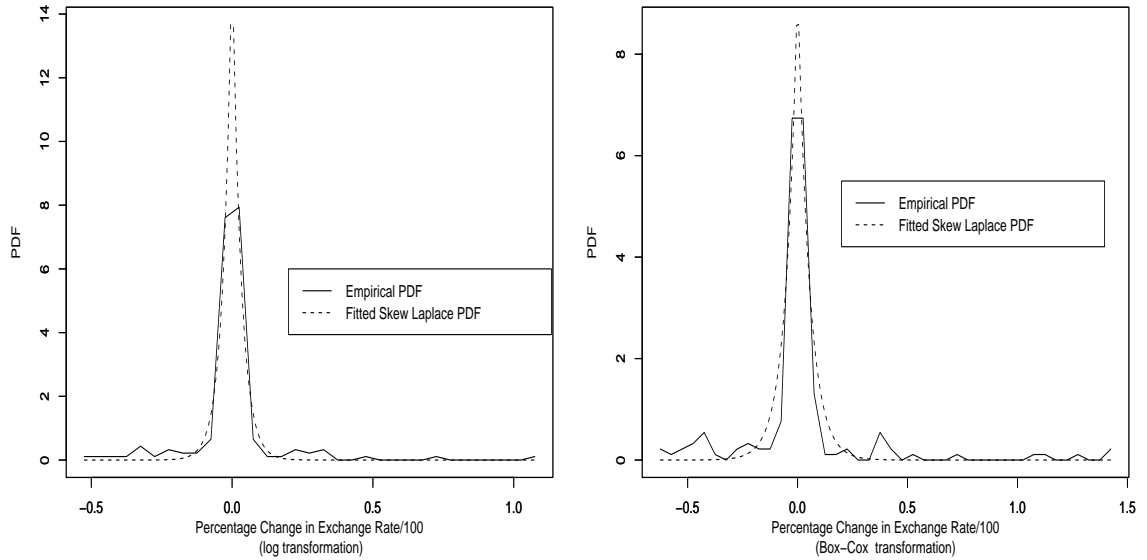


Figure 4.5: Fitting SL model for Switzerland Franc exchange rate data

Percentage Change in Exchange Rates of United Kingdom Pound to US Dollar

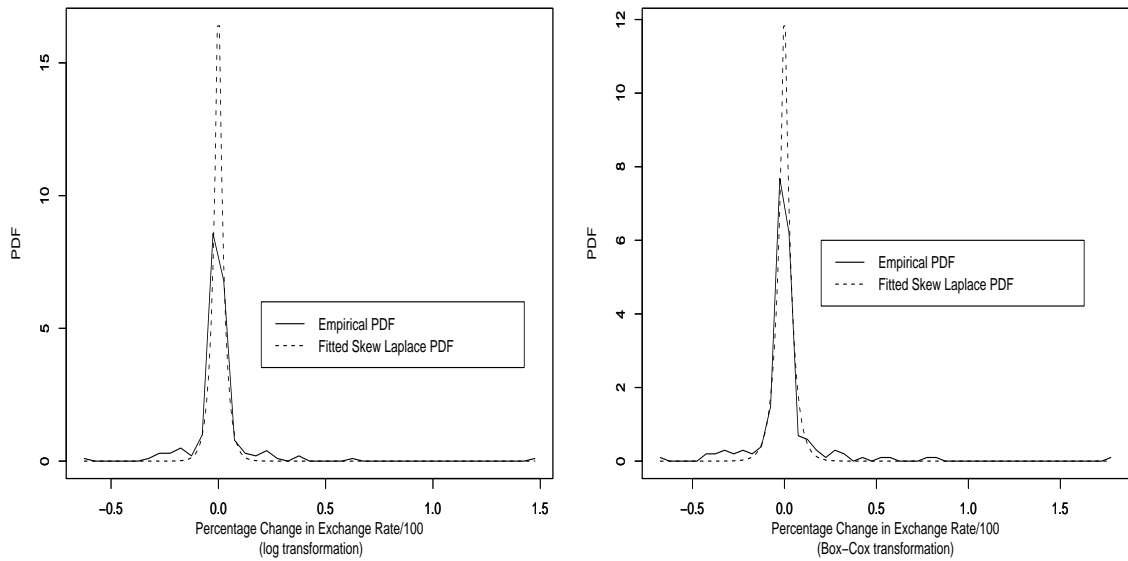


Figure 4.6: Fitting SL model for United Kingdom Pound exchange rate data

4.3 Conclusion

In the present study we have identified a real world financial data, that is, the exchange rate for six different currencies, namely, Australian Dollar, Canadian Dollar, European Euro, Japanese Yen, Switzerland Franc and United Kingdom Pound with respect to US Dollar. Traditionally the financial analysts are using the classical Gaussian pdf to model such data. We have shown that the SL pdf fits significantly better the subject data than the Gaussian and the Laplace pdf. Thus, in performing inferential analysis on the exchange rate data, one can obtain much better results which will lead to minimizing the risk in a decision making process. The goodness of fit comparisons was based on two different approach, namely, using the log transformation and using the Box-Cox transformation. Table 4.2 and table 4.3 show that in either case the SL pdf fits better the subject data than the Gaussian pdf and Laplace pdf.

Chapter 5

On the Truncated Skew Laplace Probability Distribution

5.1 Introduction

To describe a life phenomenon we will be mostly interested when the random variable is positive. Thus, we now consider the case when skew Laplace pdf is truncated to the left 0. Throughout this study, unless otherwise stated we shall assume that $\lambda > 0$. In this case we can write

$$\begin{aligned} F^*(x) = \Pr(X < x | x > 0) &= \frac{\int_0^x f(t) dt}{1 - F(0)} \\ &= \frac{F(x) - F(0)}{1 - F(0)} \end{aligned}$$

Hence, it can be shown that the cdf of the truncated Skew Laplace, TSL, random variable is given by

$$F^*(x) = 1 + \frac{\exp\left(-\frac{(1+\lambda)x}{\phi}\right) - 2(1+\lambda)\exp\left(-\frac{x}{\phi}\right)}{(2\lambda+1)} \quad (5.1.1)$$

and the corresponding probability density function by

$$f^*(x) = \begin{cases} \frac{(1+\lambda)}{\phi(2\lambda+1)} \left\{ 2\exp\left(-\frac{x}{\phi}\right) - \exp\left(-\frac{(1+\lambda)x}{\phi}\right) \right\} & \text{if } x > 0, \\ 0 & \text{otherwise} \end{cases} \quad (5.1.2)$$

Aryal *et al.* [1], proposed this probability distribution as a reliability model.

A graphical presentation of $f^*(x)$ for $\phi = 1$ and various values of λ is given in Figure 5.1.

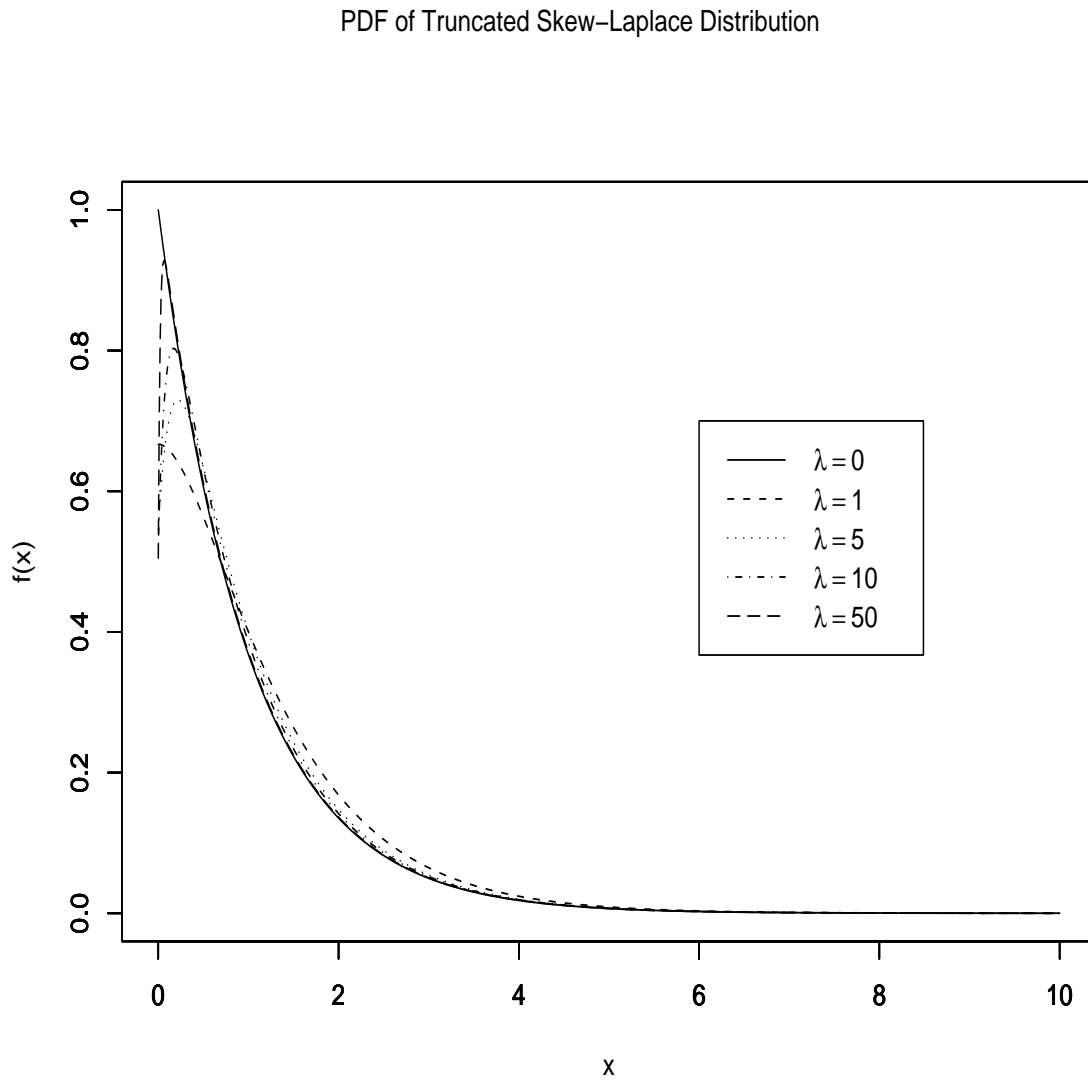


Figure 5.1: PDF of truncated skew Laplace distribution for $\phi = 1$ and $\lambda = 0, 1, 2, 5, 10, 50$

In this study we will provide a comprehensive description of the mathematical properties of (5.1.2). In particular, we shall derive the formulas for the k th moment, variance, skewness, kurtosis, moment generating function, characteristic function, cumulant generating function, the k th cumulant, mean deviation about the mean, expressions for Rényi entropy, Shannon's entropy, cumulative residual entropy. Also we will study reliability and hazard rate behavior of the subject pdf.

5.2 Moments

If X be a random variable with pdf given by (5.1.2), then using the definition of the gamma function, it is easy to show that the k th moment of X is given by

$$E(X^k) = \frac{\phi^k(1+\lambda)\Gamma(k+1)}{(2\lambda+1)} \left\{ 2 - \frac{1}{(1+\lambda)^{k+1}} \right\}. \quad (5.2.3)$$

Using the Binomial expansion and (5.2.3), the k th central moment of X can be derived to be given by

$$E\{(X - \mu)^k\} = \begin{cases} \mu^k + \sum_{j=1}^{\frac{k}{2}} \binom{k}{2j} \mu^{k-2j} \phi^{2j} \frac{(1+\lambda)\Gamma(2j+1)}{(2\lambda+1)} \\ \times \left\{ 2 - \frac{1}{(1+\lambda)^{2j+1}} \right\} \\ - \sum_{j=1}^{\frac{k}{2}} \binom{k}{2j-1} \mu^{k-2j+1} \phi^{2j-1} \frac{(1+\lambda)\Gamma(2j)}{(2\lambda+1)} \\ \times \left\{ 2 - \frac{1}{(1+\lambda)^{2j}} \right\}, & \text{if } k \text{ is even,} \\ -\mu^k - \sum_{j=1}^{\frac{k-1}{2}} \binom{k}{2j} \mu^{k-2j} \phi^{2j} \frac{(1+\lambda)\Gamma(2j+1)}{(2\lambda+1)} \\ \times \left\{ 2 - \frac{1}{(1+\lambda)^{2j+1}} \right\} \\ + \sum_{j=1}^{\frac{k+1}{2}} \binom{k}{2j-1} \mu^{k-2j+1} \phi^{2j-1} \frac{(1+\lambda)\Gamma(2j)}{(2\lambda+1)} \\ \times \left\{ 2 - \frac{1}{(1+\lambda)^{2j}} \right\}, & \text{if } k \text{ is odd.} \end{cases} \quad (5.2.4)$$

where $\mu = E(X)$ is the expectation of X . It follows from (5.2.3) and (5.2.4) that the expectation, variance, skewness and kurtosis of X are given by

$$\begin{aligned}\text{Exp}(X) &= \phi \frac{(1 + 4\lambda + 2\lambda^2)}{(1 + \lambda)(1 + 2\lambda)} \\ \text{Var}(X) &= \phi^2 \frac{(1 + 8\lambda + 16\lambda^2 + 12\lambda^3 + 4\lambda^4)}{(1 + \lambda)^2(1 + 2\lambda)^2} \\ \text{Ske}(X) &= \frac{2(1 + 12\lambda + 42\lambda^2 + 70\lambda^3 + 66\lambda^4 + 36\lambda^5 + 8\lambda^6)}{(1 + 8\lambda + 16\lambda^2 + 12\lambda^3 + 4\lambda^4)^{3/2}}\end{aligned}$$

and

$$\begin{aligned}\text{Kur}(X) &= \frac{3(176\lambda^{10} + 1408\lambda^9 + 4944\lambda^8 + 10000\lambda^7 + 12824\lambda^6 + 10728\lambda^5)}{(1 + \lambda)^2(1 + 8\lambda + 16\lambda^2 + 12\lambda^3 + 4\lambda^4)^2} \\ &\quad + \frac{(5800\lambda^4 + 1992\lambda^3 + 427\lambda^2 + 54\lambda + 3)}{(1 + \lambda)^2(1 + 8\lambda + 16\lambda^2 + 12\lambda^3 + 4\lambda^4)^2}\end{aligned}$$

A graphical representation of these statistical expressions are given in figure 5.2 as a function of the parameter λ .

Note that for $\lambda = 0, \phi = 1$ we have $\text{Exp}(X) = 1, \text{Var}(X) = 1, \text{Skewness}(X) = 2,$ and $\text{Kurtosis}(X) = 9$ yield the standard exponential distribution. Also it is clear that both expectation and variance are first increasing and then decreasing functions of λ . The expectation increases from ϕ to 1.17157ϕ as λ converges from 0 to $\frac{1}{\sqrt{2}}$ and then decreases to ϕ as λ goes to ∞ . On the other hand the variance increases from ϕ^2 to $1.202676857\phi^2$ as λ increases from 0 to 0.3512071921 and then decreases to ϕ^2 as λ goes to ∞ .

Behavior of Expectation, Variance, Skewness and Kurtosis

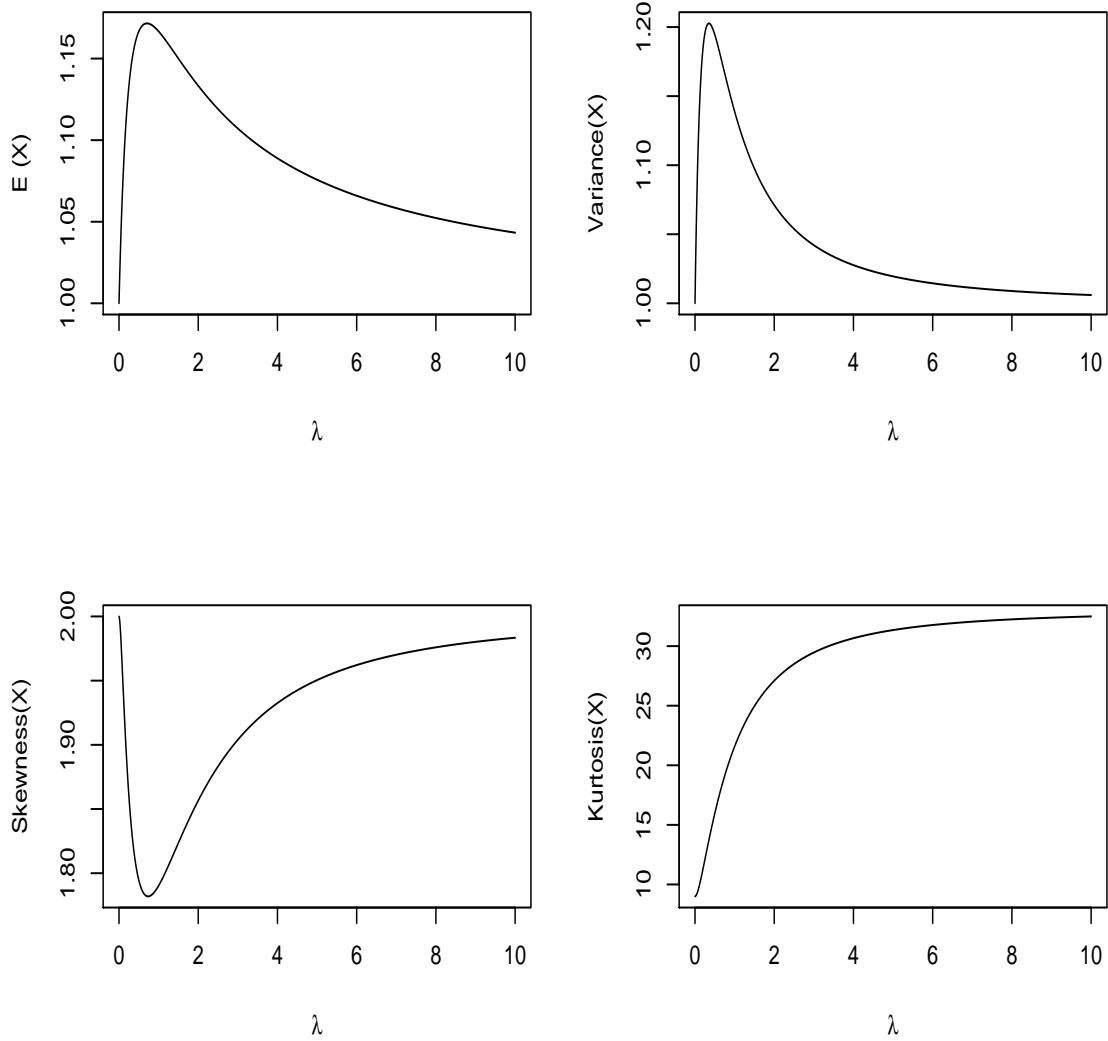


Figure 5.2: Behavior of expectation, variance, skewness and kurtosis of TSL distribution

5.3 MGF and Cumulants

The moment generating function of a random variable X is defined by $M(t) = E(\exp(tX))$. When X has the pdf (5.1.2) direct integration yields that

$$M(t) = \frac{(1 + \lambda)}{(1 + 2\lambda)} \left(\frac{(1 + 2\lambda - \phi t)}{(1 - \phi t)(1 + \lambda - \phi t)} \right)$$

for $t < 1/\phi$. Thus, the characteristic function defined by $\psi(t) = E(\exp(itX))$ and the cumulant generating function defined by $K(t) = \log(M(t))$ take the forms

$$\psi(t) = \frac{(1 + \lambda)}{(1 + 2\lambda)} \left(\frac{(1 + 2\lambda - i\phi t)}{(1 - i\phi t)(1 + \lambda - i\phi t)} \right)$$

and

$$K(t) = \log \left(\frac{1 + \lambda}{1 + 2\lambda} \right) + \log \left(\frac{(1 + 2\lambda - \phi t)}{(1 - \phi t)(1 + \lambda - \phi t)} \right),$$

respectively, where $i = \sqrt{-1}$ is the complex number. By expanding the cumulant generating function as

$$K(t) = \sum_{k=1}^{\infty} a_k \frac{(t)^k}{k!},$$

one obtains the cumulants a_k given by

$$a_k = (k - 1)! \phi^k \left(1 + \frac{1}{(1 + \lambda)^k} - \frac{1}{(1 + 2\lambda)^k} \right)$$

5.4 Percentiles

As mention in chapter 3 we are always interested to compute the percentiles. The 100 p th percentile x_p is defined by $F(x_p) = p$, where F is given by (5.1.1). Then x_p is the solution of the transcendental equation

$$1 + \frac{\exp\left(-\frac{(1+\lambda)x_p}{\phi}\right) - 2(1+\lambda)\exp\left(-\frac{x_p}{\phi}\right)}{(2\lambda+1)} = p$$

Substituting $y_p = \frac{2(1+\lambda)\exp(-\frac{x_p}{\phi})}{(1+2\lambda)(1-p)}$, this equation can be reduced to

$$1 - y_p + \frac{(2\lambda+1)^\lambda(1-p)^\lambda}{[2(1+\lambda)]^{1+\lambda}} y_p^{1+\lambda} = 0,$$

which takes the form of (3.1.14). Thus, using (3.1.15), y_p is given by

$$y_p = 1 + \sum_{j=1}^{\infty} \binom{(1+\lambda)j}{j-1} \frac{1}{j} \frac{[(1+2\lambda)(1-p)]^{\lambda j}}{(2+2\lambda)^{(1+\lambda)j}}$$

and hence the solution for x_p is given by

$$x_p = -\phi \log \left\{ \frac{(1-p)(1+2\lambda)}{2(1+\lambda)} \left(1 + \sum_{j=1}^{\infty} \binom{(1+\lambda)j}{j-1} \frac{1}{j} \frac{[(1+2\lambda)(1-p)]^{\lambda j}}{(2+2\lambda)^{(1+\lambda)j}} \right) \right\} \quad (5.4.5)$$

5.5 Mean Deviation

As mention in chapter 3, if we are interested to find the amount of scatter in a population is evidently measured to some extent by the totality of deviations from the mean. This is known as the mean deviation about the mean and it is defined by

$$\delta_1(X) = \int_0^{\infty} |x - \mu| f(x) dx$$

where $\mu = E(X)$. This measures can be calculated using the relationships that

$$\delta_1(X) = \int_0^\mu (\mu - x)f(x)dx + \int_\mu^\infty (x - \mu)f(x)dx.$$

Thus, for a TSL random variable X we have

$$\begin{aligned} \delta_1(X) = & \frac{4\phi(1+\lambda)}{(1+2\lambda)} \exp\left(-\frac{(1+4\lambda+2\lambda^2)}{(1+\lambda)(1+2\lambda)}\right) \\ & - \frac{2\phi}{(1+\lambda)(1+2\lambda)} \exp\left(-\frac{(1+4\lambda+2\lambda^2)}{(1+2\lambda)}\right). \end{aligned} \quad (5.5.6)$$

5.6 Entropy

As mention in chapter 3, we study the entropy to measure the variation of the uncertainty. Rényi entropy is defined by

$$\mathcal{J}_R(\gamma) = \frac{1}{1-\gamma} \log \left\{ \int f^\gamma(x) dx \right\}, \quad (5.6.7)$$

where $\gamma > 0$ and $\gamma \neq 1$ (Rényi,) [19]. For the pdf (5.1.2), note that

$$\begin{aligned} \mathcal{J}_R(\gamma) = & \frac{1}{(1-\gamma)} \left\{ \log \left(\frac{(1+\lambda)}{\phi(1+2\lambda)} \right)^\gamma \right\} \\ & + \left\{ \log \left(\int_0^\infty \left(2 \exp\left(-\frac{x}{\phi}\right) - \exp\left(-\frac{(1+\lambda)x}{\phi}\right) \right)^\gamma dx \right) \right\}. \end{aligned}$$

Further calculation yields the entropy in terms of incomplete beta function as

$$\mathcal{J}_R(\gamma) = \frac{\gamma}{1-\gamma} \log \left(\frac{1+\lambda}{\phi(1+2\lambda)} \right) + \frac{1}{1-\gamma} \log \left(\frac{\phi}{\lambda} 2^{\gamma(1+\frac{1}{\lambda})} B_{1/2} \left(\frac{\gamma}{\lambda}, \gamma+1 \right) \right).$$

The Shanon entropy of a a distribution F is defined by

$$H(F) = - \sum_i p_i \log p_i$$

where p_i 's are the probabilities computed from the distribution F . However, the extension of the discrete case to the continuous case with distribution F and density f

is called differential entropy and is given by

$$H(F) = E(-\log f(x)) = - \int f(x) \log f(x) dx$$

However this extension has a few drawbacks as pointed out by Rao *et al.* [23], like, it may assume any value on the extended real line, it is defined for the distributions with densities only.

Rao *et al.* [23] introduced the *cumulative residual entropy* (CRE) defined by

$$\mathcal{E}(X) = - \int \Pr(|X| > x) \log \Pr(|X| > x) dx,$$

which is more general than Shannon's entropy in that the definition is valid in the continuous and discrete domains. For the pdf (5.1.2), we can write

$$\Pr(|X| > x) = \frac{2(1+\lambda) \exp(-\frac{x}{\phi}) - \exp(-\frac{(1+\lambda)x}{\phi})}{(1+2\lambda)}.$$

Hence,

$$\begin{aligned} \mathcal{E}(X) &= \frac{1}{(1+2\lambda)} \int_0^\infty \exp(-\frac{(1+\lambda)x}{\phi}) \log \left(\frac{2(1+\lambda) \exp(-\frac{x}{\phi}) - \exp(-\frac{(1+\lambda)x}{\phi})}{(1+2\lambda)} \right) dx \\ &\quad - \frac{2(1+\lambda)}{(1+2\lambda)} \int_0^\infty \exp(-\frac{x}{\phi}) \log \left(\frac{2(1+\lambda) \exp(-\frac{x}{\phi}) - \exp(-\frac{(1+\lambda)x}{\phi})}{(1+2\lambda)} \right) dx. \end{aligned}$$

Using Taylor expansion and on integrating we have the CRE given by

$$\begin{aligned} \mathcal{E}(X) &= \frac{\phi(1+6\lambda+6\lambda^2+2\lambda^3)}{(1+\lambda)^2(1+2\lambda)} + \frac{\phi(1+4\lambda+2\lambda^2) \log(1+2\lambda)}{(1+\lambda)(1+2\lambda)} \\ &\quad - \frac{\phi}{(1+2\lambda)} \sum_{k=1}^{\infty} \frac{2^{-k}(1+\lambda)^{-k}}{k(\lambda k + \lambda + 1)} + \frac{2\phi(1+\lambda)}{(1+2\lambda)} \sum_{k=1}^{\infty} \frac{2^{-k}(1+\lambda)^{-k}}{k(\lambda k + 1)} \\ &\quad - \frac{\phi(1+4\lambda+2\lambda^2) \log(2+2\lambda)}{(1+\lambda)(1+2\lambda)} \end{aligned} \tag{5.6.8}$$

Note that $\mathcal{E}(X)$ in (5.6.8) is positive and each series is convergent and is bounded above by 1. Thus, we have

$$\mathcal{E}(X) \leq \frac{\phi(1 + 6\lambda + 6\lambda^2 + 2\lambda^3)}{(1 + \lambda)^2(1 + 2\lambda)} + \frac{\phi(1 + 4\lambda + 2\lambda^2) \log(1 + 2\lambda)}{(1 + \lambda)(1 + 2\lambda)} + \frac{2\phi(1 + \lambda)}{(1 + 2\lambda)}.$$

5.7 Estimation

Given a random sample X_1, \dots, X_n from (5.1.2), we are interested in estimating the inherent parameters using the method of moments. By equating the theoretical expressions for $E(X)$ and $E(X^2)$ with the corresponding sample estimates, one obtains the equations:

$$m_1 = \phi \frac{(1 + 4\lambda + 2\lambda^2)}{(1 + \lambda)(1 + 2\lambda)} \quad (5.7.9)$$

and

$$m_2 = 2\phi^2 \frac{(1 + 6\lambda + 6\lambda^2 + 2\lambda^3)}{(1 + \lambda)^2(1 + 2\lambda)} \quad (5.7.10)$$

where

$$m_1 = \frac{1}{n} \sum_{i=1}^n x_i$$

and

$$m_2 = \frac{1}{n} \sum_{i=1}^n x_i^2.$$

On solving this system of equations we have

$$(8m_1^2 - 4m_2)\lambda^4 + (28m_1^2 - 16m_2)\lambda^3 + (36m_1^2 - 20m_2)\lambda^2 + (16m_1^2 - 8m_2)\lambda + (2m_1^2 - m_2) = 0$$

which contains the parameter λ only. One can solve this equation for λ . Using either (5.7.9) or (5.7.10) we can get the corresponding value of ϕ .

The method of maximum likelihood to estimate the parameters is described below:

The underlying likelihood function for a complete sample of size n is given by

$$L(\lambda, \phi; x_1, x_2, \dots, x_n) = \frac{(1 + \lambda)^n}{\phi^n (1 + 2\lambda)^n} \prod_{i=1}^n \left\{ 2 \exp\left(-\frac{x_i}{\phi}\right) - \exp\left(-\frac{(1 + \lambda)x_i}{\phi}\right) \right\} \quad (5.7.11)$$

The corresponding log-likelihood function is given by:

$$\ln L(\lambda, \phi) = n \ln(1 + \lambda) - n \ln \phi - n \ln(1 + 2\lambda) - \frac{1}{\phi} \sum_{i=1}^n x_i + \sum_{i=1}^n \ln\left[2 - \exp\left(-\frac{\lambda x_i}{\phi}\right)\right].$$

Taking the derivative of the above function with respect to λ and ϕ and equating each equations to zero we obtain

$$\frac{n}{1 + \lambda} - \frac{2n}{1 + 2\lambda} + \frac{1}{\phi} \sum_{i=1}^n x_i \frac{\exp\left(-\frac{\lambda x_i}{\phi}\right)}{\left[2 - \exp\left(-\frac{\lambda x_i}{\phi}\right)\right]} = 0 \quad (5.7.12)$$

and

$$-\frac{n}{\phi} + \frac{1}{\phi^2} \sum_{i=1}^n x_i - \frac{\lambda}{\phi^2} \sum_{i=1}^n x_i \frac{\exp\left(-\frac{\lambda x_i}{\phi}\right)}{\left[2 - \exp\left(-\frac{\lambda x_i}{\phi}\right)\right]} = 0. \quad (5.7.13)$$

These equations can not be solved analytically but statistical software can be used to find the maximum likelihood estimators of λ and ϕ .

5.8 Reliability and Hazard Rate Functions

The subject pdf can be a useful characterization of failure time of a given system because of the analytical structure. Thus, The reliability function $R(t)$, which is the probability of an item not failing prior to some time t , is defined by $R(t) = 1 - F(t)$. The reliability function for TSL(λ, ϕ) probability distribution is given by

$$R(t) = \frac{2(1 + \lambda) \exp\left(-\frac{t}{\phi}\right) - \exp\left(-\frac{(1+\lambda)t}{\phi}\right)}{(2\lambda + 1)}. \quad (5.8.14)$$

The hazard rate function, also known as instantaneous failure rate function is defined by

$$h(t) = \lim_{\Delta t \rightarrow 0} \frac{Pr(t < T \leq t + \Delta t | T > t)}{\Delta t} = \frac{f(t)}{R(t)}.$$

It is immediate from (5.1.2) and (5.1.1) that the hazard rate function for TSL distribution is given

$$h(t) = \frac{(1 + \lambda)}{\phi} \frac{\left\{2 - \exp\left(\frac{-\lambda t}{\phi}\right)\right\}}{\left\{2 + 2\lambda - \exp\left(\frac{-\lambda t}{\phi}\right)\right\}}.$$

It is clear that h is an increasing function, it increases from $\frac{1}{\phi} \frac{(1+\lambda)}{(1+2\lambda)}$ to $\frac{1}{\phi}$ as t varies from 0 to ∞ .

This is an important feature of this pdf which makes it quite different from the exponential and Weibull pdf's where the hazard rate for an exponential distribution is constant whereas the hazard rate for Weibull distribution is either strictly increasing or strictly decreasing. In the present study we will present a comparisons of TSL distribution with some other life time distribution whose probability density function are similar to that of TSL distribution.

The cumulative hazard rate function for TSL is given by

$$\begin{aligned} H(t) &= \int_0^t h(u)du = -\log(R(t)) \\ &= \frac{t}{\phi} + \log\left(\frac{1+2\lambda}{2+2\lambda-\exp(-\frac{\lambda t}{\phi})}\right). \end{aligned}$$

Also we have

$$A(t) = \frac{1}{t} \int_0^t h(\tau)d\tau,$$

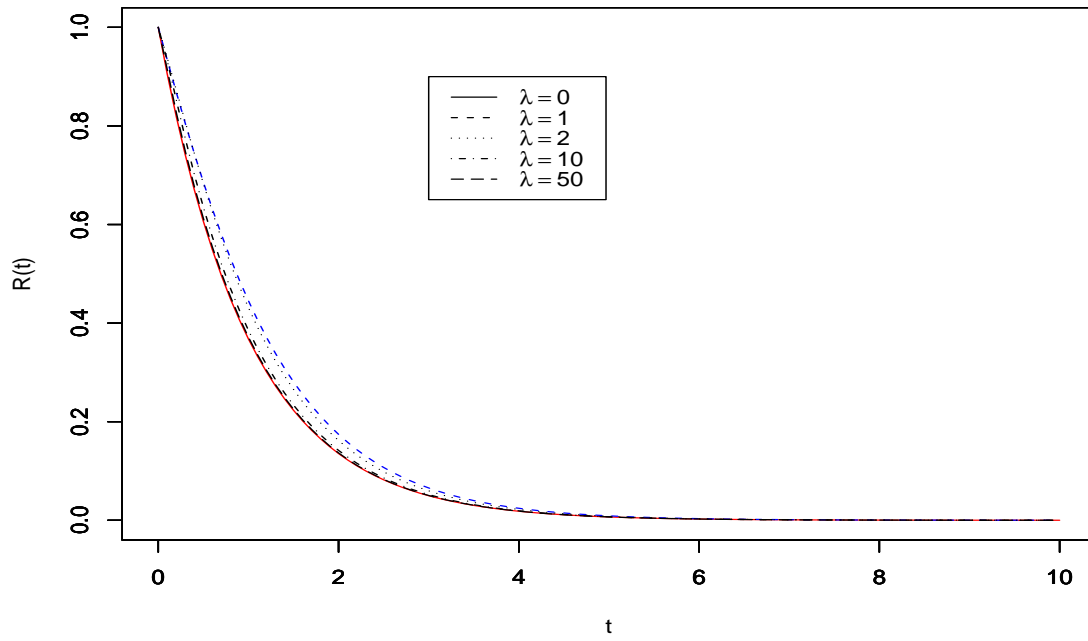
as the failure(hazard) rate average. Thus, for a TSL random variable the failure rate average is given by

$$A(t) = \frac{1}{\phi} + \frac{1}{t} \log\left(\frac{1+2\lambda}{2+2\lambda-\exp(-\frac{\lambda t}{\phi})}\right).$$

A graphical representation of the reliability $R(t)$ and the hazard rate $h(t)$ for $\phi = 1$ and various values of the parameter λ is given in Figure 5.3.

Note that TSL has increasing failure rate(IFR) hence the reliability function is decreasing. Also note from the figures that significance differences occur at the early time. In fact the hazard rate is constant for $\lambda = 0$ which means exponential pdf is a particular case of TSL pdf. Moreover, for a given value of the parameter ϕ , the reliability increases until it attains $\lambda = 1$ and then it decreases.

Reliability for TSL Distribution



Hazard rate for TSL Distribution

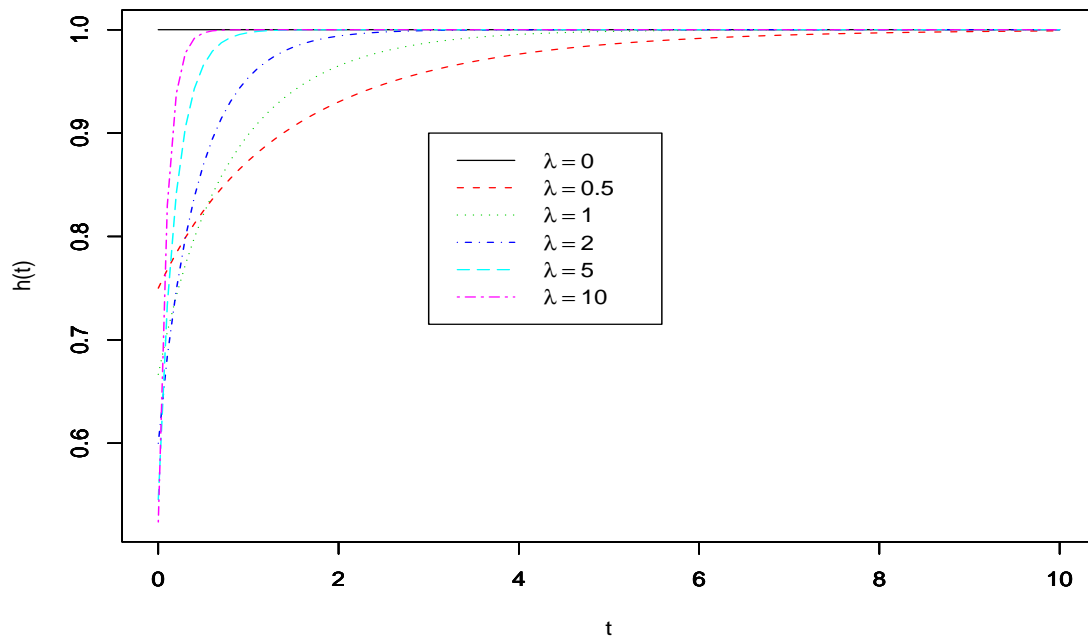


Figure 5.3: Reliability and hazard rate of TSL distribution for $\phi = 1$

5.9 Mean Residual Life Time and the Mean Time Between Failure

The mean residual life (MRL) at a given time t measures the expected remaining life time of an individual of age t . It is denoted by $m(t)$ and is defined as

$$\begin{aligned} m(t) &= E(T - t | T \geq t) \\ &= \frac{\int_t^\infty R(u) du}{R(t)} \end{aligned}$$

The cumulative hazard rate function is given by $H(t) = -\log(R(t))$ and we can express the mean residual life time in terms of H by

$$m(t) = \int_0^\infty \exp(H(t) - H(t+x)) dx \quad (5.9.15)$$

Now, if we consider the converse problem, that of expressing the failure rate in terms of the mean residual life and its derivatives we have

$$m'(t) = h(t)m(t) - 1. \quad (5.9.16)$$

Hence, the MRL for a TSL random variable is given by

$$m(t) = \frac{\phi}{(1+\lambda)} \left\{ \frac{2(1+\lambda)^2 - \exp(-\frac{\lambda t}{\phi})}{2(1+\lambda) - \exp(-\frac{\lambda t}{\phi})} \right\} \quad (5.9.17)$$

Now, we discuss the mean time between failure (MTBF) for the truncated skew-Laplace pdf. The time difference between the expected next failure time and current failure time is called the *Mean Time Between Failure* (MTBF). Many scientists and engineers consider the reciprocal of the intensity function (also called the hazard rate function) at current failure time as the MTBF. That is,

$$MTBF = \frac{1}{\nu(t)} \quad (5.9.18)$$

where $\nu(t)$ is the intensity function. Based on this definition the MTBF for TSL

distribution will be given by

$$MTBF = \frac{\phi}{(1 + \lambda)} \frac{\left\{ 2(1 + \lambda) - \exp\left(-\frac{\lambda t}{\phi}\right) \right\}}{\left\{ 2 - \exp\left(-\frac{\lambda t}{\phi}\right) \right\}}. \quad (5.9.19)$$

But the Mean time between failure is indeed the expected interval length from the current failure time, say $T_n = t_n$, to the next failure time, $T_{n+1} = t_{n+1}$. We will use $MTBF_n$ to denote the MTBF at current state $T_n = t_n$. Hence, it follows that

$$MTBF_n = \int_{t_n}^{\infty} t f_{n+1}(t|t_1, t_2, \dots, t_n) dt - t_n \quad (5.9.20)$$

where $\int_{t_n}^{\infty} t f_{n+1}(t|t_1, t_2, \dots, t_n) dt$ is the expected $(n + 1)$ th failure under the condition $T_n = t_n$.

For TSL distribution , we have

$$f_{n+1}(t|t_1, t_2, \dots, t_n) = \frac{1}{\phi} \frac{2(1 + \lambda) \exp\left(-\frac{t}{\phi}\right) - (1 + \lambda) \exp\left(-\frac{(1+\lambda)t}{\phi}\right)}{2(1 + \lambda) \exp\left(-\frac{t_n}{\phi}\right) - \exp\left(-\frac{(1+\lambda)t_n}{\phi}\right)} \quad (5.9.21)$$

Hence, the $MTBF_n$ is given by

$$\begin{aligned} MTBF_n &= \phi \frac{[2(1 + \lambda) \exp\left(-\frac{t_n}{\phi}\right) - \frac{1}{(1+\lambda)} \exp\left(-\frac{(1+\lambda)t_n}{\phi}\right)]}{[2(1 + \lambda) \exp\left(-\frac{t_n}{\phi}\right) - \exp\left(-\frac{(1+\lambda)t_n}{\phi}\right)]} \\ &= \frac{\phi}{(1 + \lambda)} \frac{\left\{ 2(1 + \lambda)^2 - \exp\left(-\frac{\lambda t_n}{\phi}\right) \right\}}{\left\{ 2(1 + \lambda) - \exp\left(-\frac{\lambda t_n}{\phi}\right) \right\}}. \end{aligned} \quad (5.9.22)$$

It is clear that for the special case $\lambda = 0$ we have $MTBF_n = \phi$.

Thus , in the case $\lambda = 0$, we have

$$MTBF_n = \frac{1}{\nu_n} \quad (5.9.23)$$

Note that, in this special case the process has a constant intensity function. For other values of λ we always have $MTBF_n < \frac{1}{\nu_n}$.

5.10 Conclusion

In this chapter we have studied all analytical aspects of TSL pdf. We have derived the analytical form of moments, moment generating function, cumulant generating function, cumulants, percentiles and entropy if a random variable follows the TSL pdf. In addition we have developed the corresponding estimations forms of the subject parameters. Also, we have developed the reliability model and its corresponding hazard rate function when the failure times are characterized by TSL pdf. In addition, we have developed the analytical forms of mean residual life times and mean time between failures and its relationship to the intensity function for systems that exhibit the characteristic of subject pdf.

Chapter 6

Comparison of TSL Probability Distribution with Other Distributions

6.1 Introduction

In the present study we will discuss a comparison of the truncated skew Laplace pdf with some other popular pdf, namely, the gamma pdf and the hypoexponential pdf whose graphical representation and characterization are similar to that of TSL pdf. In section 6.2 we will consider the comparisons with two parameter gamma model and in section 6.3 we will be comparing TSL pdf with the so called hypoexponential pdf. We will make comparisons in terms of the reliability behavior. In fact, we will simulate data from TSL pdf and check whether it has the same reliability if the data was assumed to be gamma pdf and hypoexponential pdf. Also we will consider a data consisting of the failure time of pressure vessels data which was studied by Keating *et al.*[13] using gamma pdf.

6.2 TSL Vs. Two Parameter Gamma Probability Distribution

The gamma probability distribution plays a crucial role in mathematical statistics and many applied areas. A random variable X is said to have gamma probability distribution with two parameters α and β , denoted by $G(\alpha, \beta)$, if X has the probability density function given by

$$f(x; \alpha, \beta) = \frac{1}{\Gamma(\alpha)\beta^\alpha} t^{\alpha-1} \exp\left(-\frac{t}{\beta}\right), \quad \alpha, \beta, t > 0.$$

where $\Gamma(\alpha)$ denotes the gamma function evaluated at α . The parameters α and β are the shape and scale parameters, respectively. The reliability and hazard functions are not available in closed form unless α happens to be an integer; however, they may be expressed in terms of the standard incomplete gamma function $\Gamma(a, z)$ defined by

$$\Gamma(a, z) = \int_0^z y^{a-1} \exp(-y) dy, \quad a > 0.$$

In terms of $\Gamma(a, z)$, the reliability function for the $G(\alpha, \beta)$ distribution is given by

$$R(t; \alpha, \beta) = \frac{\Gamma(\alpha) - \Gamma(\alpha, t/\beta)}{\Gamma(\alpha)},$$

and, if α is an integer, it is given by

$$R(t; \alpha, \beta) = \sum_{k=0}^{\alpha-1} \frac{(t/\beta)^k \exp(-t/\beta)}{k!}.$$

The hazard rate is given by

$$h(t; \alpha, \beta) = \frac{t^{\alpha-1} \exp(-t/\beta)}{\beta^\alpha [\Gamma(\alpha) - \Gamma(\alpha, t/\beta)]},$$

for any $\alpha > 0$ and, if α is an integer it becomes

$$h(t, \alpha, \beta) = \frac{t^{\alpha-1}}{\beta^\alpha \Gamma(\alpha) \sum_{k=0}^{\alpha-1} (t/\beta)^k / k!}.$$

The shape parameter α is of special interest since whether $\alpha - 1$ is negative, zero or positive, corresponds to a decreasing failure rate (DFR), constant, or increasing failure rate (IFR), respectively.

It is clear that the gamma model has more flexibility than that of TSL model as the former one can be used even if the data has DFR. In fact, the exponential model is a particular case of both models, that is, $TSL(0, 1)$ and $Gamma(1, 1)$ are the exponential models. But if in the gamma model $\alpha > 1$, it has IFR which appears to be the same as that of TSL model but a careful study shows that there is a significant difference in these two models even in this case.

Figure 6.1 gives a graphical comparisons of the reliability functions of TSL and Gamma pdf. It is clearly seen that $\text{Gamma}(1, 1)$ and $\text{TSL}(0, 1)$ are identical yielding the exponential $\text{EXP}(1)$ model.

Reliability of TSL and Gamma distributions

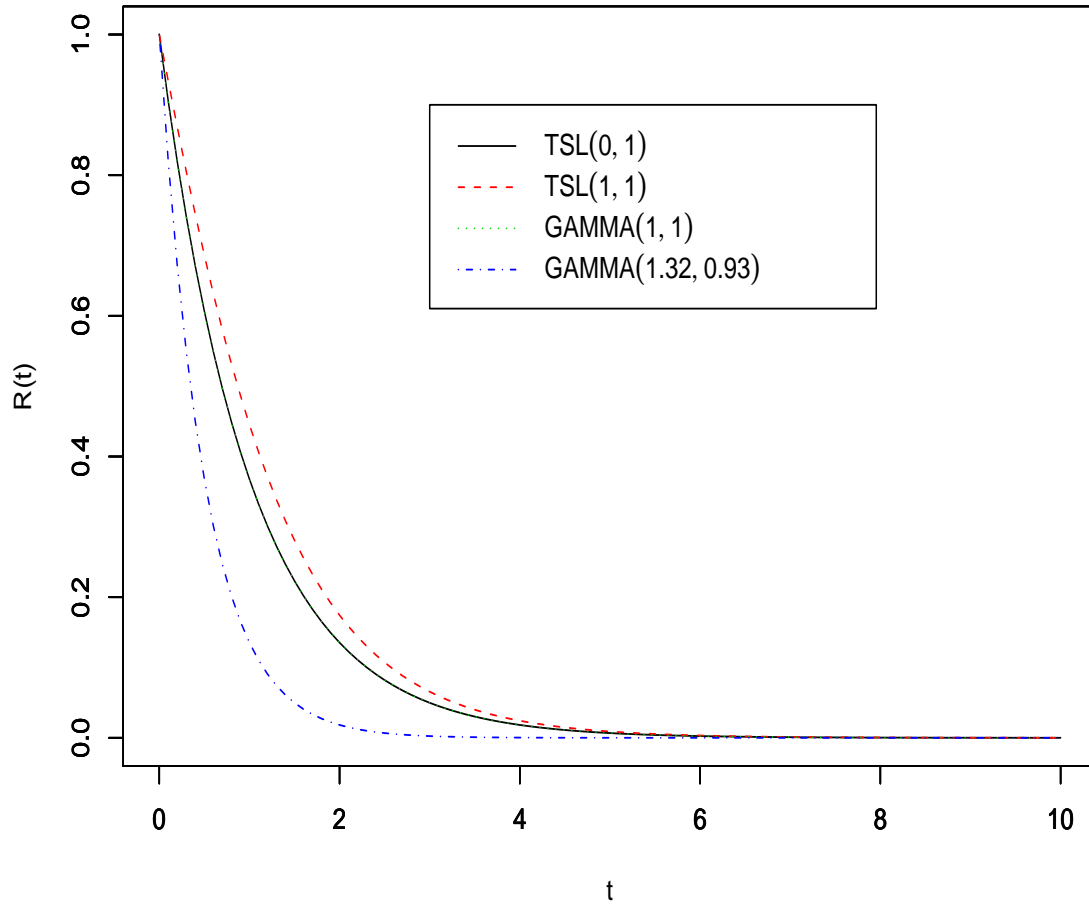


Figure 6.1: Reliability of TSL and Gamma distributions

Table 6.1 gives a comparisons of two parameter gamma model with respect to TSL model. We simulate 100 data for different values of the TSL parameters λ and ϕ as indicated in the table. We used the Newton-Raphson algorithm to get its estimates. Again we get the estimates of the parameters α and β of gamma model assuming that the data satisfy the gamma model.

ϕ	λ	$\widehat{\phi}$	$\widehat{\lambda}$	$\widehat{\alpha}$	$\widehat{\beta}$	n_1	n_2
1	0	1.045522	0.06166375	1.013032	1.085437	4	4
1	1	1.056979	1.341615	1.317944	0.9267452	3	7
1	2	1.067631	1.478364	1.379657	0.8905008	5	7
1	5	1.012352	4.362386	1.231728	0.8906503	6	6
1	10	1.011322	9.99997	1.189242	0.8872343	6	6
2	0	2.081052	0.000026	0.9989892	2.083215	5	5
2	1	2.055891	0.91154	1.380973	1.74020	6	8
2	2	2.098627	2.024318	1.313875	1.809048	5	6
2	5	1.965912	5.942018	1.20621	1.738104	6	6
2	10	2.10125	10.92677	1.250668	1.747456	4	4
5	0	4.619639	0.6206207	1.126453	4.801781	7	7
5	1	5.45453	1.144246	1.378976	4.597368	6	8
5	2	5.128795	1.554738	1.358207	4.335356	5	6
5	5	4.817129	4.976809	1.223907	4.235068	5	5
5	10	5.488563	9.635667	1.238111	4.631146	3	3

Table 6.1: Comparison between TSL and gamma models when both parameters are unknown

In Table 6.1 we have used the Kolmogrov-Smirnov non-parametric test to check whether the data generated from $TSL(\lambda, \phi)$ also satisfied the $Gamma(\alpha, \beta)$. The table shows that the TSL pdf closely resembles a two parameter gamma pdf. In the table n_1 and n_2 , respectively, denote the corresponding number of item failed before T_T and T_G . Where T_T and T_G are defined by

$$P(T \geq T_T) \geq 0.95$$

and

$$P(T \geq T_G) \geq 0.95$$

respectively. Note that T_T and T_G respectively denote the failure time assuming TSL and Gamma pdf.

Table 6.1 shows that the significance difference occurs when parameter $\lambda = 1$ and they are identical for $\lambda = 0$. Also if $\lambda > 1$ then bigger the value of λ closer the relation with gamma pdf subject to the condition that the parameter ϕ remains the same.

If the two models happen to be identical we should be able to find the parameters of one distribution knowing the parameters of the other. A usual technique is by equating the first two moments. If this is the case we must have the following system of equations

$$\begin{cases} \frac{\phi(2\tau^2 - 1)}{2\tau^2 - \tau} = \alpha\beta \\ \frac{2\phi^2(2\tau^3 - 1)}{2\tau^3 - \tau^2} = \alpha^2\beta^2 + \alpha\beta^2 \end{cases}$$

where $\tau = 1 + \lambda$. On solving the system of equations we get

$$\alpha = \frac{(2\tau^2 - 1)^2}{(4\tau^4 - 4\tau^3 + 4\tau^2 + 1)},$$

and

$$\beta = \frac{\phi(4\tau^4 - 4\tau^3 + 4\tau^2 + 1)}{(2\tau^2 - \tau)(2\tau^2 - 1)}.$$

This shows that one can get the parameters α and β of a Gamma distribution if we know the parameters λ and ϕ of a TSL distribution.

If we know or have data to estimate the parameter λ then the MLE of ϕ is given by

$$\hat{\phi} = \frac{(2\lambda + 1)(\lambda + 1)}{n(2\lambda^2 + 4\lambda + 1)} \sum_{i=1}^n X_i$$

Again we compare the TSL pdf with gamma pdf assuming that the parameter λ is known. We follow the same procedure as in the previous case and the Table 6.2

also presents the same quantities as Table 6.1 does.

ϕ	λ	$\widehat{\phi}$	$\widehat{\alpha}$	$\widehat{\beta}$	n_1	n_2
1	1	1.011927	1.200189	0.9836633	4	5
1	2	1.009196	1.102708	1.037224	4	4
1	5	0.9884295	1.111627	0.9565353	2	2
1	10	1.010180	1.166273	0.9036562	3	3
1	50	0.923165	1.015271	0.9181059	5	5
2	1	1.812826	1.168546	1.809911	2	3
2	2	2.013709	1.190309	1.917320	5	6
2	5	2.031695	1.186196	1.842538	5	5
2	10	2.008019	1.219654	1.717656	2	2
2	50	1.835290	1.217294	1.522315	4	4
5	1	5.45453	1.378976	4.597368	6	8
5	2	5.128795	1.358207	4.335356	5	6
5	5	4.817129	1.223907	4.235068	5	5
5	10	5.488563	1.238111	4.631146	3	3
5	50	4.716769	1.043489	4.565573	5	5

Table 6.2: Comparison between TSL and gamma models when one parameter is known

Table 6.2 also supports the conclusion we have drawn from the previous table. That means there is a significance difference between these two models when the parameter $\lambda = 1$. But for a large value of λ we do not see much differences in terms of reliability subject to the condition the value of the parameter ϕ remains the same.

Finally, we would like to present a real world problem where the TSL model gives a better fit than the competing gamma model.

The following data is the failure times(in hours) of pressure vessels constructed of fiber/epoxy composite materials wrapped around metal lines subjected to a certain constant pressure. This data was studied by Keating *et al.*[12].

274, 1.7, 871, 1311, 236
458, 54.9, 1787, 0.75, 776
28.5, 20.8, 363, 1661, 828
290, 175, 970, 1278, 126

Pal, N. *et al.* [20] mention that the $Gamma(1.45, 300)$ model fits for the subject data. We have run a Kolmogorove Smirnov nonparametric statistical test and observed the following results:

K-S statistics $D_{Gamma} = 0.2502$ and $D_{TSL} = 0.200$ for $Gamma(1.45, 300)$ and $TSL(5939.8, 575.5)$ distribution respectively.

Figure 6.2 exhibits the p-p plot of the pressure vessel data assuming TSL and Gamma pdf. It is evident that TSL fits better than the Gamma model. Hence we recommend that TSL is a better model for the pressure vessel data.

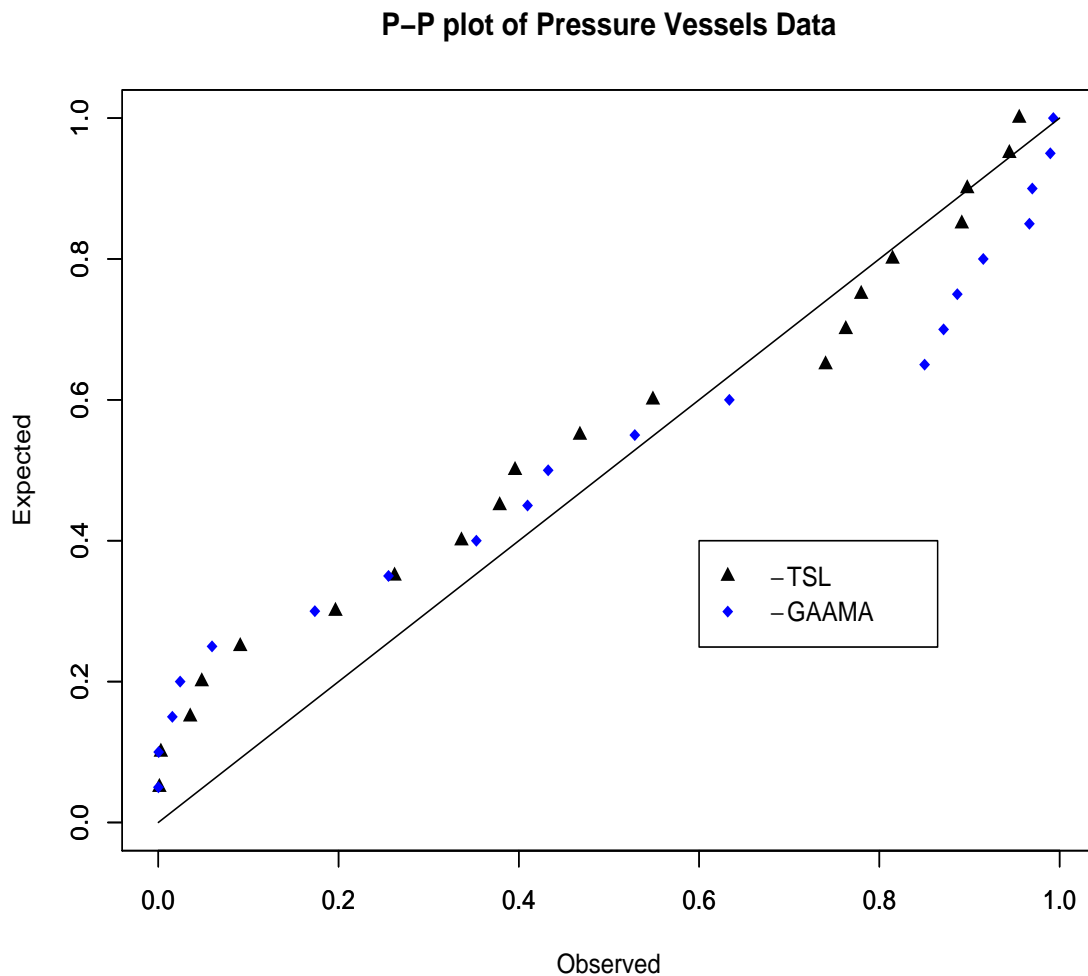


Figure 6.2: P-P Plots of Vessel data using TSL and Gamma distribution

Table 6.3 below gives the reliability estimates using TSL pdf and Gamma pdf and Figure 6.3 exhibits the reliability graphs. There is a significance differences on the estimates.

t	$\hat{R}_{TSL}(t)$	$\hat{R}_{GAMMA}(t)$	t	$\hat{R}_{TSL}(t)$	$\hat{R}_{GAMMA}(t)$
0.75	0.999	0.999	363	0.532	0.471
1.70	0.997	0.999	458	0.451	0.365
20.80	0.965	0.984	776	0.260	0.150
28.50	0.952	0.976	828	0.237	0.129
54.90	0.909	0.940	871	0.220	0.113
126.0	0.803	0.826	970	0.185	0.085
175.0	0.738	0.745	1278	0.108	0.034
236.0	0.664	0.647	1311	0.102	0.030
274.0	0.621	0.590	1661	0.056	0.010
290.0	0.604	0.567	1787	0.045	0.007

Table 6.3: The Reliability estimates of Pressure Vessels Data

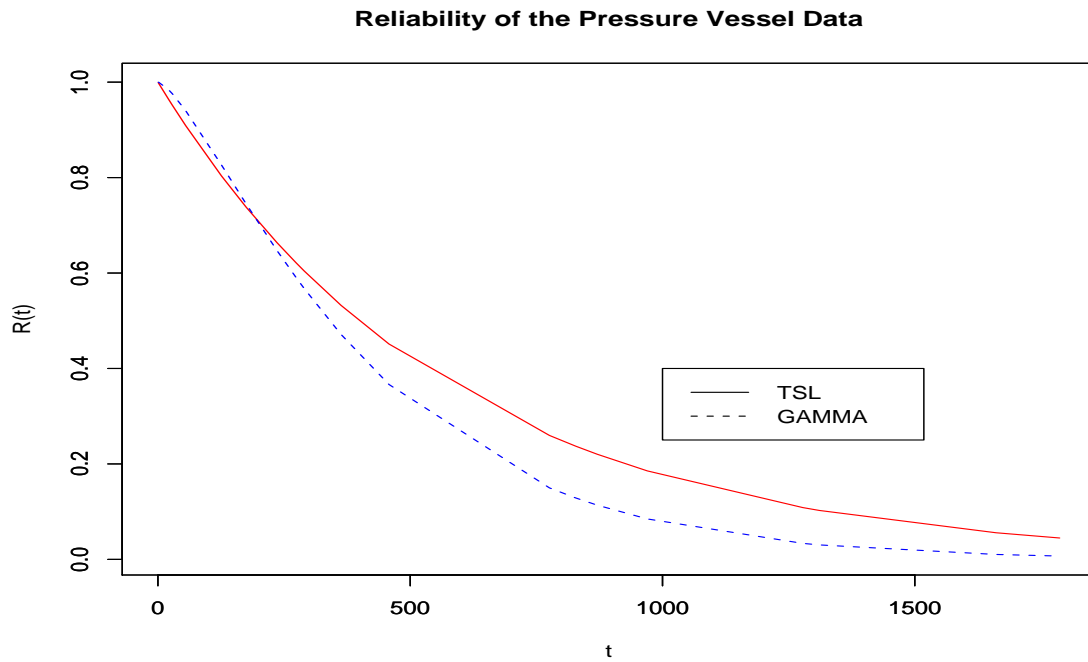


Figure 6.3: Reliability of Vessel data using TSL and Gamma distribution

6.3 TSL Vs. Hypoexponential Probability Distribution

Observing the probability structure of the truncated skew Laplace pdf it is our interest to look for an existing probability distribution which can be written as a difference of two exponential function. We will compare TSL pdf with the hypoexponential pdf. Many natural phenomenon can be divided into sequential phases. If the time the process spends in each phase is independent and exponentially distributed, then it can be shown that the overall time is hypoexponentially distributed. It has been empirically observed that the service times for input-output operations in a computer system often possess this distribution see K.S. Trivedi [25]. It will have n parameters one for each of its distinct phases. Her we are interested in a two-stage hypoexponential process. That is, if X be a random variable with parameters λ_1 and $\lambda_2 (\lambda_1 \neq \lambda_2)$, then its pdf is given by

$$f(x) = \frac{\lambda_1 \lambda_2}{\lambda_2 - \lambda_1} \{ \exp(-\lambda_1 x) - \exp(-\lambda_2 x) \} \quad x > 0.$$

We will use the notation $Hypo(\lambda_1, \lambda_2)$ to denote a hypoexponential random variable with parameters λ_1 and λ_2 , respectively. Figure 6.4 gives a graphical display of the pdf of the hypoexponential distribution for $\lambda_1 = 1$ and different values of $\lambda_2 > \lambda_1$. In fact, because of the symmetry it doesn't matter which parameter need to be bigger and which one to be smaller.

Figure 5.4 has the parameters $\lambda_1 = 1$ and $\lambda_2 = 1.5, 2, 3, 5, 10, 50$. From the figure it is clearly seen that as the value of the parameter λ_2 increases the pdf looks like TSL pdf. The corresponding cdf is given by

$$F(x) = 1 - \frac{\lambda_2}{\lambda_2 - \lambda_1} \exp(-\lambda_1 x) + \frac{\lambda_1}{\lambda_2 - \lambda_1} \exp(-\lambda_2 x) \quad x \geq 0.$$

The Reliability function $R(t)$ of a $Hypo(\lambda_1, \lambda_2)$ random variable is given by

$$R(t) = \frac{\lambda_2}{\lambda_2 - \lambda_1} \exp(-\lambda_1 t) - \frac{\lambda_1}{\lambda_2 - \lambda_1} \exp(-\lambda_2 t). \quad (6.3.1)$$

PDF of Hypoexponential Distribution

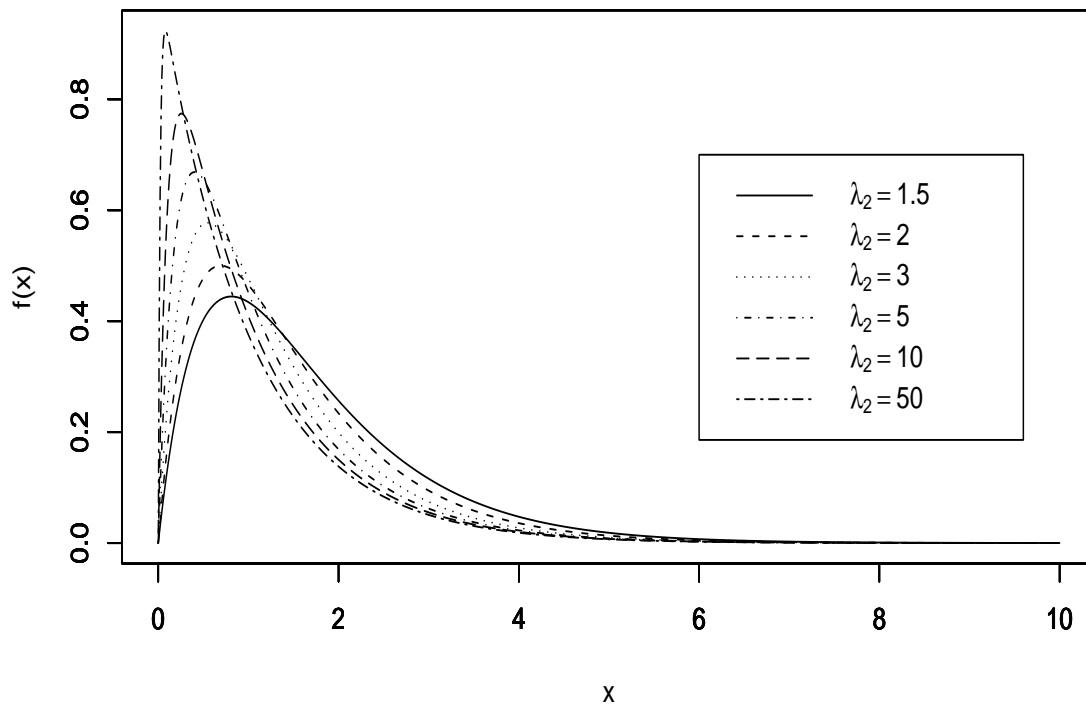


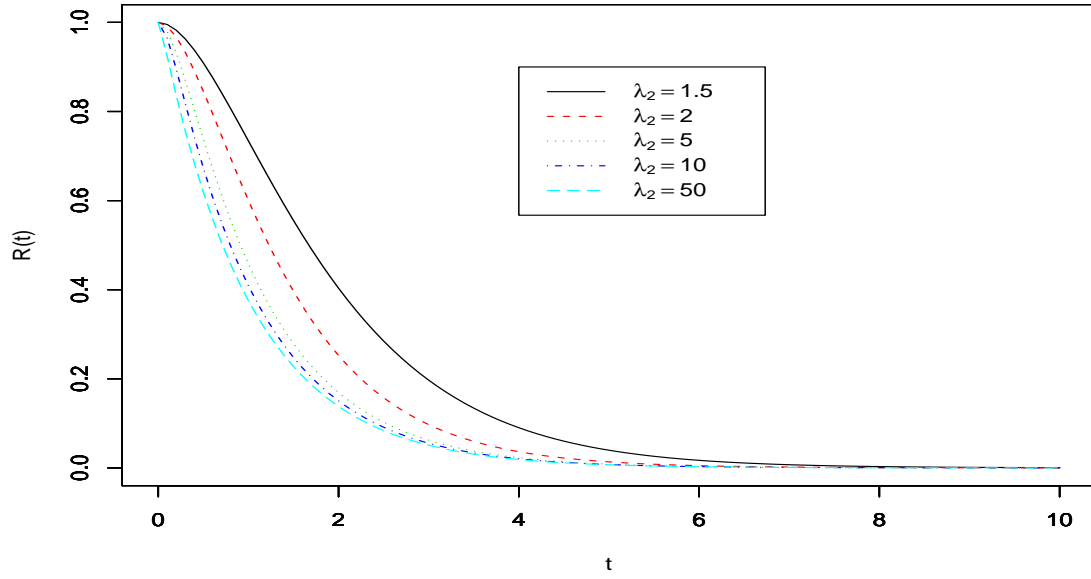
Figure 6.4: PDF of Hypoexponential Distribution for $\lambda_1 = 1$ and different values of λ_2

The hazard rate function $h(t)$ of a $Hypo(\lambda_1, \lambda_2)$ random variable is given by

$$h(t) = \frac{\lambda_1 \lambda_2 [\exp(-\lambda_1 t) - \exp(-\lambda_2 t)]}{\lambda_2 \exp(-\lambda_1 t) - \lambda_1 \exp(-\lambda_2 t)}. \quad (6.3.2)$$

It is clear that $h(t)$ is increasing function of the parameter λ_2 . It increases from 0 to $\min\{\lambda_1, \lambda_2\}$. Figure 6.5 exhibits the reliability function and hazard hazard rate function of a hypoexponential random variable with parameters $\lambda_1 = 1$ and different values of $\lambda_2 > \lambda_1$.

Reliability of Hypoexponential Distribution



Hazard rate for hypoexponential Distribution

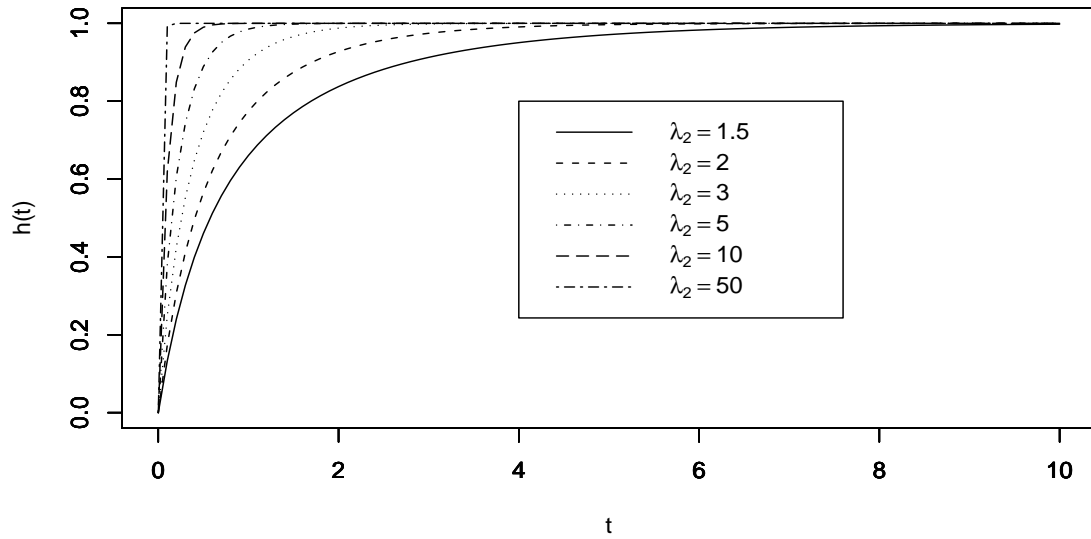


Figure 6.5: Reliability and hazard rate function of Hypoexponential distribution

Also, note that the mean residual life (MRL) time at time t for $Hypo(\lambda_1, \lambda_2)$ is given by

$$m_{Hypo}(t) = \frac{1}{\lambda_1 \lambda_2} \frac{\lambda_2^2 \exp(-\lambda_1 t) - \lambda_1^2 \exp(-\lambda_2 t)}{[\lambda_2 \exp(-\lambda_1 t) - \lambda_1 \exp(-\lambda_2 t)]}.$$

We now proceed to make a comparison between TSL and hypoexponential pdf in terms of the reliability and the mean residual life times. We will generate from the hypoexponential distributions a random samples of size 50, 100 and 500 for different values of the parameters λ_1, λ_2 and then proceed to fit the data to TSL model.

n	λ_1	λ_2	$\widehat{\lambda}_1$	$\widehat{\lambda}_2$	$\widehat{\phi}$	$\widehat{\lambda}$	M_{TSL}	M_{HYPO}
50	1	2	0.934	2.325	2.745	1.349	1.362	1.129
50	1	5	0.975	5.133	2.779	1.097	1.108	1.029
50	1	10	0.979	12.223	1.042	6.968	1.042	1.021
50	1	20	0.940	26.742	1.069	15.349	1.069	1.063
100	1	2	0.876	2.565	1.376	2.403	1.391	1.184
100	1	5	0.903	6.835	1.178	6.216	1.179	1.108
100	1	10	0.950	9.838	1.098	8.439	1.099	1.052
100	1	20	1.029	26.322	0.892	0.242	0.982	0.971
500	1	2	0.915	2.576	1.339	3.076	1.348	1.132
500	1	5	0.961	6.489	1.088	3.453	1.093	1.042
500	1	10	0.881	10.224	1.174	8.355	1.173	1.135
500	1	20	1.016	27.411	0.988	14.044	0.988	0.983

Table 6.4: Comparison between TSL and Hypoexponential Models

To create Table 6.4 we generate a random sample of sizes 50, 100 and 500 from hypoexponential pdf with parameters λ_1 and $\lambda_2 = 2, 5, 10 \& 20$ for each sample size. We have used the Newton-Raphson algorithm to estimate the maximum likelihood estimators of λ_1 and λ_2 . We expect that the data will fit the TSL model, so assuming the data satisfies TSL model we estimate the parameters ϕ and λ .

In addition we computed the mean residual life times for both the models at $T = T_{n/2}$. From the table we can observe that if the sample size is large and the difference between the two parameters λ_1 and λ_2 is large both TSL and hypoexponential model will produce the same result whereas for small size and small difference between λ_1

and λ_2 there is a significant differences. The figures(6.6-6.8) below also support this argument.

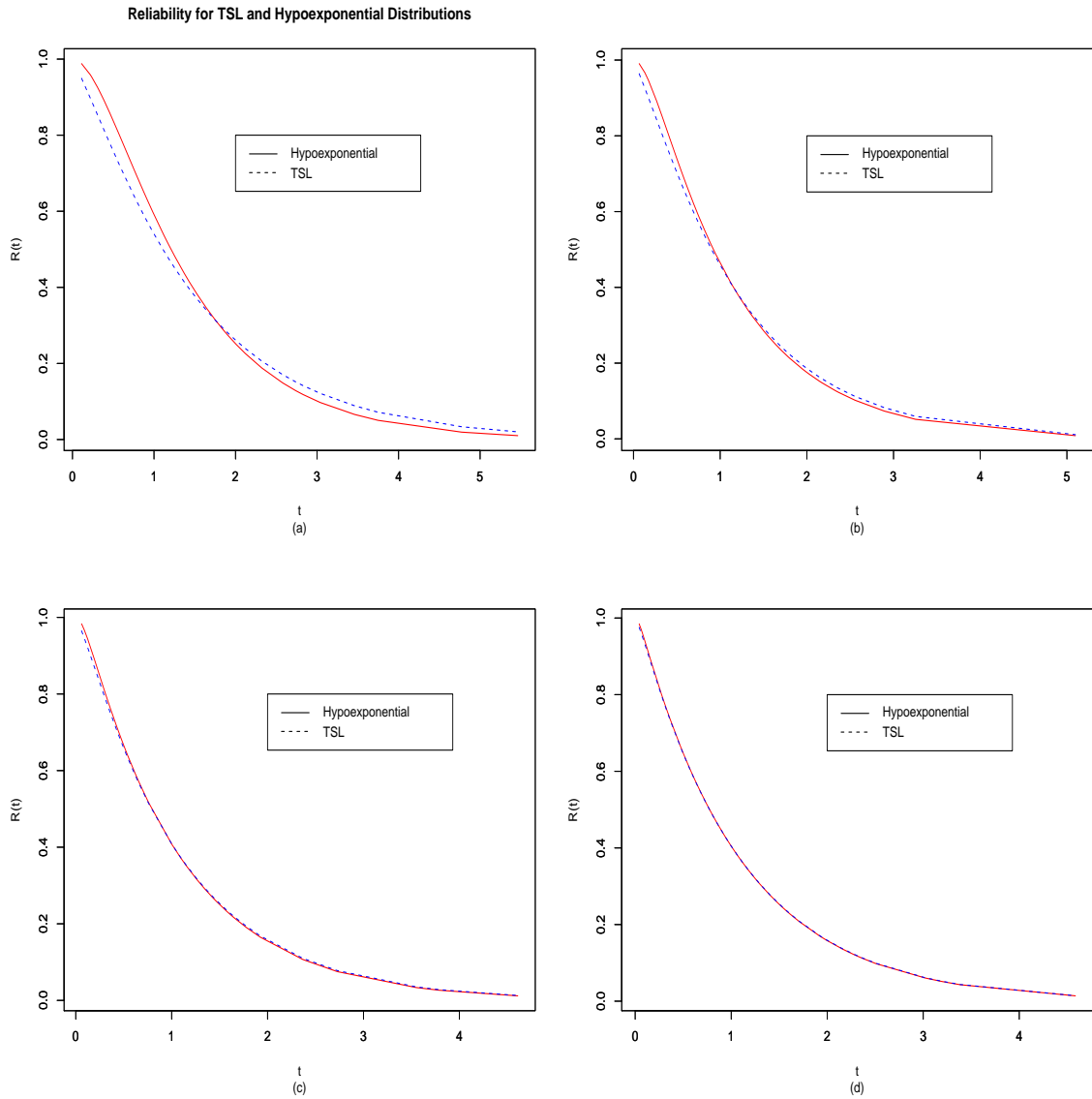


Figure 6.6: Reliability of TSL and Hypoexponential distributions for $n = 50, \lambda_1 = 1$, and (a) $\lambda_2 = 2$, (b) $\lambda_2 = 5$, (c) $\lambda_2 = 10$, (d) $\lambda_2 = 20$

Reliability for TSL and Hypoexponential Distribution

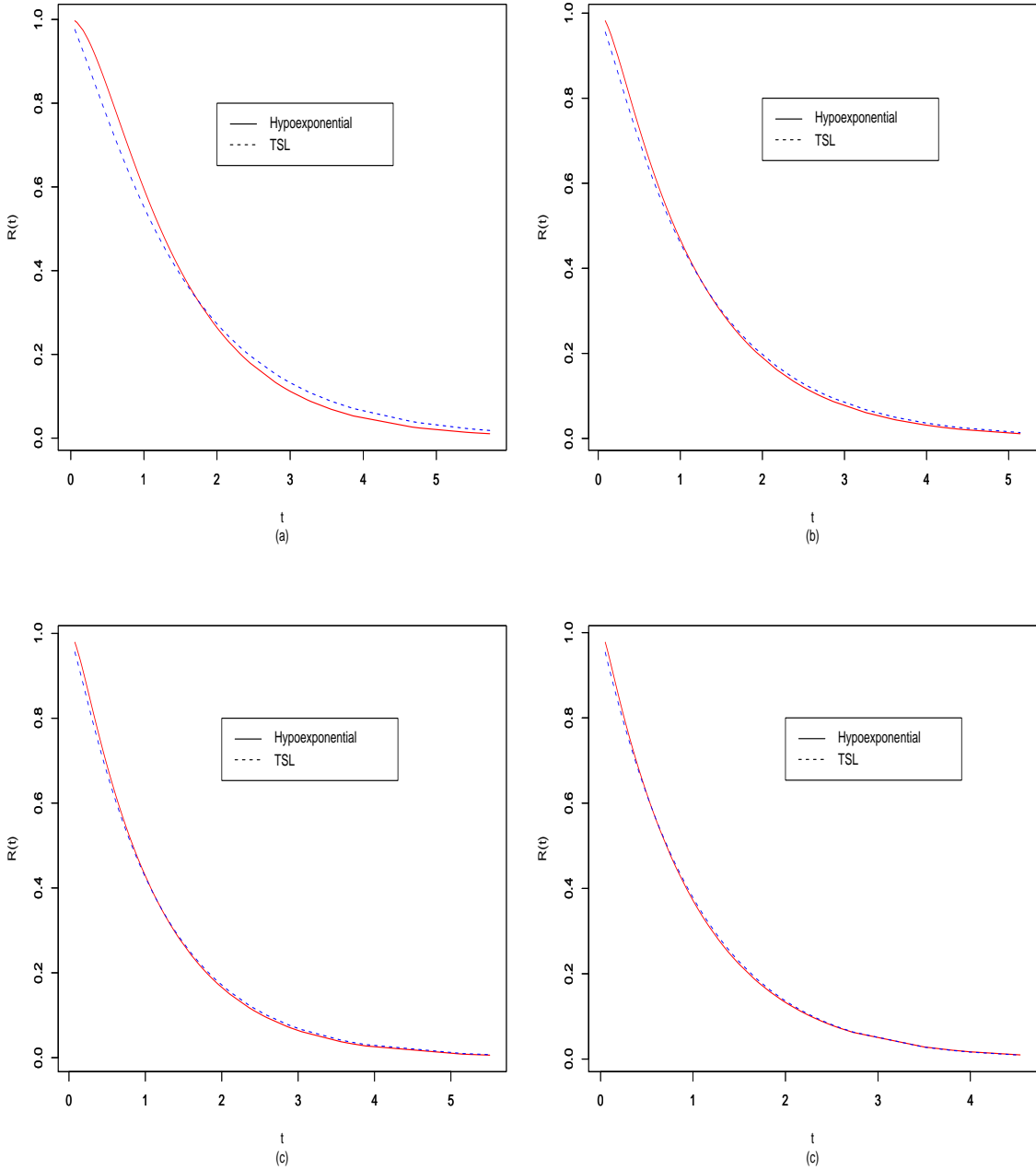


Figure 6.7: Reliability of TSL and Hypoexponential distributions for $n = 100$, $\lambda_1 = 1$, and (a) $\lambda_2 = 2$, (b) $\lambda_2 = 5$, (c) $\lambda_2 = 10$, (d) $\lambda_2 = 20$

Reliability for TSL and Hypoexponential Distribution

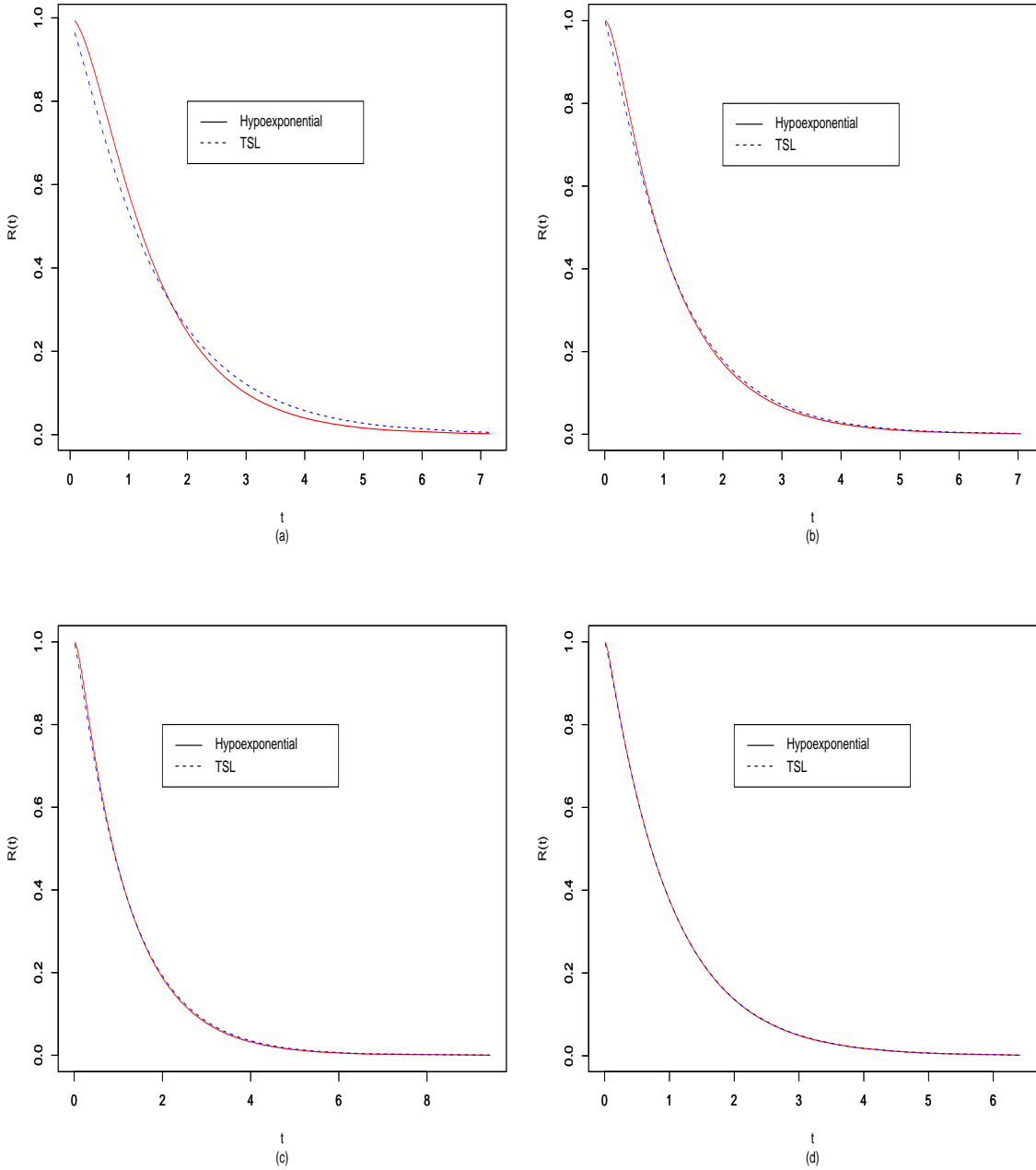


Figure 6.8: Reliability of TSL and Hypoexponential distributions for $n = 500$, $\lambda_1 = 1$, and (a) $\lambda_2 = 2$, (b) $\lambda_2 = 5$, (c) $\lambda_2 = 10$, (d) $\lambda_2 = 20$

6.4 Conclusion

In this chapter we have compared the TSL model with two commonly used existing models namely, the Gamma model and hypoexponential models in terms of their reliability behavior. We have seen that both $TSL(0, 1)$ and $Gamma(1, 1)$ are identical as both yield the exponential model but a careful study shows there are some situations where the TSL model gives a better fit than gamma model. In deed we have illustrate this with a real information of Pressure Vessels failure data.

Also we make a comparison with the hypoexponential pdf and have concluded that if the sample size is large and the difference between the two parameters λ_1 and λ_2 is also large both TSL and hypoexponential model will produce the same result whereas for small size and small difference between λ_1 and λ_2 there is a significant differences. In fact the shape of the hazard rate function of the TSL model seems like a graph of a function of the form $h(t) = 1 - \exp(-\alpha t)$. Working backward, we have derived the corresponding pdf as shown below.

Since the relation between the hazard rate function and the CDF can be expressed in terms of a differential equation given by

$$h(t) = -\frac{d}{dt} \log(1 - F(t)),$$

and on solving this equation we have

$$1 - F(t) = \exp\left(-\int_0^t h(u)du\right)$$

which yields a new distribution whose pdf and CDF are respectively

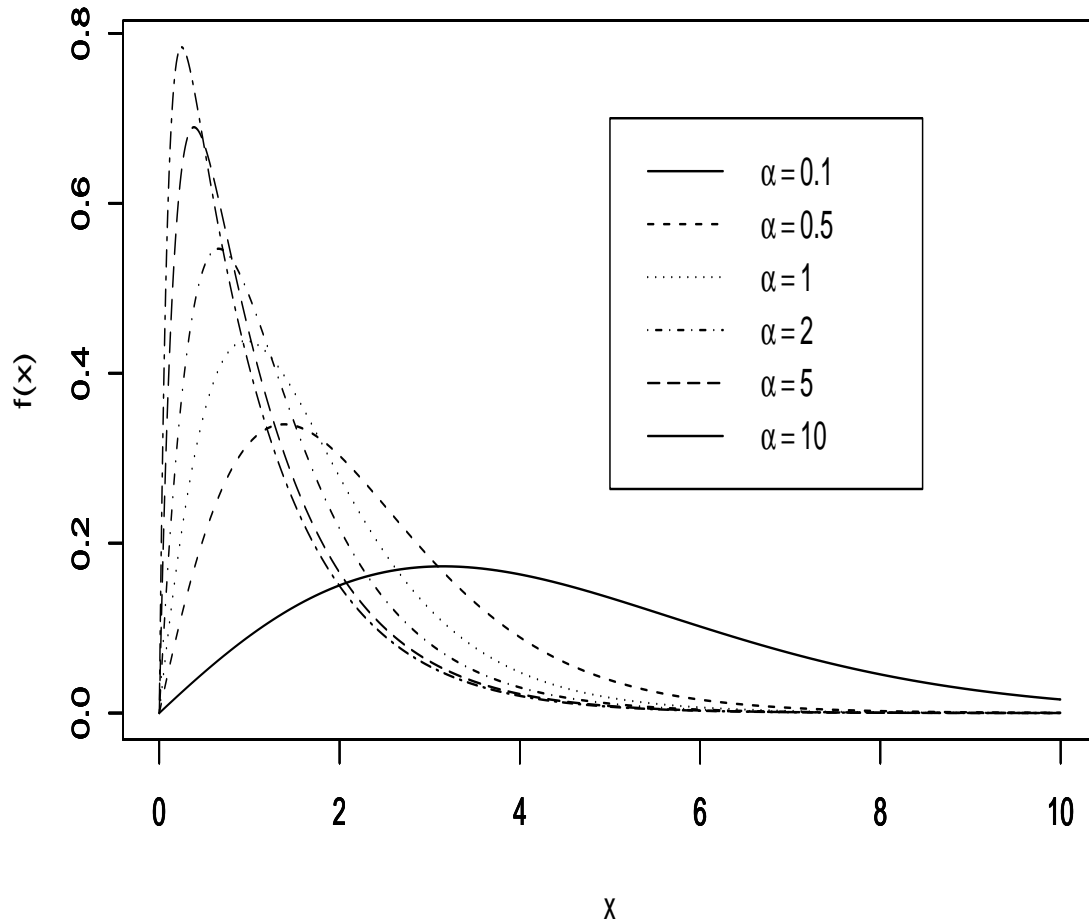
$$f(t) = (1 - \exp(-\alpha t)) \exp\left(\frac{1}{\alpha}(1 - \exp(-\alpha t)) - t\right) \quad (6.4.3)$$

and

$$F(t) = 1 - \exp\left(\frac{1}{\alpha}(1 - \exp(-\alpha t)) - t\right) \quad (6.4.4)$$

Figure below shows the probability distribution function of this distribution for different values of α .

pdf of the distribution derived from hazard rate



Chapter 7

Preventive Maintenance and the TSL Probability Distribution

7.1 Introduction

In many situations, failure of a system or unit during actual operation can be very costly or in some cases quite dangerous if the system fails. Thus, it is better to repair or replace before it fails. But on the other hand, one does not want to make too frequent replacement of the system unless it is absolutely necessary. Thus we try to develop a replacement policy that balances the cost of failures against the cost of planned replacement or maintenance.

Suppose that a unit which is to operate over a time 0 to time t , $[0,t]$ is replaced upon failure (with failure probability distribution F). We assume that the failures are easily detected and instantly replaced. A cost c_1 that includes the cost resulting from planned replacement and a cost c_2 that includes all costs resulting from failure is invested. Then the expected cost during the period $[0,t]$ is

$$C(t) = c_1E(N_1(t)) + c_2E(N_2(t)),$$

where $E(N_1(t))$ and $E(N_2(t))$ denotes the expected number of planned replacement and expected number of failures.

We would like to seek the policy minimizing $C(t)$ for a finite time span or minimizing $\lim_{t \rightarrow \infty} \frac{C(t)}{t}$ for an infinite time span. Since the TSL probability distribution has an increasing failure rate we expect this model to be useful in maintenance system.

7.2 Age Replacement Policy and TSL Probability Distribution

First we consider the so called the “Age replacement policy”. In this policy we always replace an item exactly at the time of failure or t^* hours after its installation, whichever occurs first. Age replacement policy for an infinite time span seems to have received the most attention in the literature. Morese(1958) showed how to determine the replacement interval minimizing cost per unit time. Barlow *et al.*[5] proved that if the failure distribution, F , is continuous then there exists a minimum-cost age replacement for any infinite time span.

Here we would like to determine the optimal t^* at which preventive replacement should be performed. The model determines the t^* that minimizes the total expected cost of preventive and failure maintenance per unit time. The total cost per cycle consists of the cost of preventive maintenance in addition to the cost of failure maintenance. Hence,

$$EC(t^*) = c_1(R(t^*)) + c_2(1 - R(t^*)) \quad (7.2.1)$$

where, c_1 and c_2 denote the cost of preventive maintenance and failure maintenance respectively. $R(t^*)$ is the probability the equipment survives until age t^* . The expected cycle length consists of the length of a preventive cycle plus the expected length of a failure cycle. Thus, we have

$$\text{Expected cycle length} = t^*R(t^*) + M(t^*)(1 - R(t^*)) \quad (7.2.2)$$

where,

$$M(t^*)(1 - R(t^*)) = \int_{-\infty}^{t^*} tf(t)dt$$

is the mean of the truncated distribution at time t^* . Hence, the

$$\text{Expected cost per unit time} = \frac{c_1R(t^*) + c_2[1 - R(t^*)]}{t^*R(t^*) + M(t^*)[1 - R(t^*)]}. \quad (7.2.3)$$

We assume that a system has a time to failure distribution being the truncated skew Laplace pdf. We would like to compute the optimal time t^* of preventive replacement. Hence, we have

$$R(t) = \frac{2(1 + \lambda) \exp\left(\frac{-t}{\phi}\right) - \exp\left(\frac{-(1+\lambda)t}{\phi}\right)}{(2\lambda + 1)} \quad (7.2.4)$$

and

$$M(t^*) = \frac{1}{1 - R(t^*)} \int_0^{t^*} tf(t)dt \quad (7.2.5)$$

Thus, we can write

$$\begin{aligned} \int_0^{t^*} tf(t)dt &= \frac{2\lambda_1\phi}{2\lambda_1 - 1} [1 - \exp(-t^*/\phi)] - \frac{\phi}{(2\lambda_1 - 1)\lambda_1} [1 - \exp(-\lambda_1 t^*/\phi)] \\ &+ \frac{t^*}{(2\lambda_1 - 1)} \exp(-\lambda_1 t^*/\phi) - \frac{2\lambda_1 t^*}{2\lambda_1 - 1} \exp(-t^*/\phi). \end{aligned} \quad (7.2.6)$$

where $\lambda_1 = \lambda + 1$.

On substituting from (7.2.4),(7.2.5) and (7.2.6) in (7.2.3)and simplifying the expressions we get the expected cost per unit time(ECU) given by

$$\text{ECU}(t^*) = \frac{\lambda_1 \{2\lambda_1(c_2 - c_1) \exp(-t^*/\phi) - (c_2 - c_1) \exp(-\lambda_1 t^*/\phi) - c_2(2\lambda_1 - 1)\}}{\phi \{2\lambda_1^2 \exp(-t^*/\phi) - 2\lambda_1^2 - 1 - \exp(-\lambda_1 t^*/\phi)\}}$$

Now we want to find the value of t^* which minimizes the above expression subject to the condition that $c_1 = 1$ and $c_2 = 10$. The following so called ‘‘Golden Section Method’’ is used to obtain the optimal value of t^* . The Golden section method is described as follows:

To minimize a function $g(t)$ subject to $a \leq t \leq b$ we can use so called Golden section method and the steps to use the algorithm are as follows:

Step 1. Choose an allowable final tolerance level δ and assume the initial interval where the minimum lies is $[a_1, b_1] = [a, b]$ and let

$$\lambda_1 = a_1 + (1 - \alpha)(b_1 - a_1)$$

$$\mu_1 = a_1 + \alpha(b_1 - a_1)$$

Take $\alpha = 0.618$, which is a positive root of $c^2 + c - 1 = 0$, and evaluate $g(\lambda_1)$ and $g(\mu_1)$, let $k=1$ and go to step 2.

Step 2. If $b_k - a_k \leq \delta$, stop as the optimal solution is $t^* = (a_k + b_k)/2$. otherwise, if $g(\lambda_k) > g(\mu_k)$, go to step 3; and if $g(\lambda_k) \leq g(\mu_k)$, go to step 4.

Step 3: Let $a_{k+1} = \lambda_k$ and $b_{k+1} = b_k$. Furthermore let $\lambda_{k+1} = \mu_k$ and $\mu_{k+1} = a_{k+1} + \alpha(b_{k+1} - a_{k+1})$. Evaluate $g(\mu_{k+1})$ and go to step 5.

Step 4: Let $a_{k+1} = a_k$ and $b_{k+1} = \mu_k$. Furthermore let $\mu_{k+1} = \lambda_k$ and $\lambda_{k+1} = a_{k+1} + (1 - \alpha)(b_{k+1} - a_{k+1})$. Evaluate $g(\lambda_{k+1})$ and go to step 5.

Step 5: Replace k by $k + 1$ and go to step 1.

To implement this method to our problem we proceed as below

Iteration 1

Consider $[a_1, b_1] = [0, 10]$, $\alpha = 0.618$ so that $1 - \alpha = 0.382$

$\lambda_1 = a_1 + (1 - \alpha)(b_1 - a_1) = 3.82$ and $\mu_1 = a_1 + \alpha(b_1 - a_1) = 6.18$.

$ECU(\lambda_1) = 8.561$ and $ECU(\mu_1) = 8.570$.

Since

$$ECU(\lambda_1) \leq ECU(\mu_1)$$

the next interval where the optimal solution lies is $[0, 6.18]$

Iteration 2

$$[a_2, b_2] = [0, 6.18]$$

$$\lambda_2 = 2.36, \text{ and } \mu_2 = 3.82$$

$$\text{ECU}(\lambda_2) = 8.533 \text{ and } \text{ECU}(\mu_2) = 8.561$$

Since

$$\text{ECU}(\lambda_2) \leq \text{ECU}(\mu_2)$$

the next interval where the optimal solution lies is $[0, 3.82]$

Iteration 3

$$[a_3, b_3] = [0, 3.82]$$

$$\lambda_3 = 1.459 \text{ and } \mu_3 = 2.36$$

$$\text{ECU}(\lambda_3) = 8.516 \text{ and } \text{ECU}(\mu_3) = 8.533$$

Since

$$\text{ECU}(\lambda_3) \leq \text{ECU}(\mu_3)$$

the next interval where the optimal solution lies is $[0, 2.36]$

Iteration 4

$$[a_4, b_4] = [0, 2.36]$$

$$\lambda_4 = 0.901 \text{ and } \mu_3 = 1.459$$

$$\text{ECU}(\lambda_4) = 8.613 \text{ and } \text{ECU}(\mu_3) = 8.516$$

Since

$$\text{ECU}(\lambda_4) \geq \text{ECU}(\mu_4)$$

the next interval where the optimal solution lies is $[0.901, 2.36]$.

Iteration 5

$$[a_5, b_5] = [0.901, 2.36]$$

$$\lambda_5 = 1.459 \text{ and } \mu_5 = 1.803$$

$$\text{ECU}(\lambda_5) = 8.516 \text{ and } \text{ECU}(\mu_5) = 8.517$$

Since

$$\text{ECU}(\lambda_5) \leq \text{ECU}(\mu_5)$$

the next interval where the optimal solution lies is $[0.901, 1.803]$

Iteration 6

$$[a_6, b_6] = [0.901, 1.803]$$

$$\lambda_6 = 1.246 \text{ and } \mu_6 = 1.459$$

$$\text{ECU}(\lambda_6) = 8.528 \text{ and } \text{ECU}(\mu_6) = 8.516$$

Since

$$\text{ECU}(\lambda_6) \geq \text{ECU}(\mu_6)$$

the next interval where the optimal solution lies is $[1.246, 1.803]$

Iteration 7

$$[a_7, b_7] = [1.246, 1.803]$$

$$\lambda_7 = 1.459 \text{ and } \mu_7 = 1.590$$

$$\text{ECU}(\lambda_7) = 8.516 \text{ and } \text{ECU}(\mu_7) = 8.514$$

Since

$$\text{ECU}(\lambda_7) \geq \text{ECU}(\mu_7)$$

the next interval where the optimal solution lies is $[1.459, 1.803]$

If we fix the δ level less than or equal to 0.5 we can conclude that the optimum value lies in the interval $[1.459, 1.803]$ and it is given by $\frac{1.459+1.803}{2} = 1.631$.

We have perform this numerical example assuming that the failure data follows the $TSL(1, 1)$ model and we obtain that to optimize the cost we have to schedule the maintenance time after 1.631 units of time.

7.3 Block Replacement Policy and TSL Probability Distribution

Here we consider the case of the so called “Block-Replacement Policy” or the constant interval policy. In this policy we perform preventive maintenance on the system after it has been operating a total of t^* unites of time, regardless of the number of intervening failures. In case the system has failed prior to the time t^* , minimal repair will be performed. We assume that the minimal repair won't change the failure rate of the system and the preventive maintenance renews the system and it become as good as new . Thus, we want to find the t^* that minimizes the expected repair and preventive maintenance cost. The total expected cost per unit time for preventive replacement at time t^* , denoted by $ECU(t^*)$ is given by

$$ECU(t^*) = \frac{\text{Total expected cost in the interval}(0, t^*)}{\text{Length of the interval}}.$$

The total expected cost in the interval $(0, t^*)$ equals to the cost of preventative main-tenance plus the cost of failure maintenance, that is $= c_1 + c_2M(t^*)$, where $M(t^*)$ is the expected number of failure in the interval $(0, t^*)$

Hence,

$$ECU(t^*) = \frac{c_1 + c_2M(t^*)}{t^*}.$$

But we know that the expected number of failure in the interval $(0, t^*)$ is the integral of the failure rate function ,that is

$$M(t^*) = E(N(t^*)) = H(t^*) = \int_0^{t^*} h(t)dt$$

So if the failure of the system follows the TSL distribution we know that

$$M(t^*) = \int_0^{t^*} h(t)dt = \frac{(1 + \lambda)t^*}{\phi} - \log ((2 + 2\lambda) \exp(\lambda t^*/\phi) - 1) + \log(2\lambda + 1)$$

Thus we have

$$ECU(t^*) = \frac{c_1 + c_2[\frac{(1+\lambda)t^*}{\phi} - \log ((2 + 2\lambda) \exp(\lambda t^*/\phi) - 1) + \log(2\lambda + 1)]}{t^*}.$$

Again we would like to minimize this equation subject to the condition $c_1 = 1$ and $c_2 = 10$. We shall use again so called "Golden Section Method" to obtain the value of t^* that minimizes $\text{ECU}(t^*)$

Iteration 1

consider $[a_1, b_1] = [0, 10]$, $\alpha = 0.618$ so that $1 - \alpha = 0.382$

$\lambda_1 = a_1 + (1 - \alpha)(b_1 - a_1) = 3.82$ and $\mu_1 = a_1 + \alpha(b_1 - a_1) = 6.18$.

$\text{ECU}(\lambda_1) = 9.523$ and $\text{ECU}(\mu_1) = 9.697$.

Since

$$\text{ECU}(\lambda_1) \leq \text{ECU}(\mu_1)$$

the next interval where the optimal solution lies is $[0, 6.18]$

Iteration 2

$[a_2, b_2] = [0, 6.18]$

$\lambda_2 = 2.36$, and $\mu_2 = 3.82$

$\text{ECU}(\lambda_2) = 9.30$ and $\text{ECU}(\mu_2) = 9.523$

Since

$$\text{ECU}(\lambda_2) \leq \text{ECU}(\mu_2)$$

the next interval where the optimal solution lies is $[0, 3.82]$

Iteration 3

$[a_3, b_3] = [0, 3.82]$

$\lambda_3 = 1.459$ and $\mu_3 = 2.36$

$\text{ECU}(\lambda_3) = 9.124$ and $\text{ECU}(\mu_3) = 9.30$

Since

$$\text{ECU}(\lambda_3) \leq \text{ECU}(\mu_3)$$

the next interval where the optimal solution lies is $[0, 2.36]$

Iteration 4

$[a_4, b_4] = [0, 2.36]$

$\lambda_4 = 0.901$ and $\mu_3 = 1.459$

$\text{ECU}(\lambda_4) = 9.102$ and $\text{ECU}(\mu_3) = 9.124$

Since

$$\text{ECU}(\lambda_4) \leq \text{ECU}(\mu_4)$$

the next interval where the optimal solution lies is $[0, 1.459]$

Iteration 5

$$[a_5, b_5] = [0, 1.459]$$

$$\lambda_5 = 0.557 \text{ and } \mu_5 = .901$$

$$\text{ECU}(\lambda_5) = 9.405 \text{ and } \text{ECU}(\mu_5) = 9.102$$

Since

$$\text{ECU}(\lambda_5) \geq \text{ECU}(\mu_5)$$

the next interval where the optimal solution lies is $[0.557, 1.459]$

Iteration 6

$$[a_6, b_6] = [0.557, 1.459]$$

$$\lambda_6 = 0.9015 \text{ and } \mu_6 = 1.114$$

$$\text{ECU}(\lambda_6) = 9.102 \text{ and } \text{ECU}(\mu_6) = 9.08$$

Since

$$\text{ECU}(\lambda_6) \geq \text{ECU}(\mu_6)$$

the next interval where the optimal solution lies is $[\.901, 1.459]$

Iteration 7

$$[a_7, b_7] = [0.901, 1.459]$$

$$\lambda_7 = 1.114 \text{ and } \mu_7 = 1.245$$

$$\text{ECU}(\lambda_7) = 9.08 \text{ and } \text{ECU}(\mu_7) = 9.09$$

Since

$$\text{ECU}(\lambda_7) \leq \text{ECU}(\mu_7)$$

the next interval where the optimal solution lies is $[\.901, 1.245]$

Again if we fix the δ level less than or equal to 0.5 we can conclude that the optimum value lies in the interval $[0.901, 1.245]$ and it is given by $\frac{0.901+1.245}{2} = 1.07$.

As in the case of Age replacement case in this numerical example we assume that the failure data follows the $TSL(1, 1)$ model and we have seen that to optimize the cost we have to schedule the maintenance time every 1.07 units of time.

7.4 Maintenance Over a Finite Time Span

The problem concerning the preventive maintenance over a finite time span is of great important in industry. It can be viewed in two different prospective; whether the total number of replacements (Failure+planned)times are known or not. The first case is straightforward and it is known in the literature from a long time. Barlow et al.(1967) derive the expression for this case. Let T^* be the total time span which means we would like to minimize the cost due to replacement or due to planned replacement until $T = T^*$. Let $C_n(T^*, T)$ be the expected cost in the time span 0 to T^* , $[0, T^*]$, considering only the first n replacements following a policy of replacement at interval T. It is clear that considering the case when $T^* \leq T$ is equivalent to no planned replacement. It is clear that

$$C_1(T^*, T) = \begin{cases} c_2 F(T^*) & \text{if } T^* \leq T, \\ c_2 F(T) + c_1(1 - F(T)) & \text{if } T^* \geq T \end{cases}$$

Thus, for $n = 1, 2, 3, \dots$, we have

$$C_{n+1}(T^*, T) = \begin{cases} \int_0^{T^*} [c_2 + C_n(T^* - y, T)] dF(y) & \text{if } T^* \leq T, \\ \int_0^T [c_2 + C_n(T^* - y, T)] dF(y) + C(T^*, T) & \text{otherwise.} \end{cases} \quad (7.4.7)$$

where $C(T^*, T) = [c_1 + C_n(T^* - T, T)][1 - F(T)]$.

Here we would like to develop a statistical model which can be used to predict the total cost before we actually used any item. Let T be the predetermined replacement time. We always replace an item exactly at the time of failure or T hours after its installation, whichever occurs first. Let τ denotes the first time to failure or

replacement then we have

$$E(\text{cost}) = \int_0^T [c_2 + C_T(T^* - y)]f_T(y)dy + [1 - F_T(T)][c_1 + C_T(T^* - T)]$$

where c_1 is the cost for preventive maintenance and $c_2(> c_1)$ is the cost for failure maintenance.

Thus, we can write

$$\begin{aligned} C(T) &= \int_0^T [c_2 + C(T^* - y)]f(y)dy + [1 - F(T)][c_1 + C(T^* - T)] \\ &= c_2 \times F(T) + c_1 \times R(T) + R(T) \times C(T^* - T) + \int_0^T C(T^* - y)]f(y)dy, \end{aligned}$$

and

$$\begin{aligned} C'(T) &= (c_2 - c_1) \times F'(T) + R'(T) \times C(T^* - T) - R(T) \times C'(T^* - T) \\ &\quad + C(T^* - T) \times f(T) \\ &= (c_2 - c_1) \times F'(T) + [R'(T) + f(T)] \times C(T^* - T) - R(T) \times C'(T^* - T) \end{aligned}$$

We need to solve this differential equation to find the total cost. We would like to consider a numerical example to see whether the minimum exist if we assume the failure model being $TSL(1, 1)$. We generate a random sample of size 100 from $TSL(1, 1)$ and fix a time T to perform preventive maintenance. We consider the preventive maintenance cost $c_1 = 1$ and failure replacement cost $c_2 = 1, 2$ and 10 . We repeat the process several times and computed the total cost for first 40 failures and got the table below. Table 7.1 shows the existence of minimum cost.

T	FC_{10}	FC_2	FC_1
1.00	340.55	88.95	57.50
1.01	347.25	89.65	57.45
1.02	336.95	87.75	56.60
1.03	342.95	88.15	56.30
1.04	339.15	87.15	55.65
1.05	341.25	87.25	55.50
1.06	334.40	86.40	55.40
1.07	343.75	87.35	55.30
1.08	332.15	84.95	54.05
1.09	338.55	85.81	54.22
1.10	318.48	82.67	53.19
1.11	327.68	84.04	52.59
1.12	344.76	86.48	54.19
1.13	333.70	84.50	53.35
1.14	340.40	85.20	53.30
1.15	338.86	84.68	53.90
1.16	331.28	82.90	53.86
1.17	338.27	84.09	54.31
1.18	335.24	83.05	53.52
1.19	341.90	84.00	54.76
1.20	363.90	87.50	56.95

Table 7.1: Comparisons of costs for different values of preventive maintenance times

In Table 7.1, $FC_i, i = 1, 2 \& 10$ represents the total cost due to preventive maintenance cost $c_1 = 1$ and the failure replacement cost $c_2 = i, i = 1, 2 \& 10$. Table shows that the minimum FC_i exists about at $T = 1.1$ units of time.

7.5 Conclusion

In this chapter we have studied the analytical behavior of the TSL pdf when it is used to model preventative maintenance strategies in both the age replacement and block replacement polices . We also have developed all essential estimates of the parameters that are inherited in such analysis.

Using numerical data we have illustrated the usefulness of determining the optimum cost utilizing a preventive maintenance that was initially modeled by a differential equation.

Chapter 8

Future Research

In this Chapter we shall identify some important research problems that resulted from the present study, that we shall investigate in the near future.

It should be noted that in the present study we are restricted to a univariate skew Laplace probability distribution. It is of great interest to extend this distribution to the multivariate cases. In fact, a random vector $\mathbf{z} = (Z_1, \dots, Z_p)^T$ is a p-dimensional skew Laplace random variable denoted by $\mathbf{z} \sim \text{SL}_p(\boldsymbol{\Omega}, \boldsymbol{\lambda})$, if it is continuous with pdf given by

$$f(\mathbf{z}) = 2g_p(\mathbf{z}; \boldsymbol{\Omega})G_p(\boldsymbol{\lambda}^T \mathbf{z}), \quad \mathbf{z} \in \mathbb{R}^p,$$

where $g_p(\mathbf{z}; \boldsymbol{\Omega})$ and $G_p(\boldsymbol{\lambda}^T \mathbf{z})$ denotes the pdf and cdf of the p-dimensional multivariate Laplace probability distribution with the correlation matrix $\boldsymbol{\Omega}$ and $\boldsymbol{\lambda}$ is the vector of shape parameters. All analytical developments in the present study, we believe, can be extended, however, it may brings some difficulties.

It should be also noted that in the present study we introduced several real world data strictly for the purpose of identifying the goodness of fit of the different types of probability density functions that we have introduced. Furthermore we compared the fitness with the fitness of the actual probability density that was used to analyze the subject data. We have demonstrated that the developed probability distribution gives a better probabilistic characterization on this real world phenomenon.

Thus, it is the aim of the future research projects to statistically fully analyze and model this real world data using the proposed analytical results we have developed.

We anticipate that our analysis will result in better decisions and estimation of the various unknowns related to each of the projects, because our analytical methods gave better fits than the one which were used to analyze the subject data.

We have presented two real world data namely the currency exchange data and the pressure vessels data in Chapter 4 and Chapter 6 respectively. In the currency exchange data we were mainly interested on whether our proposed models, the skew Laplace probability distribution, fits the exchange rate data better than the traditionally used Gaussian model. We have observed graphically and statistically that the goodness of fit of the SL pdf fits much better. Now we are interested to study the possible impact of choosing this model on the financial analysis and decision making of this data. More specifically, we are interested on estimation and the inferential statistical study of the data.

We shall also investigate analytically and by simulation the relationship between the skew Laplace probability distribution and the skew Normal probability distribution. We will use this currency exchange rate data to verify the relation.

In Chapter 6 we have observed, graphically and statistically that the by goodness of fit of the truncated skew Laplace (TSL) probability distribution fitted the pressure vessels failure data better than the two parameter gamma distribution that was used to analyze the data. Now it is of interest to investigate the inferential study of the TSL model to this data and compare the findings.

Finally, in Chapter 7 we have observed that the TSL probability distribution can be used in the preventive maintenance over an infinite and over a finite time span. We shall study the existence, instability behavior of the delay differential equation that was resulted on computing the expected cost over a finite time span. Furthermore,

we shall study the most appropriate numerical technique to obtain the estimates of the solution of the nonlinear differential, delay equation and apply quasi linearization methodology to reduce the subject differential system into a quasi-linear form so that we can obtain an exact analytical solution of the system. These two approaches will be compared to determine their effectiveness.

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