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SUBADDITIVITY OF HOMOGENEOUS NORMS ON CERTAIN NILPOTENT LIE GROUPS

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ABSTRACT. Let N be a Lie group with its Lie algebra generated by the left-invariant vector fields $X_1, \dots, X_k$ on N. An explicit fundamental solution for the (hypoelliptic) operator $L = X_1^2 + \cdots + X_k^2$ on N has been obtained for the Heisenberg group by Folland [1] and for the nilpotent (Iwasawa) groups of isometries of rank-one symmetric spaces by Kaplan and Putz [2]. Recently Kaplan [3] introduced a (still larger) class of step-2 nilpotent groups N arising from Clifford modules for which similar explicit solutions exist. As in the case of L being the ordinary Laplacian on $N = \mathbf{R}^k$, these solutions are of the form $g \mapsto \text{const}\|g\|^{2-m}$, $g \in N$, where the "norm" function $\| \ \|$ satisfies a certain homogeneity condition. We prove that the above norm is also subadditive.

Let $\mathfrak{u}$, $\mathfrak{v}$ be real finite-dimensional vector spaces each equipped with a positive definite quadratic form $| \ |^2$. Let μ: $\mathfrak{u} \times \mathfrak{v} \to \mathfrak{v}$ be a *composition of* these *quadratic forms* [3, p. 148] normalized in the sense that $\mu(u_0, v) = v$ for some $u_0 \in \mathfrak{u}$. Define ϕ: $\mathfrak{v} \times \mathfrak{v} \to \mathfrak{u}$ by demanding $\langle u, \phi(v, v')\rangle = \langle \mu(u, v), v'\rangle$, $u \in \mathfrak{u}$; $v, v' \in \mathfrak{v}$, relative to the inner products $\langle \ , \ \rangle$ induced by the given quadratic forms. Let $\mathfrak{z}$ denote the orthogonal complement to $\mathbf{R}u_0$ in $\mathfrak{u}$ and π: $\mathfrak{u} \to \mathfrak{z}$ the orthogonal projection. Now set $\mathfrak{n} = \mathfrak{v} \times \mathfrak{z}$ and define a bracket on $\mathfrak{n}$ by $[(v, z), (v', z')] = (0, \pi \circ \phi(v, v'))$. On the simply connected analytic group N, corresponding to the Lie algebra $\mathfrak{n}$ (i.e. on Kaplan's *type* **H** *group*) we define a *norm function* by $\|n\| = (|v|^4 + 16|z|^2)^{1/4}$, where $n = \exp(v + z)$, $v \in \mathfrak{v}$, $z \in \mathfrak{z}$; $\mathfrak{n} \cong \mathfrak{v} \oplus \mathfrak{z}$. We now prove

THEOREM. *The norm function* $\| \ \|$ *is subadditive, i.e.*

$$\|nn'\| \leq \|n\| + \|n'\|, \qquad n, n' \in N.$$

PROOF. We have

$$\begin{aligned}\|nn'\|^4 &= \|\exp\left(v + v' + z + z' + \tfrac{1}{2}[v, v']\right)\|^4 \\ &= |v + v'|^4 + 16|z + z' + \tfrac{1}{2}[v, v']|^2.\end{aligned}$$

Now

$$\begin{aligned}|v + v'|^4 = |v|^4 + |v'|^4 + 4\langle v, v'\rangle^2 + 4|v|^2\langle v, v'\rangle \\ + 4|v'|^2\langle v, v'\rangle + 2|v|^2|v'|^2, \qquad (1)\end{aligned}$$

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$$16|z + z' + \tfrac{1}{2}[v, v']|^2 = 16|z|^2 + 16|z'|^2 + 4|[v, v']|^2 + 16\langle z, [v, v']\rangle + 16\langle z', [v, v']\rangle + 32\langle z, z'\rangle. \quad (2)$$

Since $2|v|^2|v'|^2 + 32\langle z, z'\rangle \leq 2\|n\|^2\|n'\|^2$ and

$$4|v|^2\langle v, v'\rangle + 16\langle z, [v, v']\rangle \leq 4\|n\|^2\left(\langle v, v'\rangle^2 + |[v, v']|^2\right)^{1/2},$$

we need

LEMMA. *In the notation above*

$$\langle v, v'\rangle^2 + |[v, v']|^2 \leq |v|^2|v'|^2, \qquad v, v' \in \mathfrak{v}.$$

For we have $|v|\,|v'| \leq \|n\|\,\|n'\|$, and collecting the above inequalities we obtain (1) + (2) $\leq (\|n\| + \|n'\|)^4$.

Proof of the Lemma follows from Schwarz's inequality on the hermitian form h_z on $\mathfrak{v}$ defined by

$$h_z(v, v') = \langle v, v'\rangle - \sqrt{-1}\,\langle z, \pi \circ \phi(v, v')\rangle,$$

if one regards $\mathfrak{v}$ as a complex vector space under the complex structure $J_z\colon \mathfrak{v} \to \mathfrak{v}$ given by $\langle J_z(v), v'\rangle = \langle z, \phi(v, v')\rangle$ with fixed $z \in \mathfrak{z}$, $|z| = 1$ (see [3, pp. 149, 150]), simply by putting $z = [v, v']/|[v, v']|$.

The initial proof of the Lemma was modernized by the referee to whom I am very grateful.

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