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SUBADDITIVITY OF EIGENVALUE SUMS

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ABSTRACT. Let $f(t)$ be a nonnegative concave function on $0 \leq t < \infty$ with $f(0) = 0$, and let X, Y be $n \times n$ matrices. Then it is known that $\|f(|X+Y|)\|_1 \leq \|f(|X|)\|_1 + \|f(|Y|)\|_1$, where $\|\cdot\|_1$ is the trace norm. We extend this result to all unitarily invariant norms and prove some inequalities of eigenvalue sums.

1. INTRODUCTION

The eigenvalues of an $n \times n$ Hermitian matrix H are denoted by $\lambda_i(H)$ ($i = 1, 2, \cdots, n$) and arranged in increasing order, that is, $\lambda_1(H) \leq \lambda_2(H) \leq \cdots \leq \lambda_n(H)$. The following sums are very important: for $1 \leq k \leq n$,

$$\sigma_{(k)}(H) := \sum_{i=1}^{k} \lambda_i(H), \qquad \sigma^{(k)}(H) := \sum_{i=n-k+1}^{n} \lambda_i(H).$$

These are represented as follows:

$$\sigma_{(k)}(H) = \min\{\sum_{i=1}^{k} \langle H\mathbf{e}_i, \mathbf{e}_i\rangle : \{\mathbf{e}_1, \cdots, \mathbf{e}_k\} \text{ is orthonormal}\}, \tag{1}$$

$$\sigma^{(k)}(H) = \max\{\sum_{i=1}^{k} \langle H\mathbf{e}_i, \mathbf{e}_i\rangle : \{\mathbf{e}_1, \cdots, \mathbf{e}_k\} \text{ is orthonormal}\}. \tag{2}$$

Hence $\sigma_{(k)}(H)$ (or $\sigma^{(k)}(H)$) is a concave (or convex) function of H(cf. [7]).

A norm $|||\cdot|||$ on the $n \times n$ matrices is called a *unitarily invariant norm* if

$$|||UXV||| = |||X|||$$

for all X and for all unitary matrices U and V. The operator norm $\|X\|$, Schatten p-norms

$$\|X\|_p := \{\sum_{i=1}^{n} \lambda_i(|X|)^p\}^{1/p} \quad (p \geq 1),$$

where $|X| = (X^*X)^{1/2}$, and Ky Fan k-norms $\|X\|_{(k)} := \sigma^{(k)}(|X|)$ ($k = 1, 2, \cdots, n$) are typical unitarily invariant norms. It is well known that $\sigma^{(k)}(|X|) \leq \sigma^{(k)}(|Y|)$ ($k = 1, 2, \cdots, n$) implies $|||X||| \leq |||Y|||$ for every unitarily invariant norm $|||\cdot|||$.

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$\sigma^{(k)}(|X|) \leq \sigma^{(k)}(|Y|)$ $(k = 1, 2, \cdots, n)$ means that the sequence $\{\lambda_k(|X|)\}_{k=1}^n$ is *submajorized* by $\{\lambda_k(|Y|)\}_{k=1}^n$ by definition; so one can restate some of the results in this paper by using the word "*submajorized*".

Let $f(t)$ be a nonnegative concave function on $0 \leq t < \infty$. Then Rotfel'd [8] and Thompson [9] (see also Theorem 4.2.14 of [4]) have shown that

$$\|f(|X+Y|)\|_1 \leq \|f(|X|)\|_1 + \|f(|Y|)\|_1.$$

We show in the fourth section that the above inequality holds for every unitarily invariant norm.

For Hermitian matrices A and B the inequality $A \leq B$ means $\langle A\mathbf{x}, \mathbf{x}\rangle \leq \langle B\mathbf{x}, \mathbf{x}\rangle$ for all vectors $\mathbf{x}$, where $\langle \cdot, \cdot \rangle$ is the inner product. A continuous function φ defined on an interval I is called *an operator monotone function* if for all A, B whose eigenvalues lie in I, $A \leq B$ implies $\varphi(A) \leq \varphi(B)$. Likewise, a continuous function φ on I is called an *operator convex function* if

$$\varphi(sA + (1-s)B) \leq s\varphi(A) + (1-s)\varphi(B)$$

for all A, B whose eigenvalues lie in I and for every s with $0 \leq s \leq 1$. φ is called an *operator concave function* if $-\varphi$ is operator convex.

We also give, in the third section, a simple proof of the following result [2]: *for a nonnegative operator monotone function φ on $[0, \infty)$ and for every unitarily invariant norm $|\!|\!| \cdot |\!|\!|$,*

$$|\!|\!|\varphi(A+B)|\!|\!| \leq |\!|\!|\varphi(A) + \varphi(B)|\!|\!| \quad (0 \leq A, B).$$

For details on this field we refer the readers to [4]. We appreciate the referee's useful comments.

2. Essential results

For Hermitian matrices A and B the trace of $BA^2B - AB^2A$ vanishes. But, in general, it is difficult to estimate the trace of $CBA^2BC - CAB^2AC$. The next lemma follows from the more general result shown in [5]. However, this special case is useful and worth stating, so we prove it directly.

Lemma 2.1. *Let $A \geq 0$ and $B \geq 0$, and let Q be an orthogonal projection such that $QB = BQ$. If $\inf\{\|B\mathbf{x}\| : Q\mathbf{x} = \mathbf{x}, \|\mathbf{x}\| = 1\} \geq \sup\{\|B\mathbf{x}\| : (1-Q)\mathbf{x} = \mathbf{x}, \|\mathbf{x}\| = 1\}$, then*

$$tr QBA^2BQ \geq tr\, QAB^2AQ, \tag{3}$$

$$tr(1-Q)BA^2B(1-Q) \leq tr\,(1-Q)AB^2A(1-Q). \tag{4}$$

Proof. Let

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{12}^* & A_{22} \end{bmatrix}, \quad B = \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix}$$

be the decompositions of A and B corresponding to $\mathbf{C}^n = Q\mathbf{C}^n \oplus (1-Q)\mathbf{C}^n$. Then

$$QBA^2BQ = \begin{bmatrix} B_1A_{11}^2B_1 + B_1A_{12}A_{12}^*B_1 & 0 \\ 0 & 0 \end{bmatrix}, \tag{5}$$

$$QAB^2AQ = \begin{bmatrix} A_{11}B_1^2A_{11} + A_{12}B_2^2A_{12}^* & 0 \\ 0 & 0 \end{bmatrix}. \tag{6}$$

The assumption implies there are real numbers $m_1, m_2 \geq 0$ such that $B_1^2 \geq m_1 \geq m_2 \geq B_2^2$. Since

$$\begin{aligned} tr\, B_1 A_{12} A_{12}^* B_1 = tr\, A_{12}^* B_1^2 A_{12} &\geq m_1 \cdot tr\, A_{12}^* A_{12} \\ &\geq m_2 \cdot tr\, A_{12} A_{12}^* \geq tr\, A_{12} B_2^2 A_{12}^*, \end{aligned}$$

we get (3). One can see (4) in the same way or by using (3). □

Notice that Q in the above lemma is the orthogonal projection onto the space spanned by eigenvectors of B corresponding to $\lambda_n(B), \lambda_{n-1}(B), \cdots, \lambda_{n-k+1}(B)$ for some $k : 1 \leq k \leq n$. We can see that the above lemma is right for operators on an infinite Hilbert space as well if A is of Hilbert-Schmidt class; so we used "inf (or sup)" instead of "min (or max)".

Corollary 2.2. *Let $A \geq 0$ and $B \geq 0$, and let Q be an orthogonal projection such that $QB = BQ$. Suppose the sharper inequality:*

$$\inf\{\|B\mathbf{x}\| : Q\mathbf{x} = \mathbf{x}, \|\mathbf{x}\| = 1\} > \sup\{\|B\mathbf{x}\| : (1-Q)\mathbf{x} = \mathbf{x}. \|\mathbf{x}\| = 1\}.$$

Then $tr\, QBA^2BQ = tr\, QAB^2AQ$ if and only if $QA = AQ$.

Proof. Assume $tr\, QBA^2BQ = tr\, QAB^2AQ$. Then by (5), (6) we get

$$tr\, B_1 A_{12} A_{12}^* B_1 = tr\, A_{12} B_2^2 A_{12}^*.$$

Since we can take m_1 and m_2 in the preceding proof as $m_1 > m_2$, $tr\, A_{12}A_{12}^*$ must vanish, and hence $A_{12} = 0$. This implies $QA = AQ$. The converse statement is clear. □

In this paper the terms "increasing" and "decreasing" are used in the wider sense, that is, they mean "non-decreasing" and "non-increasing", respectively.

Proposition 2.3. *Let $f(t)$ be a continuous function on $a \leq t \leq b$ with $a \geq 0$, and let A, B be $n \times n$ nonnegative Hermitian matrices such that $a \leq A + B \leq b$. If $f(t)$ is decreasing and $tf(t)$ is increasing, or if $f(t)$ is increasing and $tf(t)$ is decreasing, then for $1 \leq k \leq n$*

$$\sigma^{(k)}(A^{1/2} f(A+B) A^{1/2} + B^{1/2} f(A+B) B^{\frac{1}{2}}) \geq \sigma^{(k)}((A+B)f(A+B)).$$

Hence, for any unitarily invariant norm $\|\!|\cdot|\!\|$

$$\|\!|A^{1/2} f(A+B) A^{1/2} + B^{1/2} f(A+B) B^{\frac{1}{2}}|\!\| \geq \|\!|(A+B)f(A+B)|\!\|.$$

Proof. The second inequality immediately follows from the first one, so we only show the first inequality. To do it we first assume that $f(t)$ is decreasing and $tf(t)$ is increasing. Notice that $f(t)$ is then nonnegative. Denote the unit eigenvector of $A+B$ corresponding to $\lambda_i := \lambda_i(A+B)$ by $\mathbf{e}_i \quad (1 \leq i \leq n)$. As $(A+B)f(A+B)\mathbf{e}_i = \lambda_i f(\lambda_i)\mathbf{e}_i$ and $\lambda_i f(\lambda_i) \leq \lambda_{i+1} f(\lambda_{i+1})$, the less side of the first inequality equals $\sum_{i=n-k+1}^n \lambda_i f(\lambda_i)$. Therefore, by (2), it is sufficient to show

$$\sum_{i=n-k+1}^{n} \langle (A^{1/2} f(A+B) A^{1/2} + B^{1/2} f(A+B) B^{1/2})\mathbf{e}_i.\mathbf{e}_i \rangle \geq \sum_{i=n-k+1}^{n} \lambda_i f(\lambda_i),$$

which is equivalent to

$$tr\{PA^{1/2} f(A+B) A^{1/2} P + PB^{1/2} f(A+B) B^{1/2} P\} \geq tr P(A+B)f(A+B)P,$$

where P is the orthogonal projection onto the space spanned by $\{\mathbf{e}_{n-k+1}, \cdots, \mathbf{e}_n\}$. Since $f(\lambda_i) \geq f(\lambda_{i+1})$, by (4)

$$\begin{aligned} tr\, PA^{1/2}f(A+B)A^{1/2}P &\geq tr\, Pf(A+B)^{1/2}Af(A+B)^{1/2}P, \\ tr\, PB^{1/2}f(A+B)B^{1/2}P &\geq tr\, Pf(A+B)^{1/2}Bf(A+B)^{1/2}P. \end{aligned}$$

Summing both inequalities yields the required inequality. We next assume that $f(t)$ is increasing and $tf(t)$ is decreasing; hence $f(t) \leq 0$. To see the required inequality we may prove

$$\sigma_{(k)}(A^{1/2}f(A+B)A^{1/2} + B^{1/2}f(A+B)B^{1/2}) \leq \sigma_{(k)}((A+B)f(A+B)),$$

because the traces of the matrices on both sides are identical. Since $tf(t)$ is decreasing, by (1) it is enough to show that

$$tr\,(PA^{1/2}f(A+B)A^{1/2}P + PB^{1/2}f(A+B)B^{1/2}P) \leq \sum_{i=n-k+1}^{n} \lambda_i f(\lambda_i),$$

where P is the orthogonal projection onto the space spanned by $\{\mathbf{e}_{n-k+1}, \cdots, \mathbf{e}_n\}$. We can obtain this inequality in the same way as above. $\square$

When we study the operator convexity, we often encounter a pair of matrices X and Y with $X^*X + Y^*Y = 1$. But in this case, $XX^* + YY^*$ is not necessarily contractive; in fact, suppose $A + B$ is invertible for $A, B \geq 0$. Then $(A+B)^{-1/2}A(A+B)^{-1/2} + (A+B)^{-1/2}B(A+B)^{-1/2} = 1$; but putting $f(t) = 1/t$ in the above proposition, we get $\sigma^{(k)}(A^{1/2}(A+B)^{-1}A^{1/2} + B^{1/2}(A+B)^{-1}B^{1/2}) \geq k$. Furthermore, we get

Corollary 2.4. *Let A and B be nonnegative Hermitian matrices such that $A+B$ is invertible. Then the following are equivalent:*

(i) $H := A^{1/2}(A+B)^{-1}A^{1/2} + B^{1/2}(A+B)^{-1}B^{1/2} \leq 1$,
(ii) $H = 1$,
(iii) $AB = BA$.

Proof. (i) $\Rightarrow$ (ii). As we mentioned above, we get $\sigma^{(k)}(H) \geq \sigma^{(k)}(1)$. Thus (i) implies $\sigma^{(k)}(H) = \sigma^{(k)}(1) = k$ for $1 \leq k \leq n$ and hence $H = 1$.

(ii) $\Rightarrow$ (iii). Assume $\lambda_n(A+B) \geq \cdots \geq \lambda_{n-k+1}(A+B) > \lambda_{n-k}(A+B)$. Let $\mathbf{e}_i$ be the unit eigenvector of $A+B$ corresponding to $\lambda_i(A+B)$, and let Q be the orthogonal projection onto the space spanned by $\mathbf{e}_n, \cdots, \mathbf{e}_{n-k+1}$. Then by Lemma 2.1

$$\begin{aligned} k &= tr\, QHQ = tr\, QA^{1/2}(A+B)^{-1}A^{1/2}Q + tr\, QB^{1/2}(A+B)^{-1}B^{1/2}Q \\ &\geq tr\, Q(A+B)^{-1/2}A(A+B)^{-1/2}Q + tr\, Q(A+B)^{-1/2}B(A+B)^{-1/2}Q \\ &= tr\, Q = k, \end{aligned}$$

from which it follows that

$$tr\, QA^{1/2}(A+B)^{-1}A^{1/2}Q = tr\, Q(A+B)^{-1/2}A(A+B)^{-1/2}Q.$$

Thus by Corollary 2.2 we obtain $AQ = QA$. Therefore, one can see that A commutes to every spectral projection of $A+B$. Thus $AB = BA$. Needless to say, (i) follows from (iii). $\square$

3. Operator concave functions

Henceforth, we give some applications of Proposition 2.3 and assume every function is continuous. To start with, we give an another proof of the first statement of Ando and Zhan's theorem:

Theorem A ([2])**.** *Let $\varphi(t)$ be a nonnegative operator monotone function on $[0,\infty)$. Then for $n \times n$ Hermitian matrices $A \geq 0$ and $B \geq 0$,*

$$\sigma^{(k)}(\varphi(A)+\varphi(B)) \geq \sigma^{(k)}(\varphi(A+B)) \quad (1 \leq k \leq n).$$

Let $\psi(t)$ be a strictly increasing function on $[0,\infty)$ with $\psi(0)=0$ and $\psi(\infty)=\infty$ such that the inverse function $\psi^{-1}(t)$ is operator monotone. Then

$$\sigma^{(k)}(\psi(A)+\psi(B)) \leq \sigma^{(k)}(\psi(A+B)) \quad (1 \leq k \leq n).$$

To prove the first inequality we may assume that $A+B$ is invertible. Then, since $(A+B)^{-1/2}A^{1/2}$ is contractive, by Hansen and Pedersen's inequality [6] we have

$$\begin{aligned} \varphi(A) &= \varphi(A^{1/2}(A+B)^{-1/2}(A+B)(A+B)^{-1/2}A^{1/2}) \\ &\geq A^{1/2}(A+B)^{-1/2}\varphi(A+B)(A+B)^{-1/2}A^{1/2}, \\ \varphi(B) &\geq B^{1/2}(A+B)^{-1/2}\varphi(A+B)(A+B)^{-1/2}B^{1/2}. \end{aligned}$$

Since $\varphi(t)$ is increasing and $\varphi(t)/t$ is decreasing, by Proposition 2.3 we get

$$\begin{aligned} \sigma^{(k)}(\varphi(A)+\varphi(B)) &\geq \sigma^{(k)}(A^{1/2}(\varphi/t)(A+B)A^{1/2}+B^{1/2}(\varphi/t)(A+B)B^{1/2}) \\ &\geq \sigma^{(k)}(\varphi(A+B)) \quad (1 \leq k \leq n). \end{aligned}$$

We can similarly prove the following extension of the first statement in the above theorem:

Let $\varphi(t)$ be an operator monotone function on $[0,\infty)$. Then for $A,B,C \geq 0$ and for $k=1,\cdots,n$,

$$\sigma^{(k)}(\varphi(A+B+C)) \geq \sigma^{(k)}(\varphi(A)+\varphi(B)+\varphi(C)).$$

Recall that $\varphi(t)$ on $[0,\infty)$ is operator monotone if and only if $\varphi(t)$ is operator concave and $\varphi(\infty) > -\infty$ (cf. Proposition 3.5 of [10]). As $\varphi(t)$ in the preceding theorem is operator concave, it is natural to ask for a similar inequality related to an operator convex function.

Proposition 3.1. *Let $\psi(t)$ be a non-constant, increasing operator convex function on $[0,\infty)$. Then $\psi(t)$ is strictly increasing and its inverse function $\psi^{-1}(t)$ is operator concave on $[\psi^{-1}(0),\infty)$.*

Proof. Assume $\psi(0)=0$. Then it is known that $\psi(t)/t$ is operator monotone on $(0,\infty)$. Since the right-side limit of $\frac{\psi(t)}{t}$ at $t=0$ exists and is nonnegative, $\psi(t)/t$ has the natural extension that is nonnegative and operator monotone on $[0,\infty)$; we denote it by $\psi(t)/t$ again. Since $\psi(t)/t$ is increasing, $\psi(t)$ is strictly increasing. By Lemma 5 of [1] the inverse function of $\psi(t)=t(\psi(t)/t)$ is operator monotone and hence operator concave. Assume next $\psi(0) \neq 0$. For $\varphi(t) := \psi(t)-\psi(0)$ we get $\psi^{-1}(t)=\varphi^{-1}(t-\psi(0))$. Since φ^{-1} is operator concave, so is $\psi^{-1}(t)$. □

The converse of the previous proposition does not holds; for instance, $t^{1/3}$ is an operator concave function on $[0,\infty)$, but its inverse function t^3 is not operator convex.

The following corollary can be shown by making use of the second inequality of Theorem A.

Corollary 3.2. *Let $\psi(t)$ be a non-constant and increasing operator convex function on $[0,\infty)$ with $\psi(0)=0$. Then for $n\times n$ Hermitian matrices $A\geq 0$ and $B\geq 0$,*

$$\sigma^{(k)}(\psi(A)+\psi(B))\leq \sigma^{(k)}(\psi(A+B))\quad (1\leq k\leq n).$$

4. Main results

We notice that if X and Y are contractive such that $X^*X+Y^*Y=1$ and if the spectra of A and B are both in an interval I, then the spectrum of X^*AX+Y^*BY is in I as well. A function is said to be monotone if it is increasing or decreasing.

Proposition 4.1. *Let $f(t)$ be a concave function on an interval I, and let A,B be $n\times n$ Hermitian matrices with the spectra in I. Then for X,Y such that $X^*X+Y^*Y=1$ and for $k=1,2,\cdots,n$,*

$$\sigma_{(k)}(f(X^*AX+Y^*BY))\geq \sigma_{(k)}(X^*f(A)X+Y^*f(B)Y). \tag{7}$$

Moreover, if $f(t)$ is monotone, then

$$\lambda_k(f(X^*AX+Y^*BY))\geq \lambda_k(X^*f(A)X+Y^*f(B)Y). \tag{8}$$

Proof. Let $\{\lambda_i\}_{i=1}^n$ be the eigenvalues of X^*AX+Y^*BY so that $f(\lambda_1)\leq f(\lambda_2)\leq\cdots\leq f(\lambda_n)$, and let $\{\mathbf{e}_i\}$ be the corresponding eigenvectors. Then the left side of (7) equals $f(\lambda_1)+\cdots+f(\lambda_k)$. By the concavity of f, we have

$$\begin{aligned}
&\sum_{i=1}^k\langle (X^*f(A)X+Y^*f(B)Y)\mathbf{e}_i,\mathbf{e}_i\rangle\\
=\ &\sum_{i=1}^k\{\|X\mathbf{e}_i\|^2\langle f(A)\frac{X\mathbf{e}_i}{\|X\mathbf{e}_i\|},\frac{X\mathbf{e}_i}{\|X\mathbf{e}_i\|}\rangle+\|Y\mathbf{e}_i\|^2\langle f(B)\frac{Y\mathbf{e}_i}{\|Y\mathbf{e}_i\|},\frac{Y\mathbf{e}_i}{\|Y\mathbf{e}_i\|}\rangle\}\\
\leq\ &\sum_{i=1}^k\{\|X\mathbf{e}_i\|^2 f(\langle A\frac{X\mathbf{e}_i}{\|X\mathbf{e}_i\|},\frac{X\mathbf{e}_i}{\|X\mathbf{e}_i\|}\rangle)+\|Y\mathbf{e}_i\|^2 f(\langle B\frac{Y\mathbf{e}_i}{\|Y\mathbf{e}_i\|},\frac{Y\mathbf{e}_i}{\|Y\mathbf{e}_i\|}\rangle)\}\\
\leq\ &\sum_{i=1}^k f(\langle (X^*AX+Y^*BY)\mathbf{e}_i,\mathbf{e}_i\rangle)=\sum_{i=1}^k f(\lambda_i).
\end{aligned}$$

Thus, by (1) we get (7).

If $f(t)$ is increasing, we can arrange eigenvalues $\{\lambda_i\}_{i=1}^n$ as $\lambda_i\leq\lambda_{i+1}$ and $f(\lambda_i)\leq f(\lambda_{i+1})$. For every unit vector $\mathbf{x}$ that is a linear combination of $\mathbf{e}_1,\cdots,\mathbf{e}_k$,

$$\begin{aligned}
\langle (X^*f(A)X+Y^*f(B)Y)\mathbf{x},\mathbf{x}\rangle &\leq f(\langle (X^*AX+Y^*BY)\mathbf{x},\mathbf{x}\rangle)\\
&\leq f(\lambda_k),
\end{aligned}$$

for $\langle (X^*AX+Y^*BY)\mathbf{x},\mathbf{x}\rangle\leq\lambda_k$. From this, by the mini-max theorem, (8) follows. It can be similarly shown even if $f(t)$ is decreasing. □

Corollary 4.2. *Let $f(t)$ be a monotone and concave function on I. If $0\in I$ and $f(0)\geq 0$, then for all X such that $X^*X\leq 1$,*

$$\lambda_k(f(X^*AX))\geq\lambda_k(X^*f(A)X)\quad (1\leq k\leq n).$$

Proof. Put $B=0$ and $Y=(1-X^*X)^{1/2}$ in (8). Since $X^*f(A)X+Y^*f(0)Y\geq X^*f(A)X$, we obtain the required inequality. □

Corollary 4.3. *Let $g(t)$ be a convex function on I. Then for all $n \times n$ Hermitian matrices A, B with the spectra in I and for all X, Y such that $X^*X + Y^*Y = 1$,*

$$\sigma^{(k)}(g(X^*AX + Y^*BY)) \leq \sigma^{(k)}(X^*g(A)X + Y^*g(B)Y) \quad (1 \leq k \leq n).$$

Moreover, if $g(t)$ is monotone, then

$$\lambda_k(g(X^*AX + Y^*BY)) \leq \lambda_k(X^*g(A)X + Y^*g(B)Y) \quad (1 \leq k \leq n).$$

Proof. Since $\lambda_k(-H) = -\lambda_{n-k+1}(H)$, by putting $f = -g$ in Proposition 4.1, we get this corollary. □

Corollary 4.3 has been shown in [3] when X and Y are both real numbers. Now we are in position to show the following main theorem.

Theorem 4.4. *Let $f(t)$ be a nonnegative, continuous and concave function on $[0, \infty)$. Then for all $n \times n$ (not necessarily Hermitian) matrices A and B and for $k = 1, \cdots, n$*

$$\sigma^{(k)}(f(|A + B|)) \leq \sigma^{(k)}(f(|A|)) + \sigma^{(k)}(f(|B|)). \tag{9}$$

Proof. Since $f(t)$ is nonnegative and concave on $[0, \infty)$, $f(t)$ is increasing. Though the right-side limit of $(f(t) - f(0))/t$ at $t = 0$ is not necessarily finite, by considering $f(t + \epsilon) - f(\epsilon)$ instead of $f(t)$ for an arbitrary $\epsilon > 0$, we may assume that $f(t)$ is right diffentiable at $t = 0$ and $f(0) = 0$. Then we can extend $f(t)/t$ continuously to $[0, \infty)$ and denote the extension by $f(t)/t$ again. Since $f(t)$ is nonnegative and concave, $f(t)/t$ is decreasing on $[0, \infty)$.

We first show (9) in the case $A \geq 0$ and $B \geq 0$. In this case there is no loss of generality in assuming that $A+B$ is invertible. Since $f(t) = t \cdot f(t)/t$, by Proposition 2.3,

$$\begin{aligned}
& \sigma^{(k)}(f(A + B)) \\
\leq \; & \sigma^{(k)}(A^{1/2}(A + B)^{-1}f(A + B)A^{1/2} + B^{1/2}(A + B)^{-1}f(A + B)B^{1/2}) \\
\leq \; & \sigma^{(k)}(A^{1/2}(A + B)^{-1/2}f(A + B)(A + B)^{-1/2}A^{1/2}) \\
+ \; & \sigma^{(k)}(B^{1/2}(A + B)^{-1/2}f(A + B)(A + B)^{-1/2}B^{1/2}) \\
\leq \; & \sigma^{(k)}(f(A)) + \sigma^{(k)}(f(B)),
\end{aligned}$$

where the second inequality is due to the subadditivity of $\sigma^{(k)}$ and the last inequality follows from Corollary 4.2.

We next consider general matrices A and B. Then there are unitary matrices U and V such that

$$|A + B| \leq U^*|A|U + V^*|B|V$$

(see [9]). Hence we have $\lambda_k(f(|A + B|)) \leq \lambda_k(f(U^*|A|U + V^*|B|V)$ for $1 \leq k \leq n$. Thus from the result shown above it follows that

$$\begin{aligned}
\sigma^{(k)}(f(|A + B|)) &\leq \sigma^{(k)}(f(U^*|A|U + V^*|B|Y)) \\
&\leq \sigma^{(k)}(f(U^*|A|U)) + \sigma^{(k)}(f(V^*|B|Y)) \\
&= \sigma^{(k)}(U^*f(|A|)U) + \sigma^{(k)}(V^*f(|B|)V) \\
&= \sigma^{(k)}(f(|A|)) + \sigma^{(k)}(f(|B|)).
\end{aligned}$$

□

By considering unitarily invariant norm $|||\cdot|||$ instead of $\sigma^{(k)}$ in the above proof, we have

Corollary 4.5. *Under the same condition as Theorem* 4.4
$$|||f(|A+B|)||| \le |||f(|A|)||| + |||f(|B|)|||.$$

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