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ORIGINAL PAPER

Study of the Analytical Treatment of the (2+1)-Dimensional Zoomeron, the Duffing and the SRLW Equations via a New Analytical Approach

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Abstract In this paper, we applied the improved tan $(\Phi(\xi)/2)$-expansion scheme for the (2+1)-dimensional Zoomeron, the Duffing and the symmetric regularized long wave equations and exact particular solutions have been found. The exact particular solutions containing four types hyperbolic function solution, trigonometric function solution, exponential solution and rational solution. We obtained the further solutions comparing with other methods as sine–cosine function method (Qawasmeh in J Math Comput Sci 3:1475–1480, 2013). Recently this method is developed for searching exact travelling wave solutions of nonlinear partial differential equations. It is shown that this method, with the help of symbolic computation, provide a straightforward and powerful mathematical tool for solving nonlinear partial differential equations.

Keywords Improved tan $(\Phi(\xi)/2)$-expansion method · The (2+1)-dimensional Zoomeron, the Duffing and the symmetric regularized long wave (SRLW) equations · Traveling wave

Introduction

During the last decades, some major contributions have been made to both the theory and applications of the nonlinear partial differential equations motivated by various practical engineering and physical problems. These applications cross diverse disciplines, such as

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chemical physics [1], viscoelasticity [2], electricity [3], biomedical engineering [1], fluid mechanics [4] and other sciences. In fact, it has been found that many models in mathematics and physics are described by nonlinear partial differential equations (NPDEs). The theory of solitons, the most important side in applications to NPDEs, has contributed to understanding many experiments in mathematical physics. Thus, it is of interest to evaluate new solutions of these equations. In the present paper, based on the improved tan $(\Phi(\xi)/2)$-expansion method, we will consider an important equation, which is the (2+1)-dimensional Zoomeron [5,6] with the form

$$\left(\frac{u_{xy}}{u}\right)_{tt} - \left(\frac{u_{xy}}{u}\right)_{xx} + 2(u^2)_{xt} = 0, \tag{1}$$

where $u = u(x, y, t)$ is the unknown real function and the amplitude of the relevant wave mode. Equation (1) appears in a wide variety of physical. The Zoomeron equation was introduced by Calogero and Degasperis in 1976 [7]. The solitary wave solutions of the Zoomeron equation obtained by Abazari [8]. In [6] the reliable treatment of the (2+1)-dimensional Zoomeron equation have been surveyed by Alquran and Al-Khaled. Qawasmeh [5] have applied the sine–cosine function method to construct the traveling wave solutions for the (2+1)-dimensional Zoomeron equation, the Duffing equation and the SRLW equation. In [9], Khan and Akbar investigated solutions of the (2 + 1)-dimensional Zoomeron equation and the (2 +1)-dimensional Burgers equation by using the MSE method and the Exp-function method. Also Bekir et al. [10] used the first integral method for constructing exact solutions of the Zoomeron and Klein–Gordon–Zakharov equations. As a second example we consider the Duffing equations as follows

$$u_{tt} + \alpha u + \beta u^3 = 0, \tag{2}$$

this equation have solved by sine–cosine method [5]. Both integral and non-integral forcing terms for Duffing equation was solved by Balaji [11]. As a last example we consider the SRLW equation [5] as follows

$$u_{tt} + u_{xx} + u_{xxtt} + (uu_x)_t = 0. \tag{3}$$

Authors of [12] was established exact travelling wave solutions of the symmetric regularized long wave (SRLW) by using analytical methods. Xu [13] applied of Exp–function method to SRLW equation. Cahnd and Malik [14] have found the exact solutions of some nonlinear evolution equations by using (G'/G)-expansion method. Recently, a variety of powerful methods for seeking the explicit and exact solutions of nonlinear evolution equations have been proposed and developed. Among them are the Hirota's bilinear method [15], homotopy analysis method [16], variational iteration method [17], homotopy perturbation method [18,19], homogenous balance method [20], sine–cosine method [21], tanh-coth method [22], Bäcklund transformation [23], $(\frac{G'}{G})$-expansion method [24–27], Exp-function method [28–30], modified simple equation method [31–33], first integral method [34–37], functional variable method [38,39], and so on [40]. Here, we use of an effective method for constructing a range of exact solutions for the following NPDEs that in this article we developed solutions as well. The standard tanh method is well-known analytical method which first presented by Malfliet's [41] and developed in [41,42]. In [22], we applied the generalized tanh–coth method in for solving some NPDEs. Also in [43], the new approach of generalized (G'/G)-expansion method to obtain exact traveling wave solutions of NLEEs is presented. In this paper we explain methods which are called the generalized tanh–coth and generalized (G'/G)-expansion methods are presented to look for travelling wave solutions of nonlinear evolution equations. Authors of [44], obtained exact solutions for the integrable sixth-order

Drinfeld–Sokolov–Satsuma–Hirota system by the generalized tanh–coth and generalized (G'/G)-expansion methods. Chand and Malik [14] have applied the (G'/G)-expansion method for finding the exact solutions of some nonlinear evolution equations. For further information in about these methods refer to Ref. [45–49]. The purpose of this paper is to obtain exact solutions of the (2+1)-dimensional Zoomeron, the Duffing and the SRLW equations and to determine the accuracy of the improved tan $(\Phi(\xi)/2)$-expansion method in solving these kind of problems. The paper is organized as follows: In "Description of improved tan $(\Phi(\xi)/2)$-expansion technique" section, we describe the improved tan $(\Phi(\xi)/2)$-expansion method. In "Illustrative Examples" section, we examine the (2+1)-dimensional Zoomeron, the Duffing and the SRLW equations respectively. Also a conclusion is given in "Conclusion" section. Finally some references are given at the end of this paper.

Description of Improved tan $(\Phi(\xi)/2)$-Expansion Technique

Step 1. We suppose that the given nonlinear fractional partial differential equation for $u(x, t)$ to be in the form

$$\mathcal{N}(u, u_x, u_t, u_{xx}, u_{tt}, ...) = 0, \tag{4}$$

which can be converted to an ODE

$$\mathcal{Q}(u, u', -\mu u', u''^2 u'', ...) = 0, \tag{5}$$

the transformation $\xi = x - \mu t$, is wave variable. Also, k, m and n are constants to be determined later.

Step 2. Suppose the traveling wave solution of Eq. (5) can be expressed as follows:

$$u(\xi) = S(\Phi) = \sum_{k=0}^{m} A_k \left[p + \tan\left(\frac{\Phi(\xi)}{2}\right)\right]^k + \sum_{k=1}^{m} B_k \left[p + \tan\left(\frac{\Phi(\xi)}{2}\right)\right]^{-k}, \tag{6}$$

where $A_k (0 \leq k \leq m)$ and $B_k (1 \leq k \leq m)$ are constants to be determined, such that $A_m \neq 0, B_m \neq 0$ and $\Phi = \Phi(\xi)$ satisfies the following ordinary differential equation:

$$\Phi'(\xi) = a \sin(\Phi(\xi)) + b \cos(\Phi(\xi)) + c. \tag{7}$$

We will consider the following special solutions of Eq. (7):

Family 1: **When** $a^2 + b^2 - c^2 < 0$ **and** $b - c \neq 0$,

$$\Phi(\xi) = -2 \arctan\left[-\frac{a}{b-c} + \frac{\sqrt{c^2 - b^2 - a^2}}{b-c} \tan\left(\frac{\sqrt{c^2 - b^2 - a^2}}{2}(\xi + C)\right)\right]. \tag{8}$$

Family 2: **When** $a^2 + b^2 - c^2 > 0$ **and** $b - c \neq 0$,

$$\Phi(\xi) = -2 \arctan\left[-\frac{a}{b-c} - \frac{\sqrt{b^2 + a^2 - c^2}}{b-c} \tanh\left(\frac{\sqrt{b^2 + a^2 - c^2}}{2}(\xi + C)\right)\right]. \tag{9}$$

Family 3: **When** $a^2 + b^2 - c^2 > 0$, $b \neq 0$ **and** $c = 0$,

$$\Phi(\xi) = 2\arctan\left[\frac{a}{b} + \frac{\sqrt{b^2 + a^2}}{b}\tanh\left(\frac{\sqrt{b^2 + a^2}}{2}(\xi + C)\right)\right]. \tag{10}$$

Family 4: **When** $a^2 + b^2 - c^2 < 0$, $c \neq 0$ **and** $b = 0$,

$$\Phi(\xi) = 2\arctan\left[-\frac{a}{c} + \frac{\sqrt{c^2 - a^2}}{c}\tan\left(\frac{\sqrt{c^2 - a^2}}{2}(\xi + C)\right)\right]. \tag{11}$$

Family 5: **When** $a^2 + b^2 - c^2 > 0$, $b - c \neq 0$ **and** $a = 0$,

$$\Phi(\xi) = 2\arctan\left[\sqrt{\frac{b + c}{b - c}}\tanh\left(\frac{\sqrt{b^2 - c^2}}{2}(\xi + C)\right)\right]. \tag{12}$$

Family 6: **When** $a = 0$ **and** $c = 0$,

$$\Phi(\xi) = \arctan\left[\frac{e^{2b(\xi+C)} - 1}{e^{2b(\xi+C)} + 1}, \frac{2e^{b(\xi+C)}}{e^{2b(\xi+C)} + 1}\right]. \tag{13}$$

Family 7: **When** $b = 0$ **and** $c = 0$,

$$\Phi(\xi) = \arctan\left[\frac{2e^{a(\xi+C)}}{e^{2a(\xi+C)} + 1}, \frac{e^{2a(\xi+C)} - 1}{e^{2a(\xi+C)} + 1}\right]. \tag{14}$$

Family 8: **When** $a^2 + b^2 = c^2$,

$$\Phi(\xi) = -2\arctan\left[\frac{(b + c)(a(\xi + C) + 2)}{a^2(\xi + C)}\right]. \tag{15}$$

Family 9: **When** $a = b = c = ka$,

$$\Phi(\xi) = 2\arctan\left[e^{ka(\xi+C)} - 1\right]. \tag{16}$$

Family 10: **When** $a = c = ka$ **and** $b = -ka$,

$$\Phi(\xi) = -2\arctan\left[\frac{e^{ka(\xi+C)}}{-1 + e^{ka(\xi+C)}}\right]. \tag{17}$$

Family 11: **When** $c = a$,

$$\Phi(\xi) = -2\arctan\left[\frac{(a + b)e^{b(\xi+C)} - 1}{(a - b)e^{b(\xi+C)} - 1}\right]. \tag{18}$$

Family 12: **When** $a = c$,

$$\Phi(\xi) = 2\arctan\left[\frac{(b + c)e^{b(\xi+C)} + 1}{(b - c)e^{b(\xi+C)} - 1}\right]. \tag{19}$$

Family 13: **When** $c = -a$,

$$\Phi(\xi) = 2\arctan\left[\frac{e^{b(\xi+C)} + b - a}{e^{b(\xi+C)} - b - a}\right]. \tag{20}$$

Family 14: **When** $b = -c$,

$$\Phi(\xi) = -2\arctan\left[\frac{ae^{a(\xi+C)}}{ce^{a(\xi+C)} - 1}\right]. \tag{21}$$

Family 15: **When** $b = 0$ **and** $a = c$,

$$\Phi(\xi) = -2\arctan\left[\frac{c(\xi + C) + 2}{c(\xi + C)}\right]. \tag{22}$$

Family 16: **When** $a = 0$ **and** $b = c$,

$$\Phi(\xi) = 2\arctan\left[c(\xi + C)\right]. \tag{23}$$

Family17: **When** $a = 0$ **and** $b = -c$,

$$\Phi(\xi) = -2\arctan\left[\frac{1}{c(\xi + C)}\right], \tag{24}$$

where $A_k(k = 0, 2, ..., m)$, $B_k(k = 1, 2, ..., m)$, a, b and c are constants to be determined later. But, the positive integer m can be determined by considering the homogeneous balance between the highest order derivatives and nonlinear terms appearing in Eq. (7). If m is not an integer, then a transformation formula should be used to overcome this difficulty.

Step 3. Substituting (6) into Eq. (5) with the value of m obtained in Step 2. Collecting the coefficients of $\tan\left(\frac{\Phi(\xi)}{2}\right)^k$, $\cot\left(\frac{\Phi(\xi)}{2}\right)^k$ $(k = 0, 1, 2, ...)$, then setting each coefficient to zero, we can get a set of over-determined partial differential equations for A_0, $A_k(k = 1, 2, ..., m)$, $B_k(k = 1, 2, ..., m)$a, b, c and p with the aid of symbolic computation Maple 13.

Step 4. Solving the algebraic equations in Step 3, then substituting $A_0, A_1, B_1, ..., A_m, B_m$, μ, p in (6).

Illustrative Examples

In this section, we present several examples to illustrate the applicability of improved $\tan(\Phi(\xi)/2)$-expansion method to solve nonlinear fractional partial differential equations introduced in "Introduction" section.

The (2+1)-Dimensional Zoomeron Equation

Consider the nonlinear (2+1)-dimensional Zoomeron equation as follows

$$\left(\frac{u_{xy}}{u}\right)_{tt} - \left(\frac{u_{xy}}{u}\right)_{xx} + 2(u^2)_{xt} = 0, \tag{25}$$

by using the wave variable $\xi = \mu(x + y - mt)$ reduce it to an ODE as follows

$$\mu^2(m^2 - 1)u''^3 + Nu = 0, \tag{26}$$

where obtained by twice integrating and neglecting the constant of integration. Balancing the u'' and u^3 by using homogenous principal, we have

$$M + 2 = 3M, \qquad \Rightarrow M = 1. \tag{27}$$

Then the trail solution is

$$u(\xi) = A_0 + A_1\left[p + \tan\left(\frac{\Phi(\xi)}{2}\right)\right] + B_1\left[p + \tan\left(\frac{\Phi(\xi)}{2}\right)\right]^{-1}. \tag{28}$$

Substituting (28) and (7) into Eq. (26) and by using the well-known Maple software, we obtain the following sets of non-trivial solutions

Set I:

$$m = m,\quad \mu = \pm\frac{2(b-c)B_1}{a^2+b^2-c^2}\sqrt{\frac{m}{m^2-1}},\ A_0 = 0,\quad A_1 = 0,\ B_1 = B_1,\ p = -\frac{a}{b-c}, \tag{29}$$

$$N = \frac{2m(b-c)^2B_1^2}{a^2+b^2-c^2},\quad u(\xi) = B_1\left[-\frac{a}{b-c} + \tan\left(\frac{\Phi(\xi)}{2}\right)\right]^{-1}, \tag{30}$$

where a, b and c are arbitrary constants. By using of the (30) and **Family 1** and **4** respectively can be written as

$$\begin{aligned} u_1(\xi) &= -\frac{(b-c)B_1}{\sqrt{c^2-b^2-a^2}}\cot\left(\frac{\sqrt{c^2-b^2-a^2}}{2}(\xi+C)\right),\\ u_2(\xi) &= \frac{cB_1}{\sqrt{c^2-a^2}}\cot\left(\frac{\sqrt{c^2-a^2}}{2}(\xi+C)\right). \end{aligned} \tag{31}$$

By using of the (30) and **Family 2, 3** and respectively get

$$\begin{aligned} u_3(\xi) &= \frac{(b-c)B_1}{\sqrt{a^2+b^2-c^2}}\coth\left(\frac{\sqrt{a^2+b^2-c^2}}{2}(\xi+C)\right),\\ u_4(\xi) &= \frac{bB_1}{\sqrt{a^2+b^2}}\coth\left(\frac{\sqrt{a^2+b^2}}{2}(\xi+C)\right),\\ u_5(\xi) &= \pm B_1\sqrt{\frac{b-c}{b+c}}\coth\left(\frac{\sqrt{b^2-c^2}}{2}(\xi+C)\right). \end{aligned} \tag{32}$$

By using of the (30) and **Family 6** we get

$$u_6(\xi) = B_1\cot\left(\frac{1}{2}\arctan\left[\frac{e^{2b(\xi+C)}-1}{e^{2b(\xi+C)}+1}, \frac{2e^{b(\xi+C)}}{e^{2b(\xi+C)}+1}\right]\right). \tag{33}$$

By using of the (30) and **Family 10, 11, 12, 13** and **14** respectively give

$$\begin{aligned} &u_7(\xi) = B_1\left[\frac{1}{2} - \frac{e^{ka(\xi+C)}}{\left[e^{ka(\xi+C)}-1\right]}\right]^{-1},\quad u_8(\xi) = B_1\left[\frac{a}{a-b} - \frac{(a+b)e^{b(\xi+C)}-1}{(a-b)e^{b(\xi+C)}-1}\right]^{-1},\\ &u_9(\xi) = B_1\left[\frac{c}{b-c} + \frac{(b+c)e^{b(\xi+C)}+1}{(b-c)e^{b(\xi+C)}-1}\right]^{-1},\ u_{10}(\xi) = B_1\left[-\frac{a}{a+b} + \frac{e^{b(\xi+C)}+b-a}{e^{b(\xi+C)}-b-a}\right]^{-1},\\ &u_{11}(\xi) = B_1\left[\frac{a}{2c} - \frac{ae^{a(\xi+C)}}{ce^{a(\xi+C)}-1}\right]^{-1}. \end{aligned} \tag{34}$$

By using of the (29) and **Family 15, 16** and **17** respectively give

$$u_{12}(\xi) = -\frac{cB_1}{2}(\xi + C), \quad u_{13}(\xi) = \frac{B_1}{c(\xi + C)}, \quad u_{14}(\xi) = -B_1 c(\xi + C), \tag{35}$$

where $\xi = \pm\frac{2(b-c)B_1}{a^2+b^2-c^2}\sqrt{\frac{m}{m^2-1}}(x + y - mt)$.
Set II:

$$\mu = \mu, \quad A_0 = \pm\frac{\mu(a + p(b - c))}{2}\sqrt{\frac{m^2 - 1}{m}}, A_1 = 0,$$

$$B_1 = \mp\frac{\mu(p^2(b - c) - (b + c) + 2ap)}{2}\sqrt{\frac{m^2 - 1}{m}}, \tag{36}$$

$$m = m, p = p, N = \frac{\mu^2(m^2 - 1)}{2}, u(\xi) = A_0 + B_1\left[p + \tan\left(\frac{\Phi(\xi)}{2}\right)\right]^{-1}, \tag{37}$$

where a, b and c are arbitrary constants. By using of the (37) and **Family 1** and **4** respectively get to

$$u_{15}(\xi) = \pm\frac{\mu}{2}\sqrt{\frac{m^2 - 1}{m}}\left\{(a + p(b - c)) - (p^2(b - c) - (b + c) + 2ap)\right.$$
$$\left.\times\left[p + \frac{a}{b - c} - \frac{\sqrt{c^2 - b^2 - a^2}}{b - c}\tan\left(\frac{\sqrt{c^2 - b^2 - a^2}}{2}(\xi + C)\right)\right]^{-1}\right\},$$
$$u_{16}(\xi) = \pm\frac{\mu}{2}\sqrt{\frac{m^2 - 1}{m}}\left\{(a - pc) - (-p^2c - c + 2ap)\right.$$
$$\left.\times\left[p - \frac{a}{c} + \frac{\sqrt{c^2 - a^2}}{b}\tanh\left(\frac{\sqrt{c^2 - a^2}}{2}(\xi + C)\right)\right]^{-1}\right\}. \tag{38}$$

By using of the (37) and **Family 2, 3** and **5** respectively can be written as

$$u_{17}(\xi) = \pm\frac{\mu}{2}\sqrt{\frac{m^2 - 1}{m}}\left\{(a + p(b - c)) - (p^2(b - c) - (b + c) + 2ap)\right.$$
$$\left.\times\left[p + \frac{a}{b - c} + \frac{\sqrt{a^2 + b^2 - c^2}}{b - c}\tanh\left(\frac{\sqrt{a^2 + b^2 - c^2}}{2}(\xi + C)\right)\right]^{-1}\right\},$$
$$u_{18}(\xi) = \pm\frac{\mu}{2}\sqrt{\frac{m^2 - 1}{m}}\left\{(a + pb) - (p^2b - b + 2ap)\right.$$
$$\left.\times\left[p + \frac{a}{b} + \frac{\sqrt{a^2 + b^2}}{b}\tanh\left(\frac{\sqrt{a^2 + b^2}}{2}(\xi + C)\right)\right]^{-1}\right\},$$

$$u_{19}(\xi) = \pm\frac{\mu}{2}\sqrt{\frac{m^2-1}{m}}\left\{p(b-c) - (p^2(b-c)-(b+c))\right.$$
$$\left.\times\left[p + \frac{\sqrt{b^2-c^2}}{b-c}\tanh\left(\frac{\sqrt{c^2-a^2}}{2}(\xi+C)\right)\right]^{-1}\right\}. \quad (39)$$

By using of the (30) and **Family 6, 7** and respectively can be written as

$$u_{20}(\xi) = \pm\frac{\mu}{2}\sqrt{\frac{m^2-1}{m}}$$
$$\times\left\{pb - b(p^2-1)\left[p + \tan\left(\frac{1}{2}\arctan\left[\frac{e^{2b(\xi+C)}-1}{e^{2b(\xi+C)}+1}, \frac{2e^{b(\xi+C)}}{e^{2b(\xi+C)}+1}\right]\right)\right]^{-1}\right\},$$
$$u_{21}(\xi) = \pm\frac{\mu}{2}\sqrt{\frac{m^2-1}{m}}\left\{a-2ap\left[p + \tan\left(\frac{1}{2}\arctan\left[\frac{2e^{a(\xi+C)}}{e^{2a(\xi+C)}+1}, \frac{e^{2a(\xi+C)}-1}{e^{2a(\xi+C)}+1}\right]\right)\right]^{-1}\right\},$$
$$u_{22}(\xi) = \pm\frac{\mu}{2}\sqrt{\frac{m^2-1}{m}}\left\{(\sqrt{c^2-b^2}+p(b-c)) - (p^2(b-c)-(b+c)\right.$$
$$\left.+2p\sqrt{c^2-b^2})\left[p + \frac{2+a\xi}{(b-c)\xi}\right]^{-1}\right\}. \quad (40)$$

By using of the (30) and **Family 9, 10, 11, 12, 13** and **14** respectively can be written as

$$u_{23}(\xi) = \pm\frac{\mu}{2}\sqrt{\frac{m^2-1}{m}}\left\{ak - 2ak(p-1)\left[p-1+e^{ak(\xi+C)}\right]^{-1}\right\},$$
$$u_{24}(\xi) = \pm\frac{\mu}{2}\sqrt{\frac{m^2-1}{m}}\left\{ak(1-2p) + 2akp(p-1)\left[p - \frac{e^{ak(\xi+C)}}{-1+e^{ak(\xi+C)}}\right]^{-1}\right\},$$
$$u_{25}(\xi) = \pm\frac{\mu}{2}\sqrt{\frac{m^2-1}{m}}\left\{(a+p(b-a)) - (p^2(b-a)-(a+b)+2ap)\right.$$
$$\left.\times\left[p - \frac{(a+b)e^{b(\xi+C)}-1}{(a-b)e^{b(\xi+C)}-1}\right]^{-1}\right\},$$
$$u_{26}(\xi) = \pm\frac{\mu}{2}\sqrt{\frac{m^2-1}{m}}\left\{(c+p(b-c)) - (p^2(b-c)-(b+c)+2cp)\right.$$
$$\left.\times\left[p + \frac{(b+c)e^{b(\xi+C)}+1}{(b-c)e^{b(\xi+C)}-1}\right]^{-1}\right\},$$
$$u_{27}(\xi) = \pm\frac{\mu}{2}\sqrt{\frac{m^2-1}{m}}\left\{(-a+p(b+a)) - (p^2(b+a)-(b-a)+2cp)\right.$$

$$\times\left[p+\frac{e^{b(\xi+C)}+b-a}{e^{b(\xi+C)}-b-a}\right]^{-1}\Bigg\},$$

$$u_{28}(\xi)=\pm\frac{\mu}{2}\sqrt{\frac{m^2-1}{m}}\left\{(a-2pc)-(-2cp^2+2ap)\left[p-\frac{ae^{a(\xi+C)}}{ce^{a(\xi+C)}-1}\right]^{-1}\right\}. \quad (41)$$

By using of the (29) and **Family 15, 16** and **17** respectively can be written as

$$u_{29}(\xi)=\pm\frac{\mu}{2}\sqrt{\frac{m^2-1}{m}}\left\{(c-pc)-(-cp^2-c+2cp)\left[p-\frac{2+c(\xi+C)}{c(\xi+C)}\right]^{-1}\right\},$$

$$u_{30}(\xi)=\pm\frac{\mu}{2}\sqrt{\frac{m^2-1}{m}}\left\{2c\left[p+c(\xi+C)\right]^{-1}\right\},$$

$$u_{31}(\xi)=\pm\frac{\mu}{2}\sqrt{\frac{m^2-1}{m}}\left\{-2pc+2pc\left[p-\frac{1}{c(\xi+C)}\right]^{-1}\right\}, \quad (42)$$

where $\xi=\mu(x+y-mt)$.
Set III:

$$m=m,\quad \mu=\pm\frac{2(b-c)B_1}{a^2+b^2-c^2}\sqrt{\frac{m}{m^2-1}},\quad A_0=0,\quad A_1=-\frac{(b-c)^2B_1}{a^2+b^2-c^2},\quad B_1=B_1,$$
$$p=-\frac{a}{b-c}, \quad (43)$$

$$N=-\frac{4m(b-c)^2B_1^2}{a^2+b^2-c^2},\qquad u(\xi)=A_1\left[-\frac{a}{b-c}+\tan\left(\frac{\Phi(\xi)}{2}\right)\right]$$
$$+B_1\left[-\frac{a}{b-c}+\tan\left(\frac{\Phi(\xi)}{2}\right)\right]^{-1}, \quad (44)$$

where a, b and c are arbitrary constants. By using of the (44) and **Family 1** and **4** respectively get to

$$u_{32}(\xi)=\frac{(b-c)B_1}{\sqrt{c^2-b^2-a^2}}\left[\tan\left(\frac{\sqrt{c^2-b^2-a^2}}{2}(\xi+C)\right)-\cot\left(\frac{\sqrt{c^2-b^2-a^2}}{2}(\xi+C)\right)\right],$$

$$u_{33}(\xi)=\frac{-cB_1}{\sqrt{c^2-a^2}}\left[\tan\left(\frac{\sqrt{c^2-a^2}}{2}(\xi+C)\right)-\cot\left(\frac{\sqrt{c^2-a^2}}{2}(\xi+C)\right)\right]. \quad (45)$$

By using of the (44) and **Family 2, 3** and **5** can be written as

$$u_{34}(\xi)=\frac{(b-c)B_1}{\sqrt{a^2+b^2-c^2}}\left[\tanh\left(\frac{\sqrt{a^2+b^2-c^2}}{2}(\xi+C)\right)+\coth\left(\frac{\sqrt{a^2+b^2-c^2}}{2}(\xi+C)\right)\right],$$

$$u_{35}(\xi)=\frac{bB_1}{\sqrt{a^2+b^2}}\left[\tanh\left(\frac{\sqrt{a^2+b^2}}{2}(\xi+C)\right)+\coth\left(\frac{\sqrt{a^2+b^2}}{2}(\xi+C)\right)\right],$$

$$u_{36}(\xi)=-B_1\sqrt{\frac{b-c}{b+c}}\left[\tanh\left(\frac{\sqrt{b^2-c^2}}{2}(\xi+C)\right)-\coth\left(\frac{\sqrt{b^2-c^2}}{2}(\xi+C)\right)\right]. \quad (46)$$

By using of the (44) and **Family 6** we get

$$u_{37}(\xi) = B_1\left[\cot\left(\frac{1}{2}\arctan\left[\frac{e^{2b(\xi+C)}-1}{e^{2b(\xi+C)}+1}, \frac{2e^{b(\xi+C)}}{e^{2b(\xi+C)}+1}\right]\right) - \tan\left(\frac{1}{2}\arctan\left[\frac{e^{2b(\xi+C)}-1}{e^{2b(\xi+C)}+1}, \frac{2e^{b(\xi+C)}}{e^{2b(\xi+C)}+1}\right]\right)\right]. \quad (47)$$

By using of the (44) and **Family 10, 11, 12, 13** and **14** respectively can be written as

$$\begin{aligned}
u_{38}(\xi) &= -4B_1\left[\frac{1}{2}-\frac{e^{ka(\xi+C)}}{e^{ka(\xi+C)}-1}\right]+B_1\left[\frac{1}{2}-\frac{e^{ka(\xi+C)}}{e^{ka(\xi+C)}-1}\right]^{-1},\\
u_{39}(\xi) &= -\frac{(b-a)^2}{b^2}B_1\left[\frac{a}{a-b}-\frac{(a+b)e^{b(\xi+C)}-1}{(a-b)e^{b(\xi+C)}-1}\right]+B_1\left[\frac{a}{a-b}-\frac{(a+b)e^{b(\xi+C)}-1}{(a-b)e^{b(\xi+C)}-1}\right]^{-1},\\
u_{40}(\xi) &= -\frac{(b-c)^2}{b^2}B_1\left[\frac{c}{b-c}+\frac{(b+c)e^{b(\xi+C)}+1}{(b-c)e^{b(\xi+C)}-1}\right]+B_1\left[\frac{c}{b-c}+\frac{(b+c)e^{b(\xi+C)}+1}{(b-c)e^{b(\xi+C)}-1}\right]^{-1},\\
u_{41}(\xi) &= -\frac{(b+a)^2}{b^2}B_1\left[-\frac{a}{a+b}+\frac{e^{b(\xi+C)}+b-a}{e^{b(\xi+C)}-b-a}\right]+B_1\left[-\frac{a}{a+b}+\frac{e^{b(\xi+C)}+b-a}{e^{b(\xi+C)}-b-a}\right]^{-1},\\
u_{42}(\xi) &= -\frac{4c^2}{b^2}B_1\left[\frac{a}{2c}-\frac{ae^{a(\xi+C)}}{ce^{a(\xi+C)}-1}\right]+B_1\left[\frac{a}{2c}-\frac{ae^{a(\xi+C)}}{ce^{a(\xi+C)}-1}\right]^{-1}, \qquad (48)
\end{aligned}$$

where $\xi = \pm\frac{2(b-c)B_1}{a^2+b^2-c^2}\sqrt{\frac{m}{m^2-1}}(x+y-mt)$.

Set IV:

$$\mu = \pm\frac{2A_1}{b-c}\sqrt{\frac{m^2-1}{m}}, \quad A_0 = -\frac{A_1(a+p(b-c))}{b-c}, \quad A_1 = A_1, \quad B_1 = 0, \qquad (49)$$

$$m = m, \quad p = p, \quad N = \frac{2(a^2+b^2-c^2)mA_1^2}{(b-c)^2}, \quad u(\xi) = A_0 + A_1 p + A_1\tan\left(\frac{\Phi(\xi)}{2}\right), \qquad (50)$$

where a, b and c are arbitrary constants. By using of the (50) and **Family 1** and **4** respectively get to

$$\begin{aligned}
u_{43}(\xi) &= -\frac{A_1\sqrt{c^2-b^2-a^2}}{b-c}\tan\left(\frac{\sqrt{c^2-b^2-a^2}}{2}(\xi+C)\right),\\
u_{44}(\xi) &= \frac{A_1\sqrt{c^2-a^2}}{c}\tan\left(\frac{\sqrt{c^2-a^2}}{2}(\xi+C)\right). \qquad (51)
\end{aligned}$$

By using of the (50) and **Family 2, 3** and **Family 5** respectively can be written

$$
\begin{aligned}
u_{45}(\xi) &= \frac{A_1\sqrt{a^2+b^2-c^2}}{b-c}\tanh\left(\frac{\sqrt{a^2+b^2-c^2}}{2}(\xi+C)\right).\\
u_{46}(\xi) &= \frac{A_1\sqrt{a^2+b^2}}{b}\tanh\left(\frac{\sqrt{a^2+b^2}}{2}(\xi+C)\right),\\
u_{47}(\xi) &= \frac{A_1\sqrt{b^2-c^2}}{b-c}\tanh\left(\frac{\sqrt{b^2-c^2}}{2}(\xi+C)\right).
\end{aligned}
\tag{52}
$$

By using of the (50) and **Family 6** and **8** respectively get to

$$
\begin{aligned}
u_{48}(\xi) &= A_1\tan\left(\frac{1}{2}\arctan\left[\frac{e^{2b(\xi+C)}-1}{e^{2b(\xi+C)}+1}, \frac{2e^{b(\xi+C)}}{e^{2b(\xi+C)}+1}\right]\right),\\
u_{49}(\xi) &= -\frac{A_1}{b-c}\left[2-\sqrt{c^2-b^2}(1-(\xi+C))\right].
\end{aligned}
\tag{53}
$$

By using of the (50) and **Family 10, 11, 12, 13** and **14** respectively can be written as

$$
\begin{aligned}
&u_{50}(\xi) = \frac{1}{2}-\frac{A_1e^{ak(\xi+C)}}{-1+e^{ak(\xi+C)}}, \quad u_{51}(\xi) = \frac{aA_1}{a-b}-A_1\frac{(a+b)e^{b(\xi+C)}-1}{(a-b)e^{b(\xi+C)}-1}.\\
&u_{52}(\xi) = -\frac{cA_1}{b-c}+A_1\frac{(b+c)e^{b(\xi+C)}+1}{(b-c)e^{b(\xi+C)}-1}, u_{53}(\xi) = -\frac{aA_1}{b+a}+A_1\frac{e^{b(\xi+C)}+b-a}{e^{b(\xi+C)}-b-a},\\
&u_{54}(\xi) = \frac{aA_1}{2c}-A_1\frac{ae^{a(\xi+C)}}{ce^{a(\xi+C)}-1}.
\end{aligned}
\tag{54}
$$

By using of the (49) and **Family 15** we get

$$
u_{55}(\xi) = -\frac{2A_1}{c(\xi+C)}, \tag{55}
$$

where $\xi = \pm\frac{2A_1}{b-c}\sqrt{\frac{m^2-1}{m}}(x+y-mt)$.

The Duffing Equation

Consider the nonlinear Duffing equation as follows

$$
u_{tt}+\alpha u+\beta u^3 = 0, \tag{56}
$$

by using the wave variable $\xi = \mu(x+y-mt)$ reduce it to an ODE as follows

$$
-\mu^2m^2u''^3 = 0. \tag{57}
$$

Balancing the u'' and u^3 by using homogenous principal, we have

$$
M+2 = 3M, \qquad \Rightarrow M = 1. \tag{58}
$$

Then the trail solution is

$$u(\xi)=A_0+A_1\left[p+\tan\left(\frac{\Phi(\xi)}{2}\right)\right]+B_1\left[p+\tan\left(\frac{\Phi(\xi)}{2}\right)\right]^{-1}. \tag{59}$$

Substituting (59) and (7) into Eq. (57) and by using the well-known Maple software, we obtain the following sets of non-trivial solutions

Set I:

$$\mu=\mu,\quad m=\pm\frac{1}{\mu}\sqrt{\frac{-2\alpha}{a^2+b^2-c^2}},\quad A_0=0,\quad A_1=0,$$
$$B_1=\pm\frac{1}{b-c}\sqrt{\frac{-\alpha}{\beta}(a^2+b^2-c^2)}, \tag{60}$$
$$p=-\frac{a}{b-c},\qquad u(\xi)=B_1\left[-\frac{a}{b-c}+\tan\left(\frac{\Phi(\xi)}{2}\right)\right]^{-1}, \tag{61}$$

where a, b and c are arbitrary constants. By using of the (61) and **Family 1, 4** respectively get to

$$u_1(\xi)=\mp\sqrt{\frac{\alpha}{\beta}}\cot\left(\frac{\sqrt{c^2-b^2-a^2}}{2}(\xi+C)\right),\quad u_2(\xi)=\pm\sqrt{\frac{\alpha}{\beta}}\cot\left(\frac{\sqrt{c^2-a^2}}{2}(\xi+C)\right). \tag{62}$$

By using of the (61) and **Family 2, 3** and **5** respectively get to

$$u_3(\xi)=\pm\sqrt{\frac{\alpha}{\beta}}\coth\left(\frac{\sqrt{a^2+b^2-c^2}}{2}(\xi+C)\right),$$
$$u_4(\xi)=\pm\sqrt{\frac{\alpha}{\beta}}\coth\left(\frac{\sqrt{a^2+b^2}}{2}(\xi+C)\right),$$
$$u_5(\xi)=\pm\sqrt{\frac{\alpha}{\beta}}\coth\left(\frac{\sqrt{b^2-c^2}}{2}(\xi+C)\right). \tag{63}$$

By using of the (61) and **Family 6** we get

$$u_6(\xi)=\pm\sqrt{\frac{-\alpha}{\beta}}\cot\left(\frac{1}{2}\arctan\left[\frac{e^{2b(\xi+C)}-1}{e^{2b(\xi+C)}+1},\ \frac{2e^{b(\xi+C)}}{e^{2b(\xi+C)}+1}\right]\right). \tag{64}$$

By using of the (61) and **Family 10, 11, 12, 13** and **14** respectively can be written as

$$u_7(\xi)=\mp\sqrt{\frac{-\alpha}{\beta}}\left[\frac{1}{2}-\frac{e^{ka(\xi+C)}}{\left[e^{ka(\xi+C)}-1\right]}\right]^{-1},$$
$$u_8(\xi)=\pm\frac{b}{b-a}\sqrt{\frac{-\alpha}{\beta}}\left[\frac{a}{a-b}-\frac{(a+b)e^{b(\xi+C)}-1}{(a-b)e^{b(\xi+C)}-1}\right]^{-1}, \tag{65}$$
$$u_9(\xi)=\mp\frac{b}{b-c}\sqrt{\frac{-\alpha}{\beta}}\left[\frac{c}{b-c}-\frac{(b+c)e^{b(\xi+C)}+1}{(b-c)e^{b(\xi+C)}-1}\right]^{-1},$$

$$u_{10}(\xi) = \mp\frac{ab}{b+a}\sqrt{\frac{-\alpha}{\beta}}\left[\frac{a}{a+b} - \frac{e^{b(\xi+C)}+b-a}{e^{b(\xi+C)}-b-a}\right]^{-1},$$

$$u_{11}(\xi) = \mp\frac{a^2}{2c}\sqrt{\frac{-\alpha}{\beta}}\left[\frac{a}{2c} - \frac{ae^{a(\xi+C)}}{ce^{a(\xi+C)}-1}\right]^{-1},$$

where $\xi = \mu\left(x + y \mp \frac{1}{\mu}\sqrt{\frac{-2\alpha}{a^2+b^2-c^2}}t\right)$.

Set II:

$$m = \pm\frac{1}{\mu}\sqrt{\frac{-2\alpha}{a^2+b^2-c^2}}, \quad A_0 = \pm\sqrt{-\frac{\alpha}{3\beta}}, \quad A_1 = 0,$$

$$B_1 = \pm\frac{2}{3(b-c)}\sqrt{\frac{-\alpha}{\beta}(a^2+b^2-c^2)}, \tag{66}$$

$$\mu = \mu, \quad p = -\frac{a}{b-c} + \frac{\sqrt{a^2+b^2-c^2}}{\sqrt{3}(b-c)}, \quad u(\xi) = A_0 + B_1\left[p + \tan\left(\frac{\Phi(\xi)}{2}\right)\right]^{-1}, \tag{67}$$

where a, b and c are arbitrary constants. By using of the (67) and **Family 1, 4** respectively get to

$$u_{12}(\xi) = \pm\sqrt{-\frac{\alpha}{3\beta}} \pm \frac{2}{3}\sqrt{\frac{-\alpha}{\beta}(a^2+b^2-c^2)}\left[\frac{\sqrt{a^2+b^2-c^2}}{\sqrt{3}} - \sqrt{c^2-b^2-a^2}\tan\left(\frac{\sqrt{c^2-b^2-a^2}}{2}(\xi+C)\right)\right]^{-1},$$

$$u_{13}(\xi) = \pm\sqrt{-\frac{\alpha}{3\beta}} \pm \frac{2}{3}\sqrt{\frac{-\alpha}{\beta}(a^2-c^2)}\left[\frac{\sqrt{a^2-c^2}}{\sqrt{3}} - \sqrt{c^2-a^2}\tan\left(\frac{\sqrt{c^2-a^2}}{2}(\xi+C)\right)\right]^{-1}. \tag{68}$$

By using of the (67) and **Family 2, 3** and **Family 5** we get

$$u_{14}(\xi) = \pm\sqrt{-\frac{\alpha}{3\beta}} \pm \frac{2}{3}\sqrt{\frac{-\alpha}{\beta}}\left[\frac{1}{\sqrt{3}} + \tanh\left(\frac{\sqrt{a^2+b^2-c^2}}{2}(\xi+C)\right)\right]^{-1},$$

$$u_{15}(\xi) = \pm\sqrt{-\frac{\alpha}{3\beta}} \pm \frac{2}{3}\sqrt{\frac{-\alpha}{\beta}}\left[\frac{1}{\sqrt{3}} + \tanh\left(\frac{\sqrt{a^2+b^2}}{2}(\xi+C)\right)\right]^{-1},$$

$$u_{16}(\xi) = \pm\sqrt{-\frac{\alpha}{3\beta}} \pm \frac{2}{3}\sqrt{\frac{-\alpha}{\beta}}\left[\frac{1}{\sqrt{3}} + \tanh\left(\frac{\sqrt{b^2-c^2}}{2}(\xi+C)\right)\right]^{-1}. \tag{69}$$

By using of the (61) and **Family 6** we get

$$u_{17}(\xi) = \pm\sqrt{-\frac{\alpha}{3\beta}} \pm \frac{2}{3}\sqrt{\frac{-\alpha}{\beta}}\left[\frac{1}{\sqrt{3}} + \tan\left(\frac{1}{2}\arctan\left[\frac{e^{2b(\xi+C)}-1}{e^{2b(\xi+C)}+1}, \frac{2e^{b(\xi+C)}}{e^{2b(\xi+C)}+1}\right]\right)\right]^{-1}. \tag{70}$$

By using of the (67) and **Family 10, 11, 12, 13** and **14** respectively can be written as

$$u_{18}(\xi)=\pm\sqrt{-\frac{\alpha}{3\beta}}\mp\frac{1}{3}\sqrt{\frac{-\alpha}{\beta}}\left[\frac{1}{2}-\frac{1}{2\sqrt{3}}-\frac{e^{ak(\xi+C)}}{-1+e^{ak(\xi+C)}}\right]^{-1},$$

$$u_{19}(\xi)=\pm\sqrt{-\frac{\alpha}{3\beta}}\pm\frac{2b}{3}\sqrt{\frac{-\alpha}{\beta}}\left[-a+\frac{b}{\sqrt{3}}-\frac{(a+b)e^{b(\xi+C)}-1}{(a-b)e^{b(\xi+C)}-1}\right]^{-1},$$

$$u_{20}(\xi)=\pm\sqrt{-\frac{\alpha}{3\beta}}\pm\frac{2b}{3}\sqrt{\frac{-\alpha}{\beta}}\left[-c+\frac{b}{\sqrt{3}}+\frac{(b+c)e^{b(\xi+C)}+1}{(b-c)e^{b(\xi+C)}-1}\right]^{-1},$$

$$u_{21}(\xi)=\pm\sqrt{-\frac{\alpha}{3\beta}}\pm\frac{2b}{3}\sqrt{\frac{-\alpha}{\beta}}\left[-a+\frac{b}{\sqrt{3}}+\frac{e^{b(\xi+C)}+b-a}{e^{b(\xi+C)}-b-a}\right]^{-1},$$

$$u_{22}(\xi)=\pm\sqrt{-\frac{\alpha}{3\beta}}\mp\frac{a}{3}\sqrt{\frac{-\alpha}{\beta}}\left[\frac{a}{2}-\frac{a}{2\sqrt{3}}-\frac{ae^{a(\xi+C)}}{ce^{a(\xi+C)}-1}\right]^{-1}, \tag{71}$$

where $\xi=\mu\left(x+y\mp\frac{1}{\mu}\sqrt{\frac{-2\alpha}{a^2+b^2-c^2}}t\right)$.

Set III:

$$m=\pm\frac{1}{\mu}\sqrt{\frac{-\alpha}{2(a^2+b^2-c^2)}},\qquad A_1=\pm\frac{b-c}{2}\sqrt{\frac{-\alpha}{\beta(a^2+b^2-c^2)}},$$
$$B_1=\mp\frac{1}{2(b-c)}\sqrt{\frac{-\alpha}{\beta(a^2+b^2-c^2)}}, \tag{72}$$
$$A_0=0,\quad \mu=\mu,\quad p=-\frac{a}{b-c},\quad u(\xi)=A_1\left[p+\tan\left(\frac{\Phi(\xi)}{2}\right)\right]$$
$$+B_1\left[p+\tan\left(\frac{\Phi(\xi)}{2}\right)\right]^{-1}, \tag{73}$$

where a, b and c are arbitrary constants. By using of the (73) and **Family 1, 4** respectively get to

$$u_{23}(\xi)=\mp\frac{1}{2}\sqrt{\frac{-\alpha}{\beta(a^2+b^2-c^2)}}\left[\sqrt{c^2-b^2-a^2}\tan\left(\frac{\sqrt{c^2-b^2-a^2}}{2}(\xi+C)\right)\right.$$
$$\left.-\frac{1}{\sqrt{c^2-b^2-a^2}}\cot\left(\frac{\sqrt{c^2-b^2-a^2}}{2}(\xi+C)\right)\right],$$
$$u_{24}(\xi)=\mp\frac{1}{2}\sqrt{\frac{-\alpha}{\beta(a^2-c^2)}}\left[\sqrt{c^2-a^2}\tan\left(\frac{\sqrt{c^2-a^2}}{2}(\xi+C)\right)\right.$$
$$\left.-\frac{1}{\sqrt{c^2-a^2}}\cot\left(\frac{\sqrt{c^2-a^2}}{2}(\xi+C)\right)\right]. \tag{74}$$

By using of the (67) and **Family 2, 3** and **Family 5** we get

$$\begin{aligned} u_{25}(\xi) &= \pm\frac{1}{2}\sqrt{\frac{-\alpha}{\beta(a^2+b^2-c^2)}}\left[\sqrt{a^2+b^2-c^2}\tanh\left(\frac{\sqrt{a^2+b^2-c^2}}{2}(\xi+C)\right)\right.\\ &\quad\left.-\frac{1}{\sqrt{a^2+b^2-c^2}}\coth\left(\frac{\sqrt{a^2+b^2-c^2}}{2}(\xi+C)\right)\right],\\ u_{26}(\xi) &= \pm\frac{1}{2}\sqrt{\frac{-\alpha}{\beta(a^2+b^2)}}\left[\sqrt{a^2+b^2}\tanh\left(\frac{\sqrt{a^2+b^2}}{2}(\xi+C)\right)\right.\\ &\quad\left.-\frac{1}{\sqrt{a^2+b^2}}\coth\left(\frac{\sqrt{a^2+b^2}}{2}(\xi+C)\right)\right],\\ u_{27}(\xi) &= \pm\frac{1}{2}\sqrt{\frac{-\alpha}{\beta(b^2-c^2)}}\left[\sqrt{b^2-c^2}\tanh\left(\frac{\sqrt{b^2-c^2}}{2}(\xi+C)\right)\right.\\ &\quad\left.-\frac{1}{\sqrt{b^2-c^2}}\coth\left(\frac{\sqrt{b^2-c^2}}{2}(\xi+C)\right)\right], \end{aligned} \tag{75}$$

By using of the (61) and **Family 6** we get

$$\begin{aligned} u_{28}(\xi) &= \pm\sqrt{-\frac{\alpha}{4\beta}}\tan\left(\frac{1}{2}\arctan\left[\frac{e^{2b(\xi+C)}-1}{e^{2b(\xi+C)}+1},\ \frac{2e^{b(\xi+C)}}{e^{2b(\xi+C)}+1}\right]\right)\\ &\quad\mp\frac{1}{b^2}\sqrt{-\frac{\alpha}{4\beta}}\cot\left(\frac{1}{2}\arctan\left[\frac{e^{2b(\xi+C)}-1}{e^{2b(\xi+C)}+1},\ \frac{2e^{b(\xi+C)}}{e^{2b(\xi+C)}+1}\right]\right). \end{aligned} \tag{76}$$

By using of the (67) and **Family 10, 11, 12, 13** and **14** respectively can be written as

$$\begin{aligned} u_{29}(\xi) &= \mp\sqrt{-\frac{\alpha}{\beta}}\left[\frac{1}{2}-\frac{e^{ak(\xi+C)}}{-1+e^{ak(\xi+C)}}\right]\pm\frac{1}{4k^2a^2}\sqrt{-\frac{\alpha}{\beta}}\left[\frac{1}{2}-\frac{e^{ak(\xi+C)}}{-1+e^{ak(\xi+C)}}\right]^{-1},\\ u_{30}(\xi) &= \pm\frac{1}{2b}\sqrt{-\frac{\alpha}{\beta}}\left[-a-\frac{(a+b)e^{b(\xi+C)}-1}{(a-b)e^{b(\xi+C)}-1}\right]\mp\frac{1}{2b}\sqrt{-\frac{\alpha}{\beta}}\left[-a-\frac{(a+b)e^{b(\xi+C)}-1}{(a-b)e^{b(\xi+C)}-1}\right]^{-1},\\ u_{31}(\xi) &= \pm\frac{1}{2b}\sqrt{-\frac{\alpha}{\beta}}\left[-c-\frac{(b+c)e^{b(\xi+C)}+1}{(b-c)e^{b(\xi+C)}-1}\right]\mp\frac{1}{2b}\sqrt{-\frac{\alpha}{\beta}}\left[-c-\frac{(b+c)e^{b(\xi+C)}+1}{(b-c)e^{b(\xi+C)}-1}\right]^{-1},\\ u_{32}(\xi) &= \pm\frac{1}{2b}\sqrt{-\frac{\alpha}{\beta}}\left[-a-\frac{e^{b(\xi+C)}+b-a}{(e^{b(\xi+C)}-b-a}\right]\mp\frac{1}{2b}\sqrt{-\frac{\alpha}{\beta}}\left[-a-\frac{e^{b(\xi+C)}+b-a}{e^{b(\xi+C)}-b-a}\right]^{-1},\\ u_{33}(\xi) &= \mp\frac{1}{a}\sqrt{-\frac{\alpha}{\beta}}\left[\frac{a}{2}-\frac{ae^{a(\xi+C)}}{ce^{a(\xi+C)}-1}\right]\pm\frac{1}{2a}\sqrt{-\frac{\alpha}{\beta}}\left[\frac{a}{2}-\frac{ae^{a(\xi+C)}}{ce^{a(\xi+C)}-1}\right]^{-1}, \end{aligned} \tag{77}$$

where $\xi=\mu\left(x+y\mp\frac{1}{\mu}\sqrt{\frac{-\alpha}{2(a^2+b^2-c^2)}}t\right)$.

Set IV:

$$
\begin{aligned}
&m=\pm\frac{1}{\mu}\sqrt{\frac{\alpha}{a^2+b^2-c^2}},\qquad A_1=\pm(b-c)\sqrt{\frac{\alpha}{2\beta(a^2+b^2-c^2)}},\\
&B_1=\mp\frac{1}{2(b-c)}\sqrt{\frac{2\alpha(a^2+b^2-c^2)}{\beta}},\\
&A_0=0,\quad \mu=\mu,\quad p=-\frac{a}{b-c},
\end{aligned}\tag{78}
$$

$$
u(\xi)=A_1\left[p+\tan\left(\frac{\Phi(\xi)}{2}\right)\right]+B_1\left[p+\tan\left(\frac{\Phi(\xi)}{2}\right)\right]^{-1},\tag{79}
$$

where a, b and c are arbitrary constants. By using of the (79) and **Family 1, 4** respectively get to

$$
\begin{aligned}
&u_{33}(\xi)=\mp\sqrt{\frac{-\alpha}{2\beta}}\left[\tan\left(\frac{\sqrt{c^2-b^2-a^2}}{2}(\xi+C)\right)-\cot\left(\frac{\sqrt{c^2-b^2-a^2}}{2}(\xi+C)\right)\right],\\
&u_{34}(\xi)=\mp\sqrt{\frac{-\alpha}{2\beta}}\left[\tan\left(\frac{\sqrt{c^2-a^2}}{2}(\xi+C)\right)-\cot\left(\frac{\sqrt{c^2-a^2}}{2}(\xi+C)\right)\right].
\end{aligned}\tag{80}
$$

By using of the (79) and **Family 2, 3** and **Family 5** we get

$$
\begin{aligned}
&u_{35}(\xi)=\pm\sqrt{\frac{-\alpha}{2\beta}}\left[\tanh\left(\frac{\sqrt{a^2+b^2-c^2}}{2}(\xi+C)\right)-\coth\left(\frac{\sqrt{a^2+b^2-c^2}}{2}(\xi+C)\right)\right],\\
&u_{36}(\xi)=\pm\sqrt{\frac{-\alpha}{2\beta}}\left[\tanh\left(\frac{\sqrt{a^2+b^2}}{2}(\xi+C)\right)-\coth\left(\frac{\sqrt{a^2+b^2}}{2}(\xi+C)\right)\right],\\
&u_{37}(\xi)=\pm\sqrt{\frac{-\alpha}{2\beta}}\left[\tanh\left(\frac{\sqrt{b^2-c^2}}{2}(\xi+C)\right)-\coth\left(\frac{\sqrt{b^2-c^2}}{2}(\xi+C)\right)\right].
\end{aligned}\tag{81}
$$

By using of the (79) and **Family 6** we get

$$
\begin{aligned}
u_{38}(\xi)=&\pm\sqrt{-\frac{\alpha}{2\beta}}\tan\left(\frac{1}{2}\arctan\left[\frac{e^{2b(\xi+C)}-1}{e^{2b(\xi+C)}+1},\ \frac{2e^{b(\xi+C)}}{e^{2b(\xi+C)}+1}\right]\right)\\
&\mp\sqrt{\frac{\alpha}{2\beta}}\cot\left(\frac{1}{2}\arctan\left[\frac{e^{2b(\xi+C)}-1}{e^{2b(\xi+C)}+1},\ \frac{2e^{b(\xi+C)}}{e^{2b(\xi+C)}+1}\right]\right).
\end{aligned}\tag{82}
$$

By using of the (79) and **Family 10, 11, 12, 13** and **14** respectively can be written as

$$
\begin{aligned}
&u_{39}(\xi)=\mp\sqrt{\frac{2\alpha}{\beta}}\left[\frac{1}{2}-\frac{e^{ak(\xi+C)}}{-1+e^{ak(\xi+C)}}\right]\pm\frac{1}{4}\sqrt{\frac{2\alpha}{\beta}}\left[\frac{1}{2}-\frac{e^{ak(\xi+C)}}{-1+e^{ak(\xi+C)}}\right]^{-1},\\
&u_{40}(\xi)=\pm\frac{1}{b}\sqrt{\frac{\alpha}{2\beta}}\left[-a-\frac{(a+b)e^{b(\xi+C)}-1}{(a-b)e^{b(\xi+C)}-1}\right]\mp b\sqrt{\frac{\alpha}{2\beta}}\left[-a-\frac{(a+b)e^{b(\xi+C)}-1}{(a-b)e^{b(\xi+C)}-1}\right]^{-1},\\
&u_{41}(\xi)=\pm\frac{1}{b}\sqrt{\frac{\alpha}{2\beta}}\left[-c-\frac{(b+c)e^{b(\xi+C)}+1}{(b-c)e^{b(\xi+C)}-1}\right]\mp b\sqrt{\frac{\alpha}{2\beta}}\left[-c-\frac{(b+c)e^{b(\xi+C)}+1}{(b-c)e^{b(\xi+C)}-1}\right]^{-1},
\end{aligned}
$$

$$u_{42}(\xi) = \pm\frac{1}{b}\sqrt{\frac{\alpha}{2\beta}}\left[-a - \frac{e^{b(\xi+C)} + b - a}{(e^{b(\xi+C)} - b - a}\right] \mp \frac{1}{b}\sqrt{\frac{\alpha}{2\beta}}\left[-a - \frac{e^{b(\xi+C)} + b - a}{e^{b(\xi+C)} - b - a}\right]^{-1},$$

$$u_{43}(\xi) = \mp\sqrt{\frac{\alpha}{2\beta}}\left[1 - \frac{ae^{a(\xi+C)}}{ce^{a(\xi+C)} - 1}\right] \pm \sqrt{\frac{\alpha}{2\beta}}\left[1 - \frac{ae^{a(\xi+C)}}{ce^{a(\xi+C)} - 1}\right]^{-1}, \tag{83}$$

where $\xi = \mu\left(x + y \mp \frac{1}{\mu}\sqrt{\frac{\alpha}{a^2+b^2-c^2}}t\right)$.

Set V:

$$m = \pm\frac{1}{\mu}\sqrt{\frac{-2\alpha}{a^2+b^2-c^2}},\quad A_0 = \pm(a + p(b-c))\sqrt{-\frac{\alpha}{\beta(a^2+b^2-c^2)}},$$

$$A_1 = \pm(b-c)\sqrt{-\frac{\alpha}{\beta(a^2+b^2-c^2)}}, \tag{84}$$

$$B_1 = 0,\quad \mu = \mu,\quad p = p,\quad u(\xi) = A_0 + A_1\left[p + \tan\left(\frac{\Phi(\xi)}{2}\right)\right], \tag{85}$$

where a, b and c are arbitrary constants. By using of the (85) and **Family 1, 4** respectively get to

$$u_{44}(\xi) = \pm\sqrt{-\frac{\alpha}{\beta(a^2+b^2-c^2)}}\left[2a + 2p(b-c) - \sqrt{c^2-b^2-a^2}\tan\left(\frac{\sqrt{c^2-b^2-a^2}}{2}(\xi+C)\right)\right],$$

$$u_{45}(\xi) = \pm\sqrt{-\frac{\alpha}{\beta(a^2-c^2)}}\left[2a - 2pc - \sqrt{c^2-a^2}\tan\left(\frac{\sqrt{c^2-a^2}}{2}(\xi+C)\right)\right], \tag{86}$$

By using of the (67) and **Family 2, 3** and **Family 5** we get

$$u_{46}(\xi) = \pm\sqrt{-\frac{\alpha}{\beta(a^2+b^2-c^2)}}\left[2a + 2p(b-c) + \sqrt{a^2+b^2-c^2}\tanh\left(\frac{\sqrt{a^2+b^2-c^2}}{2}(\xi+C)\right)\right],$$

$$u_{47}(\xi) = \pm\sqrt{-\frac{\alpha}{\beta(a^2+b^2)}}\left[2a + 2pb + \sqrt{a^2+b^2}\tanh\left(\frac{\sqrt{a^2+b^2}}{2}(\xi+C)\right)\right],$$

$$u_{48}(\xi) = \pm\sqrt{-\frac{\alpha}{\beta(b^2-c^2)}}\left[2p(b-c) + \sqrt{b^2-c^2}\tanh\left(\frac{\sqrt{b^2-c^2}}{2}(\xi+C)\right)\right]. \tag{87}$$

By using of the (61) and **Family 6** we get

$$u_{49}(\xi) = \pm\sqrt{-\frac{\alpha}{\beta}}\left[2p + \tan\left(\frac{1}{2}\arctan\left[\frac{e^{2b(\xi+C)} - 1}{e^{2b(\xi+C)} + 1}, \frac{2e^{b(\xi+C)}}{e^{2b(\xi+C)} + 1}\right]\right)\right]. \tag{88}$$

By using of the (67) and **Family 10, 11, 12, 13** and **14** respectively can be written as

$$
\begin{aligned}
u_{50}(\xi) &= \pm\sqrt{-\frac{\alpha}{\beta}}\left[1-4p+2\frac{e^{ak(\xi+C)}}{-1+e^{ak(\xi+C)}}\right],\\
u_{51}(\xi) &= \pm\frac{1}{b}\sqrt{-\frac{\alpha}{\beta}}\left[a+2p(b-a)-(b-a)\frac{(a+b)e^{b(\xi+C)}-1}{(a-b)e^{b(\xi+C)}-1}\right],\\
u_{52}(\xi) &= \pm\frac{1}{b}\sqrt{-\frac{\alpha}{3\beta}}\left[c+2p(b-c)+(b-c)\frac{(b+c)e^{b(\xi+C)}+1}{(b-c)e^{b(\xi+C)}-1}\right],\\
u_{53}(\xi) &= \pm\frac{1}{b}\sqrt{-\frac{\alpha}{\beta}}\left[a+2p(b+a)+(b+a)\frac{e^{b(\xi+C)}+b-a}{e^{b(\xi+C)}-b-a}\right],\\
u_{54}(\xi) &= \pm\frac{1}{a}\sqrt{-\frac{\alpha}{\beta}}\left[a-4pc+2c\frac{ae^{a(\xi+C)}}{ce^{a(\xi+C)}-1}\right],
\end{aligned}
\tag{89}
$$

where $\xi=\mu\left(x+y\mp\frac{1}{\mu}\sqrt{\frac{-2\alpha}{a^2+b^2-c^2}}t\right)$.

The SRLW Equation

We consider last example the nonlinear SRLW equation as follows

$$u_{tt}+u_{xx}+u_{xxtt}+(uu_x)_t=0, \tag{90}$$

by using the wave variable $\xi=\mu(x-mt)$ reduce it to an ODE as follows

$$(m^2+1)u+\mu^2m^2u''-\frac{m}{2}u^2=0, \tag{91}$$

where obtained by twice integrating and neglecting the constant of integration. Balancing the u'' and u^2 by using homogenous principal, we have

$$M+2=2M, \qquad \Rightarrow M=2. \tag{92}$$

Then the trail solution is

$$
\begin{aligned}
u(\xi) = A_0 &+ A_1\left[p+\tan\left(\frac{\Phi(\xi)}{2}\right)\right]+A_2\left[p+\tan\left(\frac{\Phi(\xi)}{2}\right)\right]^2\\
&+B_1\left[p+\tan\left(\frac{\Phi(\xi)}{2}\right)\right]^{-1}+B_2\left[p+\tan\left(\frac{\Phi(\xi)}{2}\right)\right]^{-2}.
\end{aligned}
\tag{93}
$$

For simplicity we set $p=1$, then Eq. (93) is simplified as follows

$$
\begin{aligned}
u(\xi) = A_0 &+ A_1\left[1+\tan\left(\frac{\Phi(\xi)}{2}\right)\right]+A_2\left[1+\tan\left(\frac{\Phi(\xi)}{2}\right)\right]^2\\
&+B_1\left[1+\tan\left(\frac{\Phi(\xi)}{2}\right)\right]^{-1}+B_2\left[1+\tan\left(\frac{\Phi(\xi)}{2}\right)\right]^{-2}.
\end{aligned}
\tag{94}
$$

Substituting (94) and (7) into Eq. (91) and by using the well-known Maple software, we obtain the following sets of non-trivial solutions

Set I:

$$m = m, \quad \mu = \pm\frac{1}{m}\sqrt{\frac{-(m^2+1)}{a^2+b^2-c^2}},$$
$$A_0 = \frac{(m^2+1)(2b^2-6bc+6ab+2a^2+4c^2-6ac)}{m(a^2+b^2-c^2)}, \tag{95}$$
$$B_1 = 0, \quad B_2 = 0, \quad A_1 = \frac{-6(b-c)(a+b-c)(m^2+1)}{m(a^2+b^2-c^2)},$$
$$A_2 = \frac{3(b-c)^2(m^2+1)}{m(a^2+b^2-c^2)}, \tag{96}$$
$$\mathrm{u}(\xi) = \mathrm{A}_0 + \mathrm{A}_1\left[1+\tan\left(\frac{\Phi(\xi)}{2}\right)\right] + \mathrm{A}_2\left[1+\tan\left(\frac{\Phi(\xi)}{2}\right)\right]^2,$$

where a, b and c are arbitrary constants. By using of the (96) and **Family 1, 4** respectively get to

$$\begin{aligned}\mathrm{u}_1(\xi) = {} & \frac{(\mathrm{m}^2+1)}{\mathrm{m}(\mathrm{a}^2+\mathrm{b}^2-\mathrm{c}^2)}\Bigg\{2\mathrm{a}^2-4\mathrm{b}^2+6\mathrm{bc}-2\mathrm{c}^2-6(\mathrm{a}+\mathrm{b}-\mathrm{c}) \\ & \times\left[\mathrm{a}-\sqrt{\mathrm{c}^2-\mathrm{b}^2-\mathrm{a}^2}\tan\left(\frac{\sqrt{\mathrm{c}^2-\mathrm{b}^2-\mathrm{a}^2}}{2}(\xi+\mathrm{C})\right)\right] \\ & +3\left[\mathrm{a}+\mathrm{b}-\mathrm{c}-\sqrt{\mathrm{c}^2-\mathrm{b}^2-\mathrm{a}^2}\tan\left(\frac{\sqrt{\mathrm{c}^2-\mathrm{b}^2-\mathrm{a}^2}}{2}(\xi+\mathrm{C})\right)\right]^2\Bigg\}, \\ \mathrm{u}_2(\xi) = {} & \frac{(\mathrm{m}^2+1)}{\mathrm{m}(\mathrm{a}^2-\mathrm{c}^2)}\Bigg\{2\mathrm{a}^2-\mathrm{c}^2-6(\mathrm{a}-\mathrm{c})\left[\mathrm{a}-\sqrt{\mathrm{c}^2-\mathrm{a}^2}\tan\left(\frac{\sqrt{\mathrm{c}^2-\mathrm{a}^2}}{2}(\xi+\mathrm{C})\right)\right] \\ & +3\left[\mathrm{a}-\mathrm{c}-\sqrt{\mathrm{c}^2-\mathrm{a}^2}\tan\left(\frac{\sqrt{\mathrm{c}^2-\mathrm{a}^2}}{2}(\xi+\mathrm{C})\right)\right]^2\Bigg\}.\end{aligned} \tag{97}$$

By using of the (96) and **Family 2, 3** and **5** respectively get to

$$\begin{aligned}\mathrm{u}_3(\xi) = {} & \frac{(\mathrm{m}^2+1)}{\mathrm{m}(\mathrm{a}^2+\mathrm{b}^2-\mathrm{c}^2)}\Bigg\{2\mathrm{a}^2-4\mathrm{b}^2+6\mathrm{bc}-2\mathrm{c}^2-6(\mathrm{a}+\mathrm{b}-\mathrm{c}) \\ & \times\left[\mathrm{a}+\sqrt{\mathrm{a}^2+\mathrm{b}^2-\mathrm{c}^2}\tanh\left(\frac{\sqrt{\mathrm{a}^2+\mathrm{b}^2-\mathrm{c}^2}}{2}(\xi+\mathrm{C})\right)\right] \\ & +3\left[\mathrm{a}+\mathrm{b}-\mathrm{c}+\sqrt{\mathrm{a}^2+\mathrm{b}^2-\mathrm{c}^2}\tanh\left(\frac{\sqrt{\mathrm{a}^2+\mathrm{b}^2-\mathrm{c}^2}}{2}(\xi+\mathrm{C})\right)\right]^2\Bigg\}, \\ \mathrm{u}_4(\xi) = {} & \frac{(\mathrm{m}^2+1)}{\mathrm{m}(\mathrm{a}^2+\mathrm{b}^2)}\Bigg\{2\mathrm{a}^2-4\mathrm{b}^2-6(\mathrm{a}+\mathrm{b})\left[\mathrm{a}+\sqrt{\mathrm{a}^2+\mathrm{b}^2}\tanh\left(\frac{\sqrt{\mathrm{a}^2+\mathrm{b}^2}}{2}(\xi+\mathrm{C})\right)\right] \\ & +3\left[\mathrm{a}+\mathrm{b}+\sqrt{\mathrm{a}^2+\mathrm{b}^2}\tanh\left(\frac{\sqrt{\mathrm{a}^2+\mathrm{b}^2}}{2}(\xi+\mathrm{C})\right)\right]^2\Bigg\},\end{aligned} \tag{98}$$

$$u_5(\xi)=\frac{(m^2+1)}{m(b^2-c^2)}\left\{-4b^2+6bc-2c^2-6(b-c)\left[\sqrt{b^2-c^2}\tanh\left(\frac{\sqrt{b^2-c^2}}{2}(\xi+C)\right)\right]\right.$$
$$\left.+3\left[b-c+\sqrt{b^2-c^2}\tanh\left(\frac{\sqrt{b^2-c^2}}{2}(\xi+C)\right)\right]^2\right\}.$$

By using of the (96) and **Family 6** we get

$$u_6(\xi)=\frac{(m^2+1)}{m}\left\{-4-6\tan\left(\frac{1}{2}\arctan\left[\frac{e^{2b(\xi+C)}-1}{e^{2b(\xi+C)}+1},\ \frac{2e^{b(\xi+C)}}{e^{2b(\xi+C)}+1}\right]\right)\right.$$
$$\left.+3\left[1+\tan\left(\frac{1}{2}\arctan\left[\frac{e^{2b(\xi+C)}-1}{e^{2b(\xi+C)}+1},\ \frac{2e^{b(\xi+C)}}{e^{2b(\xi+C)}+1}\right]\right)\right]^2\right\}. \tag{99}$$

By using of the (96) and **Family 10, 11, 12, 13** and **14** respectively can be written as

$$u_7(\xi)=\frac{(m^2+1)}{m}\left\{-10+12\frac{e^{ka(\xi+C)}}{\left[e^{ka(\xi+C)}-1\right]}+12\left[1-\frac{e^{ka(\xi+C)}}{\left[e^{ka(\xi+C)}-1\right]}\right]^2\right\},$$
$$u_8(\xi)=\frac{(m^2+1)}{m}\left\{\frac{6ab-2a^2-4b^2}{b^2}-\frac{6(b-a)}{b}\frac{(a+b)e^{b(\xi+C)}-1}{(a-b)e^{b(\xi+C)}-1}\right.$$
$$\left.+\frac{3(b-a)^2}{b^2}\left[1-\frac{(a+b)e^{b(\xi+C)}-1}{(a-b)e^{b(\xi+C)}-1}\right]^2\right\},$$
$$u_9(\xi)=\frac{(m^2+1)}{m}\left\{\frac{6c-4b}{b}-\frac{6(b-c)}{b}\frac{(b+c)e^{b(\xi+C)}+1}{(b-c)e^{b(\xi+C)}-1}\right.$$
$$\left.+\frac{3(b-c)^2}{b^2}\left[1+\frac{(b+c)e^{b(\xi+C)}+1}{(b-c)e^{b(\xi+C)}-1}\right]^2\right\},$$
$$u_{10}(\xi)=\frac{(m^2+1)}{m}\left\{-\frac{4b+6a}{b}-\frac{6(b+a)(b+2a)}{b^2}\frac{e^{b(\xi+C)}+b-a}{e^{b(\xi+C)}-b-a}\right.$$
$$\left.+\frac{3(b+a)^2}{b^2}\left[1+\frac{e^{b(\xi+C)}+b-a}{e^{b(\xi+C)}-b-a}\right]^2\right\},$$
$$u_{11}(\xi)=\frac{(m^2+1)}{m}\left\{\frac{2a^2-12c^2}{a^2}-\frac{12c(a-2c)}{a^2}\frac{ae^{a(\xi+C)}}{ce^{a(\xi+C)}-1}\right.$$
$$\left.+\frac{12c^2}{b^2}\left[1-\frac{ae^{a(\xi+C)}}{ce^{a(\xi+C)}-1}\right]^2\right\}, \tag{100}$$

where $\xi=\frac{1}{m}\sqrt{\frac{-(m^2+1)}{a^2+b^2-c^2}}(x-mt)$.

Set II:

$$m = m, \quad \mu = \pm\frac{1}{m}\sqrt{-\frac{m^2+1}{a^2+b^2-c^2}}, \quad A_0 = \frac{-6(m^2+1)(a-c)(b-c)}{m(a^2+b^2-c^2)}, \tag{101}$$

$$B_1 = \frac{12(a-c)(a+b-c)(m^2+1)}{m(a^2+b^2-c^2)}, \quad B_2 = \frac{-12(a-c)^2(m^2+1)}{m(a^2+b^2-c^2)}, \quad A_1=0,\ A_2=0,$$

$$u(\xi) = A_0 + B_1\left[1+\tan\left(\frac{\Phi(\xi)}{2}\right)\right]^{-1} + B_2\left[1+\tan\left(\frac{\Phi(\xi)}{2}\right)\right]^{-2}, \tag{102}$$

where a, b and c are arbitrary constants. By using of the (102) and **Family 1, 4** respectively get to

$$\begin{aligned} u_{12}(\xi) = \frac{(m^2+1)}{m(a^2+b^2-c^2)}\Bigg\{&-6(a-c)(b-c)+12(a-c)(a+b-c)\left[1+\frac{a}{b-c}\right.\\ &\left.-\frac{\sqrt{c^2-b^2-a^2}}{b-c}\tan\left(\frac{\sqrt{c^2-b^2-a^2}}{2}(\xi+C)\right)\right]^{-1} - 12(a-c)^2\left[1+\frac{a}{b-c}\right.\\ &\left.-\frac{\sqrt{c^2-b^2-a^2}}{b-c}\tan\left(\frac{\sqrt{c^2-b^2-a^2}}{2}(\xi+C)\right)\right]^{-2}\Bigg\}, \end{aligned} \tag{103}$$

$$\begin{aligned} u_{13}(\xi) = \frac{(m^2+1)}{m(a^2-c^2)}\Bigg\{&6(a-c)c+12(a-c)(a-c)\left[1-\frac{a}{c}+\frac{\sqrt{c^2-a^2}}{c}\right.\\ &\left.\times\tan\left(\frac{\sqrt{c^2-a^2}}{2}(\xi+C)\right)\right]^{-1}\\ &-12(a-c)^2\left[1-\frac{a}{c}+\frac{\sqrt{c^2-a^2}}{c}\tan\left(\frac{\sqrt{c^2-a^2}}{2}(\xi+C)\right)\right]^{-2}\Bigg\}. \end{aligned}$$

By using of the (102) and **Family 2, 3** and **5** respectively get to

$$\begin{aligned} u_{14}(\xi) = \frac{(m^2+1)}{m(a^2+b^2-c^2)}\Bigg\{&-6(a-c)(b-c)+12(a-c)(a+b-c)\left[1+\frac{a}{b-c}\right.\\ &\left.+\frac{\sqrt{a^2+b^2-c^2}}{b-c}\tanh\left(\frac{\sqrt{a^2+b^2-c^2}}{2}(\xi+C)\right)\right]^{-1}\\ &-12(a-c)^2\left[1+\frac{a}{b-c}+\frac{\sqrt{a^2+b^2-c^2}}{b-c}\right.\\ &\left.\times\tanh\left(\frac{\sqrt{a^2+b^2-c^2}}{2}(\xi+C)\right)\right]^{-2}\Bigg\}, \end{aligned}$$

$$\begin{aligned} u_{15}(\xi) = \frac{(m^2+1)}{m(a^2+b^2)}\Bigg\{&-6ab+12a(a+b)\left[1+\frac{a}{b}+\frac{\sqrt{a^2+b^2}}{b}\tanh\left(\frac{\sqrt{a^2+b^2}}{2}(\xi+C)\right)\right]^{-1}\\ &+3a^2\left[1+\frac{a}{b}+\frac{\sqrt{a^2+b^2}}{b}\tanh\left(\frac{\sqrt{a^2+b^2}}{2}(\xi+C)\right)\right]^{-2}\Bigg\}, \end{aligned}$$

$$u_{16}(\xi) = \frac{(m^2+1)}{m(b^2-c^2)}\left\{6c(b-c) - 12c(b-c)\left[1+\frac{\sqrt{b^2-c^2}}{b-c}\tanh\left(\frac{\sqrt{b^2-c^2}}{2}(\xi+C)\right)\right]^{-1}\right.$$
$$\left. -12c^2\left[1+\frac{\sqrt{b^2-c^2}}{b-c}\tanh\left(\frac{\sqrt{b^2-c^2}}{2}(\xi+C)\right)\right]^{-2}\right\}. \tag{104}$$

By using of the (102) and **Family 13** and **14** respectively can be written as

$$u_{17}(\xi) = \frac{(m^2+1)}{m}\left\{-\frac{12a(a+b)}{b^2} + \frac{24a(b+2a)}{b^2}\left[1+\frac{e^{b(\xi+C)}+b-a}{e^{b(\xi+C)}-b-a}\right]^{-1}\right.$$
$$\left. -\frac{48a^2}{b^2}\left[1+\frac{e^{b(\xi+C)}+b-a}{e^{b(\xi+C)}-b-a}\right]^{-2}\right\}, \tag{105}$$
$$u_{18}(\xi) = \frac{(m^2+1)}{m}\left\{\frac{12c(a-c)}{a^2} + \frac{12(a-c)(a-2c)}{a^2}\left[1-\frac{ae^{a(\xi+C)}}{ce^{a(\xi+C)}-1}\right]^{-1}\right.$$
$$\left. -\frac{12(a-c)^2}{a^2}\left[1-\frac{ae^{a(\xi+C)}}{ce^{a(\xi+C)}-1}\right]^{2}\right\},$$

where $\xi = \frac{1}{m}\sqrt{\frac{-(m^2+1)}{a^2+b^2-c^2}}(x-mt)$.

Set III:

$$m = m,\ \mu = \pm\frac{1}{m}\sqrt{\frac{m^2+1}{a^2+b^2-c^2}},\ A_0 = \frac{2(m^2+1)(a^2+3ab-3ac+b^2+2c^2-3bc)}{m(a^2+b^2-c^2)},$$
$$B_1 = -\frac{12(a-c)(a+b-c)(m^2+1)}{m(a^2+b^2-c^2)},\ B_2 = \frac{12(a-c)^2(m^2+1)}{m(a^2+b^2-c^2)},\ A_1 = 0,\ A_2 = 0, \tag{106}$$
$$u(\xi) = A_0 + B_1\left[1+\tan\left(\frac{\Phi(\xi)}{2}\right)\right]^{-1} + B_2\left[1+\tan\left(\frac{\Phi(\xi)}{2}\right)\right]^{-2}, \tag{107}$$

where a, b and c are arbitrary constants. By using of the (107) and **Family 1, 4** respectively get to

$$u_{19}(\xi) = \frac{2(m^2+1)}{m(a^2+b^2-c^2)}\left\{a^2+3ab-3ac+b^2+2c^2-3bc-6(a-c)(a+b-c)\right.$$
$$\times\left[1+\frac{a}{b-c}-\frac{\sqrt{c^2-b^2-a^2}}{b-c}\tan\left(\frac{\sqrt{c^2-b^2-a^2}}{2}(\xi+C)\right)\right]^{-1}$$

$$+6(a-c)^2\left[1+\frac{a}{b-c}-\frac{\sqrt{c^2-b^2-a^2}}{b-c}\tan\left(\frac{\sqrt{c^2-b^2-a^2}}{2}(\xi+C)\right)\right]^{-2}\Bigg\},$$

$$u_{20}(\xi)=\frac{2(m^2+1)}{m(a^2-c^2)}\left\{a^2-3ac+2c^2-6(a-c)(a-c)\left[1-\frac{a}{c}+\frac{\sqrt{c^2-a^2}}{c}\right.\right.$$
$$\left.\times\tan\left(\frac{\sqrt{c^2-a^2}}{2}(\xi+C)\right)\right]^{-1}+6(a-c)^2\left[1-\frac{a}{c}+\frac{\sqrt{c^2-a^2}}{c}\right.$$
$$\left.\left.\times\tan\left(\frac{\sqrt{c^2-a^2}}{2}(\xi+C)\right)\right]^{-2}\right\}. \tag{108}$$

By using of the (107) and **Family 2, 3** and **5** respectively get to

$$u_{21}(\xi)=\frac{2(m^2+1)}{m(a^2+b^2-c^2)}\left\{a^2+3ab-3ac+b^2+2c^2-3bc-6(a-c)(a+b-c)\right.$$
$$\times\left[1+\frac{a}{b-c}+\frac{\sqrt{a^2+b^2-c^2}}{b-c}\tanh\left(\frac{\sqrt{a^2+b^2-c^2}}{2}(\xi+C)\right)\right]^{-1}$$
$$+6(a-c)^2\left[1+\frac{a}{b-c}+\frac{\sqrt{a^2+b^2-c^2}}{b-c}\right.$$
$$\left.\left.\times\tanh\left(\frac{\sqrt{a^2+b^2-c^2}}{2}(\xi+C)\right)\right]^{-2}\right\},$$

$$u_{22}(\xi)=\frac{2(m^2+1)}{m(a^2+b^2)}\left\{a^2+3ab+b^2+-6a(a+b)\left[1+\frac{a}{b}+\frac{\sqrt{a^2+b^2}}{b}\right.\right.$$
$$\left.\times\tanh\left(\frac{\sqrt{a^2+b^2}}{2}(\xi+C)\right)\right]^{-1}$$
$$\left.+6a^2\left[1+\frac{a}{b}+\frac{\sqrt{a^2+b^2}}{b}\tanh\left(\frac{\sqrt{a^2+b^2}}{2}(\xi+C)\right)\right]^{-2}\right\},$$

$$u_{23}(\xi)=\frac{2(m^2+1)}{m(b^2-c^2)}\left\{b^2+2c^2-3bc+6c(b-c)\left[1+\frac{\sqrt{b^2-c^2}}{b-c}\right.\right.$$
$$\left.\times\tanh\left(\frac{\sqrt{b^2-c^2}}{2}(\xi+C)\right)\right]^{-1}$$
$$\left.+6c^2\left[1+\frac{\sqrt{b^2-c^2}}{b-c}\tanh\left(\frac{\sqrt{b^2-c^2}}{2}(\xi+C)\right)\right]^{-2}\right\}. \tag{109}$$

By using of the (107) and **Family 7** we get

$$u_{24}(\xi) = \frac{2(m^2+1)}{m}\left\{1 - 6\left[1 + \tan\left(\frac{1}{2}\arctan\left[\frac{2e^{a(\xi+C)}}{e^{2a(\xi+C)}+1}, \frac{e^{2a(\xi+C)}-1}{e^{2a(\xi+C)}+1}\right]\right)\right]^{-1}\right.$$
$$\left.+6\left[1 + \tan\left(\frac{1}{2}\arctan\left[\frac{2e^{a(\xi+C)}}{e^{2a(\xi+C)}+1}, \frac{e^{2a(\xi+C)}-1}{e^{2a(\xi+C)}+1}\right]\right)\right]^{-2}\right\}. \tag{110}$$

By using of the (107) and **Family 13** and **14** respectively can be written as

$$u_{25}(\xi) = \frac{2(m^2+1)}{m}\left\{\frac{b^2+6ab+6a^2}{b^2} - \frac{12a(b+2a)}{b^2}\left[1 + \frac{e^{b(\xi+C)}+b-a}{e^{b(\xi+C)}-b-a}\right]^{-1}\right.$$
$$\left.+\frac{24a^2}{b^2}\left[1 + \frac{e^{b(\xi+C)}+b-a}{e^{b(\xi+C)}-b-a}\right]^{-2}\right\},$$
$$u_{26}(\xi) = \frac{2(m^2+1)}{m}\left\{\frac{a^2-6ac+6c^2}{a^2} - \frac{12(a-c)(a-2c)}{a^2}\left[1 + \frac{ae^{a(\xi+C)}}{ce^{a(\xi+C)}-1}\right]^{-1}\right.$$
$$\left.+\frac{12(a-c)^2}{b^2}\left[1 + \frac{ae^{a(\xi+C)}}{ce^{a(\xi+C)}-1}\right]^{-2}\right\}, \tag{111}$$

where $\xi = \frac{1}{m}\sqrt{\frac{m^2+1}{a^2+b^2-c^2}}(x - mt)$.

Conclusion

In the present work, we successfully obtained the exact solutions of the (2+1)-dimensional Zoomeron equation, the Duffing equation and the SRLW equation with the improved $\tan(\Phi(\xi)/2)$-expansion method. By new scheme we established solitary solutions are include four type namely, triangular functions solutions, exponential solutions and rational solutions. The applied method will be used in further works to establish more entirely new solutions for the (2+1)-dimensional Zoomeron equation, the Duffing equation and the SRLW equation. This paper has shown the new method is sufficient incentive to seek more new exact soliton solutions of NEEs in mathematical physical. We found in this work the obtained results for the nonlinear aforementioned equations give very good results even very further of applied method in [5]. It can be concluded that this method is a very powerful and efficient technique in finding exact solutions for wide classes of problems. The performance of this method is reliable and effective and gives more solutions.

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