STUDY OF THE DYNAMICS OF A PARTICLE IN A CONSTANT HOMOGENEOUS MAGNETIC FIELD AND A TRANSVERSE HOMOGENEOUS ROTATING ELECTRIC FIELD IN THE DEVELOPMENT OF AN X-RAY SOURCE

A. BOURDIER
Commissariat à l'Energie Atomique, CEA Bruyères-le-Châtel,
BP 12, F-91680 Bruyères-le-Châtel (FRANCE)

M. VALENTINI and J. VALAT
Laboratoire de Physique des Milieux Ionisés, Ecole Polytechnique, Centre National de la Recherche Scientifique, 91128 Palaiseau (FRANCE)

Abstract

The relativistic motion of a charged particle in a homogeneous constant magnetic field and a transverse circularly polarized electric field is reduced to an integrable form. Using canonical transformations, it is shown that the equations of motion can be derived from a one degree of freedom time-dependent Hamiltonian which has a first integral. As a consequence the system can be shown to be integrable. An equation governing the energy of the particle is obtained. A simple approximate expression for its maximum value is derived for the case when the particle is initially resonant and at rest. This gives the upper limit in frequency of the X-rays emitted when, for instance, an electron hits a high-Z material.

The relativistic motion of an electron in a constant homogeneous magnetic field and a transverse electric field is studied. This problem has already been explored by other authors[1,2]. One of the aims of this paper is to bring some enlightenment to their discussion by using the Hamiltonian formalism. Another aim is to derive a simple approximate expression for the maximum energy the particle can reach in the interesting physical situation, when the electron is initially resonant and at rest.This gives the upper limit in frequency of the X-rays emitted when the particle hits a high-Z material.

Let us reduce the motion of an electron in an homogeneous constant magnetic field and a transverse circularly polarized electric field to a problem with a single degree of freedom.

The constant magnetic field is assumed to be along the z-axis, and the electric field has the following components

$$E_x = E_0 \cos\omega_0 t,\quad E_y = E_0 \sin\omega_0 t,\quad E_z = 0, \qquad (1)$$

where E_0 and ω_0 are constants.

The following gauge is chosen for the electromagnetic field

$$A = -\left(\frac{B_0}{2} y + \frac{E_0}{\omega_0}\sin\omega_0 t\right)\hat{e}_x + \left(\frac{B_0}{2} x + \frac{E_0}{\omega_0}\cos\omega_0 t\right)\hat{e}_y. \qquad (2)$$

Assuming that the motion of an electron is in the x-y plane, its relativistic Hamiltonian is

$$H = \left[\left(P_x - \frac{eE_0}{\omega_0}\sin\omega_0 t - \frac{eB_0}{2} y\right)^2 c^2 + \left(P_y + \frac{eE_0}{\omega_0}\cos\omega_0 t + \frac{eB_0}{2} x\right)^2 c^2 + m^2c^4\right]^{1/2}. \qquad (3)$$

The Hamilton equations allow us to readily find two constants of motion

$$C_1 = P_x + \frac{eB_0}{2} y,\quad C_2 = P_y - \frac{eB_0}{2} x. \qquad (4)$$

Another constant of motion can be obtained by using Noether's theorem[3,4]. It is simple to show that the Lagrangian of the system is invariant under the following transformation

$$t \to t - \varepsilon/\omega_0,\quad x \to x + \varepsilon y,\quad y \to y - \varepsilon x, \qquad (5)$$

where ε is an infinitesimal quantity. Therefore, a third first integral is

$$C_3 = yP_x - xP_y + H/\omega_0. \qquad (6)$$

It can be noted that the two first constants are canonically conjugated

$$\left\{C_1, \frac{C_2}{eB_0}\right\} = 1. \qquad (7)$$

This property can be used to reduce the dimension of the problem by choosing the two constants as new conjugated momentum and coordinate.
The following dimensionless variables and parameters are introduced

$$\hat{x} = x\frac{\omega_0}{c},\ \hat{y} = y\frac{\omega_0}{c},\ \hat{P}_{x,y} = \frac{P_{x,y}}{mc},\ \hat{t} = \omega_0 t,\ \hat{H} = \gamma = \frac{H}{mc^2},$$
$$a = \frac{eE_0}{mc\omega_0},\ \Omega_0 = \frac{eB_0}{m\omega_0},$$

where m is the mass of the charged particle.
A first canonical transformation is then introduced: $\left(\hat{x}, \hat{y}, \hat{P}_x, \hat{P}_y\right) \to \left(\tilde{x}, \tilde{y}, \tilde{P}_x, \tilde{P}_y\right)$, given by the following type 2 generating function[3,5]

$$F_2 = \left(\tilde{P}_x - \frac{\Omega_0}{2}\hat{y}\right)\hat{x} + \tilde{P}_y\hat{y}. \tag{8}$$

A second canonical transformation is introduced: $\left(\tilde{x}, \tilde{y}, \tilde{P}_x, \tilde{P}_y\right) \to (Q_1, Q_2, P_1, P_2)$, generated by

$$F_2 = \left(P_2 + \Omega_0\tilde{x}\right)\tilde{y} + P_1\left(\tilde{x} + \frac{P_2}{\Omega_0}\right). \tag{9}$$

The product of the two transformations yield

$$\begin{aligned} \hat{x} &= Q_1 - \frac{P_2}{\Omega_0}, \\ \hat{y} &= Q_2 - \frac{P_1}{\Omega_0}, \\ \hat{P}_x &= \frac{1}{2}\left(\Omega_0 Q_2 + P_1\right), \\ \hat{P}_y &= \frac{1}{2}\left(\Omega_0 Q_1 + P_2\right). \end{aligned} \tag{10}$$

In the new variables, the Hamiltonian is

$$H = \left[\left(P_1 - a\sin\hat{t}\right)^2 + \left(\Omega_0 Q_1 + a\cos\hat{t}\right)^2 + 1\right]^{1/2}. \tag{11}$$

As expected P_2 and Q_2 are cyclic variables. The Hamiltonian depends on time and has one degree of freedom.
In these variables, the Hamilton equations are

$$\begin{aligned} \dot{P}_1 &= -\frac{\Omega_0}{\gamma}\left(\Omega_0 Q_1 + a\cos\hat{t}\right), \\ \dot{Q}_1 &= \frac{1}{\gamma}\left(P_1 - a\sin\hat{t}\right). \end{aligned} \tag{12}$$

The constant C_3 becomes

$$I = H - \frac{P_1^2}{2\Omega_0} - \frac{\Omega_0}{2}Q_1^2. \tag{13}$$

This constant allows to show that the Liouville theorem still holds, and the problem is integrable[6,7].

The integrability of the motion can be shown in a second manner. Introducing the variables

$$\begin{aligned} \bar{Q}_1 &= Q_1 + a/\Omega_0 \cos\hat{t}, \\ \bar{P}_1 &= P_1 - a\sin\hat{t}, \end{aligned} \tag{14}$$

and the complex quantity $Z = \bar{P}_1 + i\,\Omega_0\bar{Q}_1$, the Hamilton equations (Eqs.(12)) are equivalent to the following equation

$$\dot{Z} = \frac{i\Omega_0 Z}{\sqrt{1 + |Z|^2}} - a\exp(i\hat{t}), \tag{15}$$

which is the equation of a nonlinear oscillator submitted to an external force. The solution of this equation is

$$Z = A_0\exp i\left[\sigma(\hat{t}) + \delta\right] - a\int_0^{\hat{t}} \exp i\left[\sigma(\hat{t}) - \sigma(\tau) + \tau\right] d\tau, \tag{16}$$

with

$$\sigma(\hat{t}) = \Omega_0\int_0^{\hat{t}} d\tau\gamma^{-1}(\tau). \tag{17}$$

A_0 and δ are real constants.
Then

$$\begin{aligned} P_1 &= A_0\cos\left[\sigma(\hat{t}) + \delta\right] + a\sin\hat{t} - a\int_0^{\hat{t}} \cos\left[\sigma(\hat{t}) - \sigma(\tau) + \tau\right] d\tau, \\ Q_1 &= \frac{A_0}{\Omega_0}\sin\left[\sigma(\hat{t}) + \delta\right] - \frac{a}{\Omega_0}\cos\hat{t} - \frac{a}{\Omega_0}\int_0^{\hat{t}} \sin\left[\sigma(\hat{t}) - \sigma(\tau) + \tau\right] d\tau. \end{aligned} \tag{18}$$

The quantities A_0 and δ are determined so that at $\hat{t} = 0$, $A_0^2 = \gamma_0^2 - 1 = \hat{p}_{x0}^2 + \hat{p}_{y0}^2$ and $\tan\delta = \hat{p}_{y0}/\hat{p}_{x0}$ ($\hat{p} = p/mc$). $\hat{p}$ is the normalized particle momentum. The subscript 0 of variables γ and $\hat{p}$ refers to their initial values .
Taking the derivative of Eq.(11) with respect to time, and using Eqs. (18), we obtain

$$\gamma\dot{\gamma} = -a\left\{A_0\cos\left[\sigma(\hat{t}) - \hat{t} + \delta\right] - a\int_0^{\hat{t}} \cos\left[\sigma(\hat{t}) - \sigma(\tau) + \tau - \hat{t}\right] d\tau\right\}. \tag{19}$$

This equation is multiplied by $\Omega_0/\gamma - 1$ and integrated between 0 and $\hat{t}$. Then, this new expression is multiplied by $\Omega_0/\gamma - 1$ and added to the equation obtained by differentiating Eq.(19) with respect to time. The resulting equation is multiplied by $\gamma\dot{\gamma}$ and integrated between 0 and $\hat{t}$. In this way, the following differential equation for the energy is derived

$$(\dot{\gamma})^2 + \frac{\gamma^2}{4} - \Omega_0\gamma + R_0 - \frac{\Gamma_0}{\gamma} - \frac{K_0}{\gamma^2} = 0, \qquad (20)$$

with

$$R_0 = aA_0 \sin\delta + \Omega_0^2 + \Omega_0\gamma_0 - \frac{\gamma_0^2}{2} - a^2, \qquad (21)$$

$$\Gamma_0 = 2\Omega_0 aA_0 \sin\delta + 2\Omega_0^2\gamma_0 - \Omega_0\gamma_0^2, \qquad (22)$$

and

$$K_0 = a^2A_0^2\cos^2\delta - \Gamma_0\gamma_0 + R_0\gamma_0^2 - \Omega_0\gamma_0^3 + \frac{\gamma_0^4}{4}. \qquad (23)$$

This result is in good agreement with the one obtained by Roberts and Buchsbaum[1].

Equation (20) describes a motion in a one-dimensional potential. It admits a solution in terms of elliptic integrals[8]· Then, Eqs.(18) prove for a second time that this problem is integrable.

Eq.(20) can also be written

$$\gamma^2\dot{\gamma}^2 = a^2(\gamma^2 - 1) - \left\{aA_0\sin\delta + (\gamma - \gamma_0)\left[\left(\frac{\gamma + \gamma_0}{2}\right) - \Omega_0\right]\right\}^2. \qquad (24)$$

Let us consider the interesting case when the particle is initially resonant and at rest $(\gamma_0 = \Omega_0 = 1)$. Letting $\gamma = 1 + \mu$, Eq.(24) becomes

$$\gamma^2\dot{\gamma}^2 = \mu\left[a^2(\mu + 2) - \frac{\mu^3}{4}\right]. \qquad (25)$$

As $\mu \geq 0$, the sign of the third order polynomial $Q(\mu) = a^2(\mu + 2) - \mu^3/4$ has to be positive. When $a < \sqrt{27}/2$, $Q(\mu)$ has only one real positive root. This implies that γ oscillates between unity and $1 + \mu m$ (μm is the positive root of $Q(\mu)$).

Assuming that a is very small $(a \ll 1)$, which is experimentally realistic, Eq.(25) yields

$$\mu_m \approx 2a^{2/3}. \qquad (26)$$

In this situation $(\gamma_0 = \Omega_0 = 1)$, equation (20) was solved numerically for different values of a. The maximum value of γ reached by the particle is compared to the one obtained through equation (26) on Fig.(1). A very good agreement between the two results is observed.

In conclusion, using canonical transformations, we have reduced the problem of relativistic motion of a charged particle in a constant homogeneous magnetic field and a transverse electric field to a time-dependent problem with a single degree of freedom. Noether's theorem was used to find a constant of motion for the system which permits one to show that the problem is integrable. An equation for the energy was derived. When the charged particle is initially resonant and at rest, it shows that the energy oscillates between two values. A tractable approximate expression for the maximum attainable energy was obtained for low values of a. This gives an upper limit for the frequency, that the particle can emit by Bremsstrahlung.

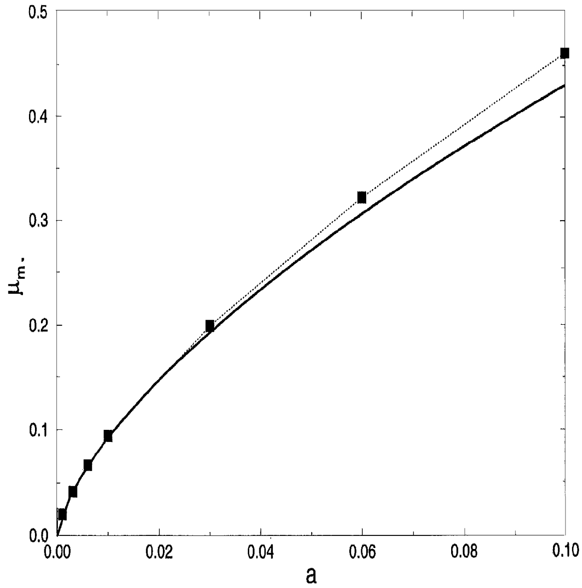

Figure 1 : Comparison between the maximum normalized kinetic energy reached by the particle calculated through Eq.(26) (full line) and Eq.(20) (full squares and dashed line).

REFERENCES

[1]C.S. Roberts and S.J. Buchsbaum, Phys. Rev. **135**, A381 (1964).

[2]H.R. Jory and A.W. Trivelpiece, J. Appl. Phys. **39**, 3053 (1968)

[3]H. Goldstein, *Classical Mechanics*, 2nd ed. (Addison - Wesley, New York,1980).

[4]R. D. Jones, Phys. Fl. **24**(3), 564 (1981).

[5]A.J. Lichtenberg and M.A. Liebermann, *Regular and Stochastic Motion* (Springer - Verlag, Berlin, 1983).

[6]S. Bouquet and A. Dewisme, *Nonlinear Evolution Equations and Dynamical Systems* (Proceedings Needs'91), edited by M. Boiti, L. Martina and F. Pe]mpinelli (World Scientific, Singapore, 1992).

[7]A. Dewisme and S. Bouquet, J. Math. Phys. **34**(3), 997 (1993).

[8]P.F. Byrd and M.D. Friedman, *Handbook of Elliptic Integrals for Engineers and Physicists* (Springer - Verlag, Berlin, 1954).