Applied Mathematical Sciences, Vol. 13, 2019, no. 5, 209 - 223
HIKARI Ltd, www.m-hikari.com
https://doi.org/10.12988/ams.2019.9111

Study of Memory Effect in an Inventory Model with Quadratic Type Demand Rate and Salvage Value

Rituparna Pakhira, Uttam Ghosh and Susmita Sarkar

Department of Applied Mathematics
University of Calcutta, Kolkata, India

Abstract

Our purpose of the paper is to propose an inventory modelwith memory effect.Generally, in deriving the solution of the classical inventory model, first order differential equation is considered. The integer order differential equation is not able to incorporate the memory or past experience of the inventory system. But in case of real life inventory problems, the past experience or memory effect has great impact. We also want to give importance on salvage value of an inventory system. A comparisonalso has beenbuilt with our previous developed paper where salvage value is not considered. The paper aims to show the result of long memory effect and short memory effect on the minimized total average cost and the optimal ordering interval. The numerical example is presented toillustrate the purpose of the paper.

Keywords: Fractional order derivative; Fractional Laplace transform method; Memory dependent inventory model

1. Introduction

Fractional calculus was developed by the mutual thoughts of Leibnitz and L'Hospital. The subject of fractional calculus did not escape from the attention of well- known scientists like Euler, J.L.Lagrange, Lacroix, P.S.Laplace [1].The first conference about fractional calculus was held in 1975 [1].Thereafter, necessary definition of fractional order derivative and fractional order integration was

developed by great scientists Grunwald, Riemann-Liouville. The developed definition of Riemann creates a problem by contradicting the ordinary calculus. The Ordinary calculus gives the derivative of constant term becomes zero but R-L fractional derivative gives derivative of constant term is non -zero. To eradicate the difficulties from the definition of fractional derivative, Caputo[2] developed a definition of fractional order derivative. Thereafter, many definition of fractional derivative was developed by Jumarie [3].One important things of all mathematical definition of derivative are geometrical interpretation which actually connect to the reality. Fractional derivative has an important physical interpretation which is memory of the system. The fractional order of fractional derivative and integration is an index of memory. So, this curious fact can be applied to the system which has any past experience effect like in the inventory system [4, 5], economic model [6, 7],and epidemic model[8].Fractional calculus has also used to generalize the model as non–integer model in the different topic of mathematics [4, 5, 8, 9],physics [9, 10]etc. But we want to include memory effect in an inventory model with quadratic type demand rate and salvage value.

Inventory model is an important research topic of operations research as well as in organizational management. It is very difficult to determine the company's profit with its optimal ordering interval without study of mathematical models. This type mathematical model is well known as inventory model from where it is found that the minimized total average cost of the total business and optimal ordering interval optimal order quantity with different condition of the demand rate. The history of Inventory problem goes back to the 20 -*th* century. First, Harris and Wilson [11, 12] developed the EOQ model. Wilson formulated the mathematical work. Then, this subject received attention from several researchers. Silver and Meal [13] proposed an approximate solution technique of a deterministic inventory model with time-dependent demand. Donaldson [14] gave an idea of the optimal algorithm for solving classical no-shortage inventory model with linear trend in demand over fixed time horizon analytically. Ritchie [15] gave a simple optimal solution for the economic order quantity with linear increasing demand. Salvage value is also an important part of discussion of the inventory system. Mishra and shah et al [16]developed an inventory model of time dependent deteriorating Items with Salvage Value. Mishra *et al*[17]also developed an inventory model with salvage value with clear numerical example. Here, an inventory model with quadratic type demand rate and salvage value has been developed with taking into account memory effect via fractional calculus.

Authors expect that inventory system is a memory affected system. For example, if an object gets its popularity in the market then its demand will increase or if it gets poor impression then its demand will gradually decrease.In some sense demand of any object depends on dealing of the shopkeeper or staff of the company with the customer i.e. the selling of any product depends on the quality as well as the shopkeeper's attitude or environment of the company or shop or public relation. The associated cost has been developed with fractional effect. Integral memory index comes from the developed associated cost.

The classical inventory models are described by the ordinary differential equations which contain integer order derivatives with respect to time. The differential equations with derivatives of integer order actually describe only the instantaneous change of the inventory level. It has no power to incorporate memory effect of the system. But order of fractional derivative and fractional integration is physically treated as an index of memory and the strength of memory depends on the fractional order. Due to the fact, we have choosed fractional order derivative and fractional order integration to include to the inventory model.

Our purpose is to compare the result of memory effect on the minimized total average cost and optimal ordering interval between an inventory model with salvage value, time varying holding cost and without salvage value[4].Almost same memory effect has found which is discussed in the section-5.

The rest part of the paper has arranged in the following way as in the section-2 review of fractional calculus is given, Classical inventory model is given by in the section-3, fractional order inventory model has presented in the section -4,Numerical example with sensitivity analysis has discussed in the section-5, the graphical presentation is given in the section-6,at last some conclusions is discussed in the section-7.

2. Review of fractional calculus

2.1Caputo fractional order derivative

M.Caputo defines the left –sided fractional derivative in the following form as follows

$$ {}_{a}^{C}D_{x}^{\alpha}\left(f(x)\right)=\frac{1}{\Gamma(n-\alpha)}\int_{a}^{x}(x-\tau)^{(n-\alpha-1)}\ f^{n}(\tau)d\tau \tag{1}$$

Where, $n-1\le\alpha<n$

The left-sided and right –sided Caputo derivatives are linear operators

$$ {}_{a}^{C}D_{x}^{\alpha}\left(f(x)+g(x)\right)={}_{a}^{C}D_{x}^{\alpha}\left(f(x)\right)+{}_{a}^{C}D_{x}^{\alpha}\left(g(x)\right) \tag{2}$$

The Caputo-fractional derivatives of constant is zero.

$ {}_{a}^{C}D_{x}^{\alpha}(M)=0$

2.2 Fractional Laplace transforms method

The Laplace transform of the function $f(t)$ is defined as

$$F(s)=L(f(t))=\int_{0}^{\infty}e^{-st}f(t)dt$$

where s>0 and s is the transform parameter

The Laplace transformation of n^{th} order derivative is defined as

$$L\left(f^n(t)\right)=s^nF(s)-\sum_{k=0}^{n-1}s^{n-k-1}f^k(0) \tag{3}$$

where $f^n(t)$ denotes *n-th* order derivative of the function *f* with respect to *t* and for non – integer m it is defined in generalized form as,

$$L\left(f^m(t)\right)=s^mF(s)-\sum_{k=0}^{n-1}s^k f^{m-k-1}(0) \tag{4}$$

Where*m is* such that $(n-1)<m\le n.$

3. Classical model

The classical inventory model is developed with the following assumptions and notations.

3.1 Assumptions

In this paper, the classical and fractional order EOQ models are developed on the basis of the following assumptions.

(i) Lead time is zero.(ii)Time horizon is infinite.(iii)There is no shortage.(iv)There is no deterioration.(v)Demand rate is $R(t)=\left(a+bt+ct^2\right)$ where $a>0,b,c\ge 0.$ (vi)Holding cost is time varying as C_1t per unit time.

3.2 Notations

$(i)R(t)$: Demand rate	$(ii)Q$: Total order quantity
$(iii)P$: Per unit cost	$(iv)C_1$ t : Inventory holding cost per unit
$(v)C_3$: Ordering cost or setup cost	$(vi)I(t)$: Stock level or inventory level
$(vii)T$: Ordering interval.	$(viii)HOC_{\alpha,\beta}$: Inventory holding cost per cycle for the classical inventory model.
$(ix)T^*$: Optimal ordering interval	$(x)TOC^{av}_{\alpha,\beta}$: Total average cost during the total time interval
$(xi)TOC^*_{\alpha,\beta}$: Minimized total average cost during the total time interval $[0,T]$ for the classical model.	$(xii)(B,.),(\Gamma,.)$ Beta function and gamma function respectively.
$(xiii)T^*_{\alpha,\beta}$: Optimal ordering interval.	

Table-1: Used symbols and items.

3.3 Classical Model Formulation

Here, we have first developed a classical inventory model depending on the above assumptions. During the total time interval *[0,T]*, the inventory level depletes due to quadratic demand rate $\mathrm{R(t)} = (a + bt + ct^2), \mathrm{a} > 0, \mathrm{b}, \mathrm{c} \geq 0$, where shortage is not allowed. The inventory level reaches zero level at time $t=T$. Therefore, inventory level at any time during the time interval *[0,T]* can be represented by the following first order ordinary differential equation as,

$$\frac{dI(t)}{dt} = -(a + bt + ct^2) \quad (5)$$

with boundary conditions $I(0) = Q$ and $I(T) = 0$.

4. Fractional order inventory model formulation with memory kernel

To study the influence of memory effects, first the differential equation (5) is written using the memory kernel function in the following form [8].

$$\frac{dI(t)}{dt} = -\int \mathrm{k}\left(t - t'\right)(a + bt' + c(t')^2)dt' \quad (6)$$

in which $k(t-t')$ plays the role of a time-dependent kernel. This type of kernel promises the existence of scaling features as it is often intrinsic in most natural phenomena. Thus, to generate the fractional order model we consider $k(t-t') = \frac{1}{\Gamma(1-\alpha)}(t-t')^{\alpha-2}$, where $0 < \alpha \leq 1$ and $\Gamma(\alpha)$ denotes the gamma function. Using the definition of fractional order derivative [8], the equation (6) can be written to the form of fractional differential equations with the Caputo-type derivative in the following form as,

$$\frac{dI(t)}{dt} = -{}_0D_t^{-(\alpha-1)}\left(\left(a + bt + ct^2\right)\right) \quad (7)$$

Now, applying fractional Caputo derivative of order $(\alpha - 1)$ on both sides of (7), and using the fact at Caputo fractional order derivative and fractional integral are inverse operators, the following fractional differential equations can be obtained for the model

${}_0^C D_t^{\alpha}\left(I(t)\right) = -\left(a + bt + ct^2\right)$ or equivalently

$$\frac{d^{\alpha}\left(\mathrm{I(t)}\right)}{dt^{\alpha}} = -\left(a + bt + ct^2\right) \qquad 0 < \alpha \leq 1.0,\ 0 \leq t \leq T \quad (8)$$

with boundary conditions $I(T) = 0$ and $I(0) = Q$.

4.1Fractional order inventory model with its analysis

Here, we consider the fractional order inventory model which will be solved by using Laplace transform method with the initial condition. In operator form the fractional differential equation (8) can be represented as

$$ {}_0^C D_t^{\alpha}\left(I(t)\right) = -\left(a + bt + ct^2\right) \quad , \qquad D^{\alpha} \equiv \frac{d^{\alpha}}{dt^{\alpha}} \quad 0 \le t \le T \tag{9}$$

where the operator D^{α} stands for the Caputo fractional derivative with the operator ($D^{\alpha} = {}_0^C D_t^{\alpha}$).

we get the inventory level for this fractional order inventory model at time *t* which can be written as

$$I(t) = \left(Q - \frac{at^{\alpha}}{\Gamma(1+\alpha)} - \frac{bt^{(\alpha+1)}}{\Gamma(2+\alpha)} - \frac{2c\ t^{(\alpha+2)}}{\Gamma(3+\alpha)} \right) \tag{10}$$

Using the boundary condition $I(T) = 0$ on the equation (10), the total order quantity is obtained as

$$Q = \left(\frac{aT^{\alpha}}{\Gamma(1+\alpha)} + \frac{b\ T^{(\alpha+1)}}{\Gamma(\alpha+2)} + \frac{2c\ T^{(\alpha+2)}}{\Gamma(3+\alpha)} \right) \tag{11}$$

and corresponding the inventory level at time *t* being,

$$I(t) = \left(\frac{a}{\Gamma(1+\alpha)}\left(T^{\alpha} - t^{\alpha}\right) + \frac{b}{\Gamma(2+\alpha)}\left(T^{\alpha+1} - t^{\alpha+1}\right) + \frac{2c}{\Gamma(3+\alpha)}\left(T^{\alpha+2} - t^{\alpha+2}\right) \right) \tag{12}$$

For the model (9), the $\beta - \left(0 < \beta \le 1.0\right)$ [4]order total inventory holding cost which is time dependent, is denoted as $HOC_{\alpha,\beta}(T)$ and defined as

$$HOC_{\alpha,\beta}(T) = C_1\left(D^{-\beta}\left(tI(t)\right)\right) = \frac{C_1}{\Gamma(\beta)}\int_0^T t(T-t)^{(\beta-1)}\left(I(t)\right)dt \quad where 0 \le t \le T$$

$$= \left(\begin{array}{l} \frac{C_1 aT^{(\alpha+\beta+1)}}{\Gamma(\alpha+1)\Gamma(\beta)}\left(B(2,\beta) - B(\alpha+2,\beta)\right) + \frac{C_1 bT^{(\alpha+\beta+2)}}{\Gamma(2+\alpha)\Gamma(\beta)}\left(B(2,\beta) - B(\alpha+3,\beta)\right) \\ \quad + \frac{2C_1 cT^{(\alpha+\beta+3)}}{\Gamma(3+\alpha)\Gamma(\beta)}\left(B(2,\beta) - B(\alpha+4,\beta)\right) \end{array} \right) \tag{13}$$

β is considered as integral memory index.

Total purchasing cost is denoted by $PC_{\alpha,\beta}$ and defined as

$$PC_{\alpha,\beta} = P \times Q = P\left(\frac{aT^{\alpha}}{\Gamma(1+\alpha)} + \frac{b\ T^{(\alpha+1)}}{\Gamma(\alpha+2)} + \frac{2c\ T^{(\alpha+2)}}{\Gamma(3+\alpha)} \right) \tag{14}$$

The β-*th* order total salvage value is denoted by $SL_{\alpha,\beta}$ and defined by

$$SL_{\alpha,\beta} = \gamma\left(Q - D^{-\beta}\left(a + bt + ct^2\right)\right) = \gamma\left(Q - \frac{1}{\Gamma(\beta)}\int_0^T (T-t)^{\beta-1}\left(a + bt + ct^2\right)dt \right)$$

$$= \gamma\left(\left(\frac{aT^{\alpha}}{\Gamma(1+\alpha)} + \frac{b\ T^{(\alpha+1)}}{\Gamma(\alpha+2)} + \frac{2c\ T^{(\alpha+2)}}{\Gamma(3+\alpha)} \right) - \left(\frac{aT^{\beta}}{\Gamma(1+\beta)} + \frac{b\ T^{(\beta+1)}B(2,\beta)}{\Gamma(\beta)} + \frac{c\ T^{(\beta+2)}B(3,\beta)}{\Gamma(\beta)} \right) \right)$$

$$\left(SL_{\alpha,\beta} \ne 0 \quad for \quad \alpha \ne \beta < 1\right) \tag{15}$$

Therefore, total average cost is as

$$TOC^{av}_{\alpha,\beta} = \frac{\left(PC_{\alpha,\beta} + HOC_{\alpha,\beta}(T) + C_3 - SL_{\alpha,\beta}\right)}{T}$$

$$= AT^{\alpha+\beta} + B_1T^{\alpha+\beta+1} + CT^{\alpha+\beta+2} + DT^{\alpha-1} + ET^{\alpha} + FT^{\alpha+1} + GT^{\beta-1} + HT^{\beta} + IT^{\beta+1} + JT^{-1} \quad (16)$$

$$A = \frac{C_1 a}{\Gamma(\beta)\Gamma(\alpha+1)}\left(B(2,\beta) - B(\alpha+2,\beta)\right), B_1 = \frac{C_1 b}{\Gamma(\beta)\Gamma(\alpha+2)}\left(B(2,\beta) - B(\alpha+3,\beta)\right)$$

$$C = \frac{2cC_1}{\Gamma(\beta)\Gamma(\alpha+3)}\left(B(2,\beta) - B(\alpha+4,\beta)\right), \mathrm{D} = \frac{(P-\gamma)a}{\Gamma(\alpha+1)}, \mathrm{E} = \frac{(P-\gamma)b}{\Gamma(\alpha+2)}, F = \frac{2c(P-\gamma)}{\Gamma(\alpha+3)},$$

$$G = \frac{a\gamma}{\Gamma(\beta+1)}, H = \frac{b\gamma B(2,\beta)}{\Gamma(\beta)}, \mathrm{I} = \frac{c\gamma B(3,\beta)}{\Gamma(\beta)}, J = \mathrm{C}_3$$

Now, we shall consider the following cases to study the behavior of this fractional order inventory model (i) $0<\alpha\le 1.0, 0<\beta\le 1.0$, (ii) $\beta = 1.0$ and $0<\alpha\le 1.0$, (iii) $\alpha = 1.0$ and $0<\beta\le 1.0$, (iv) $\alpha = 1.0, \beta = 1.0$.

(i) Case-1: $0<\alpha\le 1.0$ and $0<\beta\le 1.0$

Total average cost is

$$TOC^{av}_{\alpha,\beta} = AT^{\alpha+\beta} + B_1T^{\alpha+\beta+1} + CT^{\alpha+\beta+2} + DT^{\alpha-1} + ET^{\alpha} + FT^{\alpha+1} + GT^{\beta-1} + HT^{\beta} + IT^{\beta+1} + JT^{-1} \quad (17)$$

$$A = \frac{C_1 a}{\Gamma(\beta)\Gamma(\alpha+1)}\left(B(2,\beta) - B(\alpha+2,\beta)\right), B_1 = \frac{C_1 b}{\Gamma(\beta)\Gamma(\alpha+2)}\left(B(2,\beta) - B(\alpha+3,\beta)\right)$$

$$C = \frac{2cC_1}{\Gamma(\beta)\Gamma(\alpha+3)}\left(B(2,\beta) - B(\alpha+4,\beta)\right), \mathrm{D} = \frac{(P-\gamma)a}{\Gamma(\alpha+1)}, \mathrm{E} = \frac{(P-\gamma)b}{\Gamma(\alpha+2)}, F = \frac{2c(P-\gamma)}{\Gamma(\alpha+3)},$$

$$G = \frac{a\gamma}{\Gamma(\beta+1)}, H = \frac{b\gamma B(2,\beta)}{\Gamma(\beta)}, \mathrm{I} = \frac{c\gamma B(3,\beta)}{\Gamma(\beta)}, J = \mathrm{C}_3$$

Inventory model can be written as follows

$$\begin{cases} \mathrm{Min} TOC^{av}_{\alpha,\beta} = AT^{\alpha+\beta} + B_1T^{\alpha+\beta+1} + CT^{\alpha+\beta+2} + DT^{\alpha-1} + ET^{\alpha} + FT^{\alpha+1} + GT^{\beta-1} + HT^{\beta} + IT^{\beta+1} + JT^{-1} \\ \mathrm{Subject\ to} T \ge 0 \end{cases} \quad (18)$$

(a)Primal-Geometric programming method

To solve (18) analytically, the primal geometric programming method has been applied. The dual form of (18) has been introduced by the dual variable (w) .The corresponding primal geometric programming problem has been constructed in the following form as,

$$Max\, d(w) = \left(\frac{A}{w_1}\right)^{w_1}\left(\frac{B_1}{w_2}\right)^{w_2}\left(\frac{C}{w_3}\right)^{w_3}\left(\frac{D}{w_4}\right)^{w_4}\left(\frac{E}{w_5}\right)^{w_5}\left(\frac{F}{w_6}\right)^{w_6}\left(\frac{G}{w_7}\right)^{w_7}\left(\frac{H}{w_8}\right)^{w_8}\left(\frac{I}{w_9}\right)^{w_9}\left(\frac{J}{w_{10}}\right)^{w_{10}} \quad (19)$$

Normalized condition is as

$$w_1 + w_2 + w_3 + w_4 + w_5 + w_6 + w_7 + w_8 + w_9 + w_{10} = 1 \quad (20)$$

Orthogonal condition is as

$$\left.\begin{matrix}(\alpha+\beta)w_1+(\alpha+\beta+1)w_2+(\alpha+\beta+2)w_3+(\alpha-1)w_4+(\alpha)w_5+(\alpha+1)w_6\\+(\beta-1)w_7+(\beta)w_8+(\beta+1)w_9+(-1)w_{10}\end{matrix}\right)=0 \quad (21)$$

and the primal-dual relations are as follows

$$\left.\begin{matrix}AT^{\alpha+\beta}=w_1d(w),B_1T^{\alpha+\beta+1}=w_2d(w),CT^{\alpha+\beta+2}=w_3d(w)\\,DT^{\alpha-1}=w_4d(w),ET^{\alpha}=w_5d(w),FT^{\alpha+1}=w_6d(w),GT^{\beta-1}=w_7d(w),\\HT^{\beta}=w_8d(w),IT^{\beta+1}=w_9d(w),JT^{-1}=w_{10}d(w)\end{matrix}\right\} \quad (22)$$

Using the above primal-dual relation the followings are given by

$$\left.\begin{matrix}\frac{Aw_2}{B_1w_1}=\frac{B_1w_3}{Cw_2},\left(\frac{Aw_2}{B_1w_1}\right)^{\beta+3}=\frac{Dw_3}{Cw_4},\frac{Aw_2}{B_1w_1}=\frac{Dw_5}{Ew_4},\frac{Aw_2}{B_1w_1}=\frac{Ew_6}{Fw_5},\left(\frac{Aw_2}{B_1w_1}\right)^{(\alpha-\beta+2)}=\frac{Gw_6}{Fw_7}\\\frac{Aw_2}{B_1w_1}=\frac{Hw_9}{Iw_8},\left(\frac{Aw_2}{B_1w_1}\right)^{\beta+2}=\frac{Jw_9}{Iw_{10}}\end{matrix}\right\} \quad (23)$$

along with

$$T=\frac{Aw_2}{B_1w_1} \quad (24)$$

Solving (20), (21) and (23) the critical value $w_i^*, i=1...10$ of the dual variable $w_i, i=1...10$ an be obtained and finally the optimum value *T** of *T*hasbeen calculated from the equation of (24) substituting the critical values of the dual variable. Now the minimized total average cost $TOC^*_{\alpha,\beta}$ has been calculated by substituting T^* in (18) analytically. The minimized total average cost and optimal ordering interval is evaluated from (18) numerically using matlab minimization method.

(ii) Case-2: $\beta=1.0$ and $0<\alpha\le 1.0$

Therefore, the minimized total average cost becomes as

$$TOC^{av}_{\alpha,1}=AT^{\alpha+1}+B_1T^{\alpha+2}+CT^{\alpha+3}+DT^{\alpha-1}+ET^{\alpha}+FT^{0}+GT+HT^{2}+IT^{-1} \quad (25)$$

$$A=\frac{C_1a}{\Gamma(1)\Gamma(\alpha+1)}\left(B(2,1)-B(\alpha+2,1)\right)+\frac{2c(P-\gamma)}{\Gamma(\alpha+3)},B_1=\frac{C_1b}{\Gamma(1)\Gamma(\alpha+2)}\left(B(2,1)-B(\alpha+3,1)\right)$$

$$C=\frac{2cC_1}{\Gamma(1)\Gamma(\alpha+3)}\left(B(2,1)-B(\alpha+4,1)\right),\mathrm{D}=\frac{(P-\gamma)a}{\Gamma(\alpha+1)},\mathrm{E}=\frac{(P-\gamma)b}{\Gamma(\alpha+2)},$$

$$F=\frac{a\gamma}{\Gamma(2)},G=\frac{b\gamma B(2,1)}{\Gamma(1)},H=\frac{c\gamma B(3,1)}{\Gamma(1)},I=\mathrm{C}_3$$

The inventory model can be written as follows

$$\begin{cases}\mathrm{Min}TOC^{av}_{\alpha,1}=AT^{\alpha+1}+B_1T^{\alpha+2}+CT^{\alpha+3}+DT^{\alpha-1}+ET^{\alpha}+FT^{0}+GT+HT^{2}+IT^{-1}\\\mathrm{Subject\ to}\,T\ge 0\end{cases} \quad (26)$$

In similar case-1 using primal-geometric programming approach, the minimized total average cost and the optimal ordering interval is obtained from (26) analytically.

(iii)Case-3: $\alpha=1.0, 0<\beta\leq 1.0$.

Here, the total average cost is

$$TOC_{1,\beta}^{av} = AT^{1+\beta} + B_1T^{2+\beta} + CT^{3+\beta} + DT^{0} + ET^{1} + FT^{2} + GT^{\beta-1} + HT^{\beta} + IT^{-1} \tag{27}$$

$$A = \frac{C_1 a}{\Gamma(\beta)\Gamma(2)}\big(B(2,\beta) - B(3,\beta)\big) + \frac{c\gamma B(3,\beta)}{\Gamma(\beta)}, B_1 = \frac{C_1 b}{\Gamma(\beta)\Gamma(3)}\big(B(2,\beta) - B(4,\beta)\big)$$

$$C = \frac{2cC_1}{\Gamma(\beta)\Gamma(4)}\big(B(2,\beta) - B(5,\beta)\big), \mathrm{D} = \frac{(P-\gamma)a}{\Gamma(2)}, \mathrm{E} = \frac{(P-\gamma)b}{\Gamma(3)}, F = \frac{2c(P-\gamma)}{\Gamma(4)},$$

$$G = \frac{a\gamma}{\Gamma(\beta+1)}, H = \frac{b\gamma B(2,\beta)}{\Gamma(\beta)}, \mathrm{I} = \mathrm{C}_3$$

In this case, the inventory model can be written as

$$\begin{cases} \mathrm{Min} TOC_{1,\beta}^{av} = AT^{1+\beta} + B_1T^{2+\beta} + CT^{3+\beta} + DT^{0} + ET^{1} + FT^{2} + GT^{\beta-1} + HT^{\beta} + IT^{-1} \\ \mathrm{Subject\ to} T \geq 0 \end{cases} \tag{28}$$

In similar way as case-1, the minimized total average cost and the optimal ordering interval is obtained from(28).

(iv)Case-4: $\alpha=1.0, \beta=1.0$.

In this case, the total average cost is

$$TOC_{1,1}^{av} = AT^{2} + B_1T^{3} + CT^{4} + DT^{0} + ET^{1} + FT^{-1} \tag{29}$$

$$A = \frac{C_1 a}{\Gamma(1)\Gamma(2)}\big(B(2,1) - B(3,1)\big) + \frac{c\gamma B(3,1)}{\Gamma(1)} + \frac{2c(P-\gamma)}{\Gamma(4)}, B_1 = \frac{C_1 b}{\Gamma(1)\Gamma(3)}\big(B(2,1) - B(4,1)\big)$$

$$C = \frac{2cC_1}{\Gamma(1)\Gamma(4)}\big(B(2,1) - B(5,1)\big), \mathrm{D} = \frac{(P-\gamma)a}{\Gamma(2)} + \frac{a\gamma}{\Gamma(2)}, \mathrm{E} = \frac{(P-\gamma)b}{\Gamma(3)} + \frac{b\gamma B(2,1)}{\Gamma(1)}, F = \mathrm{C}_3$$

To find the minimum value of $TOC_{\alpha,\beta}^{av}(T)$ we propose the corresponding non-linear programming problem in the following form and solve it by primal geometric programming method.

$$\begin{cases} \mathrm{Min} TOC_{1,1}^{av} = AT^{2} + B_1T^{3} + CT^{4} + DT^{0} + ET^{1} + FT^{-1} \\ \mathrm{Subject\ to} T \geq 0 \end{cases} \tag{30}$$

In similar case-1,the minimized total average cost and the optimal ordering interval is obtained from(30).

5.Numerical Example

We consider a numerical example toillustrate the fractional order model with proper units

$a = 40, b = 20, c = 2, C_1 = 15, C_3 = 200, P = 300, \gamma = 0.1.$

α	β	$T^*_{\alpha,\beta}$	$TOC^*_{\alpha,\beta}$
0.1	1.0	2.9579	1.3761×10^4
0.2	1.0	2.6725	1.4960×10^4
0.3	1.0	2.3823	1.6035×10^4
0.4	1.0	2.0855	1.6921×10^4
0.5	1.0	1.78006	$1.7544 \times 10^4\downarrow$(decreasing)
0.6	1.0	1.4665	**1.7823×10^4(maximum value)**
0.7	1.0	1.1429	$1.7667 \times 10^4\downarrow$(decreasing)
0.8	1.0	0.8131	1.6972×10^4
0.9	1.0	0.4940	1.5624×10^4
1.0↑(growing memory effect)	1.0	0.2517	1.3569×10^4

Table-2: Minimized total average cost and the optimal ordering interval for $\beta = 1.0,$ and α varies from 0.1 to 1.0 as defined in section 4.1.

In this paper [4](table-4), the inventory holding cost is not time varying and salvage value is not considered. There is a critical memory effect ($\alpha = 0.6$) where minimized total average cost becomes maximum and then gradually decreases below and above.Table-2 shows that there is same nature of changing the result of memory effect on the minimized total average cost as paper[4]. Here,in the table-2, there is also found same effect of the differential memory index.But considering time varying holding cost and salvage value, the numerical values of the minimized total average cost is high compared to without salvage value model [4].

α	β	$T^*_{\alpha,\beta}$	$TOC^*_{\alpha,\beta}$
1.0	0.1	0.2547	1.3581×10^4
1.0	0.2	0.22534	1.3583×10^4
1.0	0.3	0.2525	1.3583×10^4
1.0	0.4	0.2519	1.3582×10^4
1.0	0.5	0.2513	1.3580×10^4
1.0	0.6	0.2513	1.3578×10^4
1.0	0.7	0.2513	1.3576×10^4
1.0	0.8↑(growing memory effect)	0.2513	1.3574×10^4
1.0	0.9	0.2515	1.3571×10^4
1.0	1.0	0.2517	1.3569×10^4

Table-3: Minimized total average cost and optimal ordering interval for $\alpha = 1.0,$ and β varies from 0.1 to 1.0 as defined in section 4.1

Holding cost is not time varying and salvage value is not considered in the paper[4].Here, in the table-3, there is also found almost same effect corresponding integral memory index. The numerical values of the minimized total average cost of paper ([4] *i.e* table-4) is low compared to the table-2 and 3.

(a)				(b)		
α	$T^*_{\alpha,\beta}$	$TOC^*_{\alpha,\beta}$ (**x10^4**)		β	$T^*_{\alpha,\beta}$	$TOC^*_{\alpha,\beta}$ (**x10^4**)
0.0951	**3.3023**	**1.3643**		**0.1**	**0.2523**	**1.361**
0.1	**3.2830**	**1.3708**		**0.166991**	**0.2507**	**1.3643**
0.2	2.8981	1.5004		0.2	0.2498	1.3653
0.3	2.5267	1.6171		0.3	0.2473	1.3673
0.4	2.1649	1.7136		0.4	0.2449	1.3683
0.5	1.8098	1.7817		**0.5**	**0.2430**	**1.3685**
0.6	**1.4597**	**1.8123**		0.6	0.2417	1.3681
0.7	1.1145	1.7954		0.7	0.2408	1.3673
0.8	0.7778	1.7207		0.8	0.2405	1.3664
0.9	0.4669	1.5776		0.9	0.2406	1.3653
1.0	**0.2409**	**1.3643**		**1.0**	**0.2409**	**1.3643**

Table-4: Minimized total average cost and optimal ordering interval for the model[4]

5.1Sensitivity analysis

We will now study the effects of changes in the values of the parameters a,b,c,C_1,C_3,P,γ on the minimized total average cost and the optimal ordering interval using the above numerical example.

It is found from the table-4 that the minimized total average cost is gradually increasing with gradually increasing value of the parameters b,c,C_1,C_3. but for gradually increasing value of a,P, the changes of the minimized total average cost for gradually increasing or decreasing feel disturbance. Here, in low memory effect, γ is not also sensitive. Here *a, P* are critical inventory parameters for the decision maker.

parameter	Parameter Change(%)	$T_{\alpha,1}^{*}$	$TOC^{*}{}_{\alpha,1}(T)$	parameter	Parameter Change (%)	$T_{\alpha,1}^{*}$	$TOC^{*}{}_{\alpha,1}(T)$
a	+50%	3.4749	$1.5990\text{x}10^{4}$	C_1	+50%	2.7523	$1.7015\text{x}10^{4}$
	+10%↑	3.0745	$1.4234\text{x}10^{4}$		+10%↑	2.9107	$1.6980\text{x}10^{4}$
	-10%	2.8327	$1.3272\text{x}10^{4}$		-10%	3.0093↓	$1.6963\text{x}10^{4}$
	-50%	2.2002↑	$1.1062\text{x}10^{4}$↑		-50%	3.2729	$1.6926\text{x}10^{4}$↑
b	+50%	2.7412	$1.7173\text{x}10^{4}$	C_3	+50%	2.9676	$1.7093\text{x}10^{4}$
	+10%↑	2.9109	$1.4449\text{x}10^{4}$		+10%↑	2.9599	$1.6996\text{x}10^{4}$
	-10%	3.0072	$1.3071\text{x}10^{4}$↑		-10%	2.9560↑	$1.6947\text{x}10^{4}$↑
	-50%	3.2296	$1.0280\text{x}10^{4}$		-50%	2.9483	$1.6847\text{x}10^{4}$
c	+50%	2.6470	$1.4691\text{x}10^{4}$	P	+50%	3.1426	$2.0208\ \text{x}10^{4}$
	+10%↑	2.8864↑	$1.3958\text{x}10^{4}$		+10%↑	3.0028	$1.5054\text{x}10^{4}$
	-10%	3.0356	$1.3558\text{x}10^{4}$↑		-10%	2.9076↑	$1.2465\text{x}10^{4}$
	-50%	3.4312	$1.2662\text{x}10^{4}$		-50%	2.6164	$7.2400\text{x}10^{3}$
γ	+50%	2.9573	$1.3763\text{x}10^{4}$				
	+10%↑	2.9578	$1.3762\text{x}10^{4}$				
	-10%	2.9581	$1.3761\text{x}10^{4}$↑				
	-50%	2.9586	$1.3760\text{x}10^{4}$				

Table-5: Sensitivity analysis on $T_{\alpha,1}^{*}$ and $TC^{*}{}_{\alpha,1}(T)$ for $\alpha = 0.1, \beta = 1.0$. *i.e* long memory effect(here↑ uses for increasing value for gradually increasing values of the parameter and ↓ uses for decreasing value for gradually increasing values of the parameter).

It is found from the table-4 that minimized total average cost is gradually increasing with gradually increasing value of the parameters $a, b, c, C_1, C_3, P, \gamma$ but for gradually increasing value of γ, the changes of the minimized total average cost is very negligible. Hence, for long memory effect, γ is not sensitive for market studies.

parameter	Parameter Change (%)	$T_{\alpha,1}^{*}$	$TOC^{*}{}_{\alpha,1}(T)$	parameter	Parameter Change (%)	$T_{\alpha,1}^{*}$	$TOC^{*}{}_{\alpha,1}(T)$
a	+50%	1.0729	$2.3531\text{x}10^{4}$	C_1	+50%	0.7937	$1.7015\text{x}10^{4}$
	+10%↑	0.8686	$1.8312\text{x}10^{4}$		+10%↑	0.8091	$1.6980\text{x}10^{4}$
	-10%	0.7560	$1.5613\text{x}10^{4}$		-10%	0.8137	$1.6963\text{x}10^{4}$
	-50%	0.5131	$1.9470\text{x}10^{3}$		-50%	0.8349	$1.6926\text{x}10^{4}$↑
b	+50%	0.6061	$1.8324\text{x}10^{4}$	C_3	+50%	0.8393	$1.7093\text{x}10^{4}$
	+10%↑	0.7617	$1.7269\text{x}10^{4}$		+10%↑	0.8185	$1.6996\text{x}10^{4}$
	-10%	0.8713	$1.6657\text{x}10^{4}$		-10%	0.8077	$1.6947\text{x}10^{4}$
	-50%	1.1994	$1.5185\text{x}10^{4}$↑		-50%	0.7853	$1.6847\text{x}10^{4}$↑
c	+50%	0.7804	$1.7057\text{x}10^{4}$	P	+50%	0.8087	$2.5290\text{x}10^{4}$
	+10%↑	0.8062	$1.6989\text{x}10^{4}$		+10%↑	0.8120	$1.8635\text{x}10^{4}$
	-10%	0.8519	$1.6954\text{x}10^{4}$		-10%	0.8145	$1.5308\text{x}10^{4}$
	-50%	0.7937	$1.6879\text{x}10^{4}$↑		-50%	0.8239	$8.6527\text{x}10^{4}$
γ	+50%	0.8130	$1.6971\text{x}10^{4}$				
	+10%↑	0.8131	$1.6972\text{x}10^{4}$				
	-10%	0.8132	$1.6972\text{x}10^{4}$				
	-50%	0.8132	$1.6972\text{x}10^{4}$				

Table-6: Sensitivity analysis on $T_{\alpha,1}^{*}$ and $TC^{*}{}_{\alpha,1}(T)$ for $\alpha = 0.8, \beta = 1.0$. *i.e* short memory effect (here↑ uses for increasing value for gradually increasing values of the parameter and ↓ uses for decreasing value for gradually increasing values of the parameter)

6. Graphical Presentation

Graphical presentation of minimized total average cost versus time horizon-T and salvage value per unit using the above numerical example.

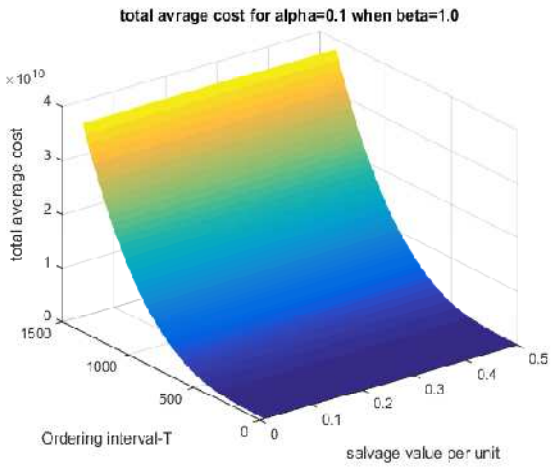

Fig-1: Total average cost versus ordering interval-T and salvage value for long memory effect(here $\alpha = 0.1, \beta = 1.0$)

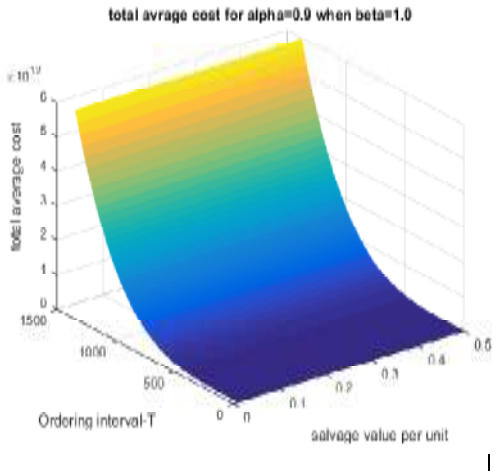

Fig-2: Total average cost versus ordering interval-T and salvage value for short memory effect(here $\alpha = 0.9, \beta = 1.0$)

7.Conclusion

This paper deals with problems of time varying holding cost with associated salvage value inventory model with taking into account memory effect. Here, a comparison between two inventory models have been proposed. In one inventory model time varying holding cost with associated salvage value is taken but in another model is not taken. Consideration of time varying holding cost with associated salvage value in the inventory modelthere is no significant change of the result of the numerical values of the minimized total average cost and the optimal ordering interval as[4].The salvage value has no sensitive effect for long memory effect or short memory effect for marketing decision. The salvage value only exist for $\alpha \neq \beta$ without considering deteriorating items. The sensitivity analysis shows that in long memory effect or short memory effect *a*, *P*are critical parameters for the decision maker.This model can be extended for deteriorating item inventory model with taking into account memory effect.

Acknowledgements. The authors would also like to thank the Department of Science and Technology, Government of India, New Delhi, for the financial assistance under AORC, Inspire fellowship Scheme towards this research work.

References

[1] K.S. Miller, B. Ross, An introduction to the Fractional calculus and Fractional Differential Equations, John Wiley's & Sons, New York, NY, USA, 1993.

[2] M. Caputo, Linear models of dissipation whose frequency independent, *Geophysical Journal of the Royal Astronomical Society*,**13**(5) (1967),529-539. https://doi.org/10.1111/j.1365-246x.1967.tb02303.x

[3] G.Jumarie, Modified Riemann-Liouville derivative and fractional Taylor series non differentiable functions further results, *Computers and Mathematics with Applications*, **51** (2006), 1367-1376. https://doi.org/10.1016/j.camwa.2006.02.001

[4] R.Pakhira, U. Ghosh, S. Sarkar, Study of Memory Effects in an Inventory Model Using Fractional Calculus, *Applied Mathematical Sciences*, Vol. **12,** no. 17 (2018), 797 – 824.https://doi.org/10.12988/ams.2018.8578

[5]R.Pakhira, U. Ghosh, S. Sarkar,Application of Memory Effects In an Inventory Model with Linear Demand and No shortage, *International Journal of Research in Advent Technology*, Vol.**6**, no. 8 (2018).

[6] V.V.Tarasova, V.E. Tarasov, Memory effects in hereditary Keynesian model, *Problems of Modern Science and Education*, no. 38 (80), (2016), 38–44. https://doi.org/10.20861/2304-2338-2016-80-001 (in Russian)

[7] V.E. Tarasov, V.V.Tarasova, Long and short memory in economics: fractional-order difference and differentiation, *IRA-International Journal of Management and Social Sciences*, Vol. **5**, no. 2, (2016), 327-334. https://doi.org/10.21013/jmss.v5.n2.p10.

[8] M.Saeedian, M.Khalighi, N.Azimi-Tafreshi, G.R.Jafari, M.Ausloos, Memory effects on epidemic evolution: The susceptible-infected-recovered epidemic model, *Physical ReviewE*, **95**(2017), 022409.https://doi.org/10.1103/physreve.95.022409

[9] L.Debnath, Fractional integral transform and Fractional equation in Fluid Mechanics, to appear.

[10] T. Das, U. Ghosh., S. Sarkar, Time independent fractional Schrodinger equation for generalized Mie-type potential in higher dimension framed with Jumarie type fractional derivative,*Journal of Mathematical Physics*,**59** (2018), 022111. https://doi.org/10.1063/1.4999262.

[11] F.W. Harris, *Operations and Cost*, A. W. Shaw Company, Chicago, 1915, pp. 48-54.

[12]R.H. Wilson, A Scientific Routine for Stock Control, *Harvard Business Review*, Vol. **13**, no. 1 (1934), 116-128.

[13] E.A. Silver and H.C. Meal, A simple modification of the EOQ for the case a varying demand rate,*Production of Inventory Management,***10** (1969), 52-65.

[14] W.A. Donaldson, Inventory replenishment policy for a linear trend in demand - an analytical solution, *Operation Research Quarterly*, **28** (1977), 663-670. https://doi.org/10.2307/3008916

[15] E. Ritchie, The optimal EOQ for linear increasing demand- a Simple optimal Solution,*Journal of the Operational Research Society*, **35** (1984), 949-952. https://doi.org/10.1057/jors.1984.18

[16] P. Mishra andN.H.Shah,Inventory Management of Time Dependent Deteriorating Items with Salvage Value, *Applied Mathematical Sciences*, Vol. **2**(2008), no. 16, 793 - 798.

[17] V.K. Mishra, Inventory Model For Time Dependent Holding Cost And Deterioration With Salvage Value And Shortages,*The Journal of Mathematics and Computer Science*, Vol. **4**, no.1 (2012), 37 – 47.
https://doi.org/10.22436/jmcs.04.01.06

[18] U. Ghosh, S.Sengupta,S. Sarkar, S. Das, Analytic Solution of linear fractional differential equation with Jumarie derivative in term of Mittag-Leffler function, *American Journal of Mathematical Analysis*,**3**(2) (2015), 32-38.

[19] I.Podlubny, Geometric and physical interpretations of fractional integral and fractional differentiation, *An international Journal of Theory and Application*, Volume **5**, no. 4 (2002).

[20] U. Ghosh, S. Sarkar, S. Das,Solutions of Linear Fractional non-Homogeneous Differential Equations with Jumarie Fractional Derivative and Evaluation of Particular Integrals,*American Journal of Mathematical Analysis.*Vol. **3**, no. 3, (2015), 54-64. https://doi.org/10.12691/ajma-3-3-1

[21] R.Pakhira, U.Ghosh. S.Sarkar, Study of Memory Effect in an Inventory Model with Linear Demand and Salvage Value, *International Journal of Applied Engineering Research*, Volume **13**, no. 20 (2018) 14741-14751.

Received: January 28, 2019; Published: February 20, 2019