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SUBADDITIVE PERIODIC FUNCTIONS

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Abstract. Some conditions under which any subadditive function is periodic are presented. It is shown that the boundedness from below in a neighborhood of a point of a subadditive periodic (s.p.) function implies its nonnegativity, and the boundedness from above in a neighborhood of a point implies it nonnegativity and global boundedness from above. A necessary and sufficient condition for existence of a subadditive periodic extension of a function $f_0 : [0, 1) \to \mathbb{R}$ is given. The continuity, differentiability of a s.p. function is discussed, and an example of a continuous nowhere differentiable s.p. function is presented. The functions which are the sums of linear functions and s.p. functions are characterized. The refinements of some known results on the continuity of subadditive functions are presented.

Keywords: subadditive function, periodic function, periodic extension, concave function, continuity, continuous nowhere differentiable function.

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1. INTRODUCTION

Subadditive functions play an important role (cf. Hille-Phillips [4], Kuczma [5]). For instance norms, seminorms, moduli of continuity measures are subadditive. They appear also in fixed point theory in connection with nonlinear contraction mappings (cf. Boyd-Wong [1], also [9]).

Assume that $f : \mathbb{R} \to \mathbb{R}$ is subadditive i.e., for all $x, y \in \mathbb{R}$,

$$f(x + y) \leq f(x) + f(y).$$

If there is a point $p \neq 0$ such that $f(p) \leq 0$ and $f(-p) \leq 0$, then

$$f(x + p) \leq f(x) + f(p) \leq f(x) \quad \text{and} \quad f(x) \leq f(x + p) + f(-p) \leq f(x + p),$$

whence $f(x + p) = f(x)$ for all $x, y \in \mathbb{R}$, that is f is periodic of period p. It turns out that, under more general conditions, subadditivity implies periodicity. We prove that *if f is subadditive and $f(p) \leq 0$, $f(q) \leq 0$, for some real numbers p, q such that $pq < 0$ and $\frac{p}{q}$ is rational, then f is periodic.* This fact shows that there is a peculiar relationship between subadditivity and periodicity.

The present paper is devoted mainly to subadditive periodic functions $f : \mathbb{R} \to \mathbb{R}$. In the first section we recall some known results about regularity of subadditive functions and we propose their refinements involving a *measure of the density of a set at a point.* In particular, the result claiming the continuity of any subadditive functions, continuous at zero and vanishing at zero is improved. In the second section we present simple conditions under which the subadditivity of a function implies its periodicity.

In section 3 we prove that *any periodic subadditive function, bounded from below in a neighborhood of a point, is nonnegative.* Moreover*, if a periodic subadditive function is bounded from above in a neighborhood of a point, then it is globally bounded* (*and nonnegative*). Some examples of discontinuous periodic subadditive functions are given.

The main result of section 4 gives a necessary and sufficient condition for existence of a subadditive periodic extension of a function $f_0 : [0, 1) \to \mathbb{R}$. Some criterions for subadditivity of periodic functions are presented and the question of the continuity is discussed.

In section 5 we show that nowhere differentiable continuous periodic function of Takagi [13] (rediscovered by van der Waerden [14]) is subadditive.

In section 6 we consider the differentiability of subadditive periodic functions. We show, among others, that if f is a subadditive periodic and differentiable at a point x_0 such that $f(x_0) = 0$, then $f(x) = 0$ for all $x \in \mathbb{R}$.

In section 7 we give simple conditions which characterize the functions being the sums of linear functions and periodic subadditive functions.

2. AUXILIARY RESULTS

In the sequel the letters $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{R}_+$ and $\mathbb{R}_-$ denote positive integers, integers, rationals, reals, nonnegative reals, and nonpositive reals, respectively.

A real function f defined on an interval $I \subset \mathbb{R}$ is said to be *subadditive* if

$$x, y, x + y \in I \Longrightarrow f(x + y) \leq f(x) + f(y),$$

and *superadditive,* if $(-f)$ is subadditive. In general, in the theory of subadditive functions, the set I is assumed to be $\mathbb{R}$, $\mathbb{R}_+$ or (a, ∞) with $a \geq 0$.

It is well known that the regularity of a subadditive function strongly depends upon its behavior at the origin. Let us remark that, given $a > 0$, any function $f : \mathbb{R} \to \mathbb{R}$ such that $a \leq f(x) \leq 2a$ is subadditive, and, of course, f can be very irregular.

One of the most important properties of subadditive functions reads as follows (Rosenbaum [11], cf. also Hille-Phillips [4] Theorem 7.8.2, 7.8.3, and Kuczma [5], Chapter XI):

Theorem 2.1. *Let $f : \mathbb{R} \to \mathbb{R}$ be a subadditive function such that $f(0) = 0$.*

(1) *If f is right-continuous at* 0 *then, for every $x \in \mathbb{R}$, there exist the one-sided limits* $f(x-)$, $f(x+)$ *and*

$$f(x+) \leq f(x) \leq f(x-).$$

(2) *If f is left-continuous at* 0 *then, for every $x \in \mathbb{R}$, there exist* $f(x-)$, $f(x+)$ *and*

$$f(x-) \leq f(x) \leq f(x+).$$

(3) *If f is continuous at* 0, *then f is continuous everywhere.*

Remark 2.2 (cf. Theorem 2.10)**.** Part (3) follows from parts (1) and (2). Moreover, it is easy to see that the continuity of f at 0 can be replaced by its upper semi-continuity at 0.

Assuming additionally the bijectivity of f we have the following result (cf. Matkowski and Świątkowski [7]).

Theorem 2.3. *If $f : \mathbb{R}_+ \to \mathbb{R}_+$ is subadditive, bijective and continuous at* 0, *then it is a homeomorphism of* $\mathbb{R}_+$.

In Matkowski-Świątkowski [8] it is shown that in this theorem the (right-) continuity of f at 0 cannot be replaced by the boundedness of f in a neighborhood of 0.

We have also (cf. Matkowski-Świątkowski [8], Theorem 2.1)

Theorem 2.4. *If $f : (0, \infty) \to \mathbb{R}_+$ is subadditive, one-to-one and $f(0+) = 0$, then it is continuous on* $(0, \infty)$.

Let us note the following (cf. Kuczma [5], Lemma 16.1.9)

Remark 2.5. Every odd subadditive function $f : I \to \mathbb{R}$ in I such that $I = -I$ is additive.

In fact, for all $x, y \in I$ such that $x + y \in I$, we have

$$f(x + y) = -f(-x - y) \geq -[f(-x) + f(-y)] = f(x) + f(y),$$

so f is additive.

There are a lot of important functions which are subadditive and even: for instance $f(x) = |x|$ for $x \in \mathbb{R}$ or, more generally, each norm in a linear space, and the moduli of continuity. As a consequence of Theorem 2.1 we get

Theorem 2.6. *If $f : \mathbb{R} \to \mathbb{R}$ is subadditive, even, right- or left-continuous at* 0 *and* $f(0) = 0$, *then it is continuous everywhere.*

Let us note the following

Lemma 2.7. *Let $a > 0$ be fixed. Suppose that $f : (a, \infty) \to \mathbb{R}$ is subadditive, nonnegative and $f(x_0) = 0$ for some $x_0 > a$. If f is continuous at x_0 then, for every $n \in \mathbb{N}$, the function f is continuous at nx_0 and $f(nx_0) = 0$.*

Proof. Let $n \in \mathbb{N}$ be arbitrarily fixed. Then $0 \leq f(nx_0) \leq nf(x_0) = 0$. For any sequence $y_k \to nx_0$, and sufficiently large $k \in \mathbb{N}$, we have

$$0 \leq f(y_k) \leq nf\left(n^{-1}y_k\right).$$

Letting $k \to \infty$ we obtain $\lim_{k\to\infty} f(y_k) = 0 = f(nx_0)$. □

For the sets $A_1, ..., A_m \subset \mathbb{R}$ put

$$\sum_{i=1}^{m} A_i := \left\{ \sum_{i=1}^{m} a_i : a_1 \in A_1, \ldots, a_m \in A_m \right\}.$$

Remark 2.8. The assumption $f(0+) = 0$ in Theorem 2.1 (1) can be replaced by the following considerably weaker one: there exist $A_1, \ldots, A_m \subset \mathbb{R}_+$ and $\delta > 0$ such that, for each $i \in \{1, \ldots, m\}$, the restriction $f|_{A_i}$ satisfies the condition $f|_{A_i}(0+) = 0$ and

$$(0, \delta) \subset \sum_{i=1}^{m} A_i.$$

To show this take a sequence $x_n \in (0, \delta)$ such that $\lim_{n\to\infty} x_n = 0$. By the assumption there are some sequences $a_{i,n}$, $n \in \mathbb{N}$, $i = 1, \ldots, m$, such that

$$x_n = a_{1,n} + ... + a_{m,n}, \qquad n \in \mathbb{N}.$$

From the subadditivity of f, for every $n \in \mathbb{N}$, we have

$$f(x_n) \leq f(a_{1,n}) + ... + f(a_{m,n}) = f|_{A_i}(a_{1,n}) + ... + f|_{A_m}(a_{m,n}),$$

whence, letting $n \to \infty$, we conclude that $f(0+) = 0$.

Let l_1 denote the one-dimensional Lebesgue measure. Given a Lebesgue measurable set $A \subset \mathbb{R}$ and $b \in \mathbb{R}$, the number

$$\lambda_b^+(A) := \lim\inf_{r\to 0+} \frac{l_1\left(A \cap (b, b+r)\right)}{r},$$

is called a *measure of the right-density of A at the point b*. Replacing here the interval $(b, b+r)$ by $(b-r, b)$ we define a *measure of the left-density of A at the point b* and denote by $\lambda_b^-(A)$, Of course we have

$$0 \leq \lambda_b^+(A) \leq 1, \qquad 0 \leq \lambda_b^-(A) \leq 1.$$

Remark 2.9. Let $A_i \subset \mathbb{R}$ be Lebesgue measurable and $b_i \in \mathbb{R}$ for $i = 1, ..., m$. If

$$\sum_{i=1}^{m} \lambda_{b_i}^+(A_i) > 1$$

then, according to the Raikov theorem [10], there exists a $\delta > 0$ such that

$$\left(\sum_{i=1}^{m} b_i, \sum_{i=1}^{m} b_i + \delta \right) \subset \sum_{i=1}^{m} A_i.$$

(The Raikov theorem may be treated as a local version of the Steinhaus theorem [12].)

Let $A \subset \mathbb{R}$ be such that $\lambda_0^+(A) > 0$. Take $m \in \mathbb{N}$ such that $m\lambda_0^+(A) > 1$. Applying Raikov Theorem with $A_i := A$ and $b_i = 0$ for $i = 1, ..., m$, we infer that, for some $\delta > 0$,

$$(0, \delta) \subset \sum_{i=1}^{m} A.$$

Hence, applying Remark 2.8, we obtain the following improvements of Theorems 2.1–2.4.

Theorem 2.10. *Let $A \subset \mathbb{R}$ be a Lebesgue measurable set. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is subadditive and such that $f(0) = 0$.*

(1) *If $\lambda_0^+(A) > 0$ and $f|_A$ is right-continuous at 0 then, for every $x \in \mathbb{R}$, there exist the one-sided limits $f(x-)$, $f(x+)$ and*

$$f(x+) \le f(x) \le f(x-).$$

(2) *If $\lambda_0^-(A) > 0$ and $f|_A$ is left-continuous at 0 then, for every $x \in \mathbb{R}$, there exist the one-sided limits $f(x-)$, $f(x+)$ and*

$$f(x-) \le f(x) \le f(x+).$$

(3) *If $\lambda_0^+(A)\lambda_0^-(A) > 0$ and $f|_A$ is continuous at 0, then f is continuous everywhere.*

Theorem 2.11. *Let $f : (0, \infty) \to \mathbb{R}$ be subadditive and $A \subset (0, \infty)$ be a Lebesgue measurable set such that $\lambda_0^+(A) > 0$. Suppose that*

$$\lim_{x \to 0+} f|_A(x) = 0.$$

Then, for every $x \in (0, \infty)$, there exist $f(x-)$, $f(x+)$ and

$$f(x+) \le f(x) \le f(x-).$$

Moreover, if f is one-to-one then f is continuous.

We omit the easy to formulate counterparts of this theorem for subadditive functions defined on $\mathbb{R}_+$, $(-\infty, 0)$ and $\mathbb{R}_-$.

Theorem 2.12. *Let $f : \mathbb{R} \to \mathbb{R}$ be even, subadditive and $A \subset (0, \infty)$ a Lebesgue measurable set such that $\lambda_0^+(A) > 0$. If $f|_A(0+) = 0$ then f is continuous in $\mathbb{R}$.*

Example 2.13. Define $f_0 : [0, 1) \to \mathbb{R}$ by

$$f_0(x) := \begin{cases} x & \text{for} \quad x \in [0, 2^{-1}] \cap \mathbb{Q}, \\ 1 - x & \text{for} \quad x \in (2^{-1}, 1] \cap \mathbb{Q}, \\ 1 & \text{for} \quad x \in (0, 1] \backslash \mathbb{Q}, \end{cases}$$

and let $f : \mathbb{R} \to \mathbb{R}$ be the periodic extension of f_0. It is easy to verify that f is subadditive, even, $f(0) = 0$, the restriction $f|_{\mathbb{Q}}$ is right- and left-continuous at 0, but f is not continuous.

This shows that if the set A is too "meagre", the continuity of the restriction $f|_A$ at 0 does not imply the continuity of the subadditive function f. Note that for $A = \mathbb{Q}$ there is no $m \in \mathbb{N}$ such that $(0, \delta) \subset \sum_{i=1}^{m} A$ for some $\delta > 0$.

Theorem 2.11 is a special case of the following

Proposition 2.14. *Let $A \subset (0, \infty)$ be a Lebesgue measurable set such that $(0, \delta) \subset \sum_{i=1}^{m} A$ for some $\delta > 0$ and $m \in \mathbb{N}$. Suppose that $f : (0, \infty) \to \mathbb{R}$ is subadditive and $f|_A(0+) = 0$. Then, for every $x \in (0, \infty)$, there exist $f(x-)$, $f(x+)$ and*

$$f(x+) \le f(x) \le f(x-).$$

Moreover, if f is one-to-one then f is continuous.

Remark 2.15. Let $C \subset \mathbb{R}$ be the Cantor set. It is well known that $C + C = [0, 2]$ (cf. Kuczma [5], Exercise 2.13). Hence, for $A := C \cap (0, \infty)$ we have $A + A = (0, 2]$. Suppose that $f : (0, \infty) \to \mathbb{R}$ is subadditive and $f|_A(0+) = 0$. Then the conditions of Proposition 2.14 are fulfilled.

Since, obviously $\lambda_0^+(C) = 0$, the assumption $\lambda_0^+(A) > 0$ in the previous theorems is not satisfied.

In the sequel we shall need the following

Lemma 2.16. *Suppose that $f : (0, \infty) \to \mathbb{R}$ is subadditive and $a \in (0, \infty)$. If $f\big|_{[a,b]}$ is bounded from above for some $b \in (a, \infty)$, then it is bounded for every $b \in (a, \infty)$.*

Proof. Assume $f(x) \le M$ for $x \in [a, b]$. Take an arbitrary $n \in \mathbb{N}$ and $y \in [a, a + n(b - a)]$. Then $y = x + k(b - a)$ with an $x \in [a, b]$ and $k \in \{0, 1, ..., n\}$, and from the subadditivity of f we get

$$f(y) \le f(x) + kf(b - a) \le M + nf(b - a).$$

Thus f is bounded from above on $[a, a + n(b - a)]$ for every $n \in \mathbb{N}$. Moreover, for $n \in \mathbb{N}$ and $x \in [a, na]$ by the subadditivity of f we have

$$f(x) \ge f((n + 1)a) - f((n + 1)a - x) \ge f((n + 1)a) - \sup f([a, na]).$$

This completes the proof. □

Example 2.17. Let $\alpha : \mathbb{R} \to \mathbb{R}$ be a discontinuous additive function. The function $f : \mathbb{R}_+ \to \mathbb{R}$,

$$f(x) := \begin{cases} |\alpha(x)| & \text{for} \quad x \in [0, 1], \\ 0 & \text{for} \quad x > 1. \end{cases}$$

is subadditive, bounded on $[1, \infty)$, and unbounded from above on a neighborhood of any point of the interval $[0, 1]$. This shows that, in Lemma 2.16, even the global boundedness of a subadditive function f in the interval $[a, \infty)$ has no influence on the behavior of f in the interval $(0, a)$.

In this connection let us note

Remark 2.18 (cf. Kuczma [5], Theorem 16.2.5)**.** Every subadditive and measurable function is locally bounded.

Recall the following criterion of subadditivity.

Remark 2.19. Let $0 < a \leq \infty$ be fixed and let $I \subset \mathbb{R}$ denote an interval of the endpoints 0 and a. Assume that $f : I \to \mathbb{R}$, and $f(0) \geq 0$ if $0 \in I$.

(i) If the function
$$(I \setminus \{0\}) \ni x \to \frac{f(x)}{x}$$
is decreasing, then f is subadditive (cf. for instance Hille-Philips [4], p. 239, where the case $a = \infty$ is considered).

(ii) If f is concave and $f(0+) \geq 0$, then f is subadditive.

To show the second part assume that $0 \in I$ and take $x, y \in I$, $0 < x < y$. From the concavity of f,
$$f(x) = f\left((1 - \frac{x}{y})0 + \frac{x}{y}y\right) \geq (1 - \frac{x}{y})f(0) + \frac{x}{y}f(y) \geq \frac{x}{y}f(y),$$
which means that the function $(I \setminus \{0\}) \ni x \to \frac{f(x)}{x}$ is decreasing and, by the first part, f is subadditive in I.

If $0 \notin I$, then we extend f on $I \cup \{0\}$ putting $f(0+)$ as the value at 0. Since the extension is concave, by the first part of the proof it is subadditive and so is f.

3. CONDITIONS UNDER WHICH SUBADDITIVITY IMPLIES PERIODICITY

The following result shows that the subadditivity and periodicity are closely related.

Theorem 3.1. *Suppose that $f : \mathbb{R} \to \mathbb{R}$ is subadditive.*

(1) *If there exist $p, q \in \mathbb{R}$ such that*
$$p < 0 < q, \qquad \frac{p}{q} \in \mathbb{Q}, \qquad f(p) \leq 0, \qquad f(q) \leq 0,$$
then f is periodic; moreover
$$f(0) = f(p) = f(q) = 0.$$

(2) *If there exist $p_1, p_2, q_1, q_2 \in \mathbb{R}$ such that*
$$p_i < 0 < q_i, \quad \frac{p_i}{q_i} \in \mathbb{Q}, \quad f(p_i) \leq 0, \quad f(q_i) \leq 0 \quad \text{for } i = 1, 2; \qquad \frac{q_2}{q_1} \notin \mathbb{Q},$$
then f is microperiodic, i.e., f has arbitrary small positive periods.

Proof. 1) By the subadditivity of f, for all $x \in \mathbb{R}$,

$$f(x+p) \le f(x) + f(p) \le f(x); \qquad f(x+q) \le f(x) + f(q) \le f(x),$$

whence, by induction,

$$f(x+kp) \le f(x); \qquad f(x+kq) \le f(x), \qquad x \in \mathbb{R},\ k \in \mathbb{N} \cup \{0\}.$$

Since $p < 0 < q$, the commensurability of p and q implies that $mp + nq = 0$ for some $m, n \in \mathbb{N}$. Putting $r := nq$ we have $mp = -r$ and, from the above inequalities

$$f(x-r) \le f(x), \qquad f(x+r) \le f(x), \qquad x \in \mathbb{R},$$

whence $f(x+r) = f(x)$ for $x \in \mathbb{R}$. From the subadditivity of f we have $0 \le f(0)$. Consequently, for all $m, n \in \mathbb{N}$,

$$0 \le f(0) = f(mp + nq) \le mf(p) + nf(q) \le 0,$$

which implies that $f(0) = f(p) = f(q) = 0$.

2) According to the first part and its proof, f is periodic; there are $m_i, n_i \in \mathbb{N}$ such that $m_i p_i + n_i q_i = 0$; and the numbers $r_i := n_i q_i$ for $i = 1, 2$, are the periods of f. Thus $f(x + kr_1 + lr_2) = f(x)$ for all $k, l \in \mathbb{Z}$ and $x \in \mathbb{R}$. Since $\frac{r_2}{r_1} = \frac{n_2}{n_1}\frac{q_2}{q_1}$ is irrational, the set $\{kr_1 + lr_2 : k, l \in \mathbb{Z}\}$ is dense in $\mathbb{R}$. The proof is complete. □

Remark 3.2. The commensurability of p and q in the above theorem is essential.

To show this consider the following

Example 3.3. For an irrational $a < 0$ put $A := \{n + ka : n \in \mathbb{N}, k \in \mathbb{Z}\}$ and fix $c > 0$. The function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) := \begin{cases} 0 & \text{for} \quad x \in A, \\ 1 & \text{for} \quad x \in \mathbb{R} \backslash (A \cup \{0\}), \\ c & \text{for} \quad x = 0, \end{cases}$$

is subadditive. Moreover, f is periodic (a-periodic) if, and only if, $c = 1$. Thus, taking $c \ne 1$ we obtain a non-periodic subadditive function f with a dense set $\mathbf{Z}(f)$ where

$$\mathbf{Z}(f) := \{x \in \mathbb{R} : f(x) = 0\}.$$

Remark 3.4. If $f : \mathbb{R} \to \mathbb{R}$ is p-periodic and subadditive then $f(p) \ge 0$.

The following result is easy to prove.

Proposition 3.5. *Suppose that $f : \mathbb{R}_+ \to \mathbb{R}$ (or $f : (0, \infty) \to \mathbb{R}$) is subadditive.*

(1) *If $f(p) = 0$ for some $p > 0$, then $f(x + np) \le f(x)$ for all $x \in \mathbb{R}_+$ (resp., for all $x > 0$) and for all $n \in \mathbb{N}$. If, moreover, f is nonnegative then $f(np) = 0$ for all $n \in \mathbb{N}$.*

(2) *If* $f(p) = f(q) = 0$ *for some* $p, q > 0$, *then* $f(x + mp + nq) \leq f(x)$ *for all* $x \in \mathbb{R}_+$ (*resp., for all* $x > 0$) *and for all* $n, m \in \mathbb{N}$. *If, moreover,* f *is nonnegative, then* $f(mp + nq) = 0$ *for all* $n, m \in \mathbb{N}$.

The periodic subadditive functions have the following easy to verify

Properties

1. Suppose that $f, g : \mathbb{R} \to \mathbb{R}$ are periodic and subadditive of periods p and q, respectively. If p and q are commensurable, i.e. $\frac{p}{q}$ is rational, then, for all $a, b \geq 0$, the function $af + bg$ is subadditive and periodic.
2. If $f : \mathbb{R} \to \mathbb{R}$ is subadditive and p-periodic and $a \in \mathbb{R}$, $a \neq 0$, then the function $g : \mathbb{R} \to \mathbb{R}$, $g(x) := f(ax)$, is subadditive and pa^{-1}-periodic.
3. If $f : \mathbb{R} \to \mathbb{R}$ is subadditive and nonnegative, then
$$x, y \in \mathbf{Z}(f) \Longrightarrow x + y \in \mathbf{Z}(f).$$
4. If $f : \mathbb{R} \to \mathbb{R}$ is subadditive, p-periodic and $g(x) := f(-x)$, then
$$x \in \mathbf{Z}(f) \Longleftrightarrow p - x \in \mathbf{Z}(g).$$
5. If $f : \mathbb{R} \to \mathbb{R}$ is subadditive, periodic and $x_0 \neq 0$ is such that $x_0, -x_0 \in \mathbf{Z}(f)$, then f is x_0-periodic.
6. If $f : \mathbb{R} \to \mathbb{R}$ is subadditive, p-periodic, non-negative and $x_0 \in \mathbf{Z}(f)$, then $\{nx_0 + kp : n \in \mathbb{N}, k \in \mathbb{Z}\} \subset \mathbf{Z}(f)$.
7. If $g, h : \mathbb{R} \to \mathbb{R}$ are p-periodic and subadditive, then
$$f(x) := \max(g(x), h(x)), \qquad x \in \mathbb{R},$$
is subadditive and p-periodic.

Theorem 3.6. *Let* $f : \mathbb{R} \to \mathbb{R}$ *be subadditive, periodic and bounded from below in a neighborhood of a point. If* $\mathbf{Z}(f) \neq \emptyset$ *and* $\mathbf{Z}(f)$ *is not dense in* $\mathbb{R}$, *then there exists* $p > 0$ *such that* $\mathbf{Z}(f) = \{kp : k \in \mathbb{Z}\}$ *and* f *is* p*-periodic.*

Proof. Let p be the infimum of all positive periods of f. Since $\mathbf{Z}(f) \neq \emptyset$ is not dense, we have $p > 0$ and f is p-periodic. (Indeed, the set P of periods - being a subgroup of $\mathbb{R}$ – would be dense, but $\mathbf{Z}(f) + P = \mathbf{Z}(f)$ and $\mathbf{Z}(f)$ is not dense.) We may assume that $p = 1$.

For an indirect argument suppose that $\mathbf{Z}(f) \cap (0, 1)$ is nonempty and put $x_0 :=$ inf $(\mathbf{Z}(f) \cap (0, 1))$. Since $\mathbf{Z}(f)$ is not dense, we have $0 < x_0 < 1$. Of course, there exist a unique $k \in \mathbb{N}$ such that
$$\frac{1}{k+1} \leq x_0 < \frac{1}{k},$$
and a decreasing sequence $z_n \in \mathbf{Z}(f)$ such that
$$\frac{1}{k+1} \leq x_0 \leq z_n < \frac{1}{k}, \quad n \in \mathbb{N}; \qquad \lim_{n\to\infty} z_n = x_0.$$

By Theorem 4.1 given below f is non-negative and due to the 1-periodicity and subadditivity of f, we have

$$0 \le f\left((k+1)z_n - 1\right) = f\left((k+1)z_n\right) \le (k+1)f(z_n) = 0,$$

that is $(k+1)z_n - 1 \in \mathbf{Z}(f)$ for all $n \in \mathbb{N}$. Since

$$0 \le (k+1)z_n - 1 < 1, \qquad n \in \mathbb{N},$$

we have either

$$0 < (k+1)z_n - 1, \qquad n \in \mathbb{N},$$

or, for all sufficiently large n,

$$0 = (k+1)z_n - 1.$$

In the first case, from the definition of x_0, we get

$$x_0 \le (k+1)z_n - 1, \qquad n \in \mathbb{N},$$

whence, letting $n \to \infty$, we obtain $x_0 \ge \frac{1}{k}$ that is a contradiction.

In the second case we have $x_0 = \frac{1}{k+1} \in \mathbf{Z}(f)$. By Property 3, for any $m \in \mathbb{N}$, the number $\frac{m}{k+1} \in \mathbf{Z}(f)$. Hence

$$f(-x_0) = f\left(-\frac{1}{k+1}\right) = f\left(1 - \frac{1}{k+1}\right) = f\left(\frac{k}{k+1}\right) = 0,$$

that is $-x_0 \in \mathbf{Z}(f)$. By Property 5, the number x_0 is a period of f. This contradiction completes the proof. □

To show that, in the above theorem, the assumptions that $\mathbf{Z}(f)$ is not dense in $\mathbb{R}$ and the local boundedness from below are indispensable, consider the following

Example 3.7. Let $\alpha : \mathbb{R} \to \mathbb{R}$ be a discontinuous additive function such that $\alpha(1) = 0$ (cf. Kuczma [5], Corollary 5.2.2).

Then $f := \alpha$, being additive, is subadditive, is not bounded from below at any point and, as every rational number is a period of f, the set of periods of f is dense.

The function $f := |\alpha|$ is subadditive, globally bounded from below with a dense set of periods.

4. LOCALLY BOUNDED PERIODIC SUBADDITIVE FUNCTIONS

In this section we prove that, under a weak regularity condition, every periodic subadditive function must be nonnegative. We may assume, without any loss of generality, that the considered functions are 1-periodic.

We begin with the following

Theorem 4.1. *If $f : \mathbb{R} \to \mathbb{R}$ is periodic, subadditive and bounded from below in a neighborhood of a point, then f is nonnegative on $\mathbb{R}$.*

Proof. Suppose that there exists an irrational $z \in \mathbb{R}$ such that $f(z) < 0$. By Kronecker's theorem ([3], p. 69, Theorem C), the set $\{k+nz : n, k \in \mathbb{Z}\}$ is dense in $\mathbb{R}$. It turns out its subset $A := \{k+nz : n \in \mathbb{N}, k \in \mathbb{Z}\}$ is also dense in $\mathbb{R}$ (cf. [6], Lemma 4). According to the assumptions, there would exist an open interval $I \subset \mathbb{R}$ and $a \in \mathbb{R}$ such that $a \leq f(x)$ for all $x \in I$. By the density of the set A we could find a sequence of points (a_j), $a_j = k_j + n_j z \in I$ such that $a_i \neq a_j$ for $i \neq j$. Hence, applying in turn the periodicity (i.e. the equality $f(x+1) = f(x)$ for $x \in \mathbb{R}$) and subadditivity of f,

$$a \leq f(a_j) = f(k_j + n_j z) = f(n_j z) \leq n_j f(z), \qquad j \in \mathbb{N}.$$

As the sequence of positive integers (n_j) is unbounded, it follows that $a \leq -\infty$. This contradiction proves that $f(z) \geq 0$ for all $z \in \mathbb{R}\backslash\mathbb{Q}$.

Now take arbitrary $z \in \mathbb{Q}$. Then $z = \frac{m}{n}$ for some $m \in \mathbb{Z}$ and $n \in \mathbb{N}$. By the subadditivity of f we have $f(0) \geq 0$. Since f is 1-periodic, $0 \leq f(0) = f(m)$. The subadditivity of f implies that

$$0 \leq \frac{1}{n} f(m) = \frac{1}{n} f\left(n \frac{m}{n}\right) \leq \frac{1}{n}\left[n f\left(\frac{m}{n}\right)\right] = f(z),$$

which completes the proof. □

Example 4.2. Let $\alpha : \mathbb{R} \to \mathbb{R}$ be an arbitrary discontinuous additive function. Then $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \alpha(x) - \alpha(1)x$ for $x \in \mathbb{R}$, is additive (therefore, also subadditive), microperiodic (as every $p \in \mathbb{Q}$ is a period of f) and odd. Moreover the graph of f is dense in $\mathbb{R}^2$.

Remark 4.3. This example shows that the assumption of the boundedness from below of a subadditive function f in a neighborhood of a point in the above theorem is essential.

Note also that the function $|f|$ is subadditive, and, of course, nonnegative. Since the graph of $|f|$ is dense in $\mathbb{R} \times \mathbb{R}_+$, the function $|f|$ is not bounded from above in a neighborhood of any point.

However we have the following

Theorem 4.4. *If $f : \mathbb{R} \to \mathbb{R}$ is periodic, subadditive and bounded from above on a set $A \subset \mathbb{R}$ either of positive Lebesgue measure, or of second category having the property of Baire, then f is nonnegative and globally bounded on $\mathbb{R}$.*

Proof. Assume first that f is bounded from above on a set A such that $\operatorname{int} A \neq \emptyset$. By Lemma 2.16, there is $c > 0$ such that f is locally bounded in (c, ∞). The periodicity of f implies the global boundedness of f on $\mathbb{R}$. Now the nonnegativity of f results from Theorem 4.1.

Assume that $A \subset \mathbb{R}$ is of positive Lebesgue measure and $f(x) \leq M$ for all $x \in A$ and some $M > 0$. Then, for all $x, y \in A$, we have $f(x+y) \leq f(x) + f(y) \leq 2M$, so f is bounded from above on the set $A + A$. As, by Steinhaus theorem (cf. Kuczma [5], Theorem 3.7.1), the interior of the set $A + A$ is nonempty, the result follows from what has been already proved.

If $A \subset \mathbb{R}$ is of the second category having the property of Baire, we can argue similarly applying Theorem of Piccard (cf. Kuczma [5], Theorem 2.9.1). □

Applying Theorems 4.1 and 3.6 we obtain

Corollary 4.5. *Let $f : \mathbb{R} \to \mathbb{R}$ be subadditive, periodic and suppose that*

$$\inf\{f(x) : x \in \mathbb{R}\} \leq 0.$$

If f is lower semicontinuous, then f is nonnegative, $f(0) = 0$, and either $\mathbf{Z}(f)$ is dense in $\mathbb{R}$ and $f = 0$, or there is $p > 0$ such that $\mathbf{Z}(f) = \{kp : k \in \mathbb{Z}\}$ and f is p-periodic.

Proof. Assume that $\mathbf{Z}(f)$ is dense. By the lower semicontinuity of f, the set $f^{-1}((0,\infty))$ is open in $\mathbb{R}$. If it were not empty, then the set $\mathbf{Z}(f) \cap f^{-1}((0,\infty))$ would be nonempty, what is impossible. Thus $f \leq 0$ and, consequently, $f = 0$. □

To see that the lower semicontinuity of f in the above corollary is essential consider the following

Example 4.6. Let $f : \mathbb{R} \to \mathbb{R}$ be 1-periodic and such that $f(x) = x$ for $x \in (0, 1]$. To show that f is subadditive take $x, y \in \mathbb{R}$. Then $x = m + s$, $y = n + t$ for uniquely determined $m, n \in \mathbb{Z}$ and $s, t \in [0, 1)$. If $s + t \in (0, 1]$ then

$$f(x + y) = f(s + t) = s + t = f(x) + f(y).$$

If $s + t \in (1, 2)$ then

$$f(x + y) = f(s + t) = f(s + t - 1) = s + t - 1 = f(x) + f(y) - 1 \leq f(x) + f(y).$$

Thus f is subadditive. Moreover

$$\inf\{f(x) : x \in \mathbb{R}\} = 0$$

and $f(x) > 0$ for every $x \in \mathbb{R}$. Consequently $\mathbf{Z}(f) = \emptyset$ (in particular, $f(0) \neq 0$).

Remark 4.7. If $f : \mathbb{R} \to \mathbb{R}$ is subadditive and 1-periodic and there is a rational x_0 such that $f(x_0) \leq 0$, then $f(0) = 0$.

In fact, we have $x_0 = \frac{m}{n}$ for some $m \in \mathbb{Z}$, $n \in \mathbb{N}$, and, by the 1-periodicity and subadditivity of f,

$$0 \leq f(0) = f(m) = f\left(n\frac{m}{n}\right) \leq nf\left(\frac{m}{n}\right) = nf(x_0) \leq 0.$$

In this connection the following question arises.

Suppose that $f : \mathbb{R} \to \mathbb{R}$ is subadditive, periodic, nonnegative and there exists an irrational number x_0 such that $f(x_0) = 0$. Is it then true that $f(0) = 0$? To see that the answer is no consider the following

Example 4.8. For an irrational $r \in \mathbb{R}$ put $A := \{n + kr : n \in \mathbb{N},\ k \in \mathbb{Z}\}$ and define $f : \mathbb{R} \to \mathbb{R}$,

$$f(x) := \begin{cases} 0 & \text{for} \quad x \in A, \\ 1 & \text{for} \quad x \notin A. \end{cases}$$

It is easy to see that f is subadditive, 1-periodic and $f(0) = 1$.

Modifying slightly the argument applied in the proof of Theorem 4.1 we prove the following

Theorem 4.9. *If $f : \mathbb{R}_+ \to \mathbb{R}$ is subadditive, 1-periodic and bounded from below in a neighborhood of a point, then f is nonnegative on $\mathbb{R}_+$.*

Proof. As in the proof of Theorem 4.1, suppose that there is $z \in \mathbb{R}_+ \setminus \mathbb{Q}$ such that $f(z) < 0$. Take an open interval $I \subset \mathbb{R}_+$ and $\alpha \in \mathbb{R}$ such that $\alpha \leq f(x)$ for all $x \in I$, and put $A := \{k + nz : k, n \in \mathbb{N}\}$. For $a \in \mathbb{R}$ denote by $E(a)$ the entire part of a. Now the set $B := \{a - E(a) : a \in A\}$ is dense in $[0, 1]$. It follows that there exists a strictly increasing sequence $n_j \in \mathbb{N}$ and two sequences $k_j, m_j \in \mathbb{N}$ such that

$$a_j := k_j + n_j z \in m_j + I \qquad \text{for} \quad j \in \mathbb{N},$$

where $m + I := \{m + x : x \in I\}$. Hence, for all $j \in \mathbb{N}$,

$$\alpha \leq f(a_j) = f(k_j + n_j z) = f(n_j z) \leq n_j f(z),$$

whence, letting $j \to \infty$, we obtain $\alpha \leq -\infty$ that is a contradiction. For $z \in \mathbb{Q} \cap \mathbb{R}_+$ we can repeat the argument used in the proof of Theorem 4.1. □

Let us remark that Theorem 4.1 follows from this result.

From Lemma 2.16 and Theorem 4.9 we immediately obtain the following generalization of Theorem 4.4.

Theorem 4.10. *If a function $f : \mathbb{R}_+ \to \mathbb{R}$ is subadditive, periodic and bounded from above on a set $A \subset \mathbb{R}_+$ either of positive Lebesgue measure, or of second category having the property of Baire, then f is nonnegative and globally bounded on $\mathbb{R}_+$.*

Note that the measurability and the global boundedness of a subadditive periodic function do not imply its continuity.

5. NECESSARY AND SUFFICIENT CONDITIONS FOR SUBADDITIVITY AND CONTINUITY OF PERIODIC EXTENSIONS

For any function $f_0 : [0, 1) \to \mathbb{R}$ there is a unique periodic function $f : \mathbb{R} \to \mathbb{R}$ such that $f|_{[0,1)} = f_0$. The function f is called a periodic extension of f_0 on $\mathbb{R}$. In a similar way we define the periodic extension of f_0 on $\mathbb{R}_+$.

Let us mention that Bruckner [2] considered a non-periodic extension problem related to subadditive functions.

The main result of these section reads as follows:

Theorem 5.1. *Let $f : \mathbb{R} \to \mathbb{R}$ be a periodic extension of $f_0 : [0, 1) \to \mathbb{R}$. Then f is subadditive if, and only if, f_0 satisfies the following conditions:*

(1) *the function f_0 is subadditive;*

(2) *the function* $g_0 : (0,1] \to \mathbb{R}$*, defined by*

$$g_0(x) := f_0(1-x), \qquad x \in (0,1],$$

is subadditive.

Proof. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a periodic extension of $f_0 : [0,1) \to \mathbb{R}$ and f_0 satisfies both conditions. Take arbitrary $x, y \in \mathbb{R}$. Then $x = m + s$, $y = n + t$ where $m, n \in \mathbb{Z}$ and $s, t \in [0,1)$ are uniquely determined. If $0 \le s + t < 1$ then, by the definition of f and subadditivity of f_0,

$$f(x+y) = f(s+t) = f_0(s+t) \le f_0(s) + f_0(t) = f(x) + f(y).$$

If $1 \le s + t < 2$ then $0 \le s + t - 1 < 1$, $0 < 2 - (s+t) \le 1$ and by the definition of g_0 and its subadditivity, we obtain

$$\begin{aligned} f(x+y) &= f(s+t-1) = f_0(s+t-1) = f_0(1 - [(2-(s+t)]) = \\ &= g_0(2-(s+t)) = g_0((1-s)+(1-t)) \le \\ &\le g_0(1-s) + g_0(1-t) = f_0(s) + f_0(t) = f(x) + f(y), \end{aligned}$$

which completes the proof of the "if" part of the theorem.

If f is subadditive, then $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) := f(1-x)$ is also subadditive. Hence the functions $f_0 := f|_{[0,1)}$ and $g_0 := g|_{(0,1]}$ are subadditive. □

This result can be treated as a criterion of subadditivity of periodic functions.

As an application we obtain the following results.

Theorem 5.2. *Let* $f : \mathbb{R} \to \mathbb{R}$ *be a periodic extension of* $f_0 : [0,1) \to \mathbb{R}$*. Then* f *is subadditive and even if, and only if,* f_0 *is subadditive in* $[0,1)$*, and*

$$f_0(1-x) = f_0(x), \qquad x \in (0,1).$$

Proof. Assume that $f_0(1-x) = f_0(x)$ for all $x \in (0,1)$. Let f be the periodic extension of f_0. Take arbitrary $x \in \mathbb{R}$. Then $x = m + s$ for uniquely determined $m \in \mathbb{Z}$ and $s \in [0,1)$. Note that $-x = (-1-m) + (1-s)$, where $-(1+m) \in \mathbb{Z}$ and $1 - s \in (0,1]$. Hence, if $s \in (0,1)$, then $1 - s \in (0,1)$, and we have

$$f(-x) = f_0(1-s) = f_0(s) = f(x),$$

which proves that f is even. The converse implication is obvious.

Since $g_0 = f_0$, the subadditivity part results from Theorem 4.10. □

Theorem 5.3. *If* $f_0 : [0,1) \to \mathbb{R}$ *is increasing and subadditive then its periodic extension is subadditive.*

Proof. Since the function $g_0 : (0,1] \to \mathbb{R}_+$, $g_0(x) = f_0(1-x)$ is decreasing and non-negative, it is subadditive. Now the result follows from Theorem 5.1. □

Theorem 5.4. *If* $f_0 : [0,1) \to \mathbb{R}_+$ *is concave, then its periodic extension is subadditive.*

Proof. In view of Remark 2.19 the function f_0 is subadditive. Let $g_0 : (0,1] \to \mathbb{R}_+$ be defined as in Theorem 5.1. Then, for all $x, y \in (0,1]$ and $\lambda \in (0,1)$, by the concavity of f_0, we have

$$g_0(\lambda x + (1-\lambda)y) = f_0(1 - (\lambda x + (1-\lambda)y)) = f_0(\lambda(1-x) + (1-\lambda)(1-y)) \geq \\ \geq \lambda f_0(1-x)) + (1-\lambda) f_0(1-y),$$

so g_0 is concave in $(0,1]$. By Remark 2.19, g_0 is subadditive. Now the result follows from Theorem 5.1. □

Remark 5.5. Let $f_0 : [0,1) \to \mathbb{R}_+$ be a concave function that is discontinuous at 0. Then its periodic extension is not continuous.

As an immediate consequence of Theorem 2.1 we obtain

Theorem 5.6. *Let $f : \mathbb{R} \to \mathbb{R}$ be the 1-periodic and subadditive extension of $f_0 : [0,1) \to \mathbb{R}$ such that $f_0(0) = 0$. Then f is continuous if, and only if, f_0 is right-continuous at 0 and $f_0(1-) = 0$.*

Proof. Let $f_0 : [0,1) \to \mathbb{R}$ be right continuous at 0 and such that $f_0(0) = f_0(1-) = 0$. Assume that its periodic extension f is subadditive. Then f is continuous at 0. By Theorem 2.1, the function f is continuous everywhere. □

Remark 5.7. According to Theorem 2.10, in the above theorem the assumption of the right-continuity of f_0 at 0 and $f_0(1-) = 0$ can be replaced by the following: there exists a Lebesgue measurable set $A \subset [0,1)$ such that $\lambda_0^+(A)\lambda_1^-(A) > 0$ and the restriction $f_0|_A$, is right-continuous at 0 and $(f_0|_A)(1-) = 0$.

Theorem 5.8. *Let $f : \mathbb{R} \to \mathbb{R}$ be a 1-periodic and subadditive extension of $f_0 : [0,1) \to \mathbb{R}$ such that $f_0(0) = 0$. Then f is even and continuous if, and only if, $f_0(1-x) = f_0(x)$ for all $x \in (0,1)$ and f_0 is right continuous at 0.*

Remark 5.9. In Theorem 5.8 the assumption of right-continuity of f_0 at 0 can be replaced by the following one: there exists a Lebesgue measurable set $A \subset [0,1)$ such that $\lambda_0^+(A) > 0$ and the restriction $f_0|_A$, is right-continuous at 0.

Note that the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) := |\sin(\pi x)|$ is even, subadditive and 1-periodic.

Remark 5.10. Theorems 5.2 and 5.3 allow us to construct a lot of non-even discontinuous subadditive periodic functions that are either right- or left-continuous at 0 and such that $f(0) = 0$.

6. NOWHERE DIFFERENTIABLE CONTINUOUS SUBADDITIVE EVEN PERIODIC FUNCTIONS

The results and examples of the previous sections show that the class of subadditive periodic functions is large. It contains nontrivial continuous as well as very irregular functions (of dense graphs).

In this section we show the following, maybe a little unexpected,

Fact 6.1. *There exist nowhere differentiable functions which are continuous subadditive, even, periodic and vanishing at zero.*

To show this consider the following

Example 6.2. Let $f : \mathbb{R} \to \mathbb{R}$ be the periodic extension of the function $f_0 : [0, 1) \to \mathbb{R}$,

$$f_0(x) := \begin{cases} x & \text{for} \quad x \in \left[0, \frac{1}{2}\right], \\ 1 - x & \text{for} \quad x \in \left(\frac{1}{2}, 1\right). \end{cases}$$

Note that f_0 is concave, $f_0(1 - x) = f_0(x)$ for all $x \in (0, 1)$ and f_0 is right continuous at 0. By Theorems 5.2 and 5.4 the function f is subadditive, even and continuous. It follows that, for any $n \in \mathbb{N}$, the function $f_n : \mathbb{R} \to \mathbb{R}$,

$$f_n(x) := \sum_{k=1}^{n} 4^{-k} f\left(4^k x\right)$$

is subadditive, 1-periodic and even. The uniform convergence implies that $h : \mathbb{R} \to \mathbb{R}$ defined by

$$h(x) := \sum_{k=1}^{\infty} 4^{-k} f\left(4^k x\right)$$

is continuous. Obviously, it is also subadditive, even and 1-periodic. Note that h is the classical example of a nowhere differentiable function due to Takagi [13] (and rediscovered by van der Waerden [14]).

Let us remark that, by Theorem 2.1, the continuity of h results from its subadditivity and the continuity at 0.

Remark 6.3. Obviously, the functions f_n, $n \in \mathbb{N}$, are not monotonic. However, for any $n \in \mathbb{N}$, there is $a_n > 0$ such that the function

$$\mathbb{R} \ni x \to a_n x + f_n(x)$$

is increasing in $\mathbb{R}$, and it is not true for the function h.

7. DIFFERENTIABLE PERIODIC SUBADDITIVE FUNCTIONS

Let us quote the following

Lemma 7.1 (cf. M. Kuczma, [5], Theorem 16.3.3)**.** *Let $f : \mathbb{R} \to \mathbb{R}$ be a measurable subadditive function, and let*

$$a := \inf_{t<0} \frac{f(t)}{t}, \qquad b := \sup_{t>0} \frac{f(t)}{t}.$$

If a resp. b is finite, then

$$a = \lim_{h \to 0-} \frac{f(h)}{h} \quad \textit{resp.} \quad b = \lim_{h \to 0+} \frac{f(h)}{h};$$

and these formulas remain valid for a or b infinite under the additional assumption that $\lim_{x\to 0} f(x) = 0$, or $\liminf_{x\to 0} f(x) > 0$. Moreover, in any case,

$$a \le b.$$

Remark 7.2. This lemma remains valid on replacing the measurability of f by its local boundedness from above.

Proposition 7.3. *Let $f : \mathbb{R} \to \mathbb{R}$ be a subadditive function. If there exists $\delta > 0$ and a function $\phi : (-\delta, \delta) \to \mathbb{R}$ such that $f(x) \le \phi(x)$ for $x \in (-\delta, \delta)$, $\phi(0) = 0$ and ϕ is differentiable at 0, then $f(x) = f(1)x$ for all $x \in \mathbb{R}$.*

Proof. From the subadditivity of f we have $f(0) \ge 0$. On the other hand, $f(0) \le \phi(0) = 0$. Hence $f(0) = 0$. Moreover we have

$$\frac{f(x)}{x} \ge \frac{\phi(x)}{x} \quad \text{for} \quad x \in (-\delta, 0); \qquad \frac{f(x)}{x} \le \frac{\phi(x)}{x} \quad \text{for} \quad x \in (0, \delta),$$

whence, by the above lemma,

$$\inf\left\{\frac{f(x)}{x} : x < 0\right\} \ge \phi'(0); \qquad \sup\left\{\frac{f(x)}{x} : x > 0\right\} \le \phi'(0),$$

and, consequently,

$$f(x) \le \phi'(0)x \quad \text{for} \quad x \in \mathbb{R}.$$

Hence, by the subadditivity of f, for any $x \in \mathbb{R}$,

$$0 = f(0) = f(x + (-x)) \le f(x) + f(-x) \le f(x) + \phi'(0)(-x),$$

whence we obtain

$$\phi'(0)x \le f(x) \quad \text{for} \quad x \in \mathbb{R},$$

which completes the proof. □

Theorem 7.4. *Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a subadditive and periodic function. If there is a point x_0 such that $f(x_0) = 0$ and f is differentiable at x_0, then $f(x) = 0$ for all $x \in \mathbb{R}$.*

Proof. The differentiability of f at x_0 implies that f is bounded in a neighborhood of x_0. In view of Theorem 3.6, there is $r > 0$ such that $\{kr : k \in \mathbb{Z}\} \subset Z(f)$, f is r-periodic and $x_0 = kr$ for some $k \in \mathbb{Z}$. The differentiability of f at x_0 implies the differentiability of f at 0. Now the result follows from the above proposition with $\phi := f$. □

Remark 7.5. Simple examples show that, in the above result, the assumption of differentiability of f at x_0 cannot be replaced by left- or right-differentiability at x_0. However, a similar reasoning proves that, if f is both left- and right-differentiable at x_0, then the result remains valid.

We generalize the last result as follows.

Proposition 7.6. *Let $g : \mathbb{R} \to \mathbb{R}$ be subadditive. Suppose that there exists $x_0 \in \mathbb{R}$ and a function $\gamma : \mathbb{R} \to \mathbb{R}$ such that $g(x) \leq \gamma(x)$ in some neighborhood of x_0, $\gamma(x_0) = g(x_0)$ and $\gamma(x_0) \leq -g(-x_0)$. If γ is differentiable at x_0, then $g(x) = g(1)x$ for all $x \in \mathbb{R}$.*

Proof. By assumption there exists $\delta > 0$ such that

$$g(x) \leq \gamma(x) \quad \text{for} \ \ x \in x_0 + (-\delta, \delta).$$

If $x_0 = 0$, the result is a consequence of Proposition 7.3. Assume that $x_0 \neq 0$. Then the function

$$f(x) := g(x) - \frac{g(x_0)}{x_0} x, \qquad x \in \mathbb{R},$$

is subadditive and $f(x_0) = 0$. Moreover, by the assumptions on γ,

$$f(-x_0) = g(-x_0) + g(x_0) = g(-x_0) + \gamma(x_0) \leq g(-x_0) - g(-x_0) = 0.$$

On the other hand, by the subadditivity of f, we have

$$0 \leq f(0) = f((-x_0) + x_0) \leq f(-x_0) + f(x_0) = f(-x_0),$$

whence $f(-x_0) = 0$ and $f(0) = 0$. It follows that f is x_0-periodic (cf. Theorem 3.1 and Property 5). Moreover, for $x \in (-\delta, \delta)$, we have

$$f(x) = f(x + x_0) = g(x + x_0) - \frac{g(x_0)}{x_0}(x + x_0) \leq \gamma(x + x_0) - \frac{g(x_0)}{x_0}(x + x_0) = \\ = \phi(x),$$

where

$$\phi(x) := \gamma(x + x_0) - \frac{g(x_0)}{x_0}(x + x_0), \qquad x \in (-\delta, \delta),$$

is differentiable at 0. Since $\phi(0) = 0$, applying Proposition 7.3, we conclude that $f(x) = f(1)x$ for all $x \in \mathbb{R}$. This completes the proof. □

Applying the above result with $\gamma = g$ we obtain

Theorem 7.7. *Suppose that $g : \mathbb{R} \to \mathbb{R}$ be subadditive. If there exists $x_0 \in \mathbb{R}$ such that $g(x_0) \leq -g(-x_0)$ and g is differentiable at x_0, then $g(x) = g(1)x$ for all $x \in \mathbb{R}$.*

8. CHARACTERIZATION OF FUNCTIONS BEING THE SUMS OF LINEAR FUNCTIONS AND PERIODIC SUBADDITIVE FUNCTIONS

We begin with

Remark 8.1. Suppose that $f, h : \mathbb{R} \to \mathbb{R}$ satisfy the inequality $f(x) \leq h(x)$ for all $x \in \mathbb{R}$. If f is subadditive and h is odd, then $h = f$.

Proof. Since, for all $x \in \mathbb{R}$,

$$0 \le f(-x+x) \le f(-x) + f(x) \le h(-x) + f(x) = -h(x) + f(x),$$

we get $h(x) \le f(x)$ for all $x \in \mathbb{R}$, whence $h = f$. $\square$

Theorem 8.2. *Let $g : \mathbb{R} \to \mathbb{R}$ be subadditive. Suppose that there are some $p, r \in \mathbb{R}$ such that*

$$p < 0 < r \qquad \text{and} \qquad \frac{g(r)}{r} \le \frac{g(p)}{p}.$$

(1) *If $\frac{p}{r} \in \mathbb{Q}$ then there exist a subadditive periodic function $f : \mathbb{R} \to \mathbb{R}$ and $a \in \mathbb{R}$ such that $f(0) = 0$ and*

$$g(x) = ax + f(x), \qquad x \in \mathbb{R}.$$

(2) *If $\frac{p}{r} \notin \mathbb{Q}$ and g is continuous at 0 then*

$$g(x) = g(1)x, \qquad x \in \mathbb{R}.$$

Proof. Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) := g(x) - \frac{g(r)}{r}x, \qquad x \in \mathbb{R}.$$

Of course f is subadditive and $f(r) = 0$. By assumption we have $rg(p) \le pg(r)$. Hence

$$f(p) = g(p) - \frac{g(r)}{r}p = \frac{1}{r}\left[rg(p) - pg(r)\right] \le 0.$$

In view of Theorem 3.1, if $\frac{p}{r} \in \mathbb{Q}$, the function f is periodic and $f(0) = 0$. Setting $a := r^{-1}g(r)$ we get $g(x) = ax + f(x)$ for all $x \in \mathbb{R}$.

To prove (2) note that, by the subadditivity of f, we have

$$f(mp + nr) \le mf(p) + nf(r) \le 0, \qquad m, n \in \mathbb{N}.$$

Since $p < 0 < r$, and $\frac{p}{q}$ is rational, it follows that the set $\{mp+nr : m, n \in \mathbb{N}\}$ is dense in $\mathbb{R}$ (cf. [6], Lemma 4). In view of Theorem 2.1, the function f, being continuous at 0, is continuous everywhere. Hence we infer that $f(x) \le 0$ for all $x \in \mathbb{R}$. Now it is enough to apply the above remark with $h = 0$. $\square$

Remark 8.3. Let H be a Hamel base of the linear space $\mathbb{R}$ over the field $\mathbb{Q}$ such that $1, \sqrt{2} \in H$. Take a function $\alpha : H \to \mathbb{R}$ such that $\alpha(1) = \alpha\left(\sqrt{2}\right) = 0$ and $\alpha(h) \ne 0$ for all $h \in H \setminus \left\{1, \sqrt{2}\right\}$. Let $\phi : \mathbb{R} \to \mathbb{R}$ be the additive extension of α (cf. Kuczma [5], Theorem 5.2.2). Put $g := \phi$ (or $g := |\phi|$), $p := -1$, $r := \sqrt{2}$. Then, except the continuity of g at 0, all the conditions of part (2) of Theorem 8.2 are fulfilled. Thus the assumption of the continuity of g at 0 is indispensable.

9. REMARKS ON A PARTIALLY "PEXIDERIZED" SUBADDITIVITY

We end this paper with some remarks on the inequality $f(x+y) \le f(x) + g(y)$ that is a partial Pexider-type generalization of subadditivity.

Proposition 9.1. *Suppose that the functions $f, g : \mathbb{R} \to \mathbb{R}$ satisfy the inequality*

$$f(x+y) \le f(x) + g(y), \qquad x, y \in \mathbb{R}. \tag{9.1}$$

If

$$g(-x) \le -g(x), \qquad x \in \mathbb{R}, \tag{9.2}$$

and

$$f(0) = 0, \tag{9.3}$$

then $g = f$ and f is additive.

Proof. Setting $x = y = 0$ in (9.1) and $x = 0$ in (9.2) we get $g(0) = 0$. Setting $y = -x$ in (9.1) and applying (9.3) and (9.2) we get

$$0 = f(x + (-x)) \le f(x) + g(-x) \le f(x) - g(x),$$

whence $g(x) \le f(x)$ for all $x \in \mathbb{R}$. Taking $x = 0$ in (9.1), by (9.3), we obtain $f(y) \le g(y)$ for all $y \in \mathbb{R}$. Thus

$$g = f \tag{9.4}$$

and, by (9.1),

$$f(x+y) \le f(x) + f(y), \qquad x, y \in \mathbb{R}, \tag{9.5}$$

that is f is subadditive. Setting here $y = -x$, by (9.3), we get

$$0 \le f(x) + f(-x), \qquad x \in \mathbb{R},$$

and from (9.4) and (9.2) we have

$$f(x) + f(-x) \le 0, \qquad x \in \mathbb{R}.$$

Thus f is an odd function. By (9.5) and Remark 2.5 the function f is additive. □

Let us note the following

Remark 9.2. Suppose that the functions $f, g : \mathbb{R} \to \mathbb{R}$ satisfy inequality (9.1). If g is r-periodic with some $r \ne 0$ and $f(0) = g(0) = 0$, then f is r-periodic.

Proof. The periodicity of g and $g(0) = 0$ imply that $g(r) = g(-r) = 0$. Taking $y = r$ in (9.1), we get, for all $x \in \mathbb{R}$,

$$f(x+r) \le f(x) + g(r) = f(x).$$

Taking $y = -r$ in (9.1), we get, for all $x \in \mathbb{R}$,

$$f(x-r) \le f(x) + g(-r) = f(x),$$

whence, replacing x by $x + r$,

$$f(x) \le f(x+r)$$

for all $x \in \mathbb{R}$. This completes the proof. □

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