



TITLE:

# Study of Many-State Classical Spin Systems by the Renormalization Group, Duality and Monte Carlo Simulation

AUTHOR(S):

Nishimori, Hidetoshi

---

CITATION:

Nishimori, Hidetoshi. Study of Many-State Classical Spin Systems by the Renormalization Group, Duality and Monte Carlo Simulation. 物性研究 1979, 32(4): 315-353

ISSUE DATE:

1979-07-20

URL:

<http://hdl.handle.net/2433/89851>

RIGHT:

---

修士論文

---

Study of Many-State Classical Spin Systems  
by the Renormalization Group, Duality  
and Monte Carlo Simulation

Hidetoshi Nishimori\*)

Department of Physics, Faculty of Science,  
University of Tokyo

(1979年3月22日受理)

**Abstract**

A class of classical spin systems in two dimensions with symmetry of a cyclic group is studied in three ways: by the real space renormalization group technique, by the duality transformation and by a numerical experiment. The former two methods are shown to be in a close relation to each other. Useful information is obtained on the shape of the critical surface, on the behavior near the criticality and on thermodynamic quantities such as the internal energy and the magnetic susceptibility. Some of these results are compared with expected properties of the planar model, to which the present model reduces if an appropriate limit is taken.

**Contents**

- I. Introduction
- II. Duality transformation
  - § 2.1 Formulation
  - § 2.2 Application
- III. Renormalization group transformation by Migdal and Kadanoff
  - § 3.1 Formulation
  - § 3.2 Application
- IV. Monte Carlo simulation
- V. Conclusion
- Appendix A. Derivation of the Migdal-Kadanoff Transformation
- Appendix B. Monte Carlo simulation and static and dynamic critical behavior of the plane rotator model (Prog. Theor. Phys. **60** (1978) 1669)

---

\*) 西森秀稔

## I. Introduction

Recently two-dimensional spin systems began to draw attention theoretically and experimentally. From the theoretical point of view, two-dimensional systems with short range interactions are quite interesting because of their marginal properties; some two-dimensional systems are supposed<sup>1),2)</sup> to have a new type of low temperature phase without long range order, while three-dimensional systems clearly undergo phase transitions accompanying onsets of long range order, and one-dimensional ones have no critical points. Developments in experimental techniques have made it possible<sup>3)~5)</sup> to test some predictions<sup>6),18)</sup> concerning this marginality.

In this context we have theoretically investigated the properties of a class of two dimensional systems by making use of three methods: the duality transformation which gives a constraint on the shape of the critical surface, the renormalization group which mainly concerns the critical phenomena and Monte Carlo simulation which gives a clear picture on the behavior of thermodynamic quantities in a wide range of temperatures.

Our Hamiltonian to be treated in this thesis is expressed as

$$H = \sum_{\langle i,j \rangle} V(\xi_i - \xi_j) \quad (1.1)$$

where  $\xi_i$  is a spin variable which takes an integer value between 1 and  $n$ :  $\xi_i = 1, 2, \dots, n$ . The indices  $i$  and  $j$  run over all pairs of nearest neighboring lattice sites. By the spin variable  $\xi_i$ , we merely mean some degree of freedom attributed to the site  $i$ .  $V$  is a periodic and even function:  $V(\xi) = V(-\xi) = V(\xi + kn)$  ( $k$  is an arbitrary integer). The temperature factor  $-1/kT$  is already included in  $V$ . The reason why models of this type are interesting can be made clear if we take some special forms of  $V$ .

First let us put

$$V(\xi_i - \xi_j) = \frac{J}{kT} \cos \left( 2\pi \frac{\xi_i - \xi_j}{n} \right). \quad (1.2)$$

This is the discrete vector model<sup>7)~12),14)</sup> (sometimes called the planar Potts model<sup>13)</sup>). In the limit  $n \rightarrow \infty$  it reduces to the planar model

$$V(\theta_i - \theta_j) = \frac{J}{kT} \cos(\theta_i - \theta_j) \quad (1.3)$$

The planar model is experimentally a lattice gas model of the superfluid Helium<sup>15),16)</sup>

in which  $\theta_j$  represents the phase of the order parameter. The two-dimensional case in this model is particularly interesting because it is believed to have a critical point at some finite temperature<sup>1),2)</sup> although the absence of long range order at any finite temperatures has been proved rigorously.<sup>17)</sup> Miyashita et al.<sup>18)</sup> studied the discretized version (1.2) of the planar model (1.3) by performing a Monte Carlo simulation. José et al.<sup>19)</sup> regarded (1.2) as an effective Hamiltonian of the planar model (1.3) when the symmetry breaking crystal field  $H'$

$$H' = \sum_i h \cos(n\theta_i)$$

is relevant to critical phenomena, and they treated the models (1.2) and (1.3) by a renormalization group transformation. Both of these two papers by Miyashita et al. and by José et al. suggest that the discrete model (1.2) behaves like the continuous model (1.3) if  $n$  is large enough and the temperature is not too low. The results of the present work also support their suggestion. From the experimental point of view, the planar model (1.3) is most interesting. However, it is very difficult to study it directly. The extrapolation from the discrete model is a possible strategy.

If we take another form of  $V$  as

$$V(\xi) = \sum_{k=-\infty}^{\infty} \delta_{\xi, kn} \times \frac{\text{const}}{kT},$$

then, it is the Potts model.<sup>8)</sup> Potts regarded this model as an extension of the Ising model (in the special case  $n=2$  this reduces to the Ising model) and found that a duality relation holds. (The duality relation which he found relates the free energy in the high temperature region to that in the low temperature region. Therefore it is possible to locate the transition point as the boundary of these two regions.) The Potts model was the first extension of the Ising model as a system for which the duality holds.

The three ways of our approach, namely, the duality transformation, the renormalization group and Monte Carlo simulation, are presented successively in chapters II, III and IV. In chapter II the Wu-Wang duality relation<sup>20)</sup> is explained, and it is found that its applicability to the determination of the distribution of singularities of the free energy is limited. This limit is complemented by the renormalization group approach proposed by Migdal<sup>21)</sup> and re-interpreted later by Kadanoff<sup>22)</sup>, as is shown in chapter III. Two of the

西森秀稔

results of this renormalization group calculation are of particular interest. That is, (i) although it is already known<sup>8),14)</sup> that the location of the transition point of the discrete vector model (1.2) with  $n$  less than six is exactly calculable, it is possible to obtain, by using the Migdal–Kadanoff transformation, the transition points of models even for  $n \geq 6$ . (ii) As to the critical phenomena, the results on the discrete vector model obtained so far<sup>7),14),18),19)</sup> are well reproduced and some new aspects of the general model (1.1) are found by means of the Migdal–Kadanoff transformation. In chapter IV we refer to the results of the Monte Carlo simulation by Miyashita et al. and comment on the validity of the finite- $n$  approximation (1.2) to the planar model (1.3). Chapter V is devoted to general discussions.

In this thesis we confine ourselves to the two-dimensional square lattice for simplicity.

## II. Duality transformation

A duality transformation relates the free energy of a system in the high temperature region to that in the low temperature region. Thus it is possible to identify the location of the transition point as the boundary of these two regions. A duality relation was first discovered by Kramers and Wannier<sup>35)</sup> for the two-dimensional Ising model, and it was extended to the Potts model by Potts<sup>8)</sup> and Kihara et al.<sup>9)</sup> Here we adopt an expression derived by Wu and Wang<sup>20)</sup> for the general model (1.1).

### §2.1 Formulation<sup>20)</sup>

We can express the partition function of the system (1.1) in two different ways. The first one is defined by

$$Z = \sum_{\{\xi\}} \exp \left( \sum_{\langle i,j \rangle} V(\xi_i - \xi_j) \right) \quad , \quad (2.1)$$

where the summation  $\sum_{\{\xi\}}$  is taken over all values of the variables  $\{\xi\}$ . For later convenience, we rewrite Eq. (2.1) in a slightly different form. Let us take the set  $\{\xi_i - \xi_j\}$  in Eq. (2.1) as a new set of variables. Clearly not all of these new variables are inde-

pendent; along a closed circuit on the square lattice (Fig. 1), the summation of new

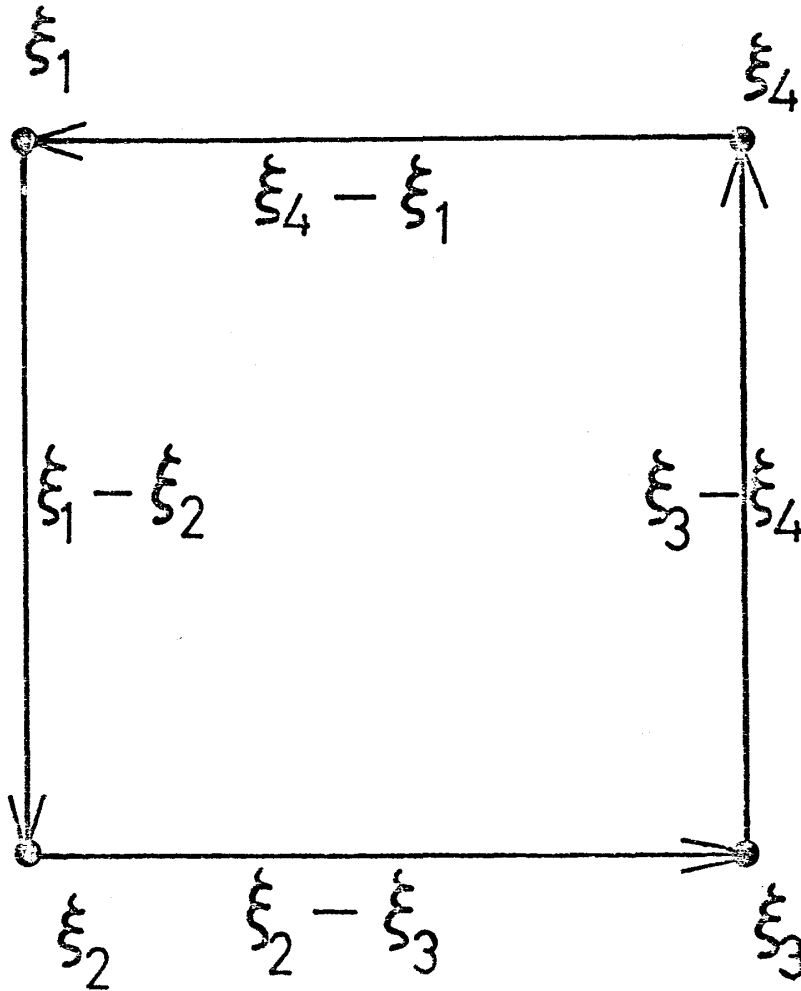


Fig. 1. The set of differences of the original variables  $[\xi_i - \xi_j]$  is taken as a new set of variables. Only  $N$  (the number of lattice sites) of the new variables are independent.

variables  $\{\xi_i - \xi_j\}$  vanishes (e.g., in Fig. 1,  $(\xi_1 - \xi_2) + (\xi_2 - \xi_3) + (\xi_3 - \xi_4) + (\xi_4 - \xi_1) = 0$ ). It should be noted that this condition for vanishing holds only when we choose the order of the subtraction  $\xi_i - \xi_j$  as follows, that is,  $\xi_i$  and  $\xi_j$  are variables at the tail and head of the arrow respectively, where the arrow connects the two neighboring lattice points  $i$  and  $j$ , and it is directed to the way along which we go around the circuit. By using this new set of variables  $\{\eta_{ij}\}$  ( $\eta_{ij} \equiv \xi_i - \xi_j$ ), which are defined for all bonds, we can write the partition function as

$$Z = \sum'_{\{\eta\}} \exp \left( \sum_{\langle i,j \rangle} V(\eta_{ij}) \right) \quad (2.2)$$

西森秀稔

where the primed sigma denotes the sum over all values of the new variables  $[\eta]$  which satisfy the condition for vanishing.

Another expression of the partition function is obtained by expanding the Boltzmann factor  $\exp [V(\xi_i - \xi_j)]$  in Eq. (2.1) in the following Fourier series:

$$\exp [V(\xi_i - \xi_j)] = \sum_{\lambda_{ij}=1}^n \exp [2\pi i \lambda_{ij} \frac{\xi_i - \xi_j}{n}] u_{\lambda_{ij}} . \quad (2.3)$$

This Fourier expansion has only  $n$  terms since the interaction  $V$  is periodic:  $V(\xi) = V(\xi+n)$ . Note that the variable  $\lambda_{ij}$  in Eq. (2.3) can be regarded to be attached to the bond connecting the two sites  $i$  and  $j$ . As is shown in the following, the substitution of Eq. (2.3) into Eq. (2.1) and the summation over  $\{\xi\}$  in Eq. (2.1) yield the second expression of the partition function:

$$Z = n^{2N} \sum'_{\{\lambda\}} \prod_{\langle i,j \rangle} u_{\lambda_{ij}} . \quad (2.4)$$

Here the primed summation has the same meaning as in (2.2) and  $N$  represents the total

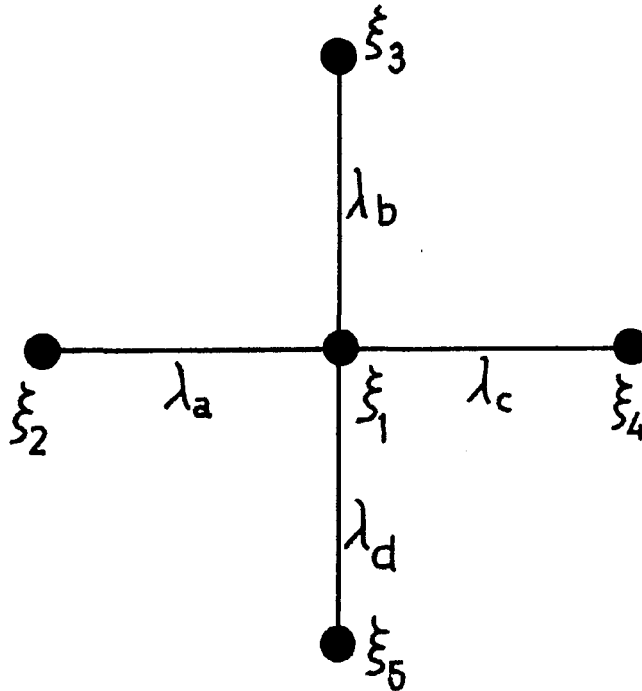


Fig. 2. An assignment of the original variables  $\{\xi\}$  and the variables  $\{\lambda\}$  in the Fourier-transformed expression.

number of sites on the lattice. We show below how the primed summation over  $\lambda$  appears in Eq. (2.4). Let us consider the configuration of the variables shown in Fig. 2. By making use of Eq. (2.3), the trace of the product of the Boltzmann factors  $\exp V(\xi_1 - \xi_2)$ ,  $\exp V(\xi_1 - \xi_3)$ ,  $\exp V(\xi_1 - \xi_4)$  and  $\exp V(\xi_1 - \xi_5)$  over  $\xi_1$  results in

$$\begin{aligned} & \sum_{\xi_1=1}^n \exp [V(\xi_1 - \xi_2) + V(\xi_1 - \xi_3) + V(\xi_1 - \xi_4) + V(\xi_1 - \xi_5)] \\ &= n \sum'_{\substack{\lambda_a \lambda_b \\ \lambda_c \lambda_d}} u_{\lambda_a} u_{\lambda_b} u_{\lambda_c} u_{\lambda_d} \exp \frac{2\pi i}{n} (-\lambda_a \xi_2 + \lambda_b \xi_3 + \lambda_c \xi_4 - \lambda_d \xi_5), \end{aligned}$$

where the prime means the constraint on the summation that the sum of  $\lambda$ 's of the four bonds  $\lambda_a + \lambda_b + \lambda_c + \lambda_d$  should be a multiple of  $n$ . If we call  $\lambda_a$  the current attached to the bond  $a$  (and similarly for  $b$ ,  $c$  and  $d$ ), it is possible to interpret this constraint as a current conservation condition with modulo  $n$  at site 1. By taking summations over all  $\xi_i$ 's in this way we reach the representation of  $Z$  in which  $\{\lambda\}$  is a new set of summation variables which satisfies the current conservation condition. This conservation condition is clearly equivalent to the condition that the current around any closed circuit should vanish. Thus we get the expression (2.4). By comparing Eqs. (2.2) and (2.4), we find that the partition function is invariant with respect to the exchange of  $\exp V(j)$  and  $u_j$  (except the pre-factor  $n^{2N}$ ):

$$Z(e^{V(1)}, e^{V(2)}, \dots, e^{V(n)}) = (nu_n)^{2N} Z(u_1/u_n, u_2/u_n, \dots, u_n/u_n), \quad (2.5)$$

where we have normalized  $u_j$  in Eq. (2.4) by  $u_n$ , because we hereafter normalize the Boltzmann factor  $\exp V(n)$  for parallel spins in Eq. (2.2) to unity ( $e^{V(n)} = e^{V(0)} = 1$ ), and consequently the corresponding Boltzmann factor  $u_n$  in Eq. (2.4) should also be made unity by dividing all  $u_j$ 's by  $u_n$ . This normalization is not necessary in general for our discussion, but it makes calculations simpler by reducing the number of free parameters. For simplicity we denote  $x_\xi$  for  $e^{V(\xi)}$  and  $x_\lambda^*$  for  $u_\lambda$ . That is,

$$X_\lambda^* = \frac{\sum_{\xi=1}^n \exp(2\pi i \frac{\xi\lambda}{n}) X_\xi}{\sum_{\xi=1}^n X_\xi}, \quad (2.6)$$

as can be verified from the definition of  $u_\lambda$  in Eq. (2.3). Then Eq. (2.5) is re-expressed as



$$Z(x_1, x_2, \dots, x_{n-1}) = (nu_0)^{2N} Z(x_1^*, x_2^*, \dots, x_{n-1}^*) . \quad (2.7)$$

This is the duality relation we have intended to derive.

From Eq. (2.6) it directly follows that all  $x_\lambda^*$ 's are monotonically decreasing functions of  $x_\xi$ 's. If we note that the high temperatures correspond to the values of  $x_\xi$ 's near 1 ( $x_\xi = e^{V/\xi} \sim e^0 = 1$ ) and the low temperatures to  $x_\xi \sim 0$  ( $x_\xi \sim e^{-\infty} = 0$ ) and similarly for  $x_\lambda^*$ 's, this duality relation (2.7) can be interpreted to relate the free energy (or the partition function) in the high temperature region to that in the low temperature region. The transformation (2.6) from  $(x_1, x_2, \dots, x_{n-1})$  to  $(x_1^*, x_2^*, \dots, x_{n-1}^*)$  is called the duality transformation. We denote it by  $D$ .

Since we treat an anisotropic interaction characterized by  $V_x$  and  $V_y$ , in the next chapter (including the isotropic interaction  $V_x = V_y = V$  as a special case), some generalization of the duality transformation is necessary. As can be seen from the arguments hitherto, the duality transformation for the anisotropic case can also be derived by changing the state variables from the original spin variables  $\{\xi\}$  to the set of parameters  $\{\lambda\}$  appearing in the Fourier expansion of the Boltzmann factor:

$$\exp (V_{\text{anis.}}(\eta)) = \sum_{\lambda=1}^n \exp \left( \frac{2\pi i \lambda \eta}{n} \right) u_\lambda$$

where  $\xi = \xi_1 - \xi_2$  (Fig. 3). Each of these parameters  $\{\lambda\}$  can be regarded to be assigned to the bond connecting two sites on the original lattice. Thus, by taking into account

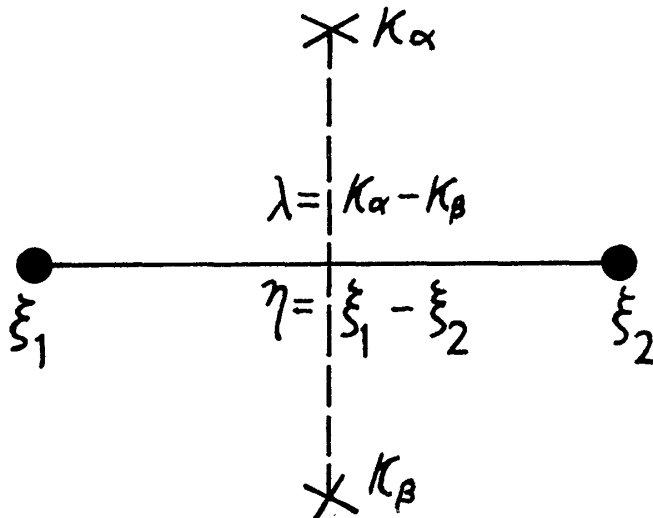


Fig. 3. Both of the variables  $\eta$  and  $\lambda$  can be expressed by the differences  $\xi_1 - \xi_2$  and  $\kappa_\alpha - \kappa_\beta$ .

the current conservation property of  $\{\lambda\}$ , it is possible to interpret  $\lambda$  as a difference  $\kappa_\alpha - \kappa_\beta$  of two parameters  $\kappa_\alpha$  and  $\kappa_\beta$  assigned to the two sites on the dual lattice (a lattice with sites each of which is located in the midst of four neighboring sites of the original lattice). Hence the duality transformation for an anisotropic model brings the interaction in the  $x$ -direction to that in the  $y$ -direction and vice versa. In this way the duality transformation (2.6) and the duality relation (2.7) are generalized as

$$y_{\lambda}^* = \frac{\sum_{\xi=1}^n \exp\left(\frac{2\pi i \lambda \xi}{n}\right) x_{\xi}}{\sum_{\xi=1}^n x_{\xi}}$$

$$x_{\lambda}^* = \frac{\sum_{\xi=1}^n \exp\left(\frac{2\pi i \lambda \xi}{n}\right) y_{\xi}}{\sum_{\xi=1}^n y_{\xi}}$$
(2.8)

and

$$Z(x_1, x_2, \dots, x_{n-1}; y_1, y_2, \dots, y_{n-1})$$

$$= n^{2N} (u_{0x} u_{0y})^N Z(y_1^*, y_2^*, \dots, y_{n-1}^*; x_1^*, x_2^*, \dots, x_{n-1}^*)$$
(2.9)

where

$$u_{0x} = \frac{1}{n} \sum_{\xi=1}^n \exp(V_x(\xi))$$

and

$$u_{0y} = \frac{1}{n} \sum_{\xi=1}^n \exp(V_y(\xi))$$

(These expressions of  $u_{0x}$  and  $u_{0y}$  have been obtained by generalizing the definition (2.3) of  $u_0$  in the isotropic case.) In this chapter we treat only the isotropic duality given in Eqs. (2.6) and (2.7).

Since a "ferromagnetic" interaction is assumed, that is,  $0 = V(0) > V(1) > \dots > V([n/2])$  ( $[n/2]$  is the largest integer not exceeding  $n/2$ ) and  $V(n) = V(0) > V(n-1) > \dots > V([n/2])$ , all of the  $x_{\xi} = e^{V(\xi)}$  are less than unity. Furthermore as mentioned in the previous chapter, we assume that  $V$  already includes the temperature factor  $-kT/J$ . From these two facts it follows that the point  $(x_1, x_2, \dots, x_{n-1})$  in the  $n-1$  dimensional space draws a line connecting  $(1, 1, \dots, 1)$  and  $(0, 0, \dots, 0)$  as we vary the temperature  $T$

西森秀稔

from infinity to zero. Correspondingly the dual Boltzmann factors  $(x_1^*, x_2^*, \dots, x_{n-1}^*)$ , as a set of monotonically decreasing functions of  $x_\xi$ 's, draw the corresponding trajectory from  $(1, 1, \dots, 1)$  to  $(0, 0, \dots, 0)$  as  $T$  decreases. Therefore it is clear from Eq. (2.7) that if the free energy is singular at some temperature  $T_c$ , then it is singular also at the corresponding point  $(x_1^*(x_1(T_c), x_2(T_c), \dots, x_{n-1}(T_c)), \dots, x_{n-1}^*(x_1(T_c), x_2(T_c), \dots, x_{n-1}(T_c)))$ . Now let us regard  $\{x_\xi\}$  as mutually independent variables and not as functions of the temperature. Then the duality relation (2.7) still holds for  $Z$ , as can be verified by re-tracing the derivation of Eq. (2.7), and it gives a constraint on the way in which the singularities of the function  $Z$  distribute; if  $Z$  is singular at  $(x_{1c}, x_{2c}, \dots, x_{n-1,c})$ , it is singular also at the dual point  $(x_1^*(x_{1c}, x_{2c}, \dots, x_{n-1,c}), \dots, x_{n-1}^*(x_{1c}, x_{2c}, \dots, x_{n-1,c}))$ . Since a physical system of the type of Eq. (1.1) is represented by a line connecting two points  $(0, 0, \dots, 0)$  and  $(1, 1, \dots, 1)$  in the  $n-1$  dimensional space, we can determine the location of the transition point of any system of the type of Eq. (1.1) by finding the distribution of singularities of the function  $Z(x_1, x_2, \dots, x_{n-1})$ : the intersection of the set of singularities (i.e., the critical surface) and the line representing the physical system is the transition point. Thus the problem of finding the transition point can be solved by determining the distribution of singularities of the function  $Z$  with mutually independent variables  $\{x_\xi\}$ . The duality (2.7) offers a clue in this procedure by limiting the possible distributions. The next section is devoted to the illustration of this idea.

## §2.2 Application

Before giving some examples of the application of the idea in the previous section, we should make two remarks.

The first one is on the reduction of the degrees of freedom. Since our interaction  $V$  is an even and periodic function, the following relation holds:

$$x_\xi = e^{V(\xi)} = e^{V(-\xi)} = e^{V(n-\xi)} = x_{n-\xi}. \quad (2.10)$$

Similarly, from the definition of the dual Boltzmann factors (2.6), we can obtain the same relation for  $x_\lambda^*$ . Therefore, for simplicity, it is appropriate to confine ourselves to the subspace  $(x_1, x_2, \dots, x_{[n/2]})$ . In the rest of this thesis all functions of  $(x_1, x_2, \dots,$

$x_{n-1}$ ) will be written as if they were functions of  $(x_1, x_2, \dots, x_{[n/2]})$ . We make the second remark on the role played by the fixed point of the duality transformation  $D$ . If there is a fixed point of  $D$  on a line connecting  $(0, 0, \dots, 0)$  and  $(1, 1, \dots, 1)$ , the assumption that this point is singular is clearly consistent with the duality (2.7). It is not only consistent but, as can be easily understood, also this assumption is rigorously justified if we assume the existence of a unique singularity on this line representing a physical system. Therefore our interest in this section is focused upon the distribution of fixed points of  $D$ . Unless mentioned explicitly, we identify a singularity (i.e., a transition point) with a fixed point.

### 2.2.1 Case 1: $n = 2$ and 3

In this case the number of degrees of freedom is unity ( $= [n/2]$ ). Thus the transition points of these models are given by, respectively,

$$x_c = x_c^* = \frac{1 - x_c}{1 + x_c}$$

and

$$x_c = x_c^* = \frac{1 - x_c}{1 + 2x_c}$$

as is well known.<sup>11)</sup>

### 2.2.2 Case 2: $n = 4$

If  $n$  is four,  $[n/2]$  is two, and, as a direct consequence of the duality transformation (2.6), the set of fixed points constitutes the line

$$x_2^c = 1 - 2x_1^c$$

The four-state Potts model<sup>10),11)</sup> is a system with  $x_1 = x_2$  and consequently its transition point is given by  $x_c = 1 - 2x_c$ , i.e.,  $x_c = 1/3$ <sup>8),9)</sup>. The four-state discrete vector model<sup>12),13)</sup> has the Boltzmann factors  $x_1 = \exp[(-1 + \cos 2\pi/4)K]$  and  $x_2 = \exp[(-1 + \cos 4\pi/4)K] = x_1^2$ . Its transition point is given by  $x_c^2 = 1 - 2x_c$ , i.e.,  $x_c = \sqrt{2} - 1$ .<sup>10),11)</sup> In the region  $x_2 > x_1$ , the situation is a little more complicated<sup>28)</sup> (the fixed point does not necessarily correspond to the critical point), and will be explained in detail in the next chapter.

### 2.2.3 Case 3: $n = 5$

In the model with  $n = 5$ , the degree of freedom ( $[n/2]$ ) is two, and the set of fixed

西森秀稔

points is a line  $x_1^c + x_2^c = (\sqrt{5}-1)/2$ , as can be verified by putting  $x_1 = x_1^* = x_1^c$  and  $x_2 = x_2^* = x_2^c$  in Eq. (2.6). Hence the five-state Potts model ( $x_1 = x_2$ ) has the transition point at  $x_c = (\sqrt{5}-1)/4^{(8),9)}$  and the five-state discrete vector model is critical at  $\exp [(-1 + \cos 2\pi/5)K_c] + \exp [(-1 + \cos 4\pi/5)K_c] = (\sqrt{5}-1)/2$ .<sup>16)</sup>

#### 2.2.4 Case 4: $n \geq 6$

It has been shown for the models with  $n$  up to five that the set of fixed points divides the parameter space  $(x_1, x_2, \dots, x_{[n/2]})$  into two parts, each of which is interpreted as the high temperature phase and the low temperature phase respectively. (see Figs. 4, 5). On the other hand, the six-state model has, in the three dimensional space  $(x_1, x_2, x_3)$ , fixed points only on the following line

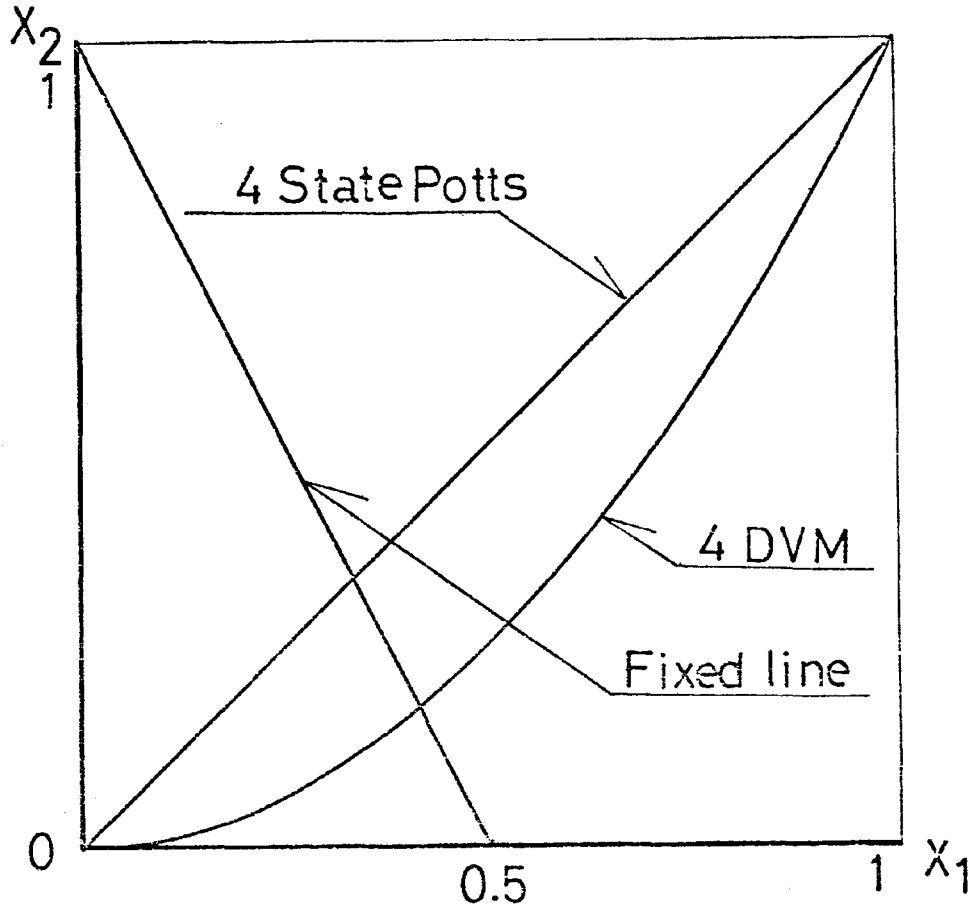


Fig. 4. The free energy of the four-state model is fixed by the duality transformation along the line  $x_2 = 1 - 2x_1$ . The two lines representing the four-state Potts model ( $x_1 = x_2$ ) and the four-state discrete vector model (4DVM,  $x_2 = x_1^2$ ) have common points with this line.

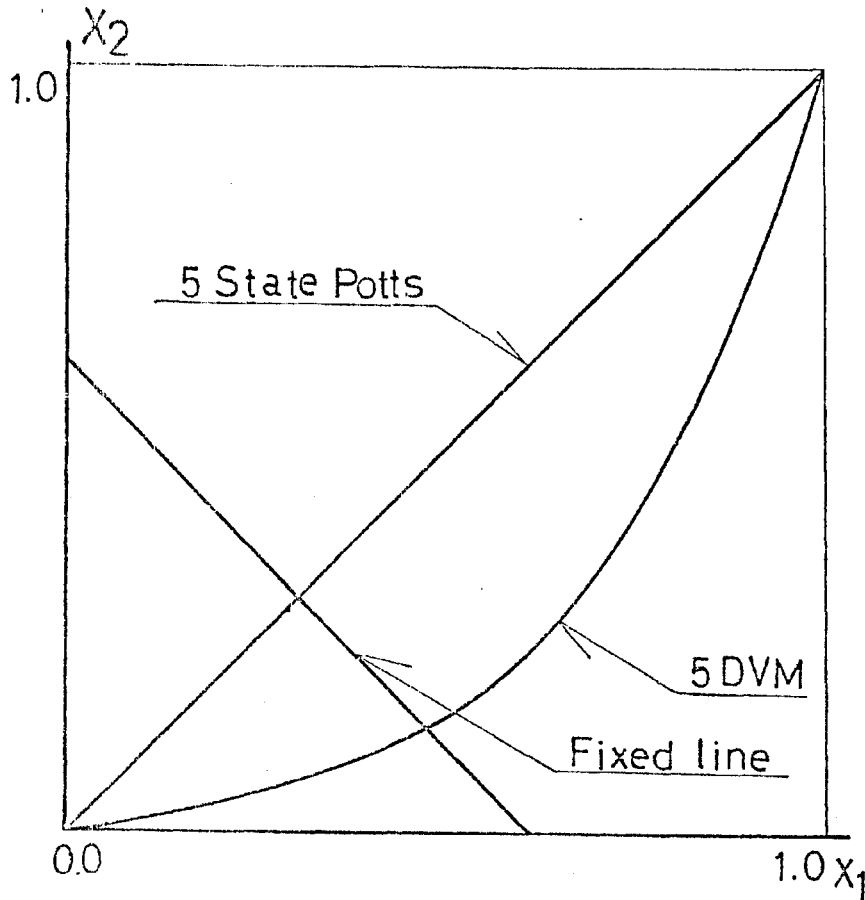


Fig. 5. All points on the line  $x_1 + x_2 = (\sqrt{5}-1)/2$  is fixed by the duality transformation of the five-state model. The transition point of any five-state model is obtained as the intersection of this invariant line and the line representing the system.

$$x_1 = \frac{\sqrt{6}}{3} - \frac{3 + \sqrt{6}}{3} x_2 \quad (2.11)$$

$$x_3 = \frac{\sqrt{6} - 3}{3} + \frac{2\sqrt{6}}{3} x_2$$

which can be obtained by putting  $x_1 = x_1^*$ ,  $x_2 = x_2^*$  and  $x_3 = x_3^*$  in Eq. (2.6). This line does not divide the three-dimensional space into two distinguishable parts, and we cannot determine the location of the transition point for an arbitrary six-state model; more precisely, a line representing a particular six-state model (e.g., the six-state discrete vector model:  $x_1 = \exp [(-1 + \cos 2\pi/6)K]$ ,  $x_2 = \exp [(-1 + \cos 4\pi/6)K]$  and  $x_3 = \exp [(-1 + \cos 6\pi/6)K]$ ) does not generally have a common point with the invariant line (2.11).

西森秀稔

Similar situation arises also for  $n \geq 7$ , and this fact makes it difficult to locate the transition point of many-state ( $n \geq 6$ ) systems only by means of the duality transformation. However, it should be noted that the absence of a fixed point of the duality transformation along a physical line connecting  $(0, 0, \dots, 0)$  and  $(1, 1, \dots, 1)$  does not immediately imply the absence of any singularities on this line; a fixed point can surely be a transition point (assuming its unique existence) but the converse is not necessarily true. The duality relation (2.7) never asserts that any singularity should be identified with a fixed point, but only tells that, if a particular point on the space  $(x_1, \dots, x_{[n/2]})$  is singular, then the point dual to it is also singular. As mentioned above, if  $n$  exceeds five, this constraint is not strong enough to locate all singularities even if we assume the uniqueness of the transition point. Thus it is necessary to introduce another approach, for example, the renormalization group transformation as in the next chapter.

Before concluding this chapter, we remark that it is possible to find the value of the transition point even for  $n$  larger than five if we confine ourselves to particular models whose trajectory in the  $[n/2]$  dimensional space intersects the set of fixed points. A most popular example is the Potts model in which  $x_1 = x_2 = \dots = x_{[n/2]} \equiv x$ . Application of Eq. (2.6) to this model yields  $x_1^* = x_2^* = \dots = x_{[n/2]}^* \equiv x^*$ . Hence the transition point exists at  $x_c = x_c^* = (1 - x_c)/(1 + (n-1)x_c)$ , namely, we obtain the well known result<sup>8)</sup>  $x_c = 1/(1 + \sqrt{n-1})$ .

### III. Renormalization group transformation by Migdal and Kadanoff

It is well known<sup>23)</sup> that the real space renormalization group technique gives an approximate value of the critical point. Among various formulations<sup>23)</sup> of the real space renormalization group transformation, the ingenious one by Kadanoff<sup>22)</sup> based on the idea of Migdal<sup>21)</sup> has a remarkable property<sup>22)</sup>: if it is applied to the Potts model (including the Ising model) on the square lattice, it gives the exact value of the transition point. In this chapter we show that this property of the Migdal-Kadanoff transformation can be extended to the general model (1.1). By extension we never mean that we have proved that the transition point obtained by the Migdal-Kadanoff formalism is exact for any

systems of the type of Eq. (1.1); we only imply that the distribution of singularities given by the Migdal-Kadanoff renormalization group transformation (MKRG) satisfies the duality constraint (2.7).

### § 3.1 Formulation

The set of Boltzmann factors (of two neighboring spins) in a system with anisotropic interaction is expressed by  $(x_1, x_2, \dots, x_{[n/2]}; y_1, y_2, \dots, y_{[n/2]})$ , where  $x_\xi = \exp V_x(\xi)$  and  $y_\xi = \exp V_y(\xi)$ . As is shown in Appendix A, MKRG transforms this set of Boltzmann factors into a new set  $(x'_1, x'_2, \dots, x'_{[n/2]}; y'_1, y'_2, \dots, y'_{[n/2]})$  with  $x'_\xi$  and  $y'_\xi$  given by

$$x'_\xi = \frac{\frac{1}{n} \sum_{\lambda} e^{i \frac{2\pi\xi\lambda}{n}} \left[ \sum_{\mu} e^{-i \frac{2\pi\lambda\mu}{n}} x_{\mu} \right]^b}{\left[ \frac{1}{n} \sum_{\lambda} \left[ \sum_{\mu} e^{-i \frac{2\pi\lambda\mu}{n}} x_{\mu} \right]^b \right]^b} \quad (3.1)$$

$$y'_\xi = \frac{\frac{1}{n} \sum_{\lambda} e^{i \frac{2\pi\xi\lambda}{n}} \left[ \sum_{\mu} e^{-i \frac{2\pi\mu\lambda}{n}} (y_{\mu})^b \right]^b}{\frac{1}{n} \sum_{\lambda} \left[ \sum_{\mu} e^{-i \frac{2\pi\mu\lambda}{n}} (y_{\mu})^b \right]^b} ,$$

where  $b$  denotes the scale factor ( $b > 1$ ). Normalization  $x_n = y_n = x'_n = y'_n = 1$  is implicit in Eq. (3.1). Resemblance of this transformation to the anisotropic duality transformation (2.9) is remarked. In fact it requires only a simple but tedious calculation to show that these two transformations are commutable:  $(x'_\xi)^* = (x_\xi')^*$  and  $(y'_\xi)^* = (y_\xi')^*$ . This commutability ensures that MKRG yields a distribution of singularities which is consistent with the duality constraint (2.9). This statement can be justified as follows.

The first step of the proof is to notice that the dual point of a fixed point is also a fixed point as a consequence of the commutability. (If  $x'_\xi = x_\xi$ , then  $(x'_\xi)^* = x_\xi^*$ , and similarly for  $y$ .) For example, MKRG, (3.1) has a trivial fixed point with all  $x$  and  $y$  zero (except  $x_n = y_n = 1$ ):  $(0, 0, \dots, 0; 0, 0, \dots, 0)$ , which is transformed into another trivial



西森秀稔

fixed point at  $(1, 1, \dots, 1; 1, 1, \dots, 1)$  by the duality transformation (2.8). It is easy to see that a non-trivial fixed point is transformed into a non-trivial fixed point. Now the second step of the proof is as follows. Let us construct an infinite series of points by starting from a point in the  $2(n-1)$  dimensional space  $(x_1, x_2, \dots, x_{n-1}; y_1, y_2, \dots, y_{n-1})$  and making the next point one after another by MKRG (3.1). The set of points dual to these points constitutes another series. If the first series flows into a non-trivial fixed point, then also the dual series does, because a dual point of a non-trivial fixed point is a non-trivial fixed point as a consequence of the commutability mentioned above. This fact, together with the ansatz of the renormalization group that all singularities are ultimately transformed into one of the non-trivial fixed points, lead to the desired result: dual point of a singularity is also a singularity within the framework of MKRG. In other words, MKRG yields a distribution of singularities (critical surface) which exactly satisfies the duality constraint (2.9).

A remark is made here. MKRG, as a renormalization group transformation, carries all the information on the properties of the system, including the value of the transition point. Our proof above never asserts that this value of  $T_c$  is always exact. MKRG is an approximation while the duality is exact. Therefore the above information obtained only by MKRG except the duality (e.g., the location of the transition point of the six-state discrete vector model, and for details see §2. 2. 4) is nothing but a conjecture. Our claim based on the above proof is that the conjecture by MKRG on the distribution of singularity of the free energy is reliable because it exactly satisfies the duality. In this statement it is assumed that the duality relation (2.7) or (2.9) is a very strong constraint on the shape of the critical surface. This assumption is believed to be reliable from the fact that the duality relation exactly determines the critical surface if  $n \leq 5$ .

Another important fact should be pointed out. As has been noted by Kadanoff<sup>(22)</sup>, the anisotropy in the recursion relation (3.1) is manifestly an artifact of the approximation. Reliable information on the distribution of singularities is obtained only in the limit  $b \rightarrow 1 + 0$  (infinitesimal transformation), in which  $x'_\xi = y'_\xi$  if  $x_\xi = y_\xi$ , and Eq. (3.1) then reduces to

$$\frac{dx_\xi}{dl} = x_\xi \ln x_\xi + \frac{1}{n} \sum_{\lambda=1}^n e^{\frac{2\pi i \lambda \mu}{n}} F_\lambda \ln F_\lambda - \frac{1}{n} x_\xi \sum_{\lambda=1}^n F_\lambda \ln F_\lambda \quad (3.2)$$

with

$$F_\lambda = \sum_{\mu=1}^n e^{i \frac{2\pi\lambda\mu}{n}} x_\mu$$

and

$$l = \ln b$$

As an example of the effect of the anisotropy of the recursion relation (3.1), in the isotropic Ising model ( $V_x = V_y$  and  $n = 2$ ), the commutability of the duality transformation and MKRG for arbitrary  $b$  guarantees that the location of the transition point obtained by MKRG satisfies the duality constraint,  $x_c = (1 - y_c) / (1 + y_c)^*$  (which can be obtained from Eq. (2.8)), but each value of  $x_c$  and  $y_c$  is different<sup>22)</sup> from the exact solution<sup>24)</sup> except in the isotropic limit  $b \rightarrow 1 + 0$ . For this reason, in the application in the next section, only the isotropic transformation (3.2) will be treated unless mentioned explicitly.

### § 3.2 Application

According to the discussion in the previous section, MKRG (3.2) should reproduce the results on the system (1.1) obtained in chapter II. As is shown in the following, it does reproduce exactly the distribution of singularities for the systems with  $n$  up to five. Furthermore it gives some conjecture concerning the systems with  $n$  exceeding five. This conjecture has a good reason which is reliable as explained in the previous section.

#### 3.2.1 Case 1: $n = 2$ and 3

The recursion relation (3.2) for these models are solved to give the transition points (the fixed points) at

$$x_c \equiv x_{1c} = \sqrt{2} - 1$$

and

$$x_c \equiv x_{1c} = (\sqrt{3} - 1) / 2$$

respectively, both of which are exact as expected. Linearization of the recursion relation near the criticality yields the values of the critical exponent

---

\*) Note that an isotropic system ( $V_x = V_y$ ) is transformed into an anisotropic one ( $V'_x \neq V'_y$ ) by MKRG(3.1) if  $b > 1$ .

$$\nu = \frac{1}{\sqrt{2} \ln(\sqrt{2} - 1) + 2} \approx 1.327 \quad (3.3)$$

and

$$\nu = \frac{\sqrt{3}}{2 \ln\left(\frac{\sqrt{3} - 1}{2}\right) + 2\sqrt{3}} \approx 1.191 \quad (3.4)$$

for  $n = 2$  and  $3$  respectively. It is well known<sup>25)</sup> that the critical exponent  $\nu$  assumes unity in the two-dimensional Ising model. The above result (3.3) exceeds the exact estimation by 33%, which exhibits the limit of validity of MKRG for the purpose of investigating the critical behavior. Nevertheless the fact that  $\nu$  decreases as  $n$  increases from two to three is qualitatively in agreement with other approximate estimations: for  $n = 3$ ,  $\nu = 0.8415$  by Marland<sup>26)</sup> (RG by decimation) and  $\nu = 0.852$  by Zwanzig and Ramshaw<sup>27)</sup> (series expansion).

### 3.2.2 Case 2: $n = 4$ and $5$

In these models the degree of freedom  $[n/2]$  is two, and therefore the flow diagram

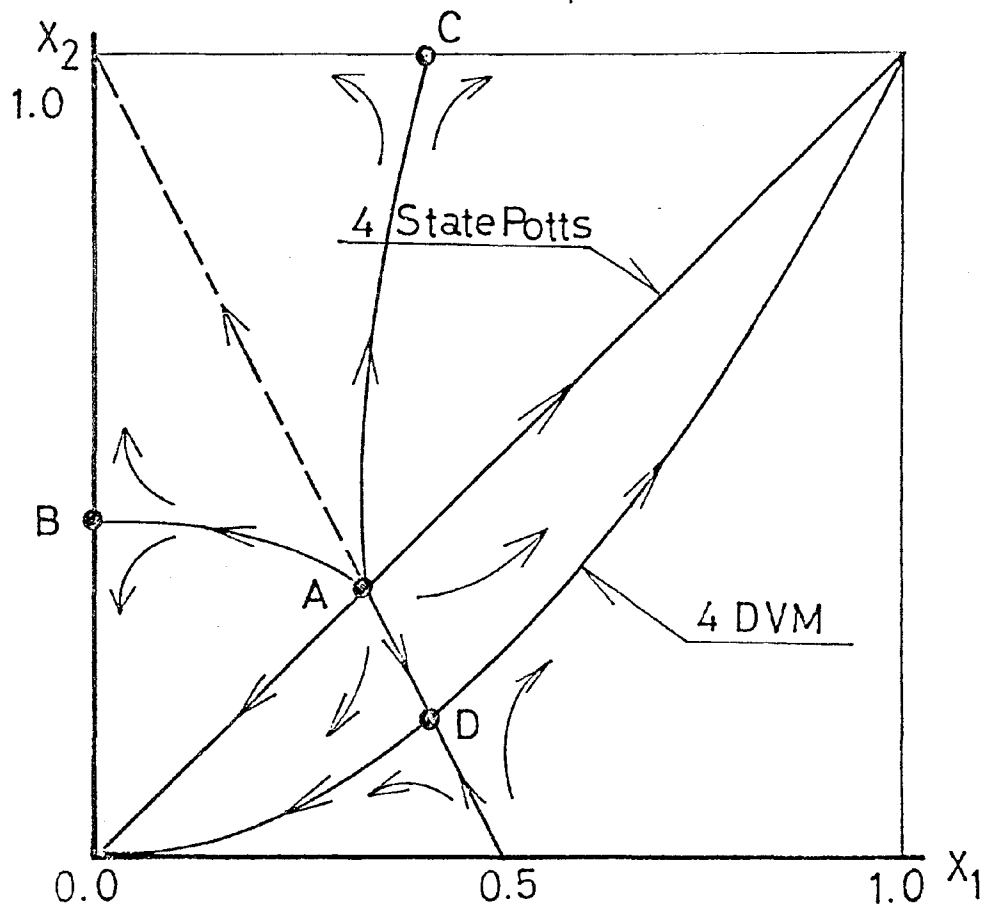


Fig. 6. The flow diagram of the four-state model. Four non-trivial fixed points (•) have been found.

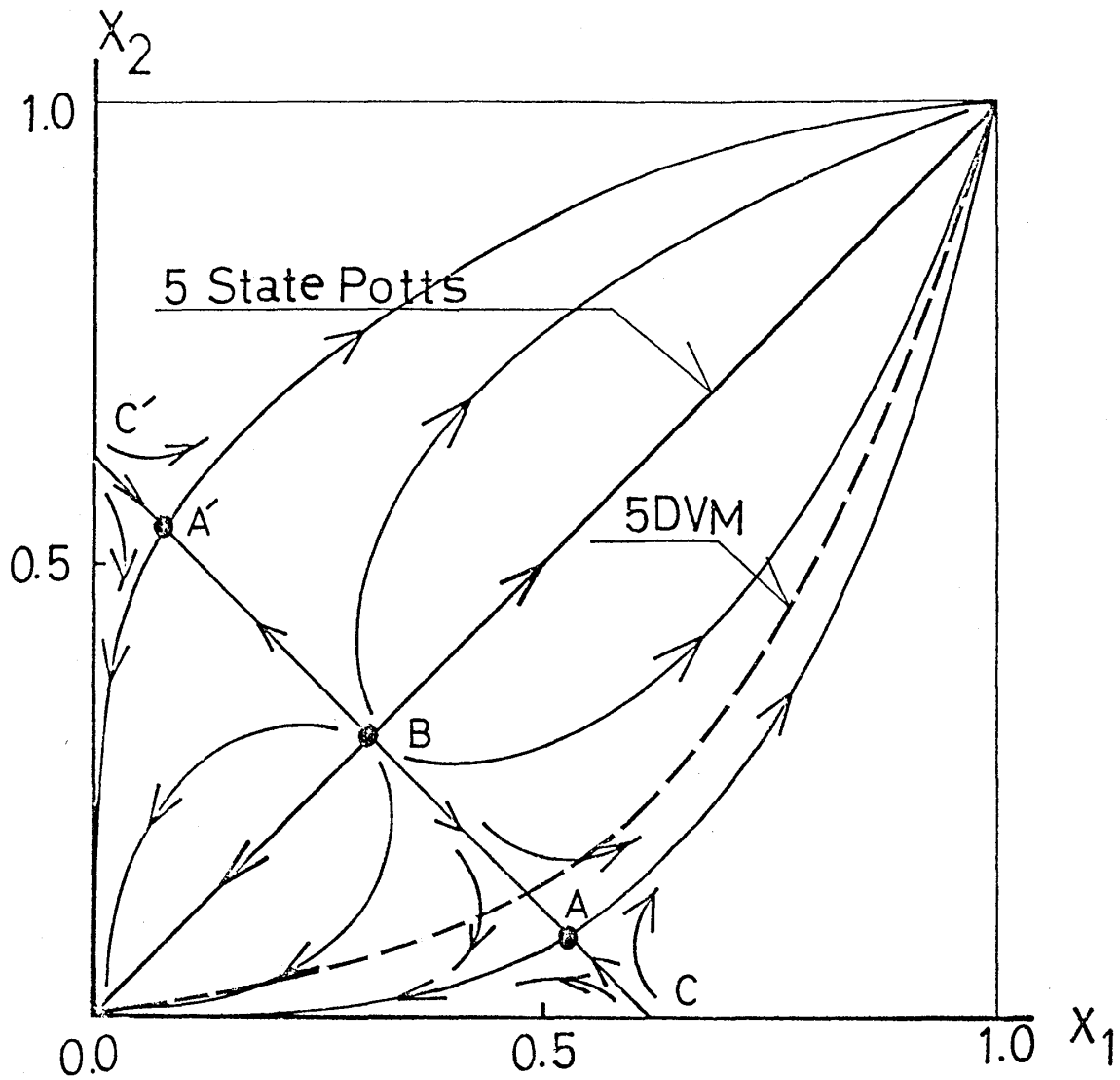


Fig. 7. The flow diagram of the five-state model. Three non-trivial fixed points (●) have been found.

of MKRG is written in a two-dimensional space as in Figs. 6 and 7.

The four-state model treated here is a special case of the Ashkin-Teller model.<sup>7)</sup> Knops<sup>28)</sup> investigated this system by the real space renormalization group in the form of the block spin transformation technique<sup>23)</sup> to obtain a flow diagram quite similar to Fig. 6. He pointed out that a branch point exists (corresponding to A in Fig. 6) in the phase boundary (critical surface) in agreement with our result. Our picture differs from his only in quantitative details: the exact loci of the transition points A, B, C and D have been given by MKRG. It is instructive to note here the relation between the duality result (Fig. 4) and the MKRG

西森秀稔

result (Fig. 6). All points on the line  $x_2 = 1 - 2x_1$  are fixed by the duality transformation, but this line is the phase boundary only between A and D (Fig. 6) according to MKRG. The phase boundary continues from A to C and from A to B, and these two segments A-C and A-B are mutually dual. This is an example in which the fixed point of the duality transformation is not a singularity of the free energy. As to the critical phenomena, MKRG (3.2) for  $n = 4$  gives the value of the critical exponent  $\nu$  as

$$\nu = \frac{1}{\sqrt{2} \ln(\sqrt{2} - 1) + 2} \simeq 1.327$$

at B, C and D (Fig. 6) and

$$\nu = \frac{1}{2 - \ln 3} \simeq 1.109$$

at A. As expected<sup>28)</sup>, three fixed points B, C and D have the same critical property as the transition point of the Ising model (see §3.2.1).

The five-state model<sup>14)</sup> has the flow diagram shown in Fig. 7. Three non-trivial fixed points appear at A(0.5129, 0.1051), A'(0.1051, 0.5129) and B( $1/(\sqrt{5} + 1)$ ,  $1/(\sqrt{5} + 1)$ ), which divide this five-state model into three universality classes. As a matter of fact, the recursion relation (3.1) (or (3.2)) is invariant under the exchange  $x_1 \leftrightarrow x_2$  (and  $y_1 \leftrightarrow y_2$ ), and therefore the fixed points A and A' represent the same critical phenomena. Thus the five-state model has actually two universality classes, one of which is of the Potts model and the other including the five-state discrete vector model. It is known<sup>29),30)</sup> that the Potts model with the number of states greater than four undergoes a phase transition of the first order. MKRG fails to explain this first order nature although it gives the value of the transition point exactly: the fixed point B at ( $1/(\sqrt{5} + 1)$ ,  $1/(\sqrt{5} + 1)$ ) is a usual fixed point of the second order transition according to MKRG. This fact has already pointed out by Kadanoff<sup>22)</sup> himself. Critical exponent  $\nu$  characterizing the fixed point A has been estimated numerically:  $\nu \simeq 2.46$ . In this five-state model, the set of fixed points of the duality transformation (which constitutes a line  $x + y = (\sqrt{5} - 1) / 2$ ) has been found to be the phase boundary from C to C' in Fig. 7 by MKRG.

### 3.2.3 Case 3: $n = 6$

The free energy of the six-state model has three degrees of freedom ( $[n/2] = 3$ ). The flow diagram of MKRG (3.2) should be written in the three-dimensional space (Fig. 8). We

have found eleven fixed points, four of which are trivial and the rest seven are non-trivial. The trivial fixed points are at A(0, 0, 0), B(1, 1, 1), C(0, 1, 0) and D(0, 0, 1), and the non-trivial ones at E(0, 0,  $\sqrt{2} - 1$ ), F(0,  $(\sqrt{3} - 1)/2$ , 0), G( $\sqrt{2} - 1$ , 1,  $\sqrt{2} - 1$ ), H( $(\sqrt{3} - 1)/2$ ,  $(\sqrt{3} - 1)/2$ , 0), I( $(\sqrt{2} - 1)$ ,  $(\sqrt{3} - 1)/2$ ,  $(\sqrt{3} - 1)/2$ ,  $\sqrt{2} - 1$ ), J( $1/(\sqrt{6} + 1)$ ,

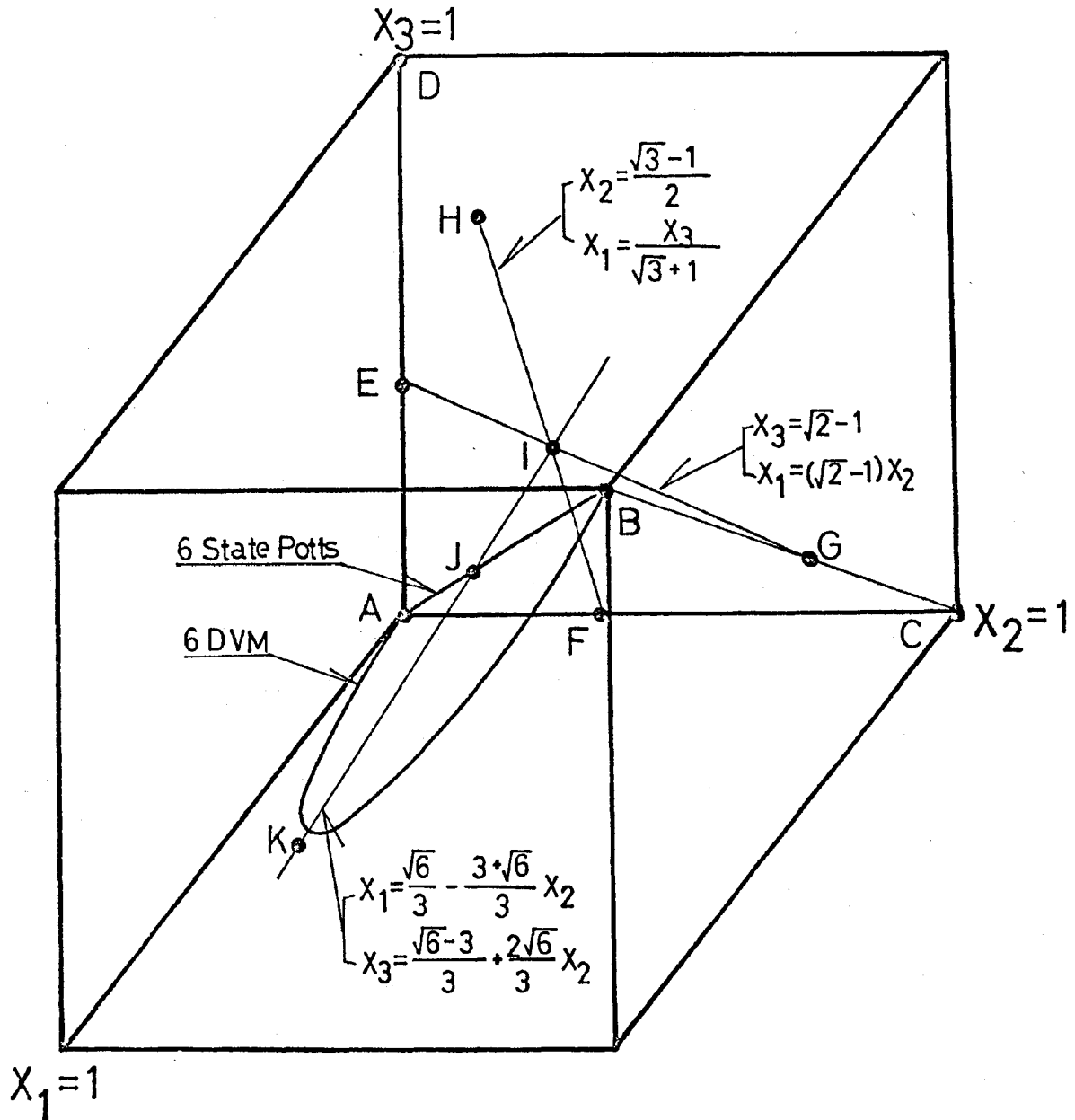


Fig. 8. The flow diagram of the six-state model has eleven fixed points (A~K). The directions of the flow are omitted for simplicity. See also Fig. 9.

西森秀稔

$1/(\sqrt{6} + 1)$ ,  $1/(\sqrt{6} - 1)$ ) and  $K(0.5847, 0.1275, 0.0249)$ . The universality classes which these non-trivial fixed points represent are, respectively, E and G—the Ising model type, F and H—the three-state model type, I—the mixed type of the Ising model and the three-state model, J—the six-state Potts model type and K—the six-state discrete vector model type.

In order to elucidate this interpretation of the fixed points, we first point out that, if  $x_1 = x_2 x_3$ , the one-dimensional transfer matrix  $T$  of the six-state model decomposes into a direct product of  $2 \times 2$  and  $3 \times 3$  matrices:

$$T = \begin{array}{c} \begin{array}{c} \xi_{i+1} \\ \xi_i \end{array} \begin{array}{c} 6 \\ 4 \\ 2 \\ 3 \\ 1 \\ 5 \end{array} \end{array} \begin{array}{c} \left( \begin{array}{cccccc} 1 & x_2 & x_2 & x_3 & x_1 & x_1 \\ x_2 & 1 & x_2 & x_1 & x_3 & x_1 \\ x_2 & x_2 & 1 & x_1 & x_1 & x_3 \\ x_3 & x_1 & x_1 & 1 & x_2 & x_2 \\ x_1 & x_3 & x_1 & x_2 & 1 & x_2 \\ x_1 & x_1 & x_3 & x_2 & x_2 & 1 \end{array} \right) \end{array} = \begin{array}{c} \left( \begin{array}{cc} 1 & x_3 \\ x_3 & 1 \end{array} \right) \otimes \left( \begin{array}{ccc} 1 & x_2 & x_2 \\ x_2 & 1 & x_2 \\ x_2 & x_2 & 1 \end{array} \right) \end{array}$$

Thus the free energy is a sum of the free energy of the Ising model (Boltzmann factors 1 and  $x_3$ ) and that of the three-state model (Boltzmann factors 1,  $x_2$  and  $x_2$ ). It is easy to see<sup>11)</sup> that a similar decomposition is possible even in higher dimensional cases under the same condition  $x_1 = x_2 x_3$ . Therefore on the surface  $x_1 = x_2 x_3$  in the cubic space of Fig. 8, the free energy is singular along the lines  $x_3 = \sqrt{2} - 1$  and  $x_2 = (\sqrt{3} - 1) / 2$  (see §2.2.1). The former singular line is of the Ising model type (logarithmic singularity in the specific heat) and the latter is of the three-state model type (possibly stronger singularity<sup>26),27)</sup> than in the Ising model). The non-trivial fixed points E, F, G, H and I lie on these singular lines, and consequently the interpretation above comes about.

Along the line  $x_1 = x_2 = x_3$ , the system reduces to the six-state Potts model. It has a singularity at the point J. (We note that MKRG again predicts the second order transition of the Potts model with the number of states greater than four.)

The final fixed point K is of particular interest since it represents the universality class to which the six-state discrete vector model belongs as is explained in the following. In Fig. 9 we show the projection of the flow diagram of MKRG starting from some arbitrary points on the

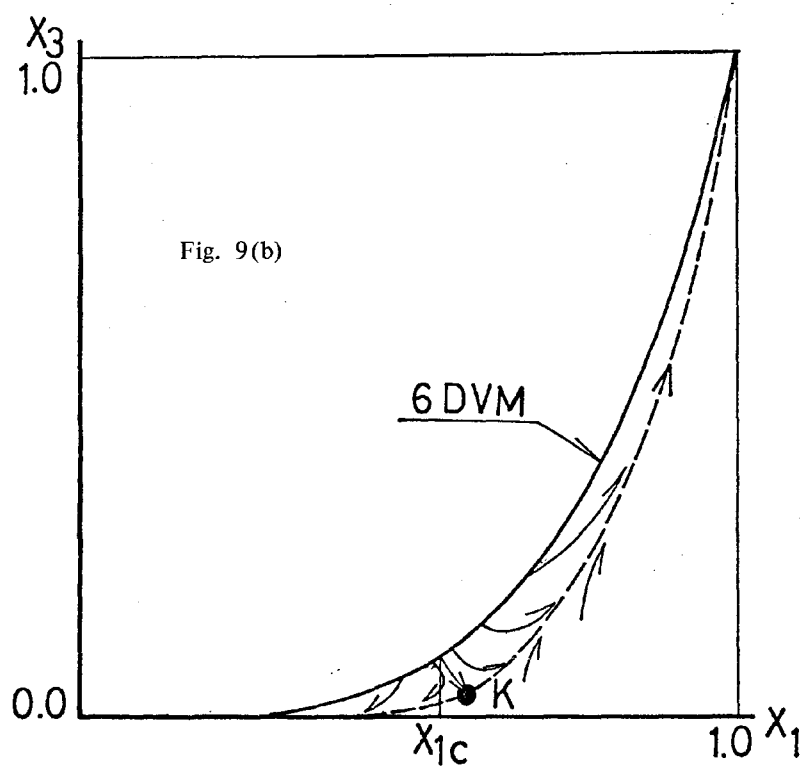
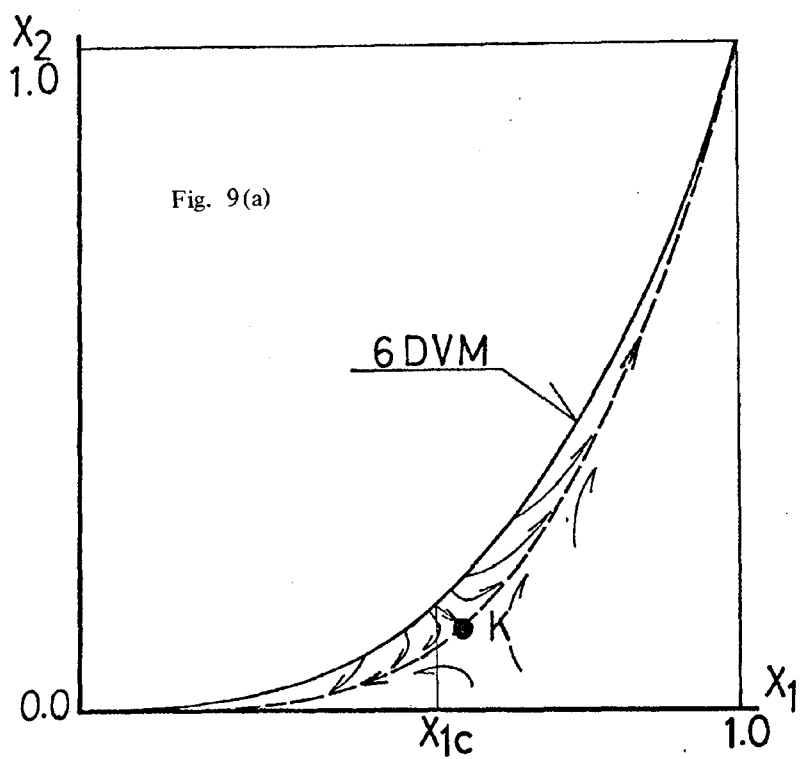


Fig. 9. The flow diagram in the neighborhood of the six-state discrete vector model. (a) is the projection to the  $x_1$ - $x_2$  plane and (b) in the next page is to the  $x_1$ - $x_3$  plane.



line representing the six-state discrete vector model. Fig. 9(a) is the projection to the  $x_1 - x_2$  plane and Fig. 9(b) is the projection to the  $x_1 - x_3$  plane. It is clearly observed that the six-state discrete vector model has two phases. The transition point is at  $x_{1c} = \exp[(-1 + \cos(2\pi/6))K_c] \simeq 0.5339$ , i. e.,  $kT_c/J \simeq 0.7968$ . This transition point flows into the non-trivial fixed point K. We have failed to locate analytically these points (the transition point of the discrete vector model and the fixed point K), but the value  $kT_c/J \simeq 0.7968$  is reasonable compared with the previous results for other discrete vector models in §2.2 (see also Fig. 13). The dashed line in Fig. 9 is the limiting interaction characterizing the present universality class. This line has been obtained after performing many steps of MKRG. In order to investigate the critical property of this universality class, we have calculated the strength of the flow  $dx_1/d\ell$  along this “universal interaction” (dashed line in Fig. 9) and plotted it in Fig. 10(a). In Figs. 10(b) and (c) the same quantities  $dx_1/d\ell$  for the four-state and five-state discrete vector models are plotted for comparison (see also Figs. 6 and 7). When  $n$  increases from five to six, the flow around this fixed point is suppressed (that is, the slope of the curve around the fixed point decreases as  $n$  increases), which is reflected to the increase of the value of the exponent  $\nu$ : numerically,  $\nu \simeq 1.3$  (four-state discrete vector model),  $\nu \simeq 2.5$  (five-state discrete vector model) and  $\nu \simeq 5.6$  (six-state discrete vector model).

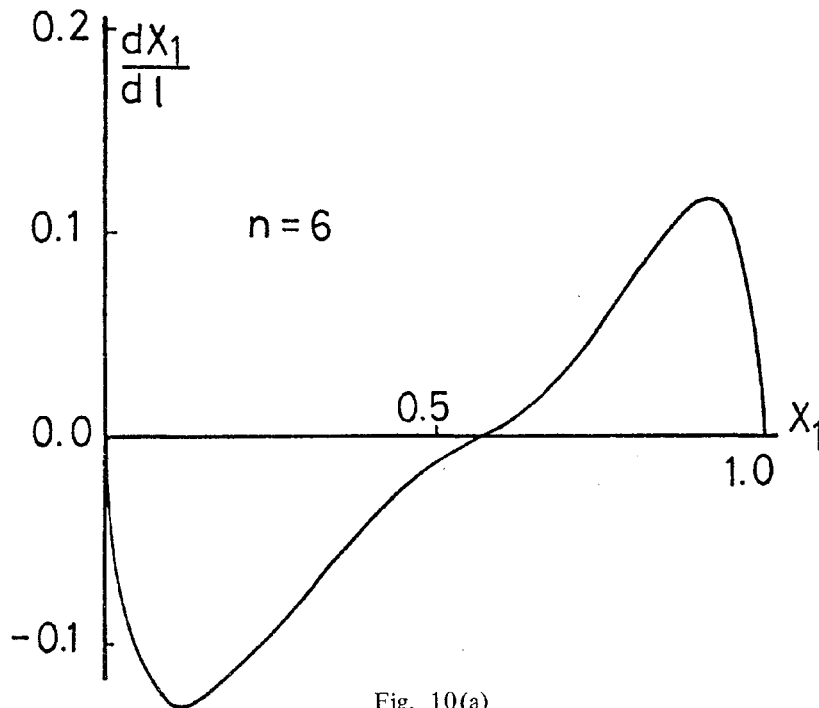


Fig. 10(a)

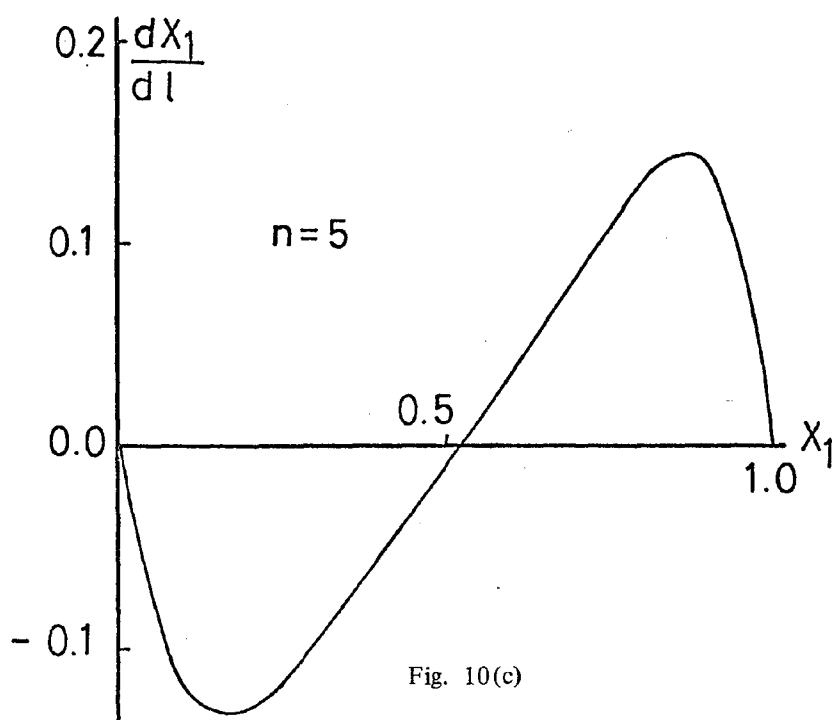
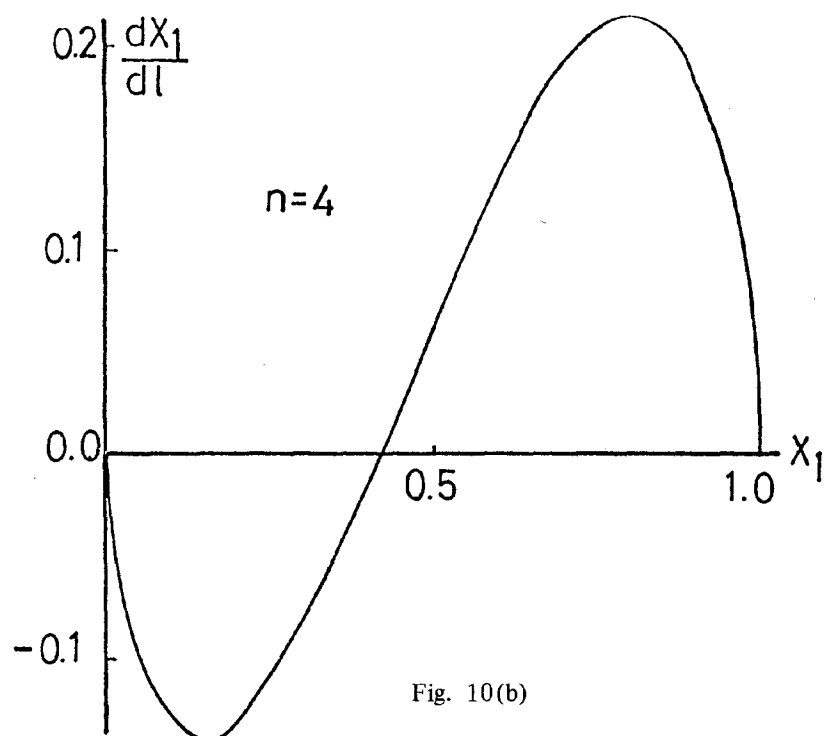


Fig. 10. The strength of the RG flow  $dx_1/dl$  is shown.

(a) is the six-state case along the universal interaction (the dashed line in Fig.9).  
(b) and (c) in the next page are the corresponding quantities in the four- and five-state models.

Another remark concerning the six-state model is made on the effect of the renormalization group transformation (3.2) applied to the points on the duality-invariant line(2.11). We have found that MKRG makes these points flow into one of the non-trivial fixed points I, J and K. The direction of the flow is from J to I, J to K, and points outside the segment I–K are attracted either to I or K. Hence MKRG predicts that the free energy is singular along the line (2.11), all points on which are fixed by the duality transformation.

#### 3.2.4 Case 4: $n$ greater than six

In the case with  $n$  exceeding six we have investigated only the Potts model and the discrete vector model. The reasons are (i) difficulty in drawing a global flow diagram in the high dimensional ( $d = [n/2]$ ) space, and (ii) existence of essentially only two universality classes (the Potts model type and the discrete vector model type) in systems with  $n$  up to six.

First we discuss the Potts model. The infinitesimal MKRG (3.2) gives the exact value of the transition point of the general  $n$ -state Potts model:

$$x_c = \frac{1}{1 + \sqrt{n}} \quad (3.5)$$

as expected from the fact that the duality relation (2.6) and (2.7) is a sufficient condition to determine  $T_c$  in this model (recall the discussion in §3.1). MKRG (3.2) again predicts the second order transition at  $x_c$  of Eq. (3.5). This is a fatal defect of MKRG if  $n \geq 5$  since a rigorous proof exists<sup>32),33)</sup> that the phase transition of the Potts model is of the first order if  $n \geq 5$

As to the discrete vector model (or its universality class), José et al.<sup>19)</sup> have already studied this system by means of the duality transformation and the renormalization group including MKRG (of scale factor  $b = 2$ ). We are particularly interested in three of their results.

(i) In the planar model ( $n$ -infinite discrete vector model) the system appears to be critical for a finite range of temperatures including the absolute zero. MKRG supports approximately this picture, and José et al. further employed another approach, namely the calculation of the vortex correction<sup>1)</sup> to the spin wave excitation,<sup>31)</sup> to confirm the validity of this picture (the critical line).

(ii) The limiting interaction after many steps of MKRG (corresponding to the dashed line in Fig. 9) for the planar model is well approximated by the Villain model<sup>32)</sup>:

$$\exp V(\theta) = \sum_{m=-\infty}^{\infty} \exp\left[-\frac{k}{2\pi} (\theta - 2\pi m)^2\right] \quad (3.6)$$

(iii) José et al. could not determine definitely how the finite- $n$  discrete vector model behaves in the critical region; one of the reasons is that they did not know the reliability of MKRG in investigating the shape of the critical surface. Their result by MKRG (scale factor  $b = 2$ ) suggests that, if  $n$  is large enough ( $n = 12$ , e. g.), the effect of the finiteness of  $n$  emerges only at very low temperatures:  $kT_c/J \simeq 1/n^2$ . They invented also a duality transformation applicable to models including the Villain model with finite number of states ( $\theta = 2\pi k/n$ ,  $k = 0, 1, \dots, n-1$  in Eq. (3.6)) which belongs to the same universality class as the discrete vector model (see below). Explicitly written, their duality relation for the finite- $n$  state Villain model is

$$Z(2\pi k) = \left(\frac{n}{2\pi k}\right)^N Z\left(\frac{n^2}{2\pi k}\right) \quad (3.7)$$

Thus, if  $Z$  has a singularity at  $K \simeq O(1)$ , then it is singular also at  $K \simeq O(n^2)$ , which is consistent with above mentioned result of MKRG.

Our investigation by the infinitesimal MKRG qualitatively supports their assertion. First, as shown in Fig. 11 the finite state Villain model belongs to the same universality class as the discrete vector model. Along the limiting universal interaction, the strength of the flow  $dx_1/d\ell$  has the tendency shown in Fig. 12 (see also Fig. 10) suggesting the approach to the planar model ( $dx_1/d\ell = 0$  below  $T_c$ ) as  $n$  increases; if  $n$  is finite but large,  $|dx_1/d\ell|$  is very small in a finite interval of  $x_1$ , and this interval extends approximately between  $kT/J \simeq O(1)$  and  $kT/J \simeq O(n^{-2})$ . However, precise numerical estimation of  $dx_1/d\ell$  shows that the transition point (the fixed point of MKRG at which  $dx_1/d\ell = 0$ ) exists uniquely for any finite value of  $n$  (Fig. 13). This estimation of  $kT_c/J$ , which is fairly reliable due to the commutability of MKRG and the duality, yields the linear dependence on  $1/n$  of  $kT_c(n)/J$  approximately (Fig. 13). It is consistent with (3.7): under the assumption of the unique existence of the singularity, the exact duality relation (3.7) for the finite-state Villain model asserts that the singularity should be located at  $kT_c/J = 2\pi/n$ . Therefore we are convinced that the present universality class (to which the discrete vector model and the finite-state Villain model belong) has only one singularity at  $kT_c/J \simeq 1/n$ , but these systems are almost critical ( $|dx_1/d\ell| \ll 1$ )

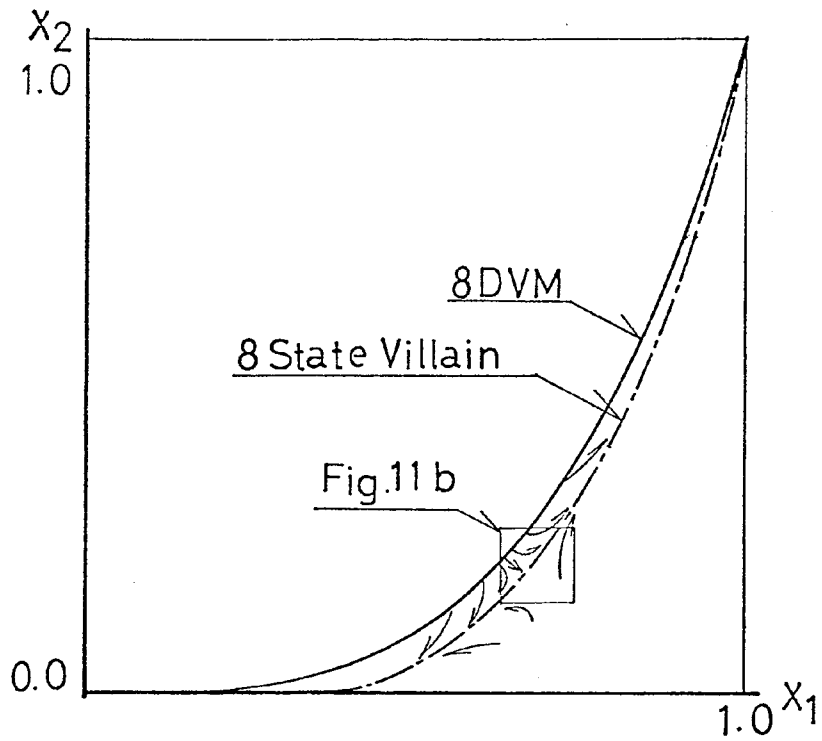


Fig. 11. (a) The line representing the finite-state Villain model lies very near the universal interaction in the space of the Boltzmann factors if  $n$  is greater than eight. This figure is the projection to the  $x_1 - x_2$  plane.

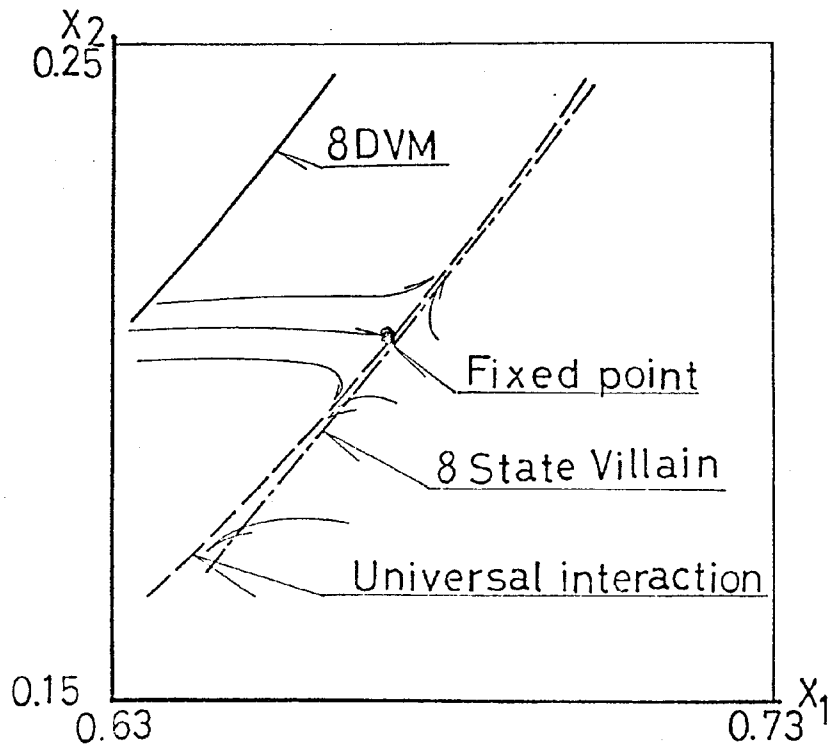


Fig. 11. (b) The same plot as in Fig. (a) but enlarged.

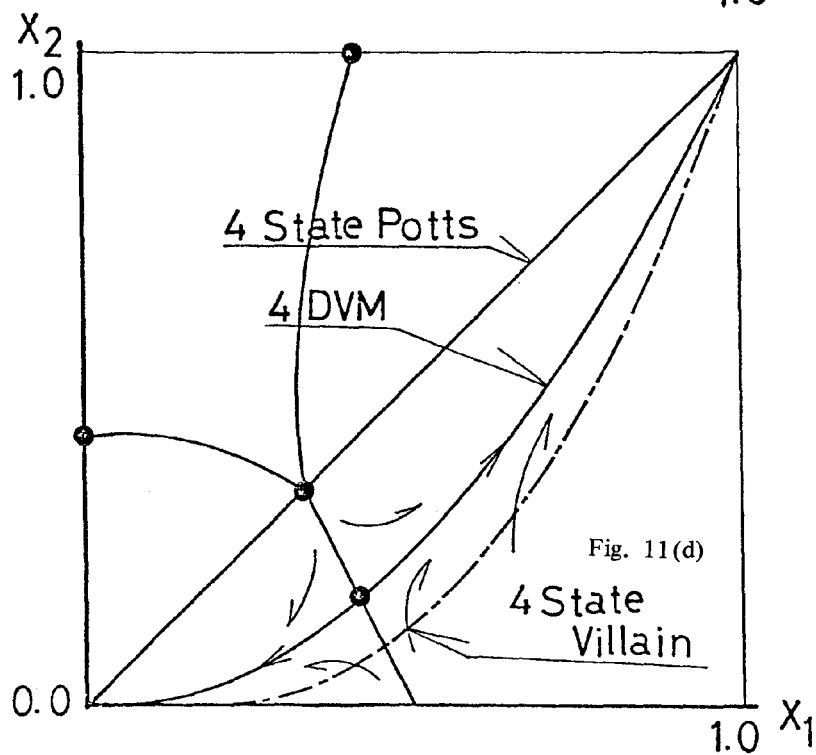
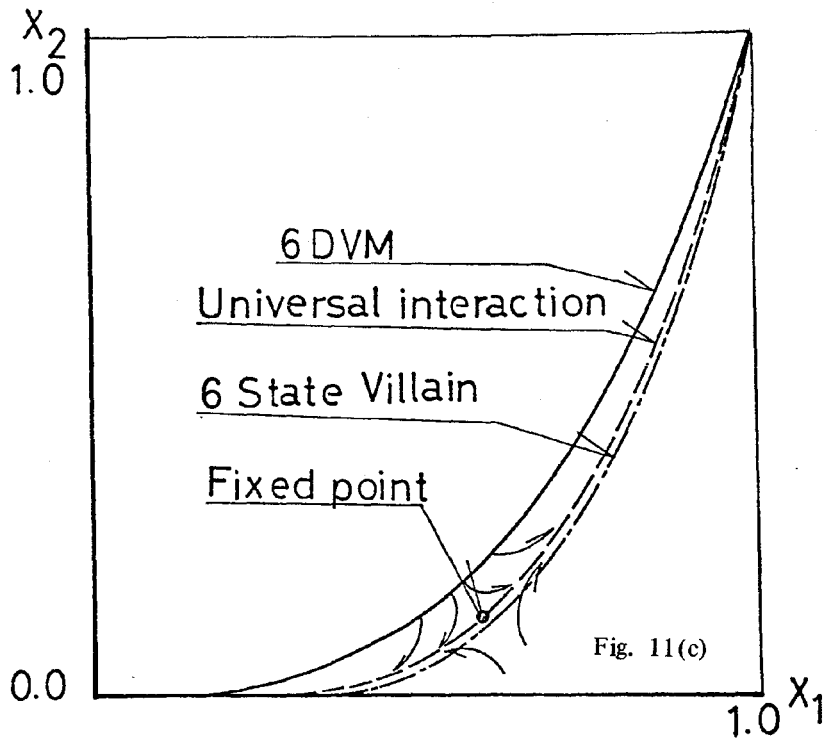


Fig. 11. (c) and (d) Even if the finite-state Villain model is not near the universal interaction, it belongs to the same universality class to the discrete vector model.

西森秀稔

in a finite range of temperatures (possibly from  $kT_c/J \simeq O(n^{-2})$  to  $O(1)$ ).

Finally, the result of our calculation of the internal energy of the discrete vector model by the renormalization group<sup>23)</sup> (3.2) is shown in Fig. 14. This curve will be compared with the result of the Monte Carlo simulation in chapter IV.

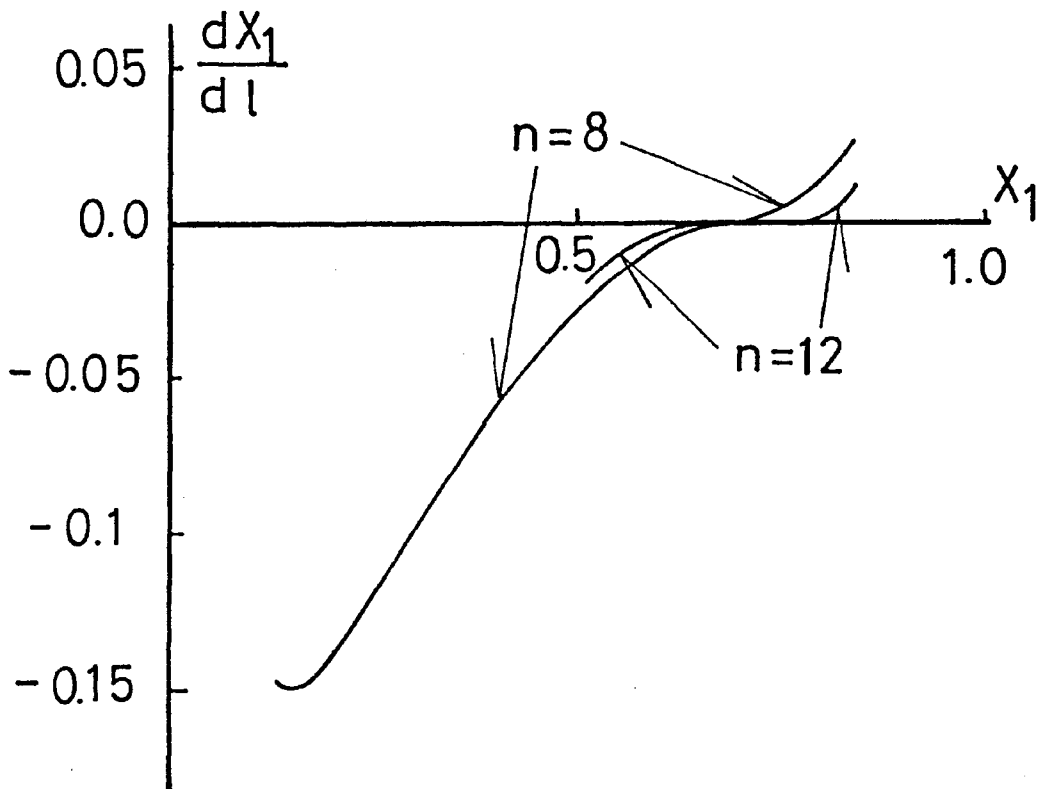


Fig. 12. If  $n$  is large enough ( $n=8, 12$  or greater), there appears an approximate critical line (finite temperature range in which  $|dx_1/dl| \ll 1$ ). Note that the abscissa of this figure is the Boltzmann factor  $x_1 = \exp [K(K(-1 + \cos(2\pi/n)))]$  and consequently the constancy of  $x_1$  when  $n$  varies does not imply the constancy of  $K$ .

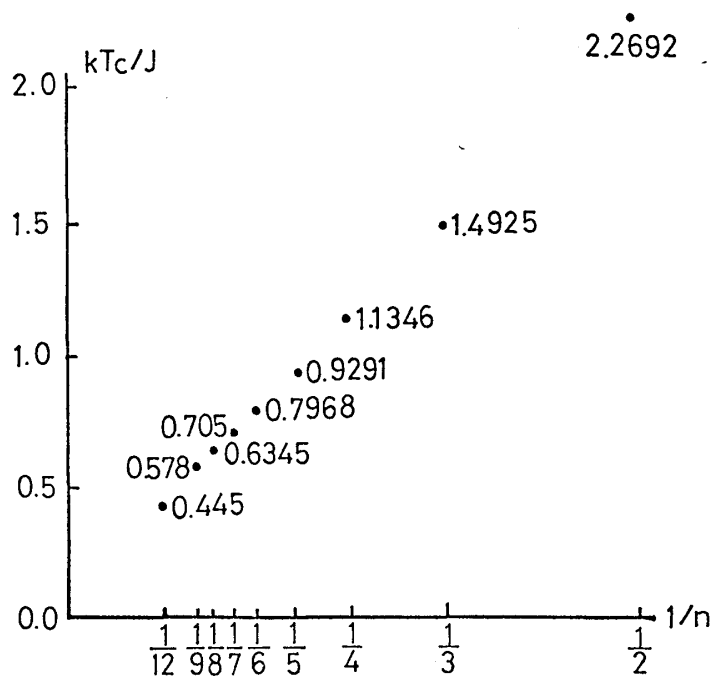


Fig. 13. Transition points of the various discrete vector models obtained by MKRG. Apparently  $kT_c/J$  is proportional to  $1/n$  rather than to other powers of  $1/n$ .

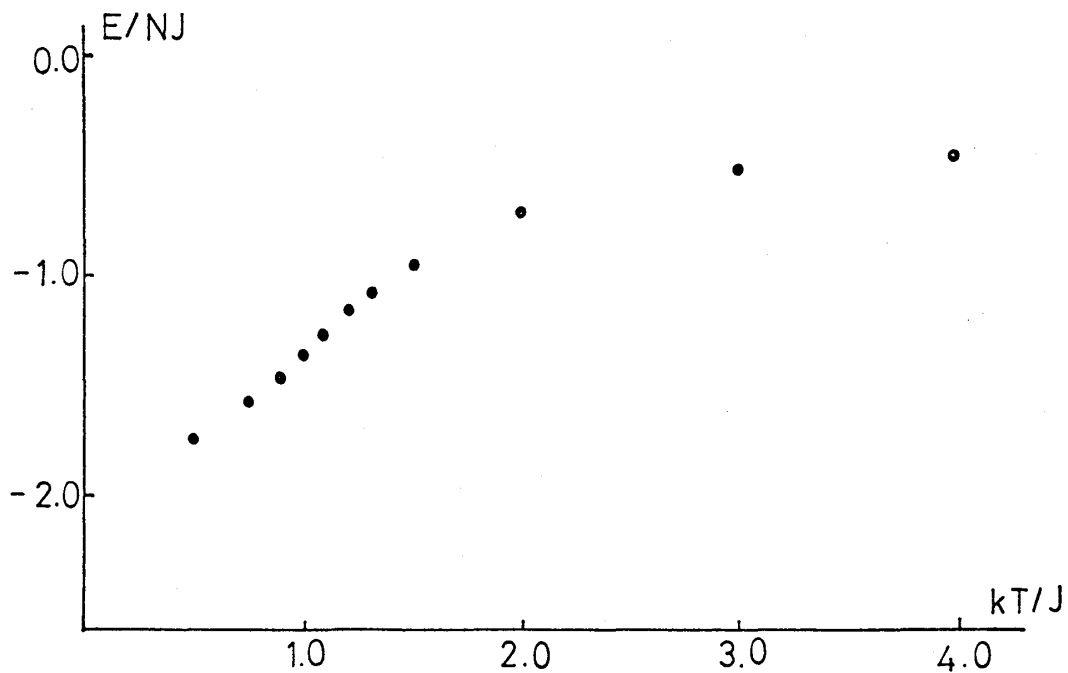


Fig. 14. The value of the internal energy of the eight-state discrete vector model obtained by MKRG is plotted.



#### IV. Monte Carlo simulation

In order to investigate the critical phenomena of the planar model ( $n$ -infinite discrete vector model) Miyashita et al.<sup>18)</sup> performed a Monte Carlo simulation (MCS) on this system by the finite- $n$  approximation ( $n = 8$  and  $12$ ). We are interested in two aspects of the results presented in their paper.

##### (i) Thermodynamic quantities

Internal energy, specific heat and magnetic susceptibility per spin have been obtained as in Fig. 15. The calculation has been done for systems of sizes  $15 \times 15$ ,  $30 \times 30$  ( $n = 8$ ) and  $15 \times 15$ ,  $30 \times 30$ ,  $50 \times 50$  ( $n = 12$ ) on the square lattice. In Fig. 15, we have plotted only the result of  $n = 8$  only. The data of the case  $n = 12$  are found in Appendix B. The susceptibility per spin (Fig. 15 (c)) is clearly dependent on the system size below  $kT/J \simeq 1.15$ , which implies that this quantity is divergent in this temperature range in the thermodynamic limit. The energy and the specific heat (Figs. 15(a) and (b)) are finite for any value of the system size. This result (the divergence of the susceptibility and the finiteness of the specific heat) is qualitatively in good agreement with other approximate estimations<sup>1),2),19),33)</sup> on the planar model. (The validity of finite- $n$  approximation will be discussed later.)

In Fig. 15(a) the energy calculated by MKRG (§3.2.4) is also plotted for comparison. The results of MCS and MKRG well coincide in the low temperature region. If  $kT/J$  is of the order of unity, these two methods give a little different values of energy, which is due to the approximate nature of MKRG<sup>22)</sup> (i. e., the manipulation “potential moving” in the derivation of MKRG is valid in the totally ordered state as mentioned in Appendix A).

##### (ii) Transition point

The susceptibility appears to be divergent below  $kT_c/J \simeq 1.15$  in the thermodynamic limit (Fig. 15(c)). Thus it is possible to identify this temperature ( $kT_c/J \simeq 1.15$ ) as the transition point. On the other hand, in §3.2.4 we have estimated the values of the transition points of the eight- and twelve-state discrete vector models as  $kT_c \simeq 0.635$  and  $0.445$  respectively (see Fig. 13). This discrepancy between MCS and MKRG is originated from the existence of an approximate critical line as mentioned in §3.2.4; the strength of the MKRG flow  $dx_1/d\ell$  is very weak ( $|dx_1/d\ell| \ll 1$ ) within a finite range of temperatures around the critical point, although our precise numerical investigation reveals that only one fixed point exists actually.

We have found numerically that the high temperature edge of this approximate critical line is at  $kT/J \simeq 1$ . This agreement of  $kT/J \simeq 1$  with  $kT/J \simeq 1.15$  by MCS supports the claim that the slow flow along the approximate critical line makes the susceptibility very large below  $kT/J \simeq O(1)$ . The low temperature edge of the approximate critical line is at  $kT/J \simeq O(n^{-2})$ . Thus it is natural to suppose that in the limit  $n \rightarrow \infty$  the low temperature edge approaches the absolute zero and the high temperature limit is fairly stable against the increase of  $n$  and finally becomes the transition point of the planar model. This picture is partly supported by the approximate coincidence of the value of  $kT_c/J$  ( $\simeq 1.15$ ) by MCS with other estimates<sup>1),19),34)</sup> for the planar model.

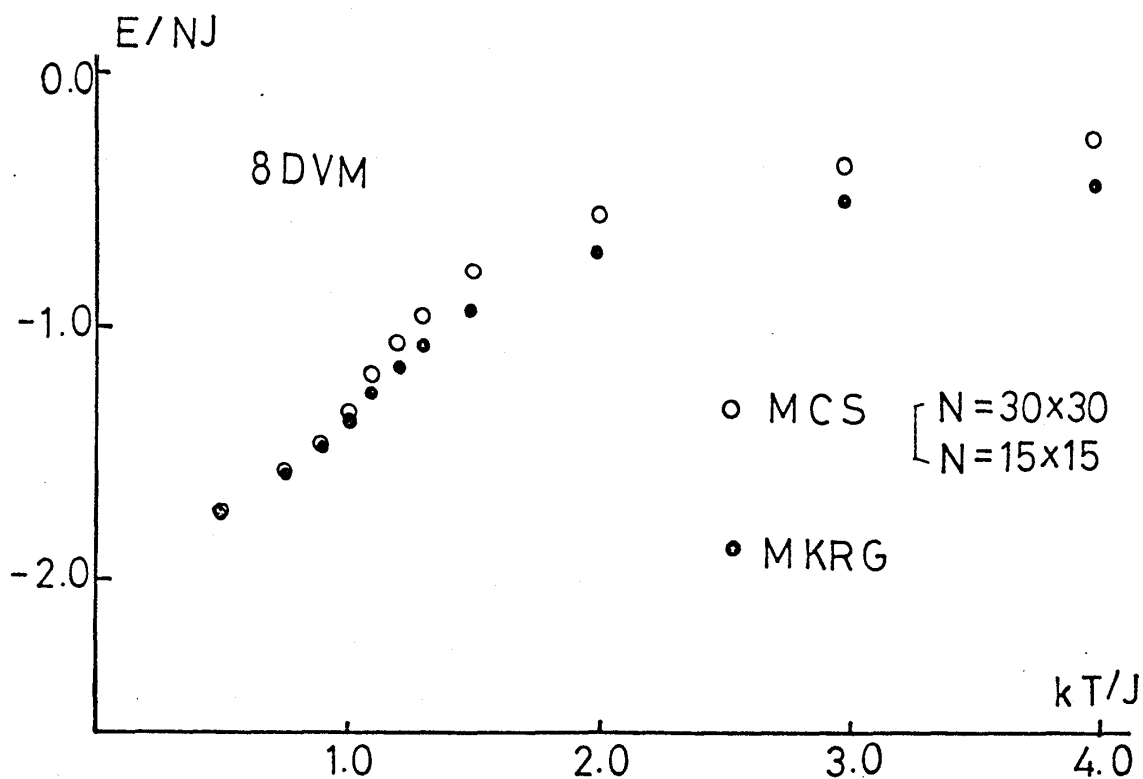


Fig. 15. (a) Internal energy of the 8DVM by MCS (Miyashita et al.) is compared with the MKRG result.

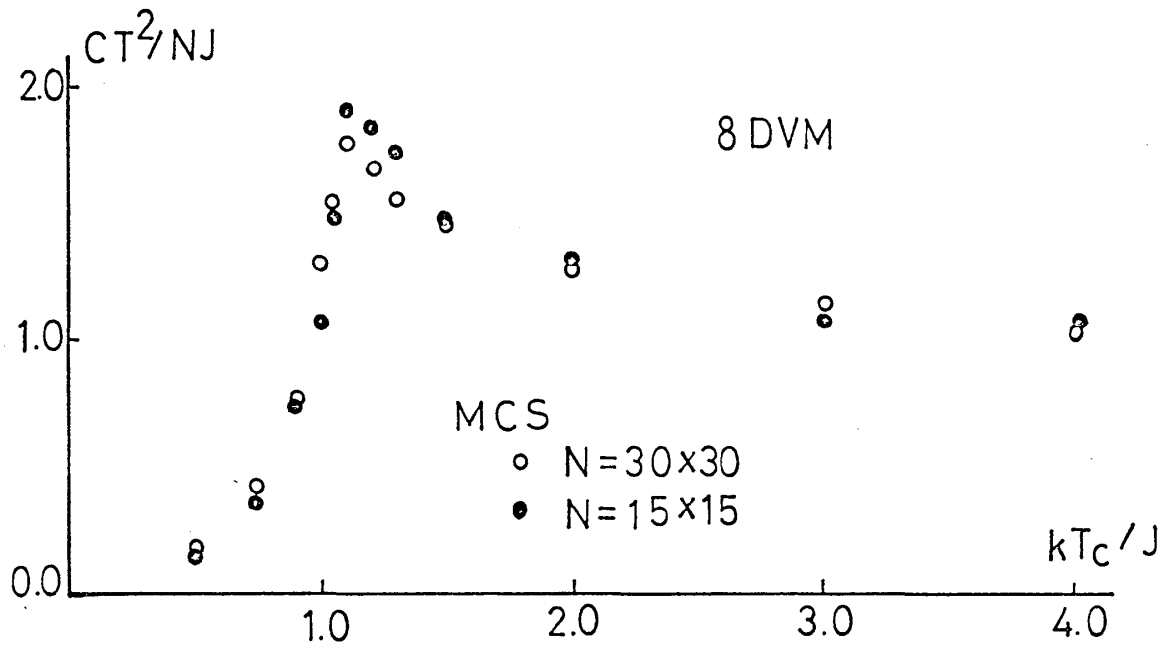


Fig. 15. (b) Specific heat obtained by MCS. This quantity looks to be finite for any values of  $N$  (the system size).

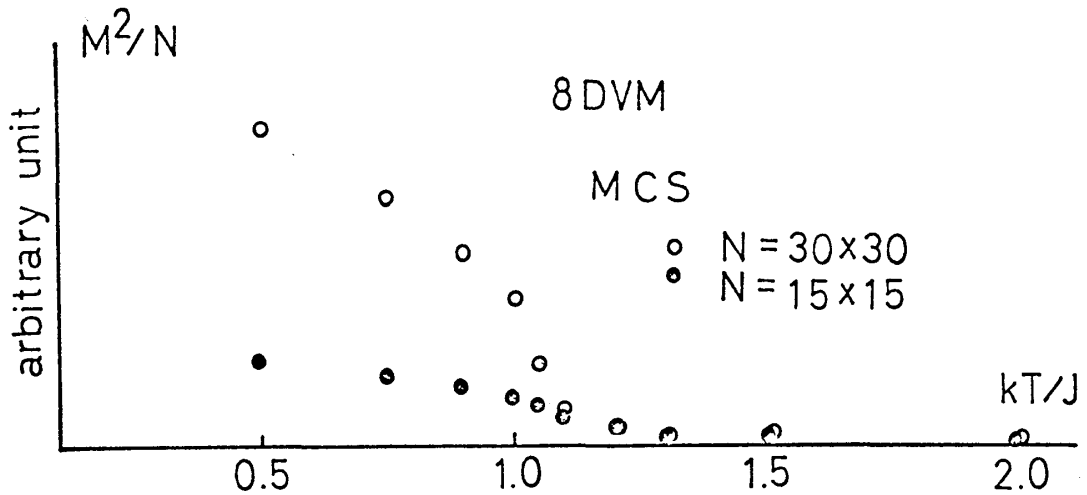


Fig. 15. (c) Susceptibility per spin divided by the temperature is equal to  $\langle M^2 \rangle$  because  $\langle M \rangle = 0$  if  $kT/J \geq kT_c/J \approx 0.6345$  for 8DVM. The system size dependence of  $\langle M^2 \rangle/N$  below  $kT/J \approx 1.15$  implies the very large value in the limit  $N \rightarrow \infty$ .

## V. Conclusion

It has been pointed out that the Wu-Wang duality relation provides a strong constraint on the shape of the critical surface of the model(1.1). If the number of states  $n$  is less than six, this constraint is a sufficient condition to determine the critical surface (except the special situation in the four-state model). If  $n$  is greater than or equal to six, we have to ask help to MKRG whose prediction on the shape of the critical surface is exactly consistent with the duality relation. In this way we have obtained the transition point of the discrete vector model analytically( $n \leq 5$ ) or numerically ( $n \geq 6$ ). Any other models of the type of Eq. (1.1) can be treated in the same way.

The internal energy of the discrete vector model has been calculated by MKRG. The result is in agreement with the MCS estimation within the precision of the approximation.

MKRG is not necessarily reliable for the purpose of the investigation of critical phenomena. For instance, MKRG cannot derive the exact result that the phase transition of the Potts model with the number of states greater than four is not of the second order. Nevertheless the critical behavior of the discrete vector model is explained qualitatively quite well by MKRG. The existence of the approximate critical line in spite of the unique transition point gives a reasonable picture of the discrete vector model. This statement is asserted by MKRG and supported by MCS performed by Miyashita et al. Furthermore it has been found that within the model (1.1), only two types of the universality classes exist if  $n \leq 6$  (the Potts model type and the discrete vector model type).

It is not easy to improve MKRG as a renormalization group transformation without destroying the commutability with the duality transformation. The extension of the present method to systems other than the type of Eq.(1.1) is also a future problem.

## Acknowledgments

I wish to acknowledge valuable discussions by Prof. M. Suzuki on various aspects of this study. I am also grateful to Dr. F. Tanaka, Mr. S. Miyashita, Dr. A. Kuroda and Mr. H. Takano for their useful discussions and comments. Dr. K. Kitahara read the manuscript critically

and gave some suggestive comments.

### Appendix A. Derivation of the Migdal-Kadanoff transformation

The Migdal-Kadanoff renormalization group transformation<sup>21),22)</sup> (3.1) is derived in this Appendix. First the idea is illustrated by the special case of the scale factor  $b = 2$ .

The reduction of the degree of freedom and the scale transformation are performed in two steps (Fig. 16). (i) Potential moving: Remove every other interaction in the  $y$ -direction  $V_y$  and make the remaining interaction in the  $y$ -direction twice ( $V_y \rightarrow 2V_y$ ). (ii) Decimation: Take a trace of the spin variables which are connected to the neighboring variables only by the interaction in the  $x$ -direction after the potential moving. By this manipulation, the interaction in the  $x$ -direction  $V_x$  becomes a new one  $(V_x)'$ . The same processes should be performed by exchanging  $x$  and  $y$  in (1) and (2) ((1)' and (2)' in Fig. 16).

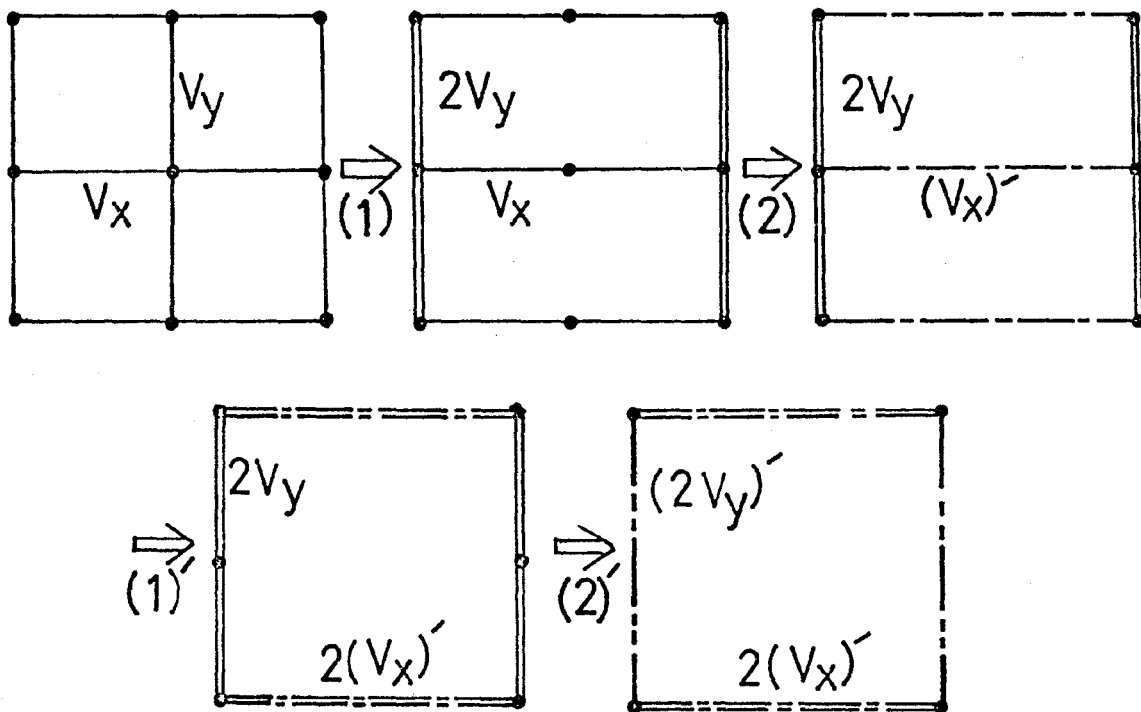


Fig. 16. The potential moving (1), (1)' and the decimation (2), (2)' are the fundamental steps of the MKRG manipulation.

The interaction of the new lattice is now  $2(V_x)'$  in the  $x$ -direction and  $(2V_y)'$  in the  $y$ -direction. If we express this transformation of the interaction in terms of the transformation of the corresponding Boltzmann factors ( $x_\xi = \exp V_x(\xi)$ ,  $y_\xi = \exp V_y(\xi)$ ),

$$x'_\xi = \left[ \frac{1}{n} \sum_{\lambda=1}^n e^{i \frac{2\pi\xi\lambda}{n}} \left[ \sum_{\mu=1}^n e^{-i \frac{2\pi\mu\lambda}{n}} x_\mu \right]^2 \right]^2 \quad (\text{A.1})$$

$$y'_\xi = \frac{1}{n} \sum_{\lambda=1}^n e^{i \frac{2\pi\xi\lambda}{n}} \left[ \sum_{\mu=1}^n e^{-i \frac{2\pi\mu\lambda}{n}} (y_\mu)^2 \right]^2 ,$$

because the potential moving (e. g.,  $V_y \rightarrow 2V_y$ ) makes the Boltzmann factor squared ( $y_\mu \rightarrow (y_\mu)^2$ ) and the decimation (e. g.,  $V_x \rightarrow (V_x)'$ ) is equivalent to the potential moving in the Fourier-transformed space

$$\begin{aligned} & \left( \sum_{\mu=1}^n \exp\left(-i \frac{2\pi\mu\lambda}{n}\right) x_\mu \rightarrow \left[ \sum_{\mu=1}^n \exp\left(-i \frac{2\pi\mu\lambda}{n}\right) x_\mu \right]^2 \right) : \\ & \sum_{\xi=1}^n \exp[V_x(\xi_1 - \xi) + V_x(\xi - \xi_2)] = \sum_{\xi=1}^n \sum_{\mu_1, \mu_2=1}^n e^{\frac{2\pi i}{n} [\mu_1(\xi_1 - \xi) + \mu_2(\xi - \xi_2)]} u_{\mu_1} u_{\mu_2} \\ & = n \sum_{\mu=1}^n e^{\frac{2\pi i}{n} \mu(\xi_1 - \xi_2)} (u_\mu)^2 \equiv n \exp V'_x(\xi_1 - \xi_2) \end{aligned}$$

The normalization ( $x_n = y_n = x'_n = y'_n$ ) can be taken into account by dividing the right hand side of Eq. (A.1) by  $x'_n$  or  $y'_n$ :

$$\begin{aligned} x'_\xi &= \frac{\left[ \frac{1}{n} \sum_{\lambda=1}^n e^{i \frac{2\pi\xi\lambda}{n}} \left[ \sum_{\mu=1}^n e^{-i \frac{2\pi\mu\lambda}{n}} x_\mu \right]^2 \right]^2}{\left[ \frac{1}{n} \sum_{\lambda=1}^n \left[ \sum_{\mu=1}^n e^{-i \frac{2\pi\mu\lambda}{n}} x_\mu \right]^2 \right]^2} \\ y'_\xi &= \frac{\frac{1}{n} \sum_{\lambda=1}^n e^{i \frac{2\pi\xi\lambda}{n}} \left[ \sum_{\mu=1}^n e^{-i \frac{2\pi\mu\lambda}{n}} (y_\mu)^2 \right]^2}{\frac{1}{n} \sum_{\lambda=1}^n \left[ \sum_{\mu=1}^n e^{-i \frac{2\pi\mu\lambda}{n}} (y_\mu)^2 \right]^2} \end{aligned} \quad (\text{A.2})$$

西森秀稔

If the scale factor is not two but  $b$  generally, only the replacement (something)<sup>2</sup>  $\rightarrow$  (something)<sup>b</sup> is necessary in (A.2). In this way we get Eq. (3.1).

The error caused by this renormalization group transformation originates in the potential moving process. If the interaction energy between neighboring spins is spatially uniform, the potential moving causes no trouble. In general the interaction energy of a spin pair is different from that of another pair at any moment because of different spin configurations. The uniformity assumption is valid in the low temperature limit. (If  $T \rightarrow 0$ , only the ground state is permitted to appear.) Therefore MKRG is a low temperature approximation.

It should be emphasized that the derivation presented here never guarantees that MKRG is more than an approximate RG transformation. Nevertheless it happens that the critical surface obtained by MKRG exactly satisfies the duality constraint as shown in chapter IV.

## References

1. J. M. Kosterlitz and D. J. Thouless, *J. Phys.* **C6** (1973) 1181.
2. H. E. Stanley and T. A. Kaplan, *Phys. Rev. Lett.* **17** (1966) 913.
3. I. Rudnick, *Phys. Rev. Lett.* **40** (1978) 1454.
4. D. J. Bishop and J. D. Reppy, *Phys. Rev. Lett.* **40** (1978) 1727.
5. K. Hirakawa and H. Yoshizawa, *J. Phys. Soc. Japan* (submitted).
6. D. R. Nelson and J. M. Kosterlitz, *Phys. Rev. Lett.* **39** (1977) 1201.
7. J. Ashkin and E. Teller, *Phys. Rev.* **64** (1943) 178.
8. R. B. Potts, *Proc. Camb. Phil. Soc.* **8** (1952) 106.
9. T. Kihara, Y. Midzuno and T. Shizume, *J. Phys. Soc. Japan* **9** (1954) 681.
10. D. D. Betts, *Can. J. Phys.* **42** (1964) 1564.
11. M. Suzuki, *Prog. Theor. Phys.* **37** (1967) 770.
12. L. Mittag and M. J. Stephen, *J. Math. Phys.* **37** (1971) 441.
13. C. Domb, *J. Phys.* **A7** (1974) 1335.
14. H. Nishimori, *Physica A* (to be published)
15. V. G. Vaks and A. I. Larkin, *Sov. Phys. JETP* **22** (1966) 676.
16. T. Matsubara and H. Matsuda, *Prog. Theor. Phys.* **16** (1956) 569.

17. N. D. Mermin and H. Wagner, Phys. Rev. Lett. **17** (1966) 1133.
18. S. Miyashita, H. Nishimori, A. Kuroda and M. Suzuki, Prog. Theor. Phys. **60** (1978) 1669. See Appendix B.
19. J. V. José, L. P. Kadanoff, S. Kirkpatrick and D. R. Nelson, Phys. Rev. **B16** (1977) 1217.
20. F. Y. Wu and Y. K. Wang, J. Math. Phys. **17** (1975) 439.
21. A. A. Migdal, Sov. Phys. JETP **42** (1976) 743.
22. L. P. Kadanoff, Ann. Phys. **100** (1976) 359.
23. Th. Neimeijer and J. M. J. van Leeuwen, in *Phase Transitions and Critical Phenomena* vol. 6, ed. by C. Domb and M. S. Green, (Academic Press).
24. L. Onsager, Phys. Rev. **65** (1944) 117.
25. M. E. Fisher, J. Math. Phys. **5** (1964) 944.
26. L. G. Marland, J. Phys. **C11** (1978) L617.
27. R. Zwanzig and J. D. Ramshaw, J. Phys. **A10** (1977) 65.
28. H. J. F. Knops, J. Phys. **A8** (1975) 1508.
29. R. J. Baxter, J. Phys. **C6** (1973) L445.
30. R. J. Baxter, H. N. V. Temperley and S. E. Ashley, Proc. Roy. Soc. London **A358** (1978) 535.
31. V. L. Berezinskii, Sov. Phys. JETP **32** (1971) 493.
32. J. Villain, J. Phys. (Paris) **36** (1975) 581.
33. J. M. Kosterlitz, J. Phys. **C7** (1974) 1046.
34. R. J. Myerson, Phys. Rev. **B16** (1977) 3203.
35. H. A. Kramers and G. H. Wannier, Phys. Rev. **60** (1941) 252.