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Subalgebras of C^*-algebras III: Multivariable operator theory

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Introduction

This paper concerns function theory and operator theory relative to the unit ball in complex d-space $\mathbf{C}^d$, $d=1,2,\ldots$. A d-contraction is a d-tuple $(T_1,\ldots,T_d)$ of mutually commuting operators acting on a common Hilbert space H satisfying

$$\|T_1\xi_1+\ldots+T_d\xi_d\|^2 \leqslant \|\xi_1\|^2+\ldots+\|\xi_d\|^2,$$

for every $\xi_1,\ldots,\xi_d\in H$. This inequality simply means that the "row operator" defined by the d-tuple, viewed as an operator from the direct sum of d copies of H to H, is a contraction. It is essential that the component operators commute with one another.

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We show that there exist d-contractions which are not polynomially bounded in the sense that there is no constant K satisfying

$$\|f(T_1,...,T_d)\| \leqslant K \sup\{|f(z_1,...,z_d)| : |z_1|^2+...+|z_d|^2 \leqslant 1\}$$

for every polynomial f. In fact, we single out a particular d-contraction $(S_1,...,S_d)$ (called the d-shift) which is not polynomially bounded but which gives rise to the appropriate version of von Neumann's inequality with constant 1: for every d-contraction $(T_1,...,T_d)$ one has

$$\|f(T_1,...,T_d)\| \leqslant \|f(S_1,...,S_d)\|$$

for every polynomial f. Indeed the indicated homomorphism of commutative operator algebras is completely contractive.

The d-shift acts naturally on a space of holomorphic functions defined on the open unit ball $B_d \subseteq \mathbf{C}^d$, which we call H^2. This space is a natural generalization of the familiar Hardy space of the unit disk, but it differs from other "H^2"-spaces in several ways. For example, unlike the space $H^2(\partial B_d)$ associated with normalized surface area on the sphere or the space $H^2(B_d)$ associated with volume measure over the interior, H^2 is not associated with any measure on $\mathbf{C}^d$. Consequently, the associated multiplication operators (the component operators of the d-shift) do not form a subnormal d-tuple. Indeed, since the naive form of von Neumann's inequality described above fails, no effective model theory in dimension $d \geqslant 2$ could be based on subnormal operators. Thus by giving up the requirement of subnormality for models, one gains a theory in which models not only exist in all dimensions but are unique as well.

In the first part of this paper we work out the basic theory of H^2 and its associated multiplier algebra, and we show that the H^2-norm is the largest Hilbert norm on the space of polynomials which is appropriate for the operator theory of d-contractions.

In Part II we emphasize the role of "$\mathcal{A}$-morphisms". These are completely positive linear maps of the d-dimensional counterpart of the Toeplitz C^*-algebra which bear a particular relation to the d-shift. Every d-contraction corresponds to a *unique* $\mathcal{A}$-morphism, and on that observation we base a model theory for d-contractions which provides an appropriate generalization of the Sz.-Nagy–Foias theory of contractions [43] to arbitrary dimension $d \geqslant 1$ (see §8). In §7 we introduce a sequence of numerical invariants $E_n(\mathcal{S})$, $n=1,2,...$, for arbitrary operator spaces $\mathcal{S}$. We show that the d-dimensional operator space $\mathcal{S}_d$ generated by the d-shift is maximal in the sense that $E_n(\mathcal{S}_d) \geqslant E_n(\mathcal{S})$ for every $n \geqslant 1$ and for every d-dimensional operator space $\mathcal{S}$ consisting of mutually commuting operators. More significantly, we show that when $d \geqslant 2$, $\mathcal{S}_d$ is *characterized* by this maximality property. That characterization fails for single operators (i.e., one-dimensional operator spaces). We may conclude that, perhaps contrary to one's function-theoretic intuition, there is more uniqueness in dimension $d\geqslant 2$ than there is in dimension one.

Since this paper is a logical sequel to [3], [4], and so many years have passed since the publication of its two predecessors, it seems appropriate to comment on its relationship to them. On the one hand, we have come to the opinion that the program proposed in [4, Chapter 1] for carrying out dilation theory in higher dimensions must be modified. That program gives necessary and sufficient conditions for finding normal dilations in multivariable operator theory. However, the results below provide two reasons why normal dilations are inappropriate for commutative sets of operators associated with the unit ball B_d. First, they may not exist (a d-contraction need not have a normal dilation with spectrum in ∂B_d, cf. Remark 3.13) and second, when they do exist they are not unique (there can be many normal dilations of a given d-contraction which have the stated properties but which are not unitarily equivalent to each other).

On the other hand, the results of this paper also demonstrate that other aspects of the program of [3], [4] are well-suited for multivariable operator theory. For example, we will see that boundary representations, the noncommutative counterparts of Choquet boundary points in the commutative theory of function spaces, play an important role in the operator theory of B_d. Boundary representations serve to explain the notable fact that in higher dimensions there is more uniqueness than there is in dimension one (cf. Theorem 7.7 and its corollary), and they provide concrete information about the absence of inner functions for the d-shift (cf. Proposition 8.13).

We were encouraged to return to these problems by recent results in the theory of E_0-semigroups. There is a dilation theory by which, starting with a semigroup of completely positive maps of $\mathcal{B}(H)$, one obtains an E_0-semigroup as its "minimal dilation" [14], [6], [7], [8], [9], [10]. In its simplest form, this dilation theory starts with a normal completely positive map $P: \mathcal{B}(H)\rightarrow\mathcal{B}(H)$ satisfying $P(\mathbf{1})=\mathbf{1}$, and constructs from it a unique endomorphism of $\mathcal{B}(K)$ where K is a Hilbert space containing H. When one looks closely at this procedure one sees that there should be a corresponding dilation theory for sets of operators such as d-contractions.

We have reported on some of these results in a conference at the Fields institute in Waterloo in early 1995. That lecture concerned the dilation theory of semigroups of completely positive maps, $\mathcal{A}$-morphisms and the issue of uniqueness. However, at that time we had not yet reached a definitive formulation of the application to operator theory.

There is a large literature relating to von Neumann's inequality and dilation theory for sets of operators, and no attempt has been made to compile a comprehensive list of references here. More references can be found in [26], [27]. Finally, I want to thank Raúl Curto for bringing me back up to date on the literature of multivariable operator theory.

Part I. Function theory

1. Basic properties of H^2

Throughout this paper we will be concerned with function theory and operator theory as it relates to the unit ball B_d in complex d-dimensional space $\mathbf{C}^d$, $d=1,2,\ldots$,

$$B_d=\{z=(z_1,z_2,\ldots,z_d)\in\mathbf{C}^d:\|z\|<1\},$$

where $\|z\|$ denotes the norm associated with the usual inner product in $\mathbf{C}^d$,

$$\|z\|^2=|z_1|^2+|z_2|^2+\ldots+|z_d|^2.$$

In dimension $d=1$ there is a familiar Hardy space which can be defined in several ways. We begin by reiterating one of the definitions of H^2 in a form that we will generalize verbatim to higher dimensions. Let $\mathcal{P}$ be the algebra of all holomorphic polynomials f in a single complex variable z. Every $f\in\mathcal{P}$ has a finite Taylor series expansion

$$f(z)=a_0+a_1z+\ldots+a_nz^n$$

and we may define the norm $\|f\|$ of such a polynomial as the l^2-norm of its sequence of Taylor coefficients,

$$\|f\|^2=|a_0|^2+|a_1|^2+\ldots+|a_n|^2. \tag{1.1}$$

The norm $\|f\|$ is of course associated with an inner product on $\mathcal{P}$, and the completion of $\mathcal{P}$ in this norm is the Hardy space H^2. It is well known that the elements of H^2 can be realized concretely as analytic functions

$$f:\{|z|<1\}\to\mathbf{C}$$

which obey certain growth conditions near the boundary of the unit disk.

Now consider the case of dimension $d>1$. $\mathcal{P}$ will denote the algebra of all complex holomorphic polynomials f in the variable $z=(z_1,z_2,\ldots,z_d)$. Every such polynomial f has a unique expansion into a finite series

$$f(z)=f_0(z)+f_1(z)+\ldots+f_n(z) \tag{1.2}$$

where f_k is a homogeneous polynomial of degree k. We refer to (1.2) as the Taylor series of f.

Definition 1.3. Let V be a complex vector space. By a Hilbert seminorm on V we mean a seminorm which derives from a positive semidefinite inner product $\langle\cdot,\cdot\rangle$ on V by way of

$$\|x\|=\langle x,x\rangle^{1/2},\quad x\in V.$$

We will define a Hilbert seminorm on $\mathcal{P}$ by imitating formula (1.1), where a_k is replaced with f_k. To make that precise we must view the expansion (1.2) in a somewhat more formal way. The space $E=\mathbf{C}^d$ is a d-dimensional vector space having a distinguished inner product

$$\langle z, w\rangle = z_1\bar{w}_1+z_2\bar{w}_2+\ldots+z_d\bar{w}_d.$$

For each $n=1,2,\ldots$ we write E^n for the symmetric tensor product of n copies of E. E^0 is defined as the one-dimensional vector space $\mathbf{C}$ with its usual inner product. For $n\geqslant 2$, E^n is the subspace of the full tensor product $E^{\otimes n}$ consisting of all vectors fixed under the natural representation of the permutation group S_n,

$$E^n = \{\xi\in E^{\otimes n} : U_\pi\xi=\xi,\ \pi\in S_n\},$$

U_π denoting the isomorphism of $E^{\otimes n}$ defined on elementary tensors by

$$U_\pi(z_1\otimes z_2\otimes\ldots\otimes z_n) = z_{\pi^{-1}(1)}\otimes z_{\pi^{-1}(2)}\otimes\ldots\otimes z_{\pi^{-1}(n)},\quad z_1\in E.$$

For a fixed vector $z\in E$ we will use the notation

$$z^n = z^{\otimes n}\in E^n$$

for the n-fold tensor product of copies of z ($z^0\in E^0$ is defined as the complex number 1). E^n is linearly spanned by the set $\{z^n : z\in E\}$, $n=0,1,2,\ldots$.

Now every homogeneous polynomial $g: E\to\mathbf{C}$ of degree k determines a unique linear functional $\tilde{g}$ on E^k by

$$g(z)=\tilde{g}(z^k),\quad z\in E$$

(the uniqueness of $\tilde{g}$ follows from the fact that E^k is spanned by $\{z^k : z\in E\}$), and thus the Taylor series (1.2) can be written in the form

$$f(z)=\sum_{k=0}^{n}\tilde{f}_k(z^k),\quad z\in E,$$

where $\tilde{f}_k$ is a uniquely determined linear functional on E^k for each $k=0,1,\ldots,n$. Finally, if we bring in the inner product on E then E (resp. $E^{\otimes k}$) becomes a d-dimensional (resp. d^k-dimensional) complex Hilbert space. Thus the subspace $E^k\subseteq E^{\otimes k}$ is also a finite-dimensional Hilbert space in a natural way. Making use of the Riesz lemma, we find that there is a unique vector $\xi_k\in E^k$ such that

$$\tilde{f}_k(z^k)=\langle z^k,\xi_k\rangle,\quad z\in E,$$

and finally the Taylor series for f takes the form

$$f(z)=\sum_{k=0}^{n}\langle z^k,\xi_k\rangle,\quad z\in E. \tag{1.4}$$

We define a Hilbert seminorm on $\mathcal{P}$ as

$$\|f\|^2=\|\xi_0\|^2+\|\xi_1\|^2+\ldots+\|\xi_n\|^2. \tag{1.5}$$

The seminorm $\|\cdot\|$ is obviously a norm on $\mathcal{P}$ in that $\|f\|=0 \Rightarrow f=0$.

Definition 1.6. H^2_d is defined as the Hilbert space obtained by completing $\mathcal{P}$ in the norm (1.5).

When there is no possibility of confusion concerning the dimension we will abbreviate H^2_d with the simpler H^2. We first point out that the elements of H^2 can be identified with the elements of the symmetric Fock space over E,

$$\mathcal{F}_+(E)=E^0\oplus E^1\oplus E^2\oplus\ldots,$$

the sum on the right denoting the infinite direct sum of Hilbert spaces.

PROPOSITION 1.7. *For every $f\in\mathcal{P}$ let Jf be the element of $\mathcal{F}_+(E)$ defined by*

$$Jf=(\xi_0,\xi_1,\ldots),$$

where $\xi_0,\xi_1,\ldots$ is the sequence of Taylor coefficients defined in (1.4), *continued so that $\xi_k=0$ for $k>n$. Then J extends uniquely to an anti-unitary operator mapping H^2 onto $\mathcal{F}_+(E)$.*

Proof. The argument is perfectly straightforward, once one realizes that J is not linear but anti-linear. □

We can also identify the elements of H^2 in more concrete terms as analytic functions defined on the ball B_d:

PROPOSITION 1.8. *Every element of H^2 can be realized as an analytic function in B_d having a power series expansion of the form*

$$f(z)=\sum_{k=0}^{\infty}\langle z^k,\xi_k\rangle \quad z=(z_1,\ldots,z_d)\in B_d$$

where the H^2-norm of f is given by $\|f\|^2=\sum_k\|\xi_k\|^2<\infty$. Such functions f satisfy a growth condition of the form

$$|f(z)|\leqslant\frac{\|f\|}{\sqrt{1-\|z\|^2}},\quad z\in B_d.$$

Proof. Because of Proposition 1.7 the elements of H^2 can be identified with the formal power series having the form

$$f(z)=\sum_{k=0}^{\infty}\langle z^k,\xi_k\rangle, \tag{1.9}$$

where the sequence $\xi_k\in E^k$ satisfies

$$\sum_{k=0}^{\infty}\|\xi_k\|^2=\|f\|^2<\infty. \tag{1.10}$$

Because of (1.10) the series in (1.9) is easily seen to converge in B_d and satisfies the stated growth condition.

In more detail, since the norm of a vector in E^k of the form z^k, $z\in E$, satisfies

$$\|z^k\|^2=\langle z^k,z^k\rangle=\langle z,z\rangle^k=\|z\|^{2k},$$

we find that

$$|\langle z^k,\xi_k\rangle|\leqslant\|z^k\|\cdot\|\xi_k\|\leqslant\|z\|^k\|\xi_k\|,$$

and hence for all $z\in E$ satisfying $\|z\|<1$ we have

$$\sum_{k=0}^{\infty}|\langle z^k,\xi_k\rangle|\leqslant\left(\sum_{k=0}^{\infty}\|z\|^{2k}\right)^{1/2}\left(\sum_{k=0}^{\infty}\|\xi_k\|^2\right)^{1/2}=(1-\|z\|^2)^{-1/2}\|f\|,$$

as asserted. □

We will make frequent use of the following family of functions in H^2. For every $x\in B_d$ define $u_x\colon B_d\to\mathbf{C}$ by

$$u_x(z)=(1-\langle z,x\rangle)^{-1},\quad \|z\|<1. \tag{1.11}$$

$u_x(z)$ is clearly analytic in z and co-analytic in x. The useful properties of the set of functions $\{u_x : x\in B_d\}$ are summarized in the following proposition, which gives the precise sense in which H^2 is characterized in abstract terms by the positive-definite reproducing kernel $k\colon B_d\times B_d\to\mathbf{C}$,

$$k(x,y)=(1-\langle x,y\rangle)^{-1}.$$

PROPOSITION 1.12. *u_x belongs to H^2 for every $x \in B_d$, and these functions satisfy*

$$\langle u_x, u_y \rangle = (1 - \langle y, x \rangle)^{-1}. \tag{1.13}$$

H^2 is spanned by $\{u_x : x \in B_d\}$, and for every $f \in H^2$ we have

$$f(z) = \langle f, u_z \rangle, \quad z \in B_d. \tag{1.14}$$

Moreover, if K is any Hilbert space spanned by a subset of its elements $\{v_x : x \in B_d\}$ which satisfy

$$\langle v_x, v_y \rangle = (1 - \langle y, x \rangle)^{-1}, \quad x, y \in B_d,$$

then there is a unique unitary operator $W : H^2 \to K$ such that $Wu_x = v_x$, $x \in B_d$.

Proof. The proof is straightforward. For example, to see that u_x belongs to H^2 we simply examine its Taylor series

$$u_x(z) = (1 - \langle z, x \rangle)^{-1} = \sum_{k=0}^{\infty} \langle z, x \rangle^k.$$

Noting that $\langle z, x \rangle^k = \langle z^k, x^k \rangle_{E^k}$ we can write

$$u_x(z) = \sum_{k=0}^{\infty} \langle z^k, x^k \rangle_{E^k}.$$

This shows that the sequence of Taylor coefficients of u_x is

$$Ju_x = (1, x, x^2, \ldots) \in \mathcal{F}_+(E).$$

Hence u_x belongs to H^2 and we have

$$\langle u_x, u_y \rangle = \langle Ju_y, Ju_x \rangle_{\mathcal{F}_+(E)} = \sum_{k=0}^{\infty} \langle y, x \rangle^k = (1 - \langle y, x \rangle)^{-1}.$$

Formula (1.13) follows.

Similarly, a direct application of Proposition 1.7 establishes (1.14). From the latter it follows that $\{u_x : x \in B_d\}$ spans H^2. Indeed, if f is any function in H^2 which is orthogonal to every u_x then

$$f(z) = \langle f, u_z \rangle = 0 \quad \text{for every } z \in B_d,$$

and hence $f = 0$.

Finally, the second paragraph is obvious from the fact that for every finite subset $x_1, \ldots, x_n \in B_d$ and $c_1, \ldots, c_n \in \mathbf{C}$ we have

$$\|c_1 u_{x_1} + \ldots + c_n u_{x_n}\|^2 = \|c_1 v_{x_1} + \ldots + c_n v_{x_n}\|^2,$$

which is apparent after expanding both sides and comparing inner products. □

The H^2-norm is invariant under the natural action of the unitary group of $\mathbf{C}^d$, as summarized by

COROLLARY. *Let V be a unitary operator on the Hilbert space $E=\mathbf{C}^d$. Then there is a unique unitary operator $\Gamma(V)\in\mathcal{B}(H^2)$ satisfying*

$$\Gamma(V)u_x = u_{Vx}, \quad x\in B_d. \tag{1.15}$$

Γ is a strongly continuous unitary representation of $\mathcal{U}(\mathbf{C}^d)$ on H^2 whose action on functions is given by

$$\Gamma(V)f(z) = f(V^{-1}z), \quad z\in B_d,\ f\in H^2. \tag{1.16}$$

Proof. Fix $V\in\mathcal{U}(\mathbf{C}^d)$. For any $x, y\in B_d$ we have

$$\langle u_{Vx}, u_{Vy}\rangle = (1-\langle Vy, Vx\rangle)^{-1} = (1-\langle y, x\rangle)^{-1} = \langle u_x, u_y\rangle.$$

It follows from Proposition 1.12 that there is a unique unitary operator $\Gamma(V)\in\mathcal{B}(H^2)$ satisfying (1.15). It is clear from (1.15) that $\Gamma(V_1V_2)=\Gamma(V_1)\Gamma(V_2)$, and strong continuity follows from the fact that

$$\langle\Gamma(V)u_x, u_y\rangle = \langle u_{Vx}, u_y\rangle = (1-\langle y, Vx\rangle)^{-1}$$

is continuous in V for fixed $x, y\in B_d$, together with the fact that H^2 is spanned by $\{u_z : z\in B_d\}$.

Finally, from (1.14) we see that for every $f\in H^2$ and every $z\in B_d$,

$$\begin{aligned} f(V^{-1}z) &= \langle f, u_{V^{-1}z}\rangle = \langle f, \Gamma(V^{-1})u_z\rangle = \langle f, \Gamma(V)^*u_z\rangle \\ &= \langle \Gamma(V)f, u_z\rangle = (\Gamma(V)f)(z), \end{aligned}$$

proving (1.16). □

2. Multipliers and the d-dimensional shift

By a *multiplier* of H^2 we mean a complex-valued function $f\colon B_d\to\mathbf{C}$ with the property

$$f\cdot H^2\subseteq H^2.$$

The set of multipliers is a complex algebra of functions defined on the ball B_d which contains the constant functions, and since H^2 itself contains the constant function 1 it follows that every multiplier must belong to H^2. In particular, multipliers are analytic functions on B_d.

Definition 2.1. The algebra of all multipliers is denoted $\mathcal{M}$. H^∞ will denote the Banach algebra of all bounded analytic functions $f\colon B_d\to\mathbf{C}$ with norm

$$\|f\|_\infty = \sup_{\|z\|<1} |f(z)|.$$

The following result implies that $\mathcal{M}\subseteq H^\infty$, and the inclusion map of $\mathcal{M}$ in H^∞ becomes a contraction after one endows $\mathcal{M}$ with its natural norm.

PROPOSITION 2.2. *Every $f\in\mathcal{M}$ defines a unique bounded operator M_f on H^2 by way of*

$$M_f\colon g\in H^2\to f\cdot g\in H^2.$$

The natural norm in $\mathcal{M}$,

$$\|f\|_{\mathcal{M}}=\sup\{\|f\cdot g\|:g\in H^2,\ \|g\|\leqslant 1\},$$

satisfies

$$\|f\|_{\mathcal{M}}=\|M_f\|,$$

the right side denoting the operator norm in $\mathcal{B}(H^2)$, and we have

$$\|f\|_\infty\leqslant\|f\|_{\mathcal{M}},\quad f\in\mathcal{M}.$$

Proof. Fix $f\in\mathcal{M}$. Notice first that if g is an arbitrary function in H^2 then by (1.16) we have

$$\langle M_f g,u_z\rangle=\langle f\cdot g,u_z\rangle=f(z)g(z). \tag{2.3}$$

A straightforward application of the closed graph theorem (which we omit) now shows that the operator M_f is bounded.

It is clear that $\|f\|_{\mathcal{M}}=\|M_f\|$. We claim now that for each $x\in B_d$ one has

$$M_f^*u_x=\bar{f}(x)u_x. \tag{2.4}$$

Indeed, since H^2 is spanned by $\{u_y:y\in B_d\}$ it is enough to show that

$$\langle M_f^*u_x,u_y\rangle=\bar{f}(x)\langle u_x,u_y\rangle,\quad y\in B_d.$$

For fixed y the left side is

$$\langle u_x,f\cdot u_y\rangle=\overline{\langle f\cdot u_y,u_x\rangle}$$

By (1.16) the latter is

$$\overline{f(x)u_y(x)}=\bar{f}(x)\overline{(1-\langle x,y\rangle)^{-1}}=\bar{f}(x)(1-\langle y,x\rangle)^{-1}=\bar{f}(x)\langle u_x,u_y\rangle,$$

and (2.4) follows.

Finally, (2.4) implies that for every $x\in B_d$ we have

$$|f(x)|=\frac{\|M_f^*u_x\|}{\|u_x\|}\leqslant\|M_f^*\|=\|M_f\|=\|f\|_{\mathcal{M}},$$

as required. □

We turn now to the definition of the d-dimensional analogue of the unilateral shift. Let $e_1, e_2, ..., e_d$ be an orthonormal basis for $E=\mathbf{C}^d$, and define $z_1, z_2, ..., z_d \in \mathcal{P}$ by

$$z_k(z) = \langle z, e_k \rangle, \quad x \in \mathbf{C}^d.$$

Such a d-tuple of linear functionals will be called a *system of coordinate functions*. If $z_1', z_2', ..., z_d'$ is another system of coordinate functions then there is a unique unitary operator $V \in \mathcal{B}(\mathbf{C}^d)$ satisfying

$$z_k'(x) = z_k(V^{-1}x), \quad 1 \leqslant k \leqslant d, \ x \in \mathbf{C}^d. \tag{2.5}$$

PROPOSITION 2.6. *Let $z_1, z_2, ..., z_d$ be a system of coordinate functions for $\mathbf{C}^d$. Then for every complex number a and polynomials $f_1, f_2, ..., f_d \in \mathcal{P}$ we have*

$$\|a \cdot 1 + z_1 f_1 + ... + z_d f_d\|^2 \leqslant |a|^2 + \|f_1\|^2 + ... + \|f_d\|^2,$$

$\|\cdot\|$ *denoting the norm in H^2.*

Proof. We claim first that each z_k is a multiplier. Indeed, if $f \in H^2$ has Taylor series

$$f(x) = \sum_{n=0}^{\infty} \langle z^n, \xi_n \rangle$$

with $\sum_n \|\xi_n\|^2 = \|f\|^2 < \infty$ then we have

$$z_k(x) f(x) = \sum_{n=0}^{\infty} \langle x, e_k \rangle \langle x^n, \xi_n \rangle. \tag{2.7}$$

Now

$$\langle x, e_k \rangle \langle x^n, \xi_n \rangle = \langle x^{n+1}, e_k \otimes \xi_n \rangle.$$

So if $e_k \cdot \xi_n$ denotes the projection of the vector $e_k \otimes \xi_n \in E \otimes E^n$ to the subspace E^{n+1} then (2.7) becomes

$$z_k(x) f(x) = \sum_{n=0}^{\infty} \langle x^{n+1}, e_k \cdot \xi_n \rangle.$$

Since

$$\sum_{n=0}^{\infty} \|e_k \cdot \xi_n\|^2 \leqslant \sum_{n=0}^{\infty} \|\xi_n\|^2 = \|f\|^2$$

it follows that $z_k f \in H^2$ and in fact

$$\|z_k f\| \leqslant \|f\|, \quad f \in H^2.$$

Thus each multiplication operator M_{z_k} is a contraction in $\mathcal{B}(H^2)$. Consider the Hilbert space

$$K=\mathbf{C}\oplus\underbrace{H^2\oplus...\oplus H^2}_{d\text{ times}}$$

and the operator $T: K\to H^2$ defined by

$$T(a,f_1,...,f_d)=a\cdot 1+z_1f_1+...+z_df_d.$$

The assertion of Proposition 2.6 is that $\|T\|\leqslant 1$. In fact, we show that the adjoint of T, $T^*: H^2\to K$, is an isometry. A routine computation implies that for all $f\in H^2$ we have

$$T^*f=(\langle f,1\rangle, S_1^*f,...,S_d^*f)\in K,$$

where we have written S_k for the multiplication operator M_{z_k}, $k=1,...,d$. Hence $TT^*\in\mathcal{B}(H^2)$ is given by

$$TT^*=E_0+S_1S_1^*+...+S_dS_d^*,$$

where E_0 is the projection on the one-dimensional space of all constant functions in H^2. We establish the key assertion as a lemma for future reference.

LEMMA 2.8. *Let $z_1,...,z_d$ be a system of coordinate functions for $\mathbf{C}^d$, and let $S_k=M_{z_k}$, $k=1,2,...,d$. Let E_0 be the projection onto the one-dimensional space of constant functions in H^2. Then*

$$E_0+S_1S_1^*+...S_dS_d^*=\mathbf{1}.$$

Proof. Since H^2 is spanned by $\{u_z: z\in B_d\}$ it is enough to show that for all $x,y\in B_d$ we have

$$\langle E_0u_x,u_y\rangle+\sum_{k=1}^{d}\langle S_kS_k^*u_x,u_y\rangle=\langle u_x,u_y\rangle. \tag{2.9}$$

Since each S_k is a multiplication operator, formula (2.4) implies that

$$S_k^*u_x=\bar z_k(x)u_x=\langle e_k,x\rangle u_x,$$

for $x\in B_d$. Thus we can write

$$\begin{aligned}\sum_{k=1}^{d}\langle S_kS_k^*u_x,u_y\rangle&=\sum_{k=1}^{d}\langle S_k^*u_x,S_k^*u_y\rangle=\sum_{k=1}^{d}\langle e_k,x\rangle\langle y,e_k\rangle\langle u_x,u_y\rangle\\&=\langle y,x\rangle\langle u_x,u_y\rangle=\langle y,x\rangle(1-\langle y,x\rangle)^{-1}.\end{aligned}$$

On the other hand, noting that $u_0=1$ and $\|u_0\|=1$, the projection E_0 is given by $E_0(f)=\langle f,u_0\rangle u_0$, $f\in H^2$. Hence

$$\langle E_0u_x,u_y\rangle=\langle u_x,u_0\rangle\langle u_0,u_y\rangle=1$$

because $\langle u_x,u_0\rangle=(1-\langle x,0\rangle)^{-1}=1$ for every $x\in B_d$. It follows that

$$\langle E_0u_x,u_y\rangle+\sum_{k=1}^{d}\langle S_kS_k^*u_x,u_y\rangle=1+\langle y,x\rangle(1-\langle y,x\rangle)^{-1}=(1-\langle y,x\rangle)^{-1}=\langle u_x,u_y\rangle,$$

as asserted. □

That completes the proof of Proposition 2.6. □

Definition 2.10. Let $z_1,\ldots,z_d$ be a system of coordinate functions for $\mathbf{C}^d$ and let $S_k=M_{z_k}$, $k=1,2,\ldots,d$. The d-tuple of operators

$$\bar{S}=(S_1,S_2,\ldots,S_d)$$

is called the d-dimensional shift or, briefly, the d-shift.

Remarks. The component operators $S_1,\ldots,S_d$ of the d-shift are mutually commuting contractions in $\mathcal{B}(H^2)$ which satisfy

$$S_1S_1^*+\ldots+S_dS_d^*=\mathbf{1}-E_0$$

where E_0 is the projection onto the space of constant functions in H^2. In particular, we conclude from Proposition 2.6 that for any $f_1,\ldots,f_d\in H^2$,

$$\|S_1f_1+\ldots+S_df_d\|^2\leqslant\|f_1\|^2+\ldots+\|f_d\|^2.$$

Notice too that if we replace $z_1,\ldots,z_d$ with a different set of coordinate functions $z_1',\ldots,z_d'$ for $\mathbf{C}^d$ then then the operators $(S_1,\ldots,S_d)$ change to a new d-shift $(S_1',\ldots,S_d')$. However, this change is not significant by virtue of the relation between z_k and z_k'. More precisely, letting V be the unitary operator defined on $\mathbf{C}^d$ by (2.5), one finds that

$$\Gamma(V)S_k\Gamma(V)^{-1}=S_k',\quad k=1,2,\ldots,d,$$

that is, $(S_1',\ldots,S_d')$ *and* $(S_1,\ldots,S_d)$ *are unitarily equivalent by way of a natural unitary automorphism of* H^2. In this sense we may speak of *the* d-shift acting on H_d^2. In particular, we may conclude that each component operator S_i is unitarily equivalent to every other one S_j, $1\leqslant j\leqslant d$.

Finally, if f is any polynomial in $\mathcal{P}$ then we may express M_f as a polynomial in the operators $S_1, ..., S_d$ as follows. We find a polynomial function $g(w_1, ..., w_d)$ of d complex variables with the property that f is the composite function of g with the coordinate functions $z_1, ..., z_d$,

$$f(x) = g(z_1(x), ..., z_d(x)), \quad x \in B_d.$$

Once this is done the multiplication operator M_f becomes the corresponding polynomial in the operators $S_1, ..., S_d$:

$$M_f = g(S_1, ..., S_d).$$

We emphasize that in the higher-dimensional cases $d \geqslant 2$, the operator norm $\|M_f\|$ can be larger than the sup norm $\|f\|_\infty$ (see §3 below). On the other hand, in all dimensions the spectral radius $r(M_f)$ of any polynomial multiplication operator satisfies

$$r(M_f) = \sup_{z \in B_d} |f(z)|. \tag{2.11}$$

In the following result we establish the formula (2.11). That follows from a straightforward application of the Gelfand theory of commutative Banach algebras and we merely sketch the details.

PROPOSITION 2.12. *Let $\mathcal{A}$ be the norm-closed subalgebra of $\mathcal{B}(H^2)$ generated by the multiplication operators M_f, $f \in \mathcal{P}$.*

Every element of $\mathcal{A}$ is a multiplication operator M_f for some $f \in \mathcal{M}$ which extends continuously to the closed ball $\bar{B}_d$, and there is a natural homeomorphism of the closed unit ball onto the space $\sigma(\mathcal{A})$ of all complex homorphisms of $\mathcal{A}$, $x \mapsto \omega_x$, defined by

$$\omega_x(M_f) = f(x), \quad \|x\| \leqslant 1.$$

For every such $f \in \mathcal{M}$ one has

$$\lim_{n \to \infty} \|M_f^n\|^{1/n} = \sup_{\|x\| < 1} |f(x)|.$$

Proof. Since the mapping $f \in \mathcal{M} \mapsto M_f \in \mathcal{B}(H^2)$ is an isometric representation of the multiplier algebra on H^2 which carries the unit of $\mathcal{M}$ to that of $\mathcal{B}(H^2)$, it is enough to work within $\mathcal{M}$ itself. That is, we may consider $\mathcal{A}$ to be the closure in $\mathcal{M}$ of the algebra of polynomials, and basically we need to identify its maximal ideal space.

Because of the inequality $\|f\|_\infty \leqslant \|f\|_\mathcal{M}$ of Proposition 2.2, we can assert that for every polynomial f and every $x \in \mathbf{C}_d$ satisfying $\|x\| \leqslant 1$ we have

$$|f(x)| \leqslant \sup_{z \in B_d} |f(z)| = \|f\|_\infty \leqslant \|f\|_\mathcal{M}.$$

It follows that there is a unique complex homomorphism ω_x of $\mathcal{A}$ satisfying

$$\omega_x(f) = f(x), \quad f \in \mathcal{P}.$$

For all $g \in \mathcal{A}$ we now have a natural continuous extension $\tilde{g}$ of g to the closed unit ball by setting

$$\tilde{g}(x) = \omega_x(g), \quad \|x\| \leqslant 1.$$

$x \mapsto \omega_x$ is a one-to-one continuous map of the closed ball in $\mathbf{C}^d$ onto its range in $\sigma(\mathcal{A})$. To see that it is surjective, let ω be an arbitrary element of $\sigma(\mathcal{A})$. Then for every $y \in \mathbf{C}^d$ we may consider the linear functional

$$\hat{y}(z) = \langle z, y \rangle, \quad z \in \mathbf{C}^d.$$

The map $y \mapsto \hat{y}$ is an antilinear mapping of $\mathbf{C}^d$ onto the space of linear functions in $\mathcal{P}$, and we claim that $\|\hat{y}\|_{\mathcal{M}} \leqslant \|y\|$. Indeed, assuming that $y \neq 0$, the linear function

$$u(x) = \frac{\hat{y}(x)}{\|y\|} = \frac{\langle x, y \rangle}{\|y\|}$$

is part of a system of coordinates for $\mathbf{C}^d$. Proposition 2.6 implies $\|u\|_{\mathcal{M}} \leqslant 1$, and hence $\|\hat{y}\|_{\mathcal{M}} \leqslant \|y\|$. Thus, $y \mapsto \omega(\hat{y})$ defines an antilinear functional on $\mathbf{C}^d$ satisfying

$$|\omega(\hat{y})| \leqslant \|\hat{y}\|_{\mathcal{M}} \leqslant \|y\|, \quad y \in \mathbf{C}^d.$$

It follows that there is a unique vector x in the unit ball of $\mathbf{C}^d$ such that

$$\omega(\hat{y}) = \langle x, y \rangle, \quad y \in \mathbf{C}^d.$$

Thus, $\omega(f) = \omega_x(f)$ on every linear functional f. Since both ω and ω_x are continuous unital homomorphisms of $\mathcal{A}$, since $\mathcal{P}$ is the algebra generated by the linear functions and the constants, and since $\mathcal{P}$ is dense in $\mathcal{A}$, it follows that $\omega = \omega_x$, and the claim is proved.

Thus we have identified the maximal ideal space of $\mathcal{A}$ with the closed unit ball in $\mathbf{C}^d$. From the elementary theory of commutative Banach algebras we deduce that for every f in $\mathcal{A}$,

$$\lim_{n \to \infty} \|f^n\|_{\mathcal{M}}^{1/n} = r(f) = \sup\{|\omega(f)| : \omega \in \sigma(\mathcal{A})\} = \sup\{|\tilde{f}(x)| : \|x\| \leqslant 1\} = \|f\|_\infty,$$

completing the proof of Proposition 2.12. □

The realization of the d-shift as a d-tuple of multiplication operators on the function space H^2 is not always convenient for making computations. We require the following realization of $(S_1, \ldots, S_d)$ as "creation" operators on the symmetric Fock space $\mathcal{F}_+(E)$.

PROPOSITION 2.13. *Let $e_1, \dots e_d$ be an orthonormal basis for a Hilbert space E of dimension d. Define operators $A_1, \dots, A_d$ on $\mathcal{F}_+(E)$ by*

$$A_i \xi = e_i \xi, \quad \xi \in \mathcal{F}_+(E),$$

where $e_i\xi$ denotes the projection of $e_1 \otimes \xi \in \mathcal{F}(E)$ to the symmetric subspace $\mathcal{F}_+(E)$. Let $z_1, \dots, z_d$ be the system of orthogonal coordinates $z_i(x) = \langle x, e_i \rangle$, $1 \leqslant i \leqslant d$. Then there is a unique unitary operator $W: H^2 \to \mathcal{F}_+(E)$ such that $W(1) = 1$ and

$$W(z_{i_1} \dots z_{i_n}) = e_{i_1} \dots e_{i_n}, \quad n \geqslant 1,\ i_k \in \{1, 2, \dots, d\}. \tag{2.14}$$

In particular, the d-tuple of operators $(A_1, \dots, A_d)$ is unitarily equivalent to the d-shift.

Proof. For every $x \in E$ satisfying $\|x\| < 1$ define an element $v_x \in \mathcal{F}_+(E)$ by

$$v_x = 1 \oplus x \oplus x^2 \oplus x^3 \oplus \dots .$$

It is obvious that $\|v_x\|^2 = (1 - \|x\|^2)^{-1}$ and, more generally,

$$\langle v_x, v_y \rangle = (1 - \langle x, y \rangle)^{-1}, \quad \|x\|, \|y\| < 1.$$

$\mathcal{F}_+(E)$ is spanned by the set $\{v_x : \|x\| < 1\}$.

Let $\{u_x : \|x\| < 1\}$ be the set of functions in H^2 defined in (1.11), and let $*$ be the unique conjugation of E defined by $e_i^* = e_i$, that is,

$$(a_1 e_1 + \dots + a_d e_d)^* = \bar{a}_1 e_1 + \dots + \bar{a}_d e_d.$$

Then we have

$$\langle u_x, u_y \rangle = (1 - \langle y, x \rangle)^{-1} = (1 - \langle x^*, y^* \rangle)^{-1} = \langle v_{x^*}, v_{y^*} \rangle$$

for all x, y in the open unit ball of E. By Proposition 1.12 there is a unique unitary operator $W: H^2 \to \mathcal{F}_+(E)$ such that $W(u_x) = v_{x^*}$, $\|x\| < 1$.

We have $W(1) = W(u_0) = v_0 = 1$. Choose $x \in E$ satifying $\|x\| \leqslant 1$ and let f_x denote the linear functional on E defined by $f_x(z) = \langle z, x \rangle$. We have $\|f_x\|_{H^2} \leqslant 1$ and in fact $\|f_x^n\|_{H^2} \leqslant 1$ for every $n = 0, 1, 2, \dots$. Hence for every $0 \leqslant r < 1$ and every $z \in B_d$ we have

$$u_{rx}(z) = (1 - \langle z, rx \rangle)^{-1} = \sum_{n=0}^{\infty} r^n \langle z, x \rangle^n = \sum_{n=0}^{\infty} r^n f_x^n(z) \in H^2.$$

Similarly,

$$v_{rx^*} = \sum_{n=0}^{\infty} r^n (x^*)^n \in \mathcal{F}_+(E).$$

Setting $W(u_{rx})$ equal to v_{rx^*} and comparing coefficients of r^n we obtain

$$W(f_x^n) = (x^*)^n$$

for every $n=0,1,\dots$. It follows that

$$W(f_{x_1} f_{x_2} \dots f_{x_n}) = x_1^* x_2^* \dots x_n^* \tag{2.15}$$

for every $x_1, x_2, \dots, x_n \in E$. Indeed, setting

$$L(x_1, x_2, \dots, x_n) = W(f_{x_1} f_{x_2} \dots f_{x_n}),$$
$$R(x_1, x_2, \dots, x_n) = x_1^* x_2^* \dots x_n^*$$

for $x_1, x_2, \dots, x_n \in E$, we see that both L and R are symmetric n-antilinear mappings which agree when $x_1 = x_2 = \dots = x_n \in B_d$. Hence $L=R$ and (2.15) follows. We obtain (2.14) by taking $x_k = e_{i_k}$ in (2.15).

(2.14) obviously implies $WS_i = A_i W$ for $i=1,\dots,d$, so that the d-tuples $(S_1,\dots,S_d)$ and $(A_1,\dots,A_d)$ are unitarily equivalent. □

3. von Neumann's inequality and the sup norm

Definition 3.1. A d-contraction is a d-tuple of operators $\bar{T} = (T_1, \dots, T_d)$ acting on a common Hilbert space H which commute with each other and satisfy

$$\|T_1\xi_1 + \dots + T_d\xi_d\|^2 \leqslant \|\xi_1\|^2 + \dots + \|\xi_d\|^2$$

for every $\xi_1, \dots, \xi_d \in H$.

Remark 3.2. We make frequent use of the following observation. For operators $T_1, \dots, T_d$ on a common Hilbert space H, the following are equivalent:

(1) $\|T_1\xi_1 + \dots + T_d\xi_d\|^2 \leqslant \|\xi_1\|^2 + \dots + \|\xi_d\|^2$ for all $\xi_1, \dots, \xi_d \in H$.

(2) $T_1T_1^* + \dots + T_dT_d^* \leqslant \mathbf{1}$.

To see this let $d \cdot H$ denote the direct sum of d copies of H, and let $\bar{T} \in \mathcal{B}(d\cdot H, H)$ be the operator defined by $\bar{T}(\xi_1, \dots, \xi_d) = T_1\xi_1 + \dots + T_d\xi_d$. A simple computation shows that the adjoint $\bar{T}^*: H \to d\cdot H$ is given by

$$\bar{T}^*\xi = (T_1^*\xi, \dots, T_d^*\xi).$$

Thus $\bar{T}\bar{T}^*$ is the operator in $\mathcal{B}(H)$ given by $\bar{T}\bar{T}^* = T_1T_1^* + \dots + T_dT_d^*$. The equivalence of (1) and (2) follows.

Notice that the d-shift $\bar{S}=(S_1,\ldots,S_d)$ acting on H^2_d is a d-contraction. Perhaps the most natural generalization of von Neumann's inequality for d-dimensional operator theory would make the following assertion. Let $\bar{T}=(T_1,\ldots,T_d)$ be a d-contraction and let $f=f(z_1,\ldots,z_d)$ be a polynomial in d complex variables $z_1,\ldots,z_d$. Then

$$\|f(T_1,\ldots,T_d)\| \leqslant \sup_{\|z\|\leqslant 1} |f(z_1,\ldots,z_d)|.$$

In this section we show that this inequality fails rather spectacularly for the d-shift, in that there is no constant K for which

$$\|f(S_1,\ldots,S_d)\| \leqslant K \sup_{\|z\|\leqslant 1} |f(z_1,\ldots,z_d)|$$

holds for all polynomials f. It follows that the multiplier algebra $\mathcal{M}$ is a proper subalgebra of H^∞. Indeed, we exhibit continuous functions

$$f: \{z\in \mathbf{C}^d : \|z\| \leqslant 1\} \to \mathbf{C}$$

which are analytic in the interior of the unit ball and which do not belong to $\mathcal{M}$.

We will establish the appropriate version of von Neumann's inequality for dimension $d\geqslant 2$ in §8.

THEOREM 3.3. *Assume $d\geqslant 2$. Let $c_0, c_1, \ldots$ be a sequence of complex numbers having the properties*

(i) $\sum_{n=0}^\infty |c_n|=1$,

(ii) $\sum_{n=0}^\infty |c_n|^2 n^{(d-1)/2}=\infty$,

and define a function $f(z_1,\ldots,z_d)$ for $|z_1|^2+\ldots+|z_d|^2\leqslant 1$ as

$$f(z_1,\ldots,z_d)=\sum_{n=0}^\infty \frac{c_n}{s^n}(z_1 z_2 \ldots z_d)^n, \tag{3.3}$$

where s denotes the sup norm

$$s=\sup_{|z_1|^2+\ldots+|z_d|^2\leqslant 1} |z_1 z_2 \ldots z_d| = \sqrt{\frac{1}{d^d}}\,. \tag{3.4}$$

Then the power series (3.3) converges uniformly over the closed unit ball to a function f satisfying $\|f\|_\infty \leqslant 1$. The restriction of f to B_d does not belong to H^2. Letting $f_0, f_1, f_2, \ldots$ be the sequence of Taylor polynomials

$$f_N(z_1,\ldots,z_d)=\sum_{n=0}^N \frac{c_n}{s^n}(z_1 z_2 \ldots z_d)^n,$$

then we have $\|f_N\|_\infty \leqslant 1$ *for every* N *while*

$$\lim_{N\to\infty} \|f_N(S_1, ..., S_d)\| = \lim_{N\to\infty} \|f_N\|_{\mathcal{M}} = \infty. \tag{3.5}$$

Remarks. It is clear that the function f belongs to the "ball algebra", that is, the closure in the sup norm $\|\cdot\|_\infty$ of the algebra of polynomials. On the other hand, f does not belong to the multiplier algebra $\mathcal{M}$, and in particular the inclusion $\mathcal{M} \subsetneq H^\infty$ is *proper*.

Note too that it is a simple matter to give explicit examples of sequences $c_0, c_1, ...$ satisfying conditions (i) and (ii). For example, let S be any *infinite* subset of the non-negative integers which is sparse enough so that

$$\sum_{n\in S} \frac{1}{n^{(d-1)/4}} < \infty.$$

If we set $c_n = 1/n^{(d-1)/4}$ if $n \in S$ and $c_n = 0$ otherwise, then we obviously have (ii) because S is infinite, and (i) can also be achieved after multiplying the sequence by a suitable positive constant.

Proof. The formula (3.4) for the sup norm,

$$s = d^{-d/2},$$

follows from the elementary fact that

$$(|z_1|^2 |z_2|^2 \dots |z_d|^2)^{1/d} \leqslant \frac{1}{d}(|z_1|^2 + |z_2|^2 + \dots + |z_d|^2),$$

with equality if and only if $|z_1| = |z_2| = \dots = |z_d|$.

Let p be the homogeneous polynomial $p(z_1, ..., z_d) = z_1 z_2 \dots z_d$. Then for every $n = 0, 1, 2, ...$ we have

$$\|p^n\|_\infty = \|p\|_\infty^n = d^{-nd/2} = s^n.$$

It follows that the power series (3.3),

$$\sum_{n=0}^{\infty} \frac{c_n}{s^n} p(z_1, ..., z_d)^n,$$

converges uniformly over the unit ball to f. Thus it remains to establish the condition (3.5).

Now for any polynomial g in the d complex variables $z_1, ..., z_d$ we have

$$\|g(S_1, ..., S_d)\| \geqslant \|g(S_1, ..., S_d)1\|_{H^2} = \|g\|_{H^2}.$$

Thus it suffices to show that the sequence of Taylor polynomials $f_0, f_1, f_2, \ldots$ defined by the partial sums of the series diverges in the H^2-norm, that is,

$$\sup_N \|f_N\|_{H^2}^2 = \sum_{n=0}^{\infty} |c_n|^2 \frac{\|p^n\|_{H^2}^2}{\|p^n\|_\infty^2} = \infty. \tag{3.6}$$

In order to establish (3.6) we will show that there is a positive constant A such that

$$\|p^n\|_{H^2}^2 \geqslant A d^{-nd} n^{(d-1)/2}, \tag{3.7}$$

for all $n=1,2,\ldots$. In view of the fact that $\|p^n\|_\infty^2 = d^{-nd}$ and the series $\sum_n |c_n|^2 n^{(d-1)/2}$ diverges, (3.6) will follow.

The estimate (3.7) is based on the following computation. Since the result is a statement about certain norms in the symmetric Fock space over $\mathbf{C}^d$, it is likely that the result of Lemma 3.8 can be found in the literature. Since we are not aware of an appropriate reference and since the estimate (3.7) depends essentially on these formulas, we have provided the details.

LEMMA 3.8. *Let $e_1, e_2, \ldots, e_d$ be an orthonormal basis for $E=\mathbf{C}^d$. Then for every d-tuple of nonnegative integers $k=(k_1, \ldots, k_d)$ we have*

$$\|e_1^{k_1} e_2^{k_2} \ldots e_d^{k_d}\|_{E^{|k|}}^2 = \frac{k_1!\, k_2! \ldots k_d!}{|k|!}$$

where $|k| = k_1 + k_2 + \ldots + k_d$.

Remark. Regarding notation, we have written $e_1^{k_1} e_2^{k_2} \ldots e_d^{k_d}$ for the projection of the vector

$$e_1^{\otimes k_1} \otimes e_2^{\otimes k_2} \otimes \ldots \otimes e_d^{\otimes k_d} \in E^{\otimes |k|}$$

to the symmetric subspace $E^{|k|} \subseteq E^{\otimes |k|}$.

Proof. For $y_1, \ldots, y_p \in E = \mathbf{C}^d$ we use the notation $y_1 y_2 \ldots y_p$ for the projection of $y_1 \otimes y_2 \otimes \ldots \otimes y_p \in E^{\otimes p}$ to the symmetric subspace E^p. Fixing $a \in E$ and $p \geqslant 1$ we have an associated "creation operator" $A: E^{p-1} \to E^p$ defined by

$$A(x_1 x_2 \ldots x_{p-1}) = a x_1 x_2 \ldots x_{p-1}, \quad x \in E.$$

We claim first that for $p \geqslant 1$ the adjoint $A^*: E^p \to E^{p-1}$ is given by

$$A^*(y_1 y_2 \ldots y_p) = \frac{1}{p} \sum_{k=1}^{p} \langle y_k, a \rangle\, y_1 \ldots \widehat{y_k} \ldots y_p \tag{3.9}$$

where $\widehat{y_k}$ means that the term y_k is missing from the symmetric tensor product. Indeed, if ζ denotes the right side of (3.9) then for every $x \in E$ we have

$$\begin{aligned}\langle \zeta, x^{p-1}\rangle &= \frac{1}{p}\sum_{k=1}^{p}\langle y_1, x\rangle \dots \langle y_{k-1}, x\rangle\langle y_k, a\rangle\langle y_{k+1}, x\rangle \dots \langle y_p, x\rangle \\ &= \langle y_1 \otimes \dots \otimes y_p, p^{-1}(a \otimes x^{p-1} + x \otimes a \otimes x^{p-2} + \dots + x^{p-1} \otimes a)\rangle.\end{aligned}$$

Since

$$p^{-1}(a \otimes x^{p-1} + x \otimes a \otimes x^{p-2} + \dots + x^{p-1} \otimes a) = a x^{p-1} \in E^p,$$

the right side of the preceding formula becomes

$$\langle y_1 \otimes \dots \otimes y_p, a x^{p-1}\rangle = \langle y_1 y_2 \dots y_p, a x^{p-1}\rangle = \langle A^*(y_1 y_2 \dots y_p), x^{p-1}\rangle.$$

(3.9) now follows because E^{p-1} is spanned by vectors of the form x^{p-1}, $x \in E$.

To prove Lemma 3.8 we proceed by induction on the total degree $|k|$. The formula is obvious for $|k|=0$. Assuming that $|k| \geqslant 1$ and that the formula has been established for total degree $|k|-1$ then we may assume (after relabelling the basis vectors $e_1, \dots, e_d$ if necessary) that $k_1 \geqslant 1$.

Taking $a=e_1$ in (3.9) and noting that $\langle e_1, e_1\rangle = 1$ and $\langle e_1, e_j\rangle = 0$ if $j=2, \dots, d$, we find that

$$A^*(e_1^{k_1} e_2^{k_2} \dots e_d^{k_d}) = \frac{k_1}{|k|} e_1^{k_1-1} e_2^{k_2} \dots e_d^{k_d},$$

and hence

$$\langle e_1^{k_1} \dots e_d^{k_d}, e_1^{k_1} \dots e_d^{k_d}\rangle = \langle A^*(e_1^{k_1} \dots e_d^{k_d}), e_1^{k_1-1} e_2^{k_2} \dots e_d^{k_d}\rangle = \frac{k_1}{|k|}\|e_1^{k_1-1} e_2^{k_2} \dots e_d^{k_d}\|^2.$$

The required formula now follows from the induction hypothesis. □

Setting $k_1 = k_2 = \dots = k_d = n$ in Lemma 3.8, we obtain

$$\|(e_1 e_2 \dots e_d)^n\|_{E^{nd}}^2 = \frac{(n!)^d}{(nd)!}.$$

The right side is easily estimated using Stirling's formula

$$n! \sim \sqrt{2\pi}\, n^{n+1/2}\, e^{-n},$$

and after obvious cancellations we find that

$$\frac{(n!)^d}{(nd)!} \sim \left(\frac{2\pi^{d-1}}{d}\right)^{1/2} d^{-nd} n^{(d-1)/2}.$$

In order to deduce (3.7) from the latter, choose an orthonormal basis $e_1, e_2, \dots, e_d$ for $\mathbf{C}^d$ so that

$$z_k(x) = \langle x, e_k \rangle, \quad k = 1, 2, \dots, d.$$

Then

$$\|(z_1 z_2 \dots z_d)^n\|_{H^2}^2 = \|(e_1 e_2 \dots e_d)^n\|_{E^{nd}}^2$$

and (3.7) follows after choosing A to be a positive number appropriately smaller than $\sqrt{2\pi^{d-1}/d}$. That completes the proof of Theorem 3.3. □

Remark 3.10. We recall that a d-tuple of commuting operators $\bar{T}=(T_1, T_2, \dots, T_d)$ on a Hilbert space H is said to be ***subnormal*** if there is a commuting d-tuple of normal operators $\bar{N}=(N_1, N_2, \dots, N_d)$ on a larger Hilbert space $K \supseteq H$ such that

$$T_k = N_k \restriction_H, \quad k = 1, 2, \dots, d.$$

The one-dimensional unilateral shift can be extended to a unitary operator on a larger space. That situation is unique to dimension 1, as we have

COROLLARY 1. *For every $d \geqslant 2$ the d-shift is not subnormal.*

Proof. In Proposition 2.12 we identified the maximal ideal space of the unital Banach algebra generated by the d-shift with the closed unit ball in $\mathbf{C}^d$. In particular, for every polynomial f the spectral radius of $f(S_1, \dots, S_d)$ is given by

$$r(f(S_1, \dots, S_d)) = \sup_{|z_1|^2 + \dots + |z_d|^2 \leqslant 1} |f(z_1, \dots, z_d)|.$$

If the d-shift were subnormal then $f(S_1, \dots, S_d)$ would be a subnormal operator for every polynomial f, and hence its norm would equal its spectral radius [22, Problem 162], contradicting Theorem 3.3. □

The two most common Hilbert spaces associated with the unit ball B_d arise from measures. These are the spaces $H^2(\partial B_d)$ associated with normalized surface measure on the boundary of B_d and the space $H^2(B_d)$ associated with normalized volume measure on B_d [37]. It is reasonable to ask if the space H^2 can be associated with some measure on $\mathbf{C}^d$. The answer is no because that would imply that the d-shift is subnormal, contradicting Corollary 1. The details are as follows.

COROLLARY 2. *There is no positive measure μ on $\mathbf{C}^d$, $d \geqslant 2$, with the property that*

$$\|f\|_{H^2}^2 = \int_{\mathbf{C}^d} |f(z)|^2 \, d\mu(z)$$

for every polynomial f.

Proof. Suppose that such a measure μ did exist. μ must be a probability measure because $\|1\|_{H^2} = 1$, and it must have finite moments of all orders.

We claim that μ must have compact support. Indeed, if f is any linear functional on $\mathbf{C}^d$ of the form $f(x)=\langle x, e\rangle$ where e is a unit vector of $\mathbf{C}^d$ then by Lemma 3.8 we have

$$\|f^n\|_{H^2}^2 = \|e^n\|_{E^n}^2 = 1$$

for every $n=1,2,\ldots$. Hence

$$\int_{\mathbf{C}^d} |f(z)|^{2n}\, d\mu(z) = 1.$$

Taking $2n$th roots we find that the function f has norm 1 when it is considered an element in the space $L^p(\mathbf{C}^d, \mu)$ for $p=2,4,6,\ldots$. Letting X be the closed support of the measure μ we find that

$$\sup_{z\in X} |f(z)| = \lim_{n\to\infty} \left(\int_{\mathbf{C}^d} |f|^{2n}\, d\mu\right)^{1/2n} = 1.$$

This proves that for every $z\in X$ and e in the unit ball of $\mathbf{C}^d$ we have

$$|\langle z, e\rangle| \leqslant 1,$$

and thus X must be contained in the closed unit ball of $\mathbf{C}^d$.

Now we simply view the d-shift as a d-tuple of multiplication operators in the space $L^2(\mu)$. Here, S_k is multiplication by z_k acting on the closure (in $L^2(\mu)$) of the space of polynomials. This d-tuple $(S_1, \ldots, S_d)$ is obviously subnormal, contradicting Corollary 1 above. □

Remark 3.11. In the conventional approach to dilation theory one seeks normal dilations for operators or sets of operators. Theorem 3.3 implies that this approach is inappropriate for d-contractions and the unit ball of $\mathbf{C}^d$ in dimension greater than one. Indeed, if $(N_1, \ldots, N_d)$ is a d-tuple of mutually commuting normal operators whose joint spectrum is contained in the closed unit ball of $\mathbf{C}^d$, then for every polynomial $f\in\mathcal{P}$ we have

$$\|f(N_1, \ldots, N_d)\| \leqslant \sup_{z\in B_d} |f(z)|.$$

Since Theorem 3.3 implies that there are polynomials f for which the inequality

$$\|f(S_1, \ldots, S_d)\| \leqslant \sup_{z\in B_d} |f(z)|$$

fails, one cannot obtain such operators $f(S_1, \ldots, S_d)$ by compressing $f(N_1, \ldots, N_d)$ to any subspace. Thus the d-shift cannot be dilated to a normal d-tuple having its spectrum in the closed unit ball.

4. Maximality of the H^2-norm

The purpose of this section is to show that in every dimension $d=1,2,\ldots$ the H^2-norm is distinguished among all Hilbert seminorms defined on the space $\mathcal{P}$ of polynomials by being the *largest* Hilbert seminorm which is appropriate for operator theory on the unit ball of $\mathbf{C}^d$. As a consequence, we show that the function space H^2 is contained in every other Hilbert space of analytic functions on the open unit ball which has these natural properties.

Definition 4.1. Let $z_1,\ldots,z_d$ be a system of coordinate functions on $\mathbf{C}^d$. A Hilbert seminorm $\|\cdot\|$ defined on the space $\mathcal{P}$ of all polynomials is said to be contractive if for every $a\in\mathbf{C}$ and every $f_1,\ldots,f_d\in\mathcal{P}$ we have

$$\|a1+z_1f_1+\ldots+z_df_d\|^2 \leqslant |a|^2+\|f_1\|^2+\ldots+\|f_d\|^2.$$

Remarks. Proposition 2.6 asserts that *the H^2-norm is a contractive norm on $\mathcal{P}$.* From Proposition 4.2 below it follows that the Hilbert norms defined on $\mathcal{P}$ by both $H^2(B_d)$ and $H^2(\partial B_d)$ are contractive norms.

It is a simple exercise to show that if a Hilbert seminorm $\|\cdot\|$ is contractive relative to one system of coordinates $z_1,\ldots,z_d$ then it is contractive relative to every system of coordinates. Thus the definition of contractive seminorm depends only on the structure of $\mathbf{C}^d$ as a d-dimensional Hilbert space.

Notice too that if $\|\cdot\|$ is any contractive seminorm then for any system of coordinate functions $z_1,\ldots,z_d$ the multiplication operators $(M_{z_1},\ldots,M_{z_d})$ give rise to a d-contraction acting on the Hilbert space obtained by completing $\mathcal{P}$ in this seminorm. Indeed, we have the following somewhat more concrete characterization of contractive Hilbert seminorms.

PROPOSITION 4.2. *Let $\|\cdot\|$ be an arbitrary Hilbert seminorm on $\mathcal{P}$, let H be the inner product space defined by $\|\cdot\|$, and let $\mathcal{P}_0$ be the maximal ideal in $\mathcal{P}$ consisting of all polynomials f such that $f(0)=0$. Then $\|\cdot\|$ is a contractive seminorm if and only if the following two conditions are satisfied*:

(1) *$1\perp\mathcal{P}_0$ in the space H, and*

(2) *for some system of coordinate functions $z_1,\ldots,z_d$ the multiplication operators $(M_{z_1},\ldots,M_{z_d})$ define a d-contraction on H.*

Proof. Once one notes that the most general element of $\mathcal{P}_0$ is a sum of the form $z_1f_1+\ldots+z_df_d$ with $f_1,\ldots,f_d\in\mathcal{P}$, the argument is straightforward. □

We collect the following observation, which asserts that condition (2) alone is enough in the presence of minimal symmetry.

COROLLARY. *For every* λ *in the circle group* $\{z\in\mathbf{C}:|z|=1\}$ *and every* $f\in\mathcal{P}$ *set* $f_\lambda(z)=f(\lambda z)$, $z\in\mathbf{C}^d$. *Let* $\|\cdot\|$ *be a Hilbert seminorm on* $\mathcal{P}$ *which satisfies* $\|f_\lambda\|=\|f\|$ *for every* $f\in\mathcal{P}$ *and every* λ, *such that for some system of coordinate functions* $z_1, ..., z_d$, *the multiplication operators* $M_{z_1}, ..., M_{z_d}$ *give rise to a d-contraction acting on the Hilbert space* H *obtained from* $\|\cdot\|$.

Then $\|\cdot\|$ *is a contractive seminorm.*

Proof. We show that the symmetry hypothesis implies condition (1) of Proposition 4.2. For every λ in the unit circle we can define a unitary operator U_λ uniquely on H by setting

$$U_\lambda f = f_\lambda, \quad f\in\mathcal{P}.$$

It is obvious that U is a unitary representation of the circle group on H. Moreover, if f is a homogeneous polynomial of degree $n=0,1,...$ then we have

$$U_\lambda f = \lambda^n f$$

for all λ. Thus for the inner product $\langle\cdot,\cdot\rangle$ associated with $\|\cdot\|$ we have

$$\langle f,1\rangle = \langle U_\lambda f, U_\lambda 1\rangle = \lambda^n\langle f,1\rangle,$$

so that if $n\geqslant 1$ then $\langle f,1\rangle=0$. It follows that $1\perp\mathcal{P}_0$, as required. □

Following is the main result of this section.

THEOREM 4.3. *Let* $\|\cdot\|$ *be any contractive Hilbert seminorm on* $\mathcal{P}$. *Then for every* $f\in\mathcal{P}$ *we have*

$$\|f\|\leqslant k\|f\|_{H^2}$$

where $k=\|1\|$. *In particular, the* H^2*-norm is the largest contractive Hilbert seminorm which assigns norm* 1 *to the constant polynomial* $f=1$.

In particular, we see that the Hilbert norms arising from the "Hardy" space $H^2(\partial B_d)$ and the "Bergman" space $H^2(B_d)$ are both dominated by $\|\cdot\|_{H^2}$. Indeed, we have the following inclusions of the corresponding Hilbert spaces of analytic functions in the open ball B_d:

$$H^2\subseteq H^2(\partial B_d)\subseteq H^2(B_d),$$

where both inclusion maps are compact operators of norm 1. Since we do not require the latter assertion, we omit the proof. However, note that of these three function spaces, H^2 is the only one that does not contain H^∞, and it is the only one of the three for which the d-contraction defined by the multiplication operators $(M_{z_1}, ..., M_{z_d})$ *fails* to be subnormal.

Remark 4.4. Every d-contraction $(T_1, \dots, T_d)$ in $\mathcal{B}(H)$ gives rise to a normal completely positive map P on $\mathcal{B}(H)$ by way of

$$P(A) = T_1 A T_1^* + \dots + T_d A T_d^*, \quad A \in \mathcal{B}(H).$$

Because of Remark 3.2 we have $P(\mathbf{1}) = T_1 T_1^* + \dots + T_d T_d^* \leqslant \mathbf{1}$, and in fact the sequence $A_n = P^n(\mathbf{1})$ is decreasing: $A_0 = \mathbf{1} \geqslant A_1 \geqslant A_2 \geqslant \dots \geqslant 0$. Thus

$$A_\infty = \lim_{n \to \infty} P^n(\mathbf{1})$$

exists as a limit in the strong operator topology and satisfies $0 \leqslant A_\infty \leqslant \mathbf{1}$. A d-contraction $\bar{T} = (T_1, \dots, T_d)$ is called *pure* if $A_\infty = 0$. Notice that if the row norm of $\bar{T}$ is less than 1, i.e., $T_1 T_1^* + \dots + T_d T_d^* \leqslant r\mathbf{1}$ for some $0 < r < 1$, then $\|P\| = \|P(\mathbf{1})\| \leqslant r < 1$, and hence $\bar{T}$ is a pure d-contraction.

For the proof of Theorem 4.3 we require an operator-theoretic result which relates closely to the material of §6. Gelu Popescu has pointed out that the operator L is related to his Poisson kernel operator K_r of [35, §8], when $r = 1$. For completeness, we include a proof.

THEOREM 4.5. *Let $(T_1, \dots, T_d)$ be a d-contraction on a Hilbert space H, and define the operator*

$$\Delta = (\mathbf{1} - T_1 T_1^* - \dots - T_d T_d^*)^{1/2}$$

and the subspace $K = \overline{\Delta H}$. Let E be a d-dimensional Hilbert space and let

$$\mathcal{F}_+(E) = \mathbf{C} \oplus E \oplus E^2 \oplus \dots$$

be the symmetric Fock space over E.

Then for every orthonormal basis $e_1, \dots, e_d$ for E there is a unique bounded operator $L: \mathcal{F}_+(E) \otimes K \to H$ satisfying $L(1 \otimes \xi) = \Delta \xi$ and

$$L(e_{i_1} e_{i_2} \dots e_{i_n} \otimes \xi) = T_{i_1} T_{i_2} \dots T_{i_n} \Delta \xi$$

for every $i_1, \dots, i_n \in \{1, 2, \dots, d\}$, $n = 1, 2, \dots$. In general we have $\|L\| \leqslant 1$, and if $(T_1, \dots, T_d)$ is a pure d-tuple, then L is a co-isometry: $LL^* = \mathbf{1}_H$.

Proof. If there is a bounded operator L satisfying the stated condition then it is obviously unique because $\mathcal{F}_+(E)$ is spanned by the set of vectors

$$\{1, e_{i_1}, e_{i_2} e_{i_3}, e_{i_4} e_{i_5} e_{i_6}, \dots : i_k \in \{1, 2, \dots, d\}, k = 1, 2, \dots\}.$$

We define L by exhibiting its adjoint, that is, we will exhibit an operator

$$A: H \to \mathcal{F}(E) \otimes K,$$

$\mathcal{F}(E)$ denoting the full Fock space over E, and we will show that $\|A\| \leqslant 1$ and

$$\langle L(\zeta), \eta \rangle = \langle \zeta, A(\eta) \rangle \tag{4.6}$$

for ζ of the form $1 \otimes \xi$ or $e_{i_1} e_{i_2} \dots e_{i_n} \otimes \xi$ for $\xi \in K$. At that point we can *define* L to be the adjoint of $P_+ A$, P_+ denoting the projection of $\mathcal{F}(E)$ onto its subspace $\mathcal{F}_+(E)$.

For every $\eta \in H$, we define $A\eta$ as a sequence of vectors $(\zeta_0, \zeta_1, \zeta_2, \dots)$ where $\zeta_n \in E^{\otimes n} \otimes K$ is defined by

$$\zeta_n = \sum_{i_1, \dots, i_n = 1}^{d} e_{i_1} \otimes \dots \otimes e_{i_n} \otimes \Delta T_{i_n}^* \dots T_{i_1}^* \eta$$

for $n \geqslant 1$ and $\zeta_0 = 1 \otimes \Delta \eta$. Notice that since $T_1^*, \dots, T_d^*$ commute, ζ_n actually belongs to the symmetric subspace $E^n \otimes K$. We claim first that

$$\sum_{n=0}^{\infty} \|\zeta_n\|^2 \leqslant \|\eta\|^2,$$

so that in fact A maps into $\mathcal{F}(E) \otimes K$ and is a contraction. Indeed, we have

$$\|\zeta_n\|^2 = \sum_{i_1, \dots, i_n = 1}^{d} \|\Delta T_{i_n}^* \dots T_{i_1}^* \eta\|^2 = \sum_{i_1, \dots, i_n = 1}^{d} \langle T_{i_1} \dots T_{i_n} \Delta^2 T_{i_n}^* \dots T_{i_1}^* \eta, \eta \rangle.$$

Let $P(A) = T_1 A T_1^* + \dots + T_d A T_d^*$ be the completely positive map of Remark 4.4. Noting that $\Delta^2 = \mathbf{1} - P(\mathbf{1})$ we find that

$$\sum_{i_1, \dots, i_n = 1}^{d} T_{i_1} \dots T_{i_n} \Delta^2 T_{i_n}^* \dots T_{i_1}^* = P^n(\mathbf{1} - P(\mathbf{1})) = P^n(\mathbf{1}) - P^{n+1}(\mathbf{1}),$$

and hence

$$\|\zeta_n\|^2 = \langle P^n(\mathbf{1})\eta, \eta \rangle - \langle P^{n+1}(\mathbf{1})\eta, \eta \rangle.$$

The series $\|\zeta_0\|^2 + \|\zeta_1\|^2 + \dots$ therefore telescopes and we are left with

$$\sum_{n=0}^{\infty} \|\zeta_n\|^2 = \|\eta\|^2 - \langle A_\infty \eta, \eta \rangle \leqslant \|\eta\|^2, \tag{4.7}$$

where A_∞ is the positive contraction $A_\infty = \lim_{n \to \infty} P^n(\mathbf{1})$ of Remark 4.4.

We now verify (4.6) for ζ of the form $\zeta=e_{j_1}\dots e_{j_n}\otimes\xi$ for $n\geqslant 1$, $j_1,\dots,j_n\in\{1,2,\dots,d\}$ and $\xi\in K$. We have

$$\begin{aligned}\langle e_{j_1}\dots e_{j_n}\otimes\xi, A\eta\rangle &= \sum_{i_1,\dots,i_n=1}^{d}\langle e_{j_1}\dots e_{j_n}\otimes\xi, e_{i_1}\otimes\dots\otimes e_{i_n}\otimes\Delta T_{i_n}^*\dots T_{i_1}^*\eta\rangle\\ &= \sum_{i_1,\dots,i_n=1}^{d}\langle e_{j_1}\otimes\dots\otimes e_{j_n}, e_{i_1}\otimes\dots\otimes e_{i_n}\rangle\langle\xi, \Delta T_{i_n}^*\dots T_{i_1}^*\eta\rangle\\ &= \langle\xi, \Delta T_{j_1}^*\dots T_{j_1}^*\eta\rangle = \langle T_{j_1}\dots T_{j_n}\Delta\xi,\eta\rangle = \langle L(\zeta),\eta\rangle.\end{aligned}$$

For $\zeta=1\otimes\xi$ with $\xi\in K$ we have

$$\langle 1\otimes\xi, A\eta\rangle = \langle 1\otimes\xi, 1\otimes\Delta\eta\rangle = \langle\xi,\Delta\eta\rangle = \langle\Delta\xi,\eta\rangle,$$

as required. If $(T_1,\dots,T_d)$ is a pure d-tuple, then it is clear from (4.7) that A is an isometry, and hence L is a co-isometry. □

Proof of Theorem 4.3. Let H be the Hilbert space obtained by completing $\mathcal{P}$ in the seminorm $\|\cdot\|$. Choose an orthonormal basis $e_1,\dots,e_d$ for $E=\mathbf{C}^d$ and let $z_1,\dots,z_d$ be the corresponding system of coordinate functions $z_i(x)=\langle x,e_i\rangle$, $i=1,\dots,d$.

Since $\|\cdot\|$ is a contractive Hilbert seminorm the multiplication operators

$$T_k=M_{z_k},\quad k=1,\dots,d,$$

define a d-contraction $(T_1,\dots,T_d)$ in $\mathcal{B}(H)$. Set

$$\Delta=\Big(\mathbf{1}-\sum_{k=1}^{d}T_kT_k^*\Big)^{1/2},$$

let $K=\overline{\Delta H}$ be the closed range of Δ, and let $L:\mathcal{F}_+(E)\otimes K\to H$ be the contraction defined in Theorem 4.5 by the conditions $L(1\otimes\xi)=\Delta\xi$ and, for $n=1,2,\dots$,

$$L(e_{i_1}\dots e_{i_n}\otimes\xi)=T_{i_1}\dots T_{i_n}\Delta\xi, \tag{4.8}$$

$\xi\in K$, $i_1,\dots,i_n\in\{1,2,\dots,d\}$.

The constant polynomial $1\in\mathcal{P}$ is represented by a vector v in H. We claim that $\Delta v=v$. Indeed, since $\|\cdot\|$ is a contractive seminorm, condition (1) of Proposition 4.2 implies that

$$v\perp T_1H+T_2H+\dots+T_dH,$$

and hence $T_k^* v=0$ for $k=1, \dots, d$. It follows that

$$\|\Delta v\|^2 = \langle \Delta^2 v, v\rangle = \|v\|^2 - \sum_{k=1}^{d} \|T_k^* v\|^2 = \|v\|^2,$$

and hence $\Delta v = v$ because $0 \leqslant \Delta \leqslant \mathbf{1}$.

In particular, $v = \Delta v \in \overline{\Delta H} = K$. Taking $\xi = v$ in (4.8) we obtain

$$L(e_{i_1} \dots e_{i_n} \otimes v) = T_{i_1} \dots T_{i_n} v.$$

Since v is the representative of 1 in H, $T_{i_1} \dots T_{i_n} v$ is the representative of the polynomial $z_{i_1} \dots z_{i_n}$ in H, and we have

$$L(e_{i_1} \dots e_{i_n} \otimes v) = z_{i_1} \dots z_{i_n} \in H.$$

By Propostion 2.13 there is a unitary operator $W: H^2 \to \mathcal{F}_+(E)$ which carries 1 to 1 and carries $z_{i_1} \dots z_{i_n} \in H^2$ to $e_{i_1} \dots e_{i_n} \in \mathcal{F}_+(E)$. Hence

$$L(W(z_{i_1} \dots z_{i_n}) \otimes v) = z_{i_1} \dots z_{i_n}.$$

By taking linear combinations we find that for every polynomial $f \in \mathcal{P}$,

$$L(Wf \otimes v) = f$$

where f on the left is considered an element of H^2 and f on the right is considered an element of H. Since $\|L\| \leqslant 1$ and W is unitary, we immediately deduce that

$$\|f\|_H \leqslant \|Wf \otimes v\| = \|f\|_{H^2} \cdot \|v\|_H.$$

Theorem 4.3 follows after noting that $\|v\|_H = \|1\|_H$. □

Remarks. In particular, the H^2-norm is the *largest* Hilbert seminorm $\|\cdot\|$ on the space $\mathcal{P}$ of all polynomials which is contractive and is normalized so that $\|1\|=1$.

We will make use of the following extremal property of the H^2-norm below.

THEOREM 4.9. *Let $\|\cdot\|$ be a contractive Hilbert seminorm on $\mathcal{P}$ satisfying $\|1\|=1$ and let $z_1, \dots, z_d$ be a system of orthogonal coordinate functions for $E = \mathbf{C}^d$. Then for every $n=1,2,\dots$ we have*

$$\sum_{i_1,\dots,i_n=1}^{d} \|z_{i_1} z_{i_2} \dots z_{i_n}\|^2 \leqslant \frac{(n+d-1)!}{n!\,(d-1)!}, \tag{4.10}$$

with equality holding if and only if $\|f\|=\|f\|_{H^2}$ *for every polynomial* f *of degree at most* n.

Proof. Let $\{e_1,\dots,e_d\}$ be an orthonormal basis for a d-dimensional Hilbert space E. We consider the projection $P_n\in\mathcal{B}(E^{\otimes n})$ of the full tensor product onto its symmetric subspace E^n. Since $\|\cdot\|$ is a contractive seminorm, Theorem 4.3 implies that for all $i_1,\dots,i_n$ we have

$$\|z_{i_1}\dots z_{i_n}\|\leqslant\|z_{i_1}\dots z_{i_n}\|_{H^2}=\|P_n(e_{i_1}\otimes\dots\otimes e_{i_n})\|$$

and hence

$$\sum_{i_1,\dots,i_n=1}^{d}\|z_{i_1}\dots z_{i_n}\|^2\leqslant\sum_{i_1,\dots,i_n=1}^{d}\|P_n(e_{i_1}\otimes\dots\otimes e_{i_n})\|^2.$$

Since $\{e_{i_1}\otimes\dots\otimes e_{i_n}:1\leqslant i_1,\dots,i_n\leqslant d\}$ is an orthonormal basis for $E^{\otimes n}$ the term on the right is $\operatorname{trace}(P_n)=\dim(E^n)$, and (4.10) follows from the computation of the dimension of E^n in (A.5).

Let $\mathcal{P}_n$ denote the subspace of H^2 consisting of homogeneous polynomials of degree n, and let Q_n be the projection of H^2 on $\mathcal{P}_n$. The preceding observations imply that if A is any operator on H^2 which is supported in $\mathcal{P}_n$ in the sense that $A=Q_nAQ_n$ then the trace of A is given by

$$\operatorname{trace}(A)=\sum_{i_1,\dots,i_n=1}^{d}\langle Az_{i_1}\dots z_{i_n},z_{i_1}\dots z_{i_n}\rangle_{H^2}. \tag{4.11}$$

Now fix n and suppose equality holds in (4.10). Since $\|\cdot\|$ is a contractive Hilbert seminorm satisfying $\|1\|=1$, Theorem 4.3 implies that $\|f\|\leqslant\|f\|_{H^2}$ for every $f\in\mathcal{P}$, and hence there is a unique operator $H\in\mathcal{B}(H^2)$ satisfying

$$\langle f,g\rangle=\langle Hf,g\rangle_{H^2},\quad f,g\in\mathcal{P},$$

and one has $0\leqslant H\leqslant\mathbf{1}$. Considering the compression Q_nHQ_n of H to $\mathcal{P}_n$ we see from (4.11) that

$$\operatorname{trace}(Q_nHQ_n)=\dim(E^n)=\operatorname{trace}(Q_n).$$

Since $Q_n-Q_nHQ_n\geqslant0$ and the trace is faithful, we conclude that $Q_nHQ_n=Q_n$, and since H is a positive contraction it follows that $Hf=f$ for every $f\in\mathcal{P}_n$.

We claim that $Hf=f$ for every $f\in\mathcal{P}_k$ and every $k=0,1,\dots,n$. To see that, choose a linear functional $z\in\mathcal{P}$ satisfying $\|z\|_{H^2}=1$. Since $\|\cdot\|$ is a contractive seminorm we have $\|z\cdot f\|\leqslant\|f\|$ for every $f\in\mathcal{P}$, and in particular we have $\|z^n\|=\|z^{n-k}z^k\|\leqslant\|z^k\|$. Thus

$$\langle Hz^k,z^k\rangle_{H^2}=\|z^k\|^2\geqslant\|z^n\|^2=\|z^n\|^2_{H^2}.$$

Since the H^2-norm of any power of z is 1 and $0\leqslant H\leqslant\mathbf{1}$, it follows that $Hz^k=z^k$. Since every polynomial of degree at most n is a linear combination of monomials of the form z^k with z as above and $k=0,1,\dots,n$, the proof of Theorem 4.9 is complete. □

Part II. Operator theory

5. The Toeplitz C^*-algebra

Let $\bar{S}=(S_1, ..., S_d)$ be the d-shift.

Definition 5.1. The Toeplitz C^*-algebra is the C^*-algebra $\mathcal{T}_d$ generated by the operators $S_1, ..., S_d$.

Remarks. Notice that we have not included the identity operator as one of the generators of $\mathcal{T}_d$, so that $\mathcal{T}_d$ is by definition the norm-closed linear span of the set of finite products of the form $T_1T_2 \dots T_n$, $n=1, 2, ...$, where

$$T_i \in \{S_1, ..., S_d, S_1^*, ..., S_d^*\}.$$

Nevertheless, (5.5) below implies that $\mathcal{T}_d$ contains an invertible positive operator

$$(d\mathbf{1}+N)(\mathbf{1}+N)^{-1} = S_1^*S_1+...+S_d^*S_d,$$

and hence $\mathbf{1}\in\mathcal{T}_d$. Thus $\mathcal{T}_d$ is the C^*-algebra generated by all multiplication operators $M_f \in \mathcal{B}(H^2)$, $f\in\mathcal{P}$.

If one starts with the Hilbert space $H^2(\partial B_d)$ rather than H^2 then there is a natural Toeplitz C^*-algebra

$$\mathcal{T}_{\partial B_d} = C^*\{M_f : f\in\mathcal{P}\} \subseteq \mathcal{B}(H^2(\partial B_d)),$$

and similarly there is a Toeplitz C^*-algebra $\mathcal{T}_{B_d}$ on the Bergman space

$$\mathcal{T}_{B_d} = C^*\{M_f : f\in\mathcal{P}\} \subseteq \mathcal{B}(H^2(B_d)),$$

see [16]. In fact, it is not hard to show that the three C^*-algebras $\mathcal{T}_d$, $\mathcal{T}_{\partial B_d}$ and $\mathcal{T}_{B_d}$ are unitarily equivalent. In that sense, the C^*-algebra $\mathcal{T}_d$ is not new. However, we are concerned with the relationship between the d-shift and its enveloping C^*-algebra $\mathcal{T}_d$, and here there are some essential differences.

For example, in the classical case of $H^2(\partial B_d)$ one can start with a continuous complex-valued function $f\in C(\partial B_d)$ and define a Toeplitz operator T_f on $H^2(\partial B_d)$ by compressing the operator of multiplication by f (acting on $L^2(\partial B_d)$) to the subspace $H^2(\partial B_d)$. In our case, however, continuous symbols do not give rise to Toeplitz operators. Indeed, we have seen that there are continuous functions f on the closed unit ball which are uniform limits of holomorphic polynomials, but which do not belong to H^2. For such an f the "Toeplitz" operator T_f is not defined. Thus we have taken some care to develop the properties of $\mathcal{T}_d$ that we require.

Let N be the number operator acting on H^2, defined as the generator of the one-parameter unitary group

$$\Gamma(e^{it}\mathbf{1}_E)=e^{itN}, \quad t\in\mathbf{R},$$

Γ being the representation of the unitary group of E on H^2 defined in the remarks following Definition 2.10. N obviously has discrete spectrum $\{0,1,2,...\}$ and the nth eigenspace of N is the space $\mathcal{P}_n$ of homogeneous polynomials of degree n,

$$\mathcal{P}_n=\{\xi\in H^2 : N\xi=n\xi\}, \quad n=0,1,2,\dots .$$

$(\mathbf{1}+N)^{-1}$ is a compact operator, and it is a fact that for every real number $p>0$,

$$\operatorname{trace}(\mathbf{1}+N)^{-p}<\infty \;\Leftrightarrow\; p>d. \tag{5.2}$$

Since N is unitarily equivalent to the Bosonic number operator, the assertion (5.2) is probably known. We lack an appropriate reference, however, and have included a proof of (5.2) in Appendix A for the reader's convenience.

The following result exhibits the commutation relations satisfied by the d-shift.

PROPOSITION 5.3. *Suppose that $d=2,3,\dots$ and let $(S_1,\dots,S_d)$ be the d-shift. Then for all $i,j=1,\dots,d$ we have*

$$S_i^*S_j-S_jS_i^*=(\mathbf{1}+N)^{-1}(\delta_{ij}\mathbf{1}-S_jS_i^*) \tag{5.4}$$

and

$$S_1^*S_1+\dots+S_d^*S_d=(d\mathbf{1}+N)(\mathbf{1}+N)^{-1}. \tag{5.5}$$

In particular, $\|S_1^*S_1+\dots+S_d^*S_d\|=d$. The commutators $S_i^*S_j-S_jS_i^*$ belong to every Schatten class $\mathcal{L}^p(H^2)$ for $p>d$, but they do not belong to $\mathcal{L}^d(H^2)$.

Remark. It follows that if A,B are operators belonging to the unital $*$-algebra generated by $S_1,\dots,S_d$, then $AB-BA\in\mathcal{L}^p(H^2)$ for every $p>d$, and hence any product of at least $d+1$ such commutators belongs to the trace class.

Proof. To establish these formulas it is more convenient to work with the d-shift in its realization on $\mathcal{F}_+(E)$ described in Proposition 2.13. Thus, we pick an orthonormal basis $e_1,\dots,e_d$ for a d-dimensional Hilbert space E and set

$$S_i\xi=e_i\xi, \quad 1\leqslant i\leqslant d,$$

for $\xi\in\mathcal{F}_+(E)=\mathbf{C}\oplus E\oplus E^2\oplus\dots$. The number operator N acts as follows on E^n:

$$N\xi=n\xi, \quad \xi\in E^n, n=0,1,2,\dots .$$

We first establish (5.4). It suffices to verify that the operators on both sides of (5.4) agree on every finite-dimensional space E^n, $n=0,1,2,\dots$. For $n=0$ and $\lambda\in\mathbf{C}$ we have $S_i^*S_j\lambda=\lambda S_i^*e_j=\delta_{ij}\lambda$, while $S_jS_i^*\lambda=0$. Hence (5.4) holds on $\mathbf{C}$. For $n\geqslant 1$ and $\xi\in E^n$ of the form $\xi=y^n$ we see from formula (3.9) that

$$S_j^*S_i\xi=S_j^*(e_iy^n)=\frac{\delta_{ij}}{n+1}y^n+\frac{n}{n+1}\langle y,e_j\rangle e_iy^{n-1}$$

while

$$S_iS_j^*\xi=\langle y,e_j\rangle S_iy^{n-1}=\langle y,e_j\rangle e_iy^{n-1}.$$

Hence

$$S_j^*S_i\xi-S_iS_j^*\xi=\frac{1}{n+1}(\delta_{ij}\xi-S_jS_i^*\xi).$$

The latter holds for all $\xi\in E^n$ because E^n is spanned by $\{y^n:y\in E\}$, and (5.4) follows.

Formula (5.5) follows from (5.4). Indeed, for $\xi\in E^n$ we have

$$S_i^*S_i\xi=\frac{1}{n+1}\xi+\frac{n}{n+1}S_iS_i^*\xi.$$

By the remarks following (2.10) we have

$$S_1S_1^*+\dots+S_dS_d^*=\mathbf{1}-E_0, \tag{5.6}$$

E_0 denoting the projection on $\mathbf{C}$. Summing the previous formula on i we obtain

$$\sum_{i=1}^{d}S_i^*S_i\xi=\frac{d}{n+1}\xi+\frac{n}{n+1}(\xi-E_0\xi)=\frac{n+d}{n+1}\xi-\frac{n}{n+1}E_0\xi=\frac{n+d}{n+1}\xi,$$

and (5.5) follows.

Now suppose $p>d$. Because of (5.2) the operator $(\mathbf{1}+N)^{-1}$ belongs to $\mathcal{L}^p$; since $\mathcal{L}^p$ is an ideal, (5.4) implies that $S_i^*S_j-S_jS_i^*\in\mathcal{L}^p$ for all i,j.

Finally, we claim that no self-commutator $[S_i^*,S_i]=S_i^*S_i-S_iS_i^*$ belongs to $\mathcal{L}^d$. Indeed, since the operators $S_1,\dots,S_d$ are unitarily equivalent to each other (by the remarks following Definition 2.10), we see that if one $[S_i^*,S_i]$ belongs to $\mathcal{L}^d$ then they all do, and in that case we would have

$$\sum_{i=1}^{d}[S_i^*,S_i]\in\mathcal{L}^d.$$

By (5.5) and (5.6) the left side of this formula is

$$\sum_{i=1}^{d}S_i^*S_i-\sum_{i=1}^{d}S_iS_i^*=(d\mathbf{1}+N)(\mathbf{1}+N)^{-1}-(\mathbf{1}-E_0)=E_0+(d-1)(\mathbf{1}+N)^{-1}.$$

Since $(\mathbf{1}+N)^{-1}\notin\mathcal{L}^d$ by (5.2), we have a contradiction and the proof of Proposition 5.3 is complete. □

The d-shift and the canonical commutation relations. The d-shift is closely related to the creation operators $(C_1, ..., C_d)$ associated with the canonical commutation relations for d degrees of freedom. Indeed, one can think of $\bar{S}=(S_1, ..., S_d)$ as the partial isometry occurring in the polar decomposition of $\bar{C}=(C_1, ..., C_d)$ in the following way. Choose an orthonormal basis $e_1, ..., e_d$ for a d-dimensional Hilbert space E. For $k=1, ..., d$, C_k is defined on the dense subspace of $\mathcal{F}_+(E)$ spanned by E^n, $n=0, 1, ...$, as

$$C_k\xi = \sqrt{n+1}\, e_k\xi, \quad \xi\in E^n$$

(see [40]). The C_k are of course unbounded operators, and they satisfy the complex form of the canonical commutation relations

$$C_iC_j = C_jC_i, \quad C_i^*C_j - C_jC_i^* = \delta_{ij}\mathbf{1}, \quad 1\leqslant i,j\leqslant d.$$

One finds that the row operator

$$\bar{C} = (C_1, ..., C_d)\colon \underbrace{\mathcal{F}_+(E)\oplus...\oplus\mathcal{F}_+(E)}_{d \text{ times}} \to \mathcal{F}_+(E)$$

is related to the number operator N by $\bar{C}\bar{C}^*=N$, and in fact the polar decomposition of $\bar{C}$ takes the form

$$\bar{C} = N^{1/2}\bar{S}$$

where $\bar{S}=(S_1, ..., S_d)$ is the d-shift; i.e., $C_k=N^{1/2}S_k$, $k=1, ..., d$.

We have seen that the d-shift is not a subnormal d-tuple. The following result asserts that, at least, the individual operators S_k, $k=1, ..., d$, are hyponormal. Indeed, any linear combination of $S_1, ..., S_d$ is a hyponormal operator.

COROLLARY. *For every $k=1, ..., d$ we have $S_k^*S_k\geqslant S_kS_k^*$.*

Proof. Proposition 5.3 implies that

$$S_k^*S_k - S_kS_k^* = (\mathbf{1}+N)^{-1}(\mathbf{1}-S_kS_k^*).$$

Since $\|S_k\|\leqslant 1$, both factors on the right are positive operators. Let E_n be the nth spectral projection of N, $n=0, 1, ...$. Since $S_kE_n=E_{n+1}S_k$ it follows that $S_kS_k^*$ commutes with E_n. Thus $(\mathbf{1}+N)^{-1}$ commutes with $\mathbf{1}-S_kS_k^*$, and the assertion follows. □

Of course in dimension $d=1$, the commutator S^*S-SS^* is a rank-one operator and therefore belongs to every Schatten class $\mathcal{L}^p$, $p\geqslant 1$.

THEOREM 5.7. *$\mathcal{T}_d$ contains the algebra $\mathcal{K}$ of all compact operators on H^2, and we have an exact sequence of C^*-algebras*

$$0 \longrightarrow \mathcal{K} \hookrightarrow \mathcal{T}_d \xrightarrow{\pi} C(\partial B_d) \longrightarrow 0$$

where π is the unital $*$-homomorphism defined by

$$\pi(S_k) = z_k,$$

z_k being the k-th coordinate function $z_k(x) = \langle x, e_k \rangle$, $x \in \partial B_d$.

Letting $\mathcal{A}$ be the commutative algebra of polynomials in the operators $S_1, ..., S_d$ we have

$$\mathcal{T}_d = \overline{\operatorname{span}}\, \mathcal{A}\mathcal{A}^*. \tag{5.8}$$

Proof. Let E_0 be the one-dimensional projection onto the space of constants in H^2. By the remark following Definition 2.10 we have

$$E_0 = \mathbf{1} - S_1 S_1^* - ... - S_d S_d^* \in \operatorname{span} \mathcal{A}\mathcal{A}^*.$$

Thus for any polynomials f, g, the the rank-one operator

$$f \otimes \bar{g} \colon \xi \mapsto \langle \xi, g \rangle f$$

can be expressed as

$$f \otimes \bar{g} = M_f E_0 M_g^* \in \operatorname{span} \mathcal{A}\mathcal{A}^*.$$

It follows that the norm closure of $\operatorname{span} \mathcal{A}\mathcal{A}^*$ contains the algebra $\mathcal{K}$ of all compact operators.

By Proposition 5.3, the quotient $\mathcal{T}_d/\mathcal{K}$ is a commutative C^*-algebra which is generated by commuting normal elements $Z_k = \pi(S_k)$, $k = 1, ..., d$, satisfying

$$Z_1 Z_1^* + ... + Z_d Z_d^* = \mathbf{1}.$$

Because $\mathcal{T}_d$ is commutative modulo $\mathcal{K}$ and since $\overline{\operatorname{span}}\, \mathcal{A}\mathcal{A}^*$ contains $\mathcal{K}$, it follows that $\overline{\operatorname{span}}\, \mathcal{A}\mathcal{A}^*$ is closed under multiplication, and (5.8) follows.

Let X be the joint spectrum of the commutative normal d-tuple $(Z_1, ..., Z_d)$ that generates $\mathcal{T}_d/\mathcal{K}$. X is a nonvoid subset of the sphere ∂B_d, and we claim that $X = \partial B_d$. Indeed, since the unitary group $\mathcal{U}(E)$ acts transitively on ∂B_d it suffices to show that for every unitary $(d \times d)$-matrix $u = (u_{ij})$, there is a $*$-automorphism θ_u of $\mathcal{T}_d/\mathcal{K}$ such that

$$\theta_u(Z_i) = \sum_{j=1}^{d} \bar{u}_{ji} Z_j.$$

For that, consider the unitary operator U acting on E by

$$Ue_i = \sum_{j=1}^{d} \bar{u}_{ji} e_j.$$

Then $\Gamma(U)$ is a unitary operator on H^2 for which

$$\Gamma(U) S_i \Gamma(U)^* = \sum_{j=1}^{d} \bar{u}_{ji} S_j,$$

and hence θ_u is obtained by promoting the spatial automorphism $T \mapsto \Gamma(U) T \Gamma(U)^*$ of $\mathcal{T}_d$ to the quotient $\mathcal{T}_d/\mathcal{K}$.

The identification of $\mathcal{T}_d/\mathcal{K}$ with $C(\partial B_d)$ asserted by $\pi(S_i)=z_i$, $i=1, ..., d$, is now obvious. □

6. d-contractions and $\mathcal{A}$-morphisms

The purpose of this section is to make some observations about the role of $\mathcal{A}$-morphisms in function theory and operator theory.

Definition 6.1. Let $\mathcal{A}$ be a subalgebra of a unital C^*-algebra $\mathcal{B}$ which contains the unit of $\mathcal{B}$. An $\mathcal{A}$-morphism is a completely positive linear map $\phi: \mathcal{B} \to \mathcal{B}(H)$ of $\mathcal{B}$ into the operators on a Hilbert space H such that $\phi(\mathbf{1})=\mathbf{1}$ and

$$\phi(AX) = \phi(A)\phi(X), \quad A \in \mathcal{A},\, X \in \mathcal{B}.$$

$\mathcal{A}$-morphisms arose naturally in our work on the dilation theory of completely positive maps and semigroups [7], [8], [9]. Jim Agler has pointed out that they are related to his notion of hereditary polynomials and hereditary isomorphisms (for example, see [1, Theorem 1.5]). Indeed, if $\mathcal{B}$ denotes the C^*-algebra generated by a single operator T and the identity, then one can show that a completely positive map of $\mathcal{B}$ which is a hereditary isomorphism on the space of hereditary polynomials in T is an $\mathcal{A}$-morphism relative to the algebra $\mathcal{A}$ of all polynomials in the *adjoint* T^*.

In general the restriction of an $\mathcal{A}$-morphism to $\mathcal{A}$ is a completely contractive representation of the subalgebra $\mathcal{A}$ on H. Theorem 4.5 implies that every d-contraction $\bar{T}$ acting on a Hilbert space H gives rise to a contraction $L: \mathcal{F}_+(\mathbf{C}^d) \otimes K \to H$ which intertwines the action of the d-shift and $\bar{T}$. L is often a co-isometry, and that implies the following assertion about $\mathcal{A}$-morphisms.

THEOREM 6.2. *Let $\mathcal{A}$ be the subalgebra of the Toeplitz C^*-algebra $\mathcal{T}_d$ consisting of all polynomials in the d-shift $(S_1, \ldots, S_d)$. Then for every d-contraction $(T_1, \ldots, T_d)$ acting on a Hilbert space H there is a unique $\mathcal{A}$-morphism*

$$\phi: \mathcal{T}_d \to \mathcal{B}(H)$$

such that $\phi(S_k)=T_k$, $k=1, \ldots, d$.

Conversely, every $\mathcal{A}$-morphism $\phi: \mathcal{T}_d \to \mathcal{B}(H)$ gives rise to a d-contraction $(T_1, \ldots, T_d)$ on H by way of $T_k=\phi(S_k)$, $k=1, \ldots, d$.

Proof. The uniqueness assertion is immediate from (5.8), since an $\mathcal{A}$-morphism is uniquely determined on the closed linear span of the set of products $\{AB^*: A, B \in \mathcal{A}\}$.

For existence, we first show that every *pure* d-contraction $\bar{T}=(T_1, \ldots, T_d)$ defines an $\mathcal{A}$-morphism as asserted in Theorem 6.2. For that, let

$$\Delta = (\mathbf{1} - T_1 T_1^* - \ldots - T_d T_d^*)^{1/2},$$

let $K=\overline{\Delta H}$ be the closed range of Δ and let $\mathcal{F}_+(E)$ be the symmetric Fock space over $E=\mathbf{C}^d$. Choose an orthonormal basis $e_1, \ldots, e_d$ for E. Theorem 4.5 asserts that there is a unique bounded operator $L: \mathcal{F}_+(E) \otimes K \to H$ satisfying $L(1 \otimes \xi)=\Delta\xi$ for $\xi \in K$, and

$$L(e_{i_1} e_{i_2} \ldots e_{i_n} \otimes \xi) = T_{i_1} T_{i_2} \ldots T_{i_n} \Delta \xi \tag{6.3}$$

for $n=1, 2, \ldots,$ $i_1, i_2, \ldots, i_n \in \{1, \ldots, d\}$, $\xi \in K$; moreover, since $(T_1, \ldots, T_d)$ is a pure d-contraction, L is a co-isometry.

We may consider that the d-shift $(S_1, \ldots, S_d)$ is defined on $\mathcal{F}_+(E)$ by

$$S_k \xi = e_k \xi, \quad k = 1, \ldots, d.$$

(6.3) implies that

$$L(f(S_1, \ldots, S_d) \otimes \mathbf{1}_K) = f(T_1, \ldots, T_d) L \tag{6.4}$$

for every polynomial f in d variables. Let $\phi: \mathcal{T}_d \to \mathcal{B}(H)$ be the completely positive map

$$\phi(X) = L(X \otimes \mathbf{1}_K) L^*, \quad X \in \mathcal{T}_d.$$

Since L^* is an isometry we have $\phi(\mathbf{1})=\mathbf{1}_H$. (6.4) implies that for every $X \in \mathcal{T}_d$ we have

$$\phi(f(S_1, \ldots, S_d) X) = f(T_1, \ldots, T_d) \phi(X),$$

and hence ϕ is an $\mathcal{A}$-morphism having the required properties.

The general case is deduced from this by a simple device. Let $\bar{T}=(T_1, \dots, T_d)$ be any d-contraction, choose a number r so that $0<r<1$, and set

$$\bar{T}_r = (rT_1, \dots, rT_d).$$

The row norm of the d-tuple $\bar{T}_r$ is at most r. Hence $\bar{T}_r$ is a pure d-contraction (see Remark 4.4). By what was just proved there is an $\mathcal{A}$-morphism $\phi_r: \mathcal{T}_d \to \mathcal{B}(H)$ satisfying

$$\phi_r(S_k) = rT_k, \quad k = 1, \dots, d.$$

We have

$$\phi_r(f(S_1, \dots, S_d)g(S_1, \dots, S_d)^*) = f(rT_1, \dots, rT_d)g(rT_1, \dots, rT_d)^*$$

for polynomials f, g. Since operators of the form $f(S_1, \dots, S_d)g(S_1, \dots, S_d)^*$ span $\mathcal{T}_d$ and since the family of maps ϕ_r, $0<r<1$, is uniformly bounded, it follows that ϕ_r converges point-norm to an $\mathcal{A}$-morphism ϕ as $r \uparrow 1$, and $\phi(S_k)=T_k$ for all k.

It remains only to show that for every $\mathcal{A}$-morphism $\phi: \mathcal{T}_d \to \mathcal{B}(H)$, the operators $T_k=\phi(S_k)$ define a d-contraction. To see that, write

$$T_k T_k^* = \phi(S_k)\phi(S_k)^* = \phi(S_k S_k^*).$$

Then

$$\sum_{k=1}^{d} T_k T_k^* = \phi\left(\sum_{k=1}^{d} S_k S_k^*\right) \leqslant \phi(\mathbf{1}) = \mathbf{1}.$$

So by Remark 3.2, $(T_1, \dots, T_d)$ is a d-contraction. □

Remarks. We have already pointed out that in general, an $\mathcal{A}$-morphism must be a completely contractive representation of $\mathcal{A}$. Conversely, if $\mathcal{A}$ is the polynomial algebra in $\mathcal{T}_d$ and $\phi: \mathcal{A} \to \mathcal{B}(H)$ is a representation which is d-contractive in the sense that its natural promotion to $(d \times d)$-matrices over $\mathcal{A}$ is a contraction, then after noting that the operator matrix

$$A = \begin{pmatrix} S_1 & S_2 & \dots & S_d \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} \in M_d(\mathcal{T}_d)$$

satisfies $\|A\|^2 = \|AA^*\| = \|S_1 S_1^* + \dots + S_d S_d^*\| = 1$, we find that the image of A under the promotion of ϕ is a contraction, and hence $T_k = \phi(S_k)$, $k=1, \dots, d$, defines a d-contraction. Thus, we may conclude

COROLLARY 1. *Let $d=1,2,\ldots$. Every d-contractive representation ϕ of the polynomial algebra $\mathcal{A}\subseteq\mathcal{T}_d$ is completely contractive, and can be extended uniquely to an $\mathcal{A}$-morphism*

$$\tilde{\phi}\colon\mathcal{T}_d\to\mathcal{B}(H).$$

We have already seen that the unitary group $\mathcal{U}_d$ of $\mathbf{C}^d$ acts naturally on $\mathcal{T}_d$ as a group of $*$-automorhisms by way of

$$\theta_U(X)=\Gamma(U)X\Gamma(U)^*,\quad X\in\mathcal{T}_d,\, U\in\mathcal{U}_d.$$

As a straightforward application of Theorem 6.2 we show that the definition of θ can be extended to all contractions in $\mathcal{B}(\mathbf{C}^d)$ so as to obtain a semigroup of $\mathcal{A}$-morphisms acting on $\mathcal{T}_d$.

COROLLARY 2. *Let $\mathcal{A}\subseteq\mathcal{T}_d$ be the algebra of all polynomials in $S_1,\ldots,S_d$. For every contraction A acting on $\mathbf{C}^d$ there is a unique $\mathcal{A}$-morphism $\theta_A\colon\mathcal{T}_d\to\mathcal{B}(H^2)$ satisfying*

$$\theta_A(M_f)=M_{f\circ A^*}\tag{6.5}$$

for every linear functional f on $\mathbf{C}^d$, A^* denoting the adjoint of $A\in\mathcal{B}(\mathbf{C}^d)$.

Proof. Considering the polar decomposition of A, we may find a pair of orthonormal bases $u_1,\ldots,u_d$ and $u'_1,\ldots,u'_d$ for $\mathbf{C}^d$ and numbers λ_k in the unit interval such that

$$Au_k=\lambda_k u'_k,\quad k=1,\ldots,d.$$

Let $z_1,\ldots,z_d$ and $z'_1,\ldots,z'_d$ be the corresponding systems of orthogonal coordinate functions

$$z_k(x)=\langle x,u_k\rangle,$$
$$z'_k(x)=\langle x,u'_k\rangle.$$

The linear functionals z_k, z'_k are related by

$$z_k\circ A^*=\lambda_k z'_k,\quad k=1,\ldots,d.\tag{6.6}$$

Thus if we realize the d-shift $(S_1,\ldots,S_d)$ as $S_k=M_{z_k}$ and if we set $T_k=\lambda_k M_{z'_k}$, then $(T_1,\ldots,T_d)$ is a d-contraction and Theorem 6.2 implies that there is a unique $\mathcal{A}$-morphism $\theta_A\colon\mathcal{T}_d\to\mathcal{B}(H^2)$ such that $\theta_A(S_k)=T_k$ for every k. After noting that θ_A satisfies (6.5) because of (6.6) above, the proof is complete. $\square$

From (6.5) together with the uniqueness assertion of Theorem 6.2 it follows that for two contractions $A,B\in\mathcal{B}(\mathbf{C}^d)$ we have $\theta_{AB}=\theta_A\circ\theta_B$. It is routine to verify that

$\theta_A(\mathcal{T}_d)\subseteq\mathcal{T}_d$, that for every fixed $X\in\mathcal{T}_d$ the function $A\mapsto\theta_A(X)$ moves continuously in the norm of $\mathcal{T}_d$, and that θ_A agrees with the previous definition when A is unitary.

Uniqueness of representing measures. Representing measures for points in the interior of the unit ball in $\mathbf{C}^d$ are notoriously nonunique in dimension $d\geqslant 2$. Indeed, for every $\bar{t}=(t_1,...,t_d)\in B_d$ there is an uncountable family of probability measures μ_α supported in the boundary ∂B_d such that $\mu_\alpha\perp\mu_\beta$ for $\alpha\neq\beta$ and

$$\int_{\partial B_d} f(\zeta)\,d\mu_\alpha(\zeta)=f(\bar{t}),\quad f\in\mathcal{P},$$

see [37, p. 186]. The following result asserts that one can recover uniqueness by replacing measures on ∂B_d with states on the Toeplitz C^*-algebra which define $\mathcal{A}$-morphisms.

COROLLARY 3. *Assume that* $\bar{t}=(t_1,...,t_d)\in\mathbf{C}^d$ *satisfies* $|t_1|^2+...+|t_d|^2<1$ *and let* $\bar{S}=(S_1,...,S_d)$ *be the d-shift. Then there is a unique state* ϕ *of* $\mathcal{T}_d$ *satisfying*

$$\phi(f(\bar{S})g(\bar{S})^*)=f(\bar{t})\bar{g}(\bar{t}),\quad f,g\in\mathcal{P}. \tag{6.7}$$

ϕ *is the (pure) vector state*

$$\phi(A)=(1-\|\bar{t}\|^2)\langle Au_{\bar{t}},u_{\bar{t}}\rangle,\quad A\in\mathcal{T}_d,$$

where $u_{\bar{t}}(x)=(1-\langle x,\bar{t}\rangle)^{-1}$ *is the* H^2*-function defined in* (1.11).

Proof. We may consider that $\bar{t}=(t_1,...,t_d)$ is a d-contraction acting on the one-dimensional Hilbert space $\mathbf{C}$. Theorem 6.2 implies that there is a unique state $\phi:\mathcal{T}_d\to\mathbf{C}$ satisfying (6.7), and it remains only to identify ϕ. From (2.4) we have

$$\langle M_fM_g^*u_{\bar{t}},u_{\bar{t}}\rangle=\langle M_g^*u_{\bar{t}},M_f^*u_{\bar{t}}\rangle=f(\bar{t})\bar{g}(\bar{t})\|u_{\bar{t}}\|^2=(1-\|\bar{t}\|^2)^{-1}f(\bar{t})\bar{g}(\bar{t}),$$

as asserted. □

7. The d-shift as an operator space

In this section we consider the operator space $\mathcal{S}_d\subseteq\mathcal{B}(H^2)$ generated by the d-shift $(S_1,...,S_d)$,

$$\mathcal{S}_d=\{a_1S_1+...+a_dS_d:a_1,...,a_d\in\mathbf{C}\}.$$

By a commutative operator space we mean a linear subspace $\mathcal{S}\subseteq\mathcal{B}(H)$ whose operators mutually commute with one another. We introduce a sequence of numerical invariants for arbitrary operator spaces, and for dimension $d\geqslant 2$ we show that among all d-dimensional commutative operator spaces, $\mathcal{S}_d$ is distinguished by the fact that its sequence of numerical invariants is maximal (Theorem 7.7).

Given an arbitrary operator space $\mathcal{S}\subseteq\mathcal{B}(H)$, let $\bar{T}=(T_1,T_2,\dots)$ be an infinite sequence of operators in $\mathcal{S}$ such that all but a finite number of terms are 0. We write $\operatorname{seq}(\mathcal{S})$ for the set of all such sequences. Every such sequence has a "row norm" and a "column norm", depending on whether one thinks of the sequence as defining an operator in $\mathcal{B}(H^\infty,H)$ or in $\mathcal{B}(H,H^\infty)$. These two norms are familiar and easily computed,

$$\|\bar{T}\|_{\text{row}}=\Big\|\sum_k T_kT_k^*\Big\|^{1/2},$$
$$\|\bar{T}\|_{\text{col}}=\Big\|\sum_k T_k^*T_k\Big\|^{1/2}.$$

Given two sequences $\bar{T},\bar{T}'\in\operatorname{seq}(\mathcal{S})$, we can form a product sequence $(T_iT_j':i,j=1,2,\dots)$ which we may consider an element of $\operatorname{seq}(\mathcal{B}(H))$, if we wish, by relabelling the double sequence as a single sequence. Though for the computations below it will be more convenient to allow the index set to vary in the obvious way. In particular, every $\bar{T}\in\operatorname{seq}(\mathcal{S})$ can be raised to the nth power to obtain $\bar{T}^n\in\operatorname{seq}(\mathcal{B}(H))$, $n=1,2,\dots$. For each $n=1,2,\dots$ we define $E_n(\mathcal{S})\in[0,+\infty]$ as

$$E_n(\mathcal{S})=\sup\{\|\bar{T}^n\|_{\text{col}}^2:\bar{T}\in\operatorname{seq}(\mathcal{S}),\ \|\bar{T}\|_{\text{row}}\leqslant 1\}.$$

In the most explicit terms, we have

$$E_n(\mathcal{S})=\sup\bigg\{\Big\|\sum_{i_1,\dots,i_n=1}^{\infty}T_{i_1}^*\dots T_{i_n}^*T_{i_n}\dots T_{i_1}\Big\|:T_i\in\mathcal{S},\ \Big\|\sum_{i=1}^{\infty}T_iT_i^*\Big\|\leqslant 1\bigg\},$$

the sup being taken over finitely nonzero sequences $T_i\in\mathcal{S}$.

Definition 7.1. $E_1(\mathcal{S}),E_2(\mathcal{S}),\dots$ is called the energy sequence of the operator space $\mathcal{S}$.

If $\mathcal{S}$ is the one-dimensional space spanned by a single operator T of norm 1, then the energy sequence degenerates to $E_n(\mathcal{S})=\|T^n\|^2$, $n=1,2,\dots$. In general, $E_n(\mathcal{S})^{1/2}$ is the norm of the homogeneous polynomial $\bar{T}\mapsto\bar{T}^n$, considered as a map of row sequences in $\mathcal{S}$ to column sequences in $\mathcal{B}(H)$.

Remarks. We have defined the energy sequence in elementary terms. It is useful, however, to relate it to completely positive maps. Fixing an operator space $\mathcal{S}$, notice that every sequence $\bar{T}\in\operatorname{seq}(\mathcal{S})$ gives rise to a normal completely positive map $P_{\bar{T}}$ on $\mathcal{B}(H)$ as the sum of the *finite* series

$$P_{\bar{T}}(A)=T_1AT_1^*+T_2AT_2^*+\dots\,. \tag{7.2}$$

Let $\mathrm{cp}(\mathcal{S})$ denote the set of all completely positive maps of the form (7.2). The norm of $P=P_{\bar{T}}$ is given by

$$\|P\| = \|P(\mathbf{1})\| = \|\bar{T}\|_{\mathrm{row}}.$$

Now any map $P\in\mathrm{cp}(\mathcal{S})$ of the form (7.2) has an adjoint P_* which is defined as the completely positive map satisfying

$$\mathrm{trace}(P(A)B) = \mathrm{trace}(AP_*(B))$$

for all finite-rank operators A, B. One finds that if $P\in\mathrm{cp}(\mathcal{S})$ is given by the finitely nonzero sequence $\bar{T}$ then $P_*\in\mathrm{cp}(\mathcal{S}^*)$ is given by the sequence of adjoints

$$P_*(A) = T_1^*AT_1 + T_2^*AT_2 + \ldots . \tag{7.3}$$

Of course P, being a normal linear map of $\mathcal{B}(H)$, is the adjoint of a bounded linear map P_* acting on the predual of $\mathcal{B}(H)$, and the map of (7.3) is simply this preadjoint extended from the trace class operators to all of $\mathcal{B}(H)$ (note that we use the fact that the sequence $\bar{T}$ is finitely nonzero here, since in general a bounded linear map of the trace class operators can be unbounded relative to the operator norm, and thus not extendable up to $\mathcal{B}(H)$).

In any case, we find that if $P\in\mathrm{cp}(\mathcal{S})$ has the form $P=P_{\bar{T}}$ for $\bar{T}\in\mathrm{seq}(\mathcal{S})$ then

$$\|P_*\| = \|P_*(\mathbf{1})\| = \|\bar{T}\|_{\mathrm{col}}.$$

Thus the definition of $E_n(\mathcal{S})$ can be restated as

$$E_n(\mathcal{S}) = \sup\{\|P_*^n\| : P\in\mathrm{cp}(\mathcal{S}),\ \|P\|\leqslant 1\}. \tag{7.4}$$

The following result implies that for a finite-dimensional operator space $\mathcal{S}$ the terms of the energy sequence are all finite, and if $\mathcal{S}$ is *commutative* then they grow no faster than $E_n(\mathcal{S})=O(n^{d-1})$, where d is the dimension of $\mathcal{S}$.

PROPOSITION 7.5. *Let $\mathcal{S}$ be an operator space of finite dimension d. Then*

$$E_n(\mathcal{S}) \leqslant d^n,$$

and if $\mathcal{S}$ is also commutative then

$$E_n(\mathcal{S}) \leqslant \frac{d(d+1)\ldots(d+n-1)}{n!} = \frac{(n+d-1)!}{n!\,(d-1)!}.$$

Proof. Let $P\in\operatorname{cp}(\mathcal{S})$ satisfy $\|P\|\leqslant 1$, and let d be the dimension of $\mathcal{S}$. It is clear that the metric operator space [8] of P is a subspace of $\mathcal{S}$, and in particular there is a linearly independent set of $r\leqslant d$ elements $T_1, ..., T_r$ in $\mathcal{S}$ such that

$$P(A)=T_1AT_1^*+...+T_rAT_r^*,\quad A\in\mathcal{B}(H).$$

Since $\|P\|=\|P(\mathbf{1})\|=\|T_1T_1^*+...+T_rT_r^*\|\leqslant 1$ it follows that $\|T_k\|\leqslant 1$ for every $k=1,...,r$, and hence

$$\|P_*\|=\|T_1^*T_1+...+T_r^*T_r\|\leqslant r\leqslant d.$$

Thus $\|P_*^n\|\leqslant d^n$ for every $n=1,2,...$. From (7.4) we conclude that $E_n(\mathcal{S})\leqslant d^n$.

In fact, the preceding argument shows that if Q is a normal completely positive map of $\mathcal{B}(H)$ whose metric operator space is r-dimensional and which satisfies $\|Q\|\leqslant 1$, then we have $\|Q_*\|\leqslant r$.

We apply this to $Q=P^n$ as follows. By [8], the metric operator space $\mathcal{E}_n$ of P^n is a subspace of

$$\operatorname{span}\{L_1L_2\dots L_n : L_i\in\mathcal{S}\}.$$

Assuming $\mathcal{S}$ to be commutative, the latter is naturally isomorphic to a quotient of the n-fold *symmetric* tensor product of vector spaces $\mathcal{S}^n$. Since

$$\dim\mathcal{S}^n=\frac{(n+d-1)!}{n!\,(d-1)!}$$

(see formula (A.5) of Appendix A), and since $\|P^n\|\leqslant 1$, we find that

$$\|P_*^n\|=\|(P^n)_*\|\leqslant\dim\mathcal{E}_n\leqslant\frac{(n+d-1)!}{n!\,(d-1)!}.$$

The required estimate follows from the observation (7.4). □

Remark. The asserted growth rate of the binomial coefficients of Proposition 7.5 is well known, and the precise asymptotic relation is reiterated in formula (A.6).

Throughout the remainder of this section we will be concerned with finite-dimensional commutative operator spaces.

Definition 7.6. A commutative operator space $\mathcal{S}$ of finite dimension d is said to be maximal if for every $n=1,2,...$ we have

$$E_n(\mathcal{S})=\frac{(n+d-1)!}{n!\,(d-1)!}.$$

Remarks. It is obvious that the row norm of any sequence of normal operators is the same as its column norm. It follows that if $\mathcal{S}$ is a space of mutually commuting normal operators, then $E_n(\mathcal{S})=1$ for every n. Similarly, it can be shown that if $\mathcal{S}$ is a finite-dimensional space of commuting quasinilpotent operators, then

$$\lim_{n\to\infty} E_n(\mathcal{S})=0.$$

(Following the suggestion of a referee, we have included a proof in Appendix B.) Thus the maximal spaces are rather far removed from both of these types.

It is also true (though less obvious) that if $\mathcal{S}$ is a commutative operator space of dimension d for which

$$E_n(\mathcal{S})=\frac{(n+d-1)!}{n!\,(d-1)!}$$

for some particular value of $n\geqslant 2$, then

$$E_k(\mathcal{S})=\frac{(k+d-1)!}{k!\,(d-1)!}$$

for every $k=1,2,\ldots,n$. Thus for operator spaces which are not maximal, once the sequence of numbers $E_n(\mathcal{S})$ departs from the sequence of maximum possible values, it never returns. We omit the proof of the latter assertion since it is not required in the sequel.

THEOREM 7.7. *For every $d=1,2,\ldots$ the operator space $\mathcal{S}_d$ of the d-shift is maximal. Conversely, if $d\geqslant 2$ and if $\mathcal{S}$ is a d-dimensional commutative operator space which is maximal, then there is a representation π of the unital C^*-algebra $C^*(\mathcal{S})$ generated by $\mathcal{S}$ on H^2 such that $\pi(\mathcal{S})=\mathcal{S}_d$. In particular, the Toeplitz C^*-algebra $\mathcal{T}_d$ is isomorphic to a quotient of $C^*(\mathcal{S})$.*

Before giving the proof of Theorem 7.7, we deduce from it the following characterization of $\mathcal{S}_d$ as a space of essentially normal operators (by that we mean a commuting family of operators in $\mathcal{B}(H)$ whose image in the Calkin algebra consists of normal elements). We remark that both the corollary and the essential part of Theorem 7.7 are false in dimension one.

COROLLARY. *Assume $d\geqslant 2$. Up to unitary equivalence, the space $\mathcal{S}_d$ spanned by the d-shift is the only d-dimensional irreducible commutative operator space consisting of essentially normal operators, which is maximal in the sense of Definition* 7.6.

Proof of corollary. Suppose that $\mathcal{S}$ acts on a Hilbert space H, and let $\mathcal{K}$ denote the algebra of all compact operators on H. Let $\pi: C^*(\mathcal{S})\to\mathcal{B}(H^2)$ be the representation of Theorem 7.7. The operators in $\mathcal{S}$ cannot be normal because $\mathcal{S}_d=\pi(\mathcal{S})$ contains no normal operators. Since $[\mathcal{S}^*,\mathcal{S}]\subseteq\mathcal{K}\cap C^*(\mathcal{S})$ and since $C^*(\mathcal{S})$ is irreducible, it follows that

$C^*(\mathcal{S})$ contains $\mathcal{K}$. $\pi(\mathcal{K})$ cannot be $\{0\}$ because that would imply that $\pi(\mathcal{S})=\mathcal{S}_d$ consists of normal operators.

Thus π is an irreducible representation of $C^*(\mathcal{S})$ which is nonzero on $\mathcal{K}$, and hence π must be unitarily equivalent to the identity representation of $C^*(\mathcal{S})$. In particular, $\mathcal{S}$ is unitarily equivalent to $\mathcal{S}_d$. □

Proof of Theorem 7.7. The proof of Theorem 7.7 will occupy the remainder of this section. Let $(S_1, ..., S_d)$ be the d-shift, let $\mathcal{S}_d=\operatorname{span}\{S_1, ..., S_d\}$ be its associated operator space, and define $P\in\operatorname{cp}(\mathcal{S}_d)$ by

$$P(A)=S_1AS_1^*+...+S_dAS_d^*.$$

By the remark following Definition 2.10 we have $P(\mathbf{1})=\mathbf{1}-E_0$, and hence $\|P\|=1$. Thus to show that $\mathcal{S}_d$ is maximal it suffices to show that for each $n\geqslant 1$, the operator $P_*^n(\mathbf{1})$ satisfies

$$\|P_*^n\|=\|P_*^n(\mathbf{1})\|=\frac{(n+d-1)!}{n!\,(d-1)!}. \tag{7.8}$$

While (7.8) can be deduced directly from Theorem 4.9, we actually require somewhat more information about the operators $P_*^n(\mathbf{1})$ and their eigenvalue distributions.

LEMMA 7.9. *Let $N=E_1+2E_2+3E_3+...$ be the number operator acting on H^2, and for every $n=1,2,...$ let $g_n\colon[0,\infty)\to\mathbf{R}$ be the bounded continuous function*

$$g_n(x)=\prod_{k=1}^{n}\frac{x+k+d-1}{x+k}.$$

Then

$$P_*^n(\mathbf{1})=g_n(N)=\sum_{k=0}^{\infty}g_n(k)E_k.$$

The eigenvalue sequence $\{g_n(0)\geqslant g_n(1)\geqslant...\}$ of $P_*^n(\mathbf{1})$ is decreasing and we have

$$\|P_*^n(\mathbf{1})\|=g_n(0)=\frac{(n+d-1)!}{n!\,(d-1)!}.$$

If $d\geqslant 2$ then the eigenvalue sequence is strictly decreasing, $g_n(0)>g_n(1)>...$.

Proof of Lemma 7.9. The assertions follow from a direct computation, which can be organized as follows. By Proposition 5.3 we have

$$P_*(\mathbf{1})=g_1(N), \tag{7.10}$$

where N is the number operator and g_1 is the function of a real variable defined by

$$g_1(x)=\frac{x+d}{x+1},\quad x\geqslant 0.$$

More generally, if g is any bounded continuous function defined on $[0,\infty)$, then we have

$$P_*(g(N)) = \tilde{g}(N), \tag{7.11}$$

where

$$\tilde{g}(x) = g(x+1)\frac{x+d}{x+1}, \quad x \geqslant 0.$$

Indeed, (7.11) follows from the fact that if E_k denotes the kth spectral projection of N,

$$N = \sum_{k=1}^{\infty} kE_k,$$

then E_k is the projection on the subspace of homogeneous polynomials of degree k in H^2, and thus for each $i=1,...,d$ we have the commutation formulas $S_i^* E_0 = 0$, and $S_i^* E_k = E_{k-1} S_i^*$ for $k \geqslant 1$. It follows that $P_*(E_0)=0$ and $P_*(E_k)=E_{k-1}P_*(\mathbf{1})$ for $k=1,2,...$, and thus

$$P_*(g(N)) = \sum_{k=1}^{\infty} g(k) P_*(E_k) = \tilde{g}(N).$$

After iterating (7.11) we find that $P_*^n(\mathbf{1}) = g_n(N)$ where

$$g_n(x) = g_1(x)g_1(x+1)\dots g_1(x+n-1) = \prod_{k=1}^{n} \frac{x+k+d-1}{x+k}.$$

Since each g_n is a monotone decreasing function we conclude that

$$\|P_*^n(\mathbf{1})\| = g_n(0) = \frac{d(d+1)\dots(d+n-1)}{n!},$$

and (7.8) follows. It is clear from the recurrence formula for g_{n+1} in terms of g_n that when $d \geqslant 2$, $g_n(x)$ is a strictly decreasing function of x. $\square$

COROLLARY. *Let ω be a state of the Toeplitz algebra $\mathcal{T}_d$, $d \geqslant 2$, such that for some $n \geqslant 1$ we have*

$$\omega(P_*^n(\mathbf{1})) = \|P_*^n(\mathbf{1})\| = \frac{(n+d-1)!}{n!\,(d-1)!}.$$

Then ω is the ground state $\omega(X) = \langle Xv, v\rangle$, v denoting the constant function $v=1$.

Proof. Fix n. By Lemma 7.9 we have

$$P_*^n(\mathbf{1}) = \lambda_0 E_0 + \lambda_1 E_1 + \dots,$$

where $\lambda_0 > \lambda_1 > \dots > 0$ and $\lambda_0 = \|P_*^n(\mathbf{1})\|$. Thus $P_*^n(\mathbf{1})$ has the form

$$P_*^n(\mathbf{1}) = \lambda_0(E_0 + K),$$

where K is a positive operator satisfying $K=(\mathbf{1}-E_0)K(\mathbf{1}-E_0)$ and $\|K\|=\lambda_1/\lambda_0<1$. Since $\omega(P_*^n(\mathbf{1}))=\lambda_0$ we have

$$\omega(E_0)+\omega(K)=1.$$

If $\omega(E_0)<1$ then we would have

$$\omega(K)\leqslant \|K\|\omega(\mathbf{1}-E_0)=\|K\|(1-\omega(E_0))<1-\omega(E_0),$$

contradicting the preceding equation. Hence $\omega(E_0)=1$ and ω must be the ground state. $\square$

In particular, Lemma 7.9 implies that $\mathcal{S}_d$ is maximal among all d-dimensional commutative operator spaces.

In order to prove the converse assertion of Theorem 7.7, we recall one or two facts from the theory of boundary representations (see [3, 2.1.2 and 2.2.2]). By a unital operator space we mean a pair $\mathcal{S}\subseteq\mathcal{B}$ consisting of a linear subspace $\mathcal{S}$ of a unital C^*-algebra $\mathcal{B}$, which contains the unit of $\mathcal{B}$ and generates $\mathcal{B}$ as a C^*-algebra, $\mathcal{B}=C^*(\mathcal{S})$. An irreducible representation $\pi:\mathcal{B}\to\mathcal{B}(H)$ is said to be a boundary representation for $\mathcal{S}$ if $\pi\restriction_{\mathcal{S}}$ has a *unique* completely positive linear extension to $\mathcal{B}$, namely π itself. Boundary representations are the noncommutative counterpart of points in the Choquet boundary of a function space $S\subseteq C(X)$. Their key property is their functoriality; if $\mathcal{S}_1\subseteq\mathcal{B}_1$ and $\mathcal{S}_2\subseteq\mathcal{B}_2$ are unital operator spaces and $\phi:\mathcal{S}_1\to\mathcal{S}_2$ is a completely isometric linear map satisfying $\phi(\mathbf{1})=\mathbf{1}$ and $\phi(\mathcal{S}_1)=\mathcal{S}_2$, then for every boundary representation $\pi_2:\mathcal{B}_2\to\mathcal{B}(H)$ for $\mathcal{S}_2$ there is a unique boundary representation $\pi_1:\mathcal{B}_1\to\mathcal{B}(H)$ for $\mathcal{S}_1$ which satisfies

$$\pi_2(\phi(T))=\pi_1(T),\quad T\in\mathcal{S}_1. \tag{7.12}$$

LEMMA 7.13. *For $d\geqslant 2$, the identity representation of the Toeplitz algebra $\mathcal{T}_d$ is a boundary representation for the $(d+1)$-dimensional space* $\operatorname{span}\{\mathbf{1},S_1,\dots,S_d\}$.

Proof. By [4, Theorem 2.1.1] it is enough to show that the Calkin map is not isometric when promoted to the space $M_d\otimes\mathcal{S}$ of $(d\times d)$-matrices over $\mathcal{S}$. Consider the operator $A\in M_d\otimes\mathcal{S}$ defined by

$$A=\begin{pmatrix} S_1 & 0 & \dots & 0\\ S_2 & 0 & \dots & 0\\ \vdots & \vdots & & \vdots\\ S_d & 0 & \dots & 0\end{pmatrix}.$$

Then $\|A^*A\|=\|S_1^*S_1+\dots+S_d^*S_d\|=d$ by Proposition 5.3. Hence $\|A\|=\sqrt{d}$. On the other hand, by Theorem 5.7 the Calkin map carries S_k to the kth coordinate function $z_k(x)=$

$\langle x, e_k \rangle$, $x \in \partial B_d$. Hence the image of A under the promoted Calkin map is the matrix of functions on ∂B_d defined by

$$F(x) = \begin{pmatrix} z_1(x) & 0 & \dots & 0 \\ z_2(x) & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ z_d(x) & 0 & \dots & 0 \end{pmatrix}.$$

Clearly $\sup\{\|F(x)\| : x \in \partial B_d\} = 1 < \sqrt{d} = \|A\|$, as required. □

Lemma 7.13 implies that in dimension $d \geqslant 2$, the d-shift $\bar{S} = (S_1, \dots, S_d)$ can be dilated to another d-contraction $\bar{T}$ in only a trivial way as a direct summand $T_k = S_k \oplus Z_k$, where $(Z_1, \dots, Z_d)$ is some d-contraction.

LEMMA 7.14. *Suppose $d \geqslant 2$, let $(T_1, \dots, T_d)$ be a d-contraction acting on a Hilbert space H, and let $K \subseteq H$ be a subspace of H such that the compressed d-tuple*

$$(P_K T_1 \restriction_K, \dots, P_K T_d \restriction_K)$$

is unitarily equivalent to the d-shift. Then P_K commutes with $\{T_1, \dots, T_d\}$.

Proof. By hypothesis, there is an isometry $U: H_d^2 \to H$ such that $UH_d^2 = K$ and $U^* T_k U = S_k$, $k = 1, \dots, d$. By Theorem 6.2, there is an $\mathcal{A}$-morphism $\phi: \mathcal{T}_d \to \mathcal{B}(H)$ such that $\phi(S_k) = T_k$, $k = 1, \dots, d$. Define a completely positive map $\psi: \mathcal{T}_d \to \mathcal{B}(H_d^2)$ by $\psi(X) = U^* \phi(X) U$. We have $\psi(\mathbf{1}) = \mathbf{1}$, and $\psi(S_k) = U^* T_k U = S_k$, $k = 1, \dots, d$. Lemma 7.13 implies that ψ must be the identity map of $\mathcal{T}_d$.

In particular, $\psi(XY) = \psi(X)\psi(Y)$ for all $X, Y \in \mathcal{T}_d$. Multiplying the latter equation on left and right by U and U^* respectively, we obtain

$$P_K \phi(XY) P_K = P_K \phi(X) P_K \phi(Y) P_K, \quad X, Y \in \mathcal{T}_d. \tag{7.15}$$

Taking $X = Y^*$ in (7.15) and making use of the Schwarz inequality for completely positive maps we have

$$\begin{aligned} ((\mathbf{1} - P_K)\phi(Y) P_K)^* (\mathbf{1} - P_K)\phi(Y) P_K &= P_K \phi(Y)^* \phi(Y) P_K - P_K \phi(Y)^* P_K \phi(Y) P_K \\ &\leqslant P_K \phi(Y^* Y) P_K - P_K \phi(Y^*) P_K \phi(Y) P_K = 0, \end{aligned}$$

and hence $(\mathbf{1} - P_K)\phi(Y) P_K = 0$. Thus K is an invariant subspace for the self-adjoint family of operators $\phi(\mathcal{T}_d)$, and hence $P_K \in \phi(\mathcal{T}_d)' \subseteq \{T_1, \dots, T_d\}'$. □

LEMMA 7.16. *Let $\mathcal{S}\subseteq\mathcal{B}(H)$ be a commutative operator space of finite dimension $d\geqslant 2$, and suppose that $\mathcal{S}$ is maximal. Then there is a state ϱ of the unital C^*-algebra $C^*(\mathcal{S})$ generated by $\mathcal{S}$, and a d-contraction $\bar{T}=(T_1,...,T_d)$, $T_i\in\mathcal{S}$, such that*

$$\varrho(g(\bar{T})^*f(\bar{T}))=\langle f,g\rangle_{H^2}$$

for all polynomials $f,g\in\mathcal{P}$.

Proof of Lemma 7.16. The set of all d-contractions $(T_1,...,T_d)$ whose component operators belong to $\mathcal{S}$ can be regarded as a compact subset of the Cartesian product of d copies of the unit ball of $\mathcal{S}$, and of course the state space of $C^*(\mathcal{S})$ is weak*-compact. Thus, after a routine compactness argument (which we omit), the proof of Lemma 7.16 reduces to establishing the following assertion: for every $n=1,2,...$ there is a pair $(\varrho,\bar{T})$ consisting of a state ϱ of $C^*(\mathcal{S})$ and a d-contraction $\bar{T}=(T_1,...,T_d)$ whose components belong to $\mathcal{S}$ such that

$$\varrho(g(\bar{T})^*f(\bar{T}))=\langle f,g\rangle_{H^2} \tag{7.17}$$

for all polynomials $f,g\in\mathcal{P}$ of degree $\leqslant n$.

To prove the latter, since $E_n(\mathcal{S})=(n+d-1)!/n!(d-1)!$ we may find a completely positive map $P\in\mathrm{cp}(\mathcal{S})$ such that $\|P\|\leqslant 1$ and

$$\|P_*^n(\mathbf{1})\|=\frac{(n+d-1)!}{n!\,(d-1)!} \tag{7.18}$$

(note that the supremum of (7.4) is achieved here because the space $\{P\in\mathrm{cp}(\mathcal{S}):\|P\|\leqslant 1\}$ is compact). Considering that the metric operator space of P is a subspace of $\mathcal{S}$ [8] we can find a (linearly independent) set $T_1,...,T_r\in\mathcal{S}$ such that

$$P(A)=T_1AT_1^*+...+T_rAT_r^*,\quad A\in\mathcal{B}(H).$$

By appending $T_{r+1}=...=T_d=0$ to the sequence if necessary, we can assume that $r=d$. Because

$$\|P\|=\|P(\mathbf{1})\|=\|T_1T_1^*+...+T_dT_d^*\|\leqslant 1,$$

$\bar{T}=(T_1,...,T_d)$ is a d-contraction for which (7.18) holds.

Let ϱ be any state of $C^*(\mathcal{S})$ satisfying

$$\varrho(P_*^n(\mathbf{1}))=\|P_*^n(\mathbf{1})\|=\frac{(n+d-1)!}{n!\,(d-1)!},$$

and consider the positive semidefinite inner product defined on $\mathcal{P}$ by

$$\langle f,g\rangle=\varrho(g(\bar{T})^*f(\bar{T})). \tag{7.19}$$

One sees (after consideration of the GNS construction for the state ϱ) that since $\bar{T}$ is a d-contraction, the Hilbert seminorm $\|f\|^2=\varrho(f(\bar{T})^*f(\bar{T}))$ satisfies

$$\|z_1f_1+\ldots+z_df_d\|^2\leqslant\|f_1\|^2+\ldots+\|f_d\|^2$$

for all polynomials $f_1,\ldots,f_d\in\mathcal{P}$. By Proposition 4.2, $\|\cdot\|$ will be a contractive seminorm provided that

$$1\perp z_1\mathcal{P}+\ldots+z_d\mathcal{P}$$

in its associated inner product space; or equivalently, that

$$\varrho(T_kf(\bar{T}))=0,\quad k=1,\ldots,d,\ f\in\mathcal{P}.$$

Since ϱ is a state, the latter will follow if we establish

$$\varrho(T_kT_k^*)=0,\quad k=1,\ldots,d. \tag{7.20}$$

To prove 7.20, let $\phi:\mathcal{T}_d\to\mathcal{B}(H)$ be an $\mathcal{A}$-morhpism satisfying $\phi(S_k)=T_k$, $k=1,\ldots,d$ (see Theorem 6.2), and let ω be the state of $\mathcal{T}_d$ defined by $\omega=\varrho\circ\phi$. We claim that ω is the ground state of $\mathcal{T}_d$. Indeed, for every n-tuple of integers $1\leqslant i_1,\ldots,i_n\leqslant d$ we have by the Schwarz inequality

$$\phi(S_{i_1}^*\ldots S_{i_n}^*S_{i_n}\ldots S_{i_1})\geqslant\phi(S_{i_n}\ldots S_{i_1})^*\phi(S_{i_n}\ldots S_{i_1})=T_{i_1}^*\ldots T_{i_n}^*T_{i_n}\ldots T_{i_1},$$

and hence

$$\omega(S_{i_1}^*\ldots S_{i_n}^*S_{i_n}\ldots S_{i_1})\geqslant\varrho(T_{i_1}^*\ldots T_{i_n}^*T_{i_n}\ldots T_{i_1}).$$

Summing over all such n-tuples we obtain

$$\sum_{i_1,\ldots,i_n=1}^{d}\omega(S_{i_1}^*\ldots S_{i_n}^*S_{i_n}\ldots S_{i_1})\geqslant\varrho(P_*^n(\mathbf{1}))=\frac{(n+d-1)!}{n!\,(d-1)!}.$$

The corollary of Lemma 7.9 implies that ω must be the ground state of $\mathcal{T}_d$. In particular, for each $k=1,\ldots,d$ we have

$$\varrho(T_kT_k^*)=\varrho(\phi(S_k)\phi(S_k)^*)=\varrho(\phi(S_kS_k^*))=\omega(S_kS_k^*)=\|S_k^*1\|^2=0,$$

and (7.20) follows.

It is clear that the Hilbert seminorm of (7.19) is normalized so that $\|1\|^2=\varrho(\mathbf{1})=1$; so by Theorem 4.3 we have $\|f\|\leqslant\|f\|_{H^2}$ for every $f\in\mathcal{P}$. Theorem 4.9 now implies that (7.17) is satisfied, and the proof is complete. $\square$

To complete the proof of Theorem 7.7, we find a d-contraction $\bar{T}=(T_1, ..., T_d)$ and a state ϱ of $C^*(\mathcal{S})$ satisfying the conditions of Lemma 7.16. The set of operators $\{T_1, ..., T_d\}$ must be linearly independent; indeed, for every polynomial $f\neq 0$ we have

$$\varrho(f(T_1, ..., T_d)^* f(T_1, ..., T_d)) = \|f\|_{H^2}^2 \neq 0,$$

and hence $f(T_1, ..., T_d)\neq 0$. It follows that $\operatorname{span}\{T_1, ..., T_d\}=\mathcal{S}$.

The GNS construction provides a nondegenerate representation σ of $C^*(\mathcal{S})$ on a Hilbert space K and a unit vector $\xi\in K$ such that $\varrho(X)=\langle\sigma(X)\xi, \xi\rangle$, $X\in C^*(\mathcal{S})$. The key property of ϱ implies that we can define an isometry $U: H_d^2\to K$ on polynomials by

$$Uf = \sigma(f(T_1, ..., T_d))\xi, \quad f\in\mathcal{P},$$

and we have $US_k=\sigma(T_k)U$ for every $k=1, ..., d$. Thus the range UH_d^2 of U is invariant under each $\sigma(T_k)$, and the restriction of the d-contraction $(\sigma(T_1), ..., \sigma(T_d))$ to UH_d^2 is unitarily equivalent to the d-shift. By Lemma 7.14, the projection UU^* must commute with $\sigma(T_k)$, $k=1, ..., d$, and hence with the unital C^*-algebra $\sigma(C^*(\mathcal{S}))$ these operators generate.

We obtain a representation $\pi: C^*(\mathcal{S})\to\mathcal{B}(H_d^2)$ by setting $\pi(X)=U^*\sigma(X)U$. Since $\pi(T_k)=U^*\sigma(T_k)U=S_k$ for each k, it follows that $\pi(\mathcal{S})=\mathcal{S}_d$. □

8. Various applications

In this section we give several applications of the preceding results to function theory and multivariable operator theory. These are a version of von Neumann's inequality for arbitrary d-contractions, a model theory for d-contractions based on the d-shift, a discussion of the absence of inner functions in the multiplier algebra of the d-shift, and some remarks concerning C^*-envelopes.

We point out that Popescu has established versions of von Neumann's inequality for noncommutative d-tuples of operators [30], [32], [34], [35]. Here, on the other hand, we are concerned with d-contractions. The version of von Neumann's inequality that is appropriate for d-contractions is the following.

THEOREM 8.1. *Let $\bar{T}=(T_1, ..., T_d)$ be an arbitrary d-contraction acting on a Hilbert space H. Then for every polynomial f in d complex variables we have*

$$\|f(T_1, ..., T_d)\| \leqslant \|f\|_{\mathcal{M}},$$

$\|f\|_{\mathcal{M}}$ being the norm of f in the multiplier algebra $\mathcal{M}$ of H^2.

More generally, let $(S_1, ..., S_d)$ be the d-shift and let $\mathcal{A} \subseteq \mathcal{T}_d$ be the algebra of all polynomials in $S_1, ..., S_d$. Then the map $f(S_1, ..., S_d) \mapsto f(T_1, ..., T_d)$ defines a completely contractive representation of $\mathcal{A}$.

Proof. The assertions are immediate consequences of Theorem 6.2, once one observes that $\|f\|_{\mathcal{M}} = \|f(S_1, ..., S_d)\|$. □

Turning now to models, we first recall some of the literature of dilation theory in d dimensions. There are a number of positive results concerning noncommutative models for noncommuting d-tuples which satisfy the conditions of Remark 3.2. The first results along these lines are due to Frazho [21] for pairs of operators. Frazho's results were generalized by Bunce [15] to d-tuples. Popescu has clarified that work by showing that such a d-tuple can often be obtained by compressing a certain natural d-tuple of isometries acting on the full Fock space $\mathcal{F}(\mathbf{C}^d)$ over $\mathbf{C}^d$ (the left creation operators) to a co-invariant subspace of $\mathcal{F}(\mathbf{C}^d)$, and he has worked out a functional calculus for that situation [28], [29], [30], [31]. We also point out some recent work of Davidson and Pitts [18], [19], relating to the operator algebra generated by the left creation operators on the full Fock space.

There is relatively little in the literature of operator theory, however, that relates to uniqueness of dilations in higher dimensions (however, see [11]). Indeed, normal dilations for d-contractions, when they exist, are almost never unique. On the other hand, recent results in the theory of semigroups of completely positive maps do include uniqueness. Generalizing work of Parathasarathy, B. V. R. Bhat [14] has shown that a unital semigroup of completely positive maps of a von Neumann algebra M can be dilated uniquely to an E_0-semigroup acting on a larger von Neumann algebra N which contains M as a hereditary subalgebra. A similar (and simpler) result holds for single unital completely positive maps: there is a unique dilation to a unital endomorphism acting on a larger von Neumann algebra as above. In the case where $M = \mathcal{B}(H)$, the latter dilation theorem is closely related to the Bunce–Frazho theory of d-tuples by way of the metric operator space associated with a normal completely postive map of $\mathcal{B}(H)$ [8], [9]. SeLegue [42] has succeeded in unifying these results.

In the following discussion, we reformulate Theorem 6.2 as a concrete assertion about d-contractions which parallels some of the principal assertions of the Sz.-Nagy–Foias model theory of 1-contractions [43]. Much of Theorem 8.5 follows directly from Theorem 6.2 and standard lore on the representation theory of C^*-algebras. For completeness, we have given a full sketch of the argument.

We recall some elementary facts about the representation theory of C^*-algebras such as $\mathcal{T}_d$. Let $\pi: \mathcal{T}_d \to \mathcal{B}(H)$ be a nondegenerate $*$-representation of $\mathcal{T}_d$ on a separable

Hilbert space H. Because of the exact sequence of Theorem 5.7, standard results about the representations of the C^*-algebra of compact operators imply that π decomposes into a direct sum $\pi_1 \oplus \pi_2$, where π_1 is a multiple of $n=0,1,2,\ldots,\infty$ copies of the identity representation of $\mathcal{T}_d$ and π_2 is a representation which annihilates $\mathcal{K}$. π_1 and π_2 are disjoint as representations of $\mathcal{T}_d$. This decomposition is unique in the sense that if π_1' is another multiple of n' copies of the identity representation of $\mathcal{T}_d$ and π_2' annihilates $\mathcal{K}$, and if $\pi_1' \oplus \pi_2'$ is unitarily equivalent to $\pi_1 \oplus \pi_2$, then $n'=n$ and π_2' is unitarily equivalent to π_2 [5].

We will make use of these observations in a form that relates more directly to operator theory.

Definition 8.2. Let $d=1,2,\ldots$. By a spherical operator (of dimension d) we mean a d-tuple $(Z_1,\ldots,Z_d)$ of commuting normal operators acting on a common Hilbert space such that

$$Z_1^* Z_1 + \ldots + Z_d^* Z_d = \mathbf{1}.$$

Spherical operators are the higher-dimensional counterparts of unitary operators. For every spherical operator $(Z_1,\ldots,Z_d)$ acting on H there is a unique unital $*$-representation $\pi: C(\partial B_d) \to \mathcal{B}(H)$ which carries the d-tuple of canonical coordinate functions to $(Z_1,\ldots,Z_d)$. This relation between d-dimensional spherical operators and nondegenerate representations of $C(\partial B_d)$ is bijective.

If $\bar{T}=(T_1,\ldots,T_d)$ is an arbitrary d-tuple of operators acting on a common Hilbert space H and n is a nonnegative integer or $+\infty$ we will write $n\cdot\bar{T}=(n\cdot T_1,\ldots,n\cdot T_d)$ for the d-tuple of operators acting on the direct sum of n copies of H defined by

$$n\cdot T_k = \underbrace{T_k \oplus T_k \oplus \ldots}_{n \text{ times}},$$

where for $n=0$ the left side is interpreted as the *nil* operator, that is, no operator at all. The direct sum of two d-tuples of operators is defined in the obvious way as a d-tuple acting on the direct sum of Hilbert spaces. The preceding remarks are summarized as follows.

PROPOSITION 8.3. *Let $(n,\bar{Z})$ be a pair consisting of an integer $n=0,1,2,\ldots,\infty$ and a spherical operator $\bar{Z}=(Z_1,\ldots,Z_d)$ (which may be the nil d-tuple when $n\geqslant 1$). Then there is a unique nondegenerate representation π of $\mathcal{T}_d$ satisfying*

$$\pi(S_k) = n\cdot S_k \oplus Z_k, \quad k=1,\ldots,d.$$

Every nondegenerate representation of $\mathcal{T}_d$ on a separable Hilbert space arises in this way, and if $(n',\bar{Z}')$ is another such pair giving rise to a representation π', then π' is unitarily equivalent to π if and only if $n'=n$ and $\bar{Z}'$ is unitarily equivalent to $\bar{Z}$.

Remarks. Of course, if $\bar{Z}$ is the nil d-tuple then its corresponding summand in the definition of π is absent. Let $\mathcal{S}\subseteq\mathcal{B}(H)$ be a set of operators acting on a Hilbert space H. A subspace $K\subseteq H$ is said to be *co-invariant* under $\mathcal{S}$ if $\mathcal{S}^*K\subseteq K$. K is co-invariant if and only if its orthogonal complement is invariant, $\mathcal{S}K^\perp\subseteq K^\perp$. A co-invariant subspace K is called *full* if H is spanned by $\{T\xi:\xi\in K\}$ where T ranges over the C^*-algebra generated by $\mathcal{S}$. The following are equivalent for any co-invariant subace K:

(8.4.1) K is full.

(8.4.2) H is the smallest reducing subspace for $\mathcal{S}$ which contains K.

(8.4.3) For every operator T in the commutant of $\mathcal{S}\cup\mathcal{S}^*$ we have

$$TK=\{0\} \;\Rightarrow\; T=0.$$

Let $\mathcal{A}$ be the algebra generated by $\mathcal{S}$ and the identity. We will often have a situation in which the C^*-algebra generated by $\mathcal{A}$ is spanned by the set of products $\mathcal{A}\mathcal{A}^*$, and in that case the following criterion can be added to the preceding list.

(8.4.4) H is the smallest invariant subspace for $\mathcal{S}$ which contains K.

Indeed, since $C^*(\mathcal{A})$ is spanned by $\mathcal{A}\mathcal{A}^*$ we have

$$\overline{\operatorname{span}}\, C^*(\mathcal{A})K=\overline{\operatorname{span}}\,\mathcal{A}\mathcal{A}^*K=\overline{\operatorname{span}}\,\mathcal{A}K,$$

and hence (8.4.1) and (8.4.4) are equivalent.

Since the d-shift is a d-contraction, any d-tuple $(T_1,\dots,T_d)$ of the form

$$T_k=n\cdot S_k\oplus Z_k$$

described in Proposition 8.3 is a d-contraction. If K is any co-invariant subspace for $\{T_1,\dots,T_d\}$ then the d-tuple $(T_1',\dots,T_d')$ obtained by compressing to K,

$$T_k'=P_KT_k\restriction_K,$$

is also a d-contraction. Indeed, for each $k=1,\dots,d$ we have

$$T_k'T_k'^*=P_KT_kP_KT_k^*\restriction_K\leqslant P_KT_kT_k^*\restriction_K,$$

and therefore $\sum_k T_k'T_k'^*\leqslant\mathbf{1}$. The following implies that d-tuples obtained from this construction are the most general d-contractions.

THEOREM 8.5. *Let $d=1,2,\dots$, let $\bar{T}=(T_1,\dots,T_d)$ be a d-contraction acting on a separable Hilbert space and let $\bar{S}=(S_1,\dots,S_d)$ be the d-shift. Then there is a triple $(n,\bar{Z},K)$ consisting of an integer $n=0,1,2,\dots,\infty$, a spherical operator $\bar{Z}$, and a full co-invariant subspace K for the operator*

$$n\cdot\bar{S}\oplus\bar{Z}$$

such that $\bar{T}$ is unitarily equivalent to the compression of $n\cdot\bar{S}\oplus\bar{Z}$ to K.

Let $T'=(T'_1,\dots,T'_d)$ be another d-contraction associated with another such triple $(n',\bar{Z}',K')$. If $\bar{T}$ and $\bar{T}'$ are unitarily equivalent then $n'=n$, and there are unitary operators $V\in\mathcal{B}(n\cdot H^2)$ and $W: H_{\bar{Z}}\to H_{\bar{Z}'}$ such that for $k=1,\dots,d$ we have

$$VS_k=S_kV, \quad WZ_k=Z'_kW,$$

and which relate K to K' by way of $(V\oplus W)K=K'$.

Finally, the integer n is the rank of the defect operator

$$\mathbf{1}-T_1T_1^*-\dots-T_dT_d^*,$$

and $\bar{Z}$ is the nil spherical operator if and only if $\bar{T}$ is a pure d-contraction.

Remark 8.6. Notice that the situation of (8.4.4) prevails in this case, and we may conclude that for the triple $(n,\bar{Z},K)$ associated with $\bar{T}$ by Theorem 8.5, the Hilbert space $\widetilde{H}$ on which $n\cdot\bar{S}\oplus\bar{Z}$ acts is generated as

$$\widetilde{H}=\overline{\operatorname{span}}\,\{f(n\cdot S_1\oplus Z_1,\dots,n\cdot S_d\oplus Z_d)\xi:\xi\in K,\, f\in\mathcal{P}\},$$

$\mathcal{P}$ denoting the set of all polynomials in d complex variables.

Before giving the proof of Theorem 8.5 we want to emphasize the following general observation which asserts that, under certain conditions, a unitary operator which intertwines two representations of a subalgebra $\mathcal{A}$ of a C^*-algebra $\mathcal{B}$ can be extended to a unitary operator which intertwines $*$-representations of $\mathcal{B}$.

We recall a general theorem of Stinespring, which asserts that every completely positive map

$$\phi: B\to\mathcal{B}(H)$$

defined on a unital C^*-algebra B can be represented in the form $\phi(x)=V^*\pi(x)V$, where π is a representation of B on another Hilbert space H_π, and $V\in\mathcal{B}(H,H_\pi)$. The pair (V,π) is called *minimal* if

$$H_\pi=\overline{\operatorname{span}}\,[\pi(x)\xi:x\in B,\,\xi\in H].$$

One can always arrange that (V,π) is minimal by cutting down to a suitable subrepresentation of π.

LEMMA 8.6. *Let B be a C^*-algebra and let A be a (perhaps non-self-adjoint) subalgebra of B such that*

$$B=\overline{\operatorname{span}}^{\|\cdot\|}AA^*. \tag{8.7}$$

For $k=1,2$ let $\phi_k: B \to \mathcal{B}(H_k)$ be A-morphisms, and let $U: H_1 \to H_2$ be a unitary operator such that

$$U\phi_1(a) = \phi_2(a)U, \quad a \in A.$$

Let (V_k, π_k) be a minimal Stinespring pair for ϕ_k, $\phi_k(x) = V_k^* \pi_k(x) V_k$, $x \in B$. Then there is a unique unitary operator $W: H_{\pi_1} \to H_{\pi_2}$ such that

(i) $W\pi_1(x) = \pi_2(x)W$, $x \in B$, *and*

(ii) $WV_1 = V_2 U$.

Proof. Since both ϕ_1 and ϕ_2 are $\mathcal{A}$-morphisms, the hypothesis on U implies that $U\phi_1(ab^*) = \phi_2(ab^*)U$ for all $a, b \in \mathcal{A}$. Hence (8.7) implies that $U\phi_1(x) = \phi_2(x)U$ for every $x \in B$. The rest now follows from standard uniqueness assertions about minimal completely positive dilations of completely positive maps of C^*-algebras [3]. □

Remark. There are many examples of subalgebras A of C^*-algebras B that satisfy (8.7) besides the algebra $\mathcal{A}$ of polynomials in the Toeplitz algebra $\mathcal{T}_d$. Indeed, if $\mathcal{A}$ is any algebra of operators on a Hilbert space which satisfies

$$\mathcal{A}^* \mathcal{A} \subseteq \mathcal{A} + \mathcal{A}^*$$

then the linear span of $\mathcal{A}\mathcal{A}^*$ is closed under multiplication, and hence the norm-closed linear span of $\mathcal{A}\mathcal{A}^*$ is a C^*-algebra. Such examples arise in the theory of E_0-semigroups [6], and in the Cuntz C^*-algebras $\mathcal{O}_n$, $n=2, \ldots, \infty$.

Proof of Theorem 8.5. Suppose that the operators T_k act on a Hilbert space H. Let $\mathcal{A}$ be the algebra of all polynomials in the d-shift $\bar{S} = (S_1, \ldots, S_d)$. By Theorem 6.3 there is an $\mathcal{A}$-morphism

$$\phi: \mathcal{T}_d \to \mathcal{B}(H)$$

such that $\phi(S_k) = T_k$ for $k = 1, \ldots, d$. Let

$$\phi(X) = V^* \pi(X) V, \quad X \in \mathcal{T}_d,$$

be a minimal Stinespring representation of ϕ. We have

$$V^* V = V^* \pi(\mathbf{1}) V = \phi(\mathbf{1}) = \mathbf{1},$$

and hence V is an isometry.

We claim that VH is co-invariant under $\pi(\mathcal{A})$,

$$\pi(\mathcal{A})^* VH \subseteq VH. \tag{8.8}$$

Indeed, if $A\in\mathcal{A}$ and P denotes the projection $P=VV^*$ then for every $X\in\mathcal{T}_d$ we have

$$P\pi(A)P\pi(X)V=V\phi(A)\phi(X)=V\phi(AX)=P\pi(AX)V=P\pi(A)\pi(X)V,$$

and hence the operator $P\pi(A)P-P\pi(A)$ vanishes on

$$\overline{\operatorname{span}}\,[\pi(X)\xi : X\in\mathcal{T}_d,\, \xi\in H]=H_\pi.$$

Thus $\pi(A)^*P=P\pi(A)^*P$, and (8.8) follows.

Because of minimality of (V,π) it follows that the subspace $K=VH\subseteq H_\pi$ is a *full* co-invariant subspace for the operator algebra $\pi(\mathcal{A})$.

Proposition 8.3 shows that if we replace π with a unitarily equivalent representation and adjust V accordingly then we may assume that there is an integer $n=0,1,2,...,\infty$ and a (perhaps nil) spherical operator $\bar{Z}=(Z_1,...,Z_d)$ such that $H_\pi=n\cdot H^2\oplus H_{\bar{Z}}$ and

$$\pi(S_k)=n\cdot S_k\oplus Z_k,\quad k=1,...,d.$$

That proves the first paragraph of Theorem 8.5.

The second paragraph follows after a straightforward application of Lemma 8.6, once one notes that if we are given two triples $(n,\bar{Z},K)$ and $(n',\bar{Z}',K')$, and we define representations π and π' of $\mathcal{T}_d$ by

$$\begin{aligned}\pi(S_k)&=n\cdot S_k\oplus Z_k=\sigma_1(S_k)\oplus\sigma_2(S_k),\\ \pi'(S_k)&=n'\cdot S_k\oplus Z_k'=\sigma_1'(S_k)\oplus\sigma_2'(S_k),\end{aligned}$$

then σ_1 is disjoint from σ_2, σ_1' is disjoint from σ_2', while σ_k is quasi-equivalent to σ_k'. Thus, any unitary operator W which intertwines the representations π and π' must decompose into a direct sum $W=W_1\oplus W_2$ where W_1 intertwines σ_1 and σ_1', and W_2 intertwines σ_2 and σ_2'.

To prove the third paragraph, choose an integer $n=0,1,2,...,\infty$, let $\bar{Z}=(Z_1,...,Z_d)$ be a spherical operator whose component operators act on a Hilbert space L, and let $K\subseteq n\cdot H^2\oplus L$ be a full co-invariant subspace for the operator

$$n\cdot\bar{S}\oplus\bar{Z},$$

where $\bar{S}=(S_1,...,S_d)$ is the d-shift. Define $\bar{T}=(T_1,...,T_d)$ by

$$T_j=P_K(n\cdot S_j\oplus Z_j)\restriction_K,$$

$j=1,...,d$. We have to identify the multiplicity n and the existence of the spherical summand $\bar{Z}$ in terms of $\bar{T}$.

Let $P_K \in \mathcal{B}(n \cdot H^2 \oplus L)$ denote the projection on K. Since K is co-invariant under $n \cdot \bar{S} \oplus \bar{Z}$ we have

$$P_K(n \cdot S_j \oplus Z_j) = P_K(n \cdot S_j \oplus Z_j) P_K = T_j P_K$$

for every $j=1, ..., d$, and hence

$$T_j T_j^* = P_K(n \cdot S_j S_j^* \oplus Z_j Z_j^*) \restriction_K . \tag{8.9}$$

By the remarks following Definition 2.10 we may sum on j to obtain

$$\sum_{j=1}^{d} T_j T_j^* = P_K(n \cdot (\mathbf{1} - E_0) \oplus \mathbf{1}_L) \restriction_K = \mathbf{1}_K - P_K(n \cdot E_0 \oplus 0) \restriction_K \tag{8.10}$$

where $E_0 \in \mathcal{B}(H^2)$ denotes the one-dimensional projection onto the constants.

From (8.10) we find that the defect operator D has the form

$$D = \mathbf{1}_K - T_1 T_1^* - ... - T_d T_d^* = P_K(n \cdot E_0 \oplus 0) \restriction_K . \tag{8.11}$$

Now for any positive operator B we have $B\xi = 0$ if and only if $\langle B\xi, \xi \rangle = 0$. Thus the relation (8.11) between the positive operators D and $n \cdot E_0 \oplus 0$ implies that their kernels are related by

$$\{\xi \in K : D\xi = 0\} = \{\xi \in K : (n \cdot E_0 \oplus 0)\xi = 0\},$$

and hence

$$\operatorname{rank} D = \dim((n \cdot E_0 \oplus 0)K).$$

The dimension of the space $N = (n \cdot E_0 \oplus 0)K$ is easily seen to be n. Indeed, notice that if $A \in \mathcal{B}(H^2)$ is a polynomial in the operators $S_1, ..., S_d$ then we have $E_0 A = E_0 A E_0 = \langle A1, 1 \rangle E_0$, and hence $E_0 A$ is a scalar multiple of E_0. Similarly, if $B \in \mathcal{B}(n \cdot H^2 \oplus L)$ is a polynomial in the operators $n \cdot S_1 \oplus Z_1, ..., n \cdot S_d \oplus Z_d$ then $(n \cdot E_0 \oplus 0)B$ is a scalar multiple of $(n \cdot E_0 \oplus 0)$, and hence for all such B we have

$$(n \cdot E_0 \oplus 0)BK \subseteq N.$$

Because K is a full co-invariant subspace, (8.4.4) implies that $n \cdot H^2 \oplus L$ is spanned by vectors of the form $B\xi$, with B as above and $\xi \in K$. It follows that

$$(n \cdot E_0 \oplus 0)(n \cdot H^2 \oplus L) \subseteq N,$$

and therefore N is the range of the n-dimensional projection $n \cdot E_0 \oplus 0$. Hence $\dim N = n$.

Finally, we consider the case in which $\bar{T}$ is a pure d-contraction. Let Q and P be the completely positive maps on $\mathcal{B}(H^2)$ and $\mathcal{B}(K)$ given respectively by

$$P(A) = S_1 A S_1^* + \ldots + S_d A S_d^*, \quad A \in \mathcal{B}(H^2),$$
$$Q(B) = T_1 B T_1^* + \ldots + T_d B T_d^*, \quad B \in \mathcal{B}(K).$$

Formula (8.9) implies that $Q(\mathbf{1}_K) = P_K(n \cdot P(\mathbf{1}_{H^2}) \oplus \mathbf{1}_L) \restriction_K$. Similarly, using co-invariance of K repeatedly as in (8.9) we have

$$T_{j_1} \ldots T_{j_r} T_{j_r}^* \ldots T_{j_1}^* = P_K(n \cdot (S_{j_1} \ldots S_{j_r} S_{j_r}^* \ldots S_{j_1}^*) \oplus Z_{j_1} \ldots Z_{j_r} Z_{j_r}^* \ldots Z_{j_1}^*) \restriction_K$$

for every $j_1, ..., j_r \in \{1, ..., d\}$. After summing on $j_1, ..., j_r$ we obtain

$$Q^r(\mathbf{1}_K) = P_K(n \cdot P^r(\mathbf{1}_{H^2}) \oplus \mathbf{1}_L) \restriction_K, \quad r = 1, 2, \ldots .$$

Since $P^r(\mathbf{1}_{H^2}) \downarrow 0$ as $r \to \infty$, we have

$$\lim_{r \to \infty} Q^r(\mathbf{1}_K) = P_K(0 \oplus \mathbf{1}_L) \restriction_K .$$

We conclude that $\bar{T}$ is a pure d-tuple if and only if $0 \oplus L \perp K$, that is, $K \subseteq n \cdot H^2 \oplus \{0\}$. Noting that $n \cdot H^2 \oplus \{0\}$ is a reducing subspace for the operator $n \cdot \bar{S} \oplus \bar{Z}$ we see from (8.4.2) that

$$n \cdot H^2 \oplus L \subseteq n \cdot H^2 \oplus \{0\},$$

and therefore $L = \{0\}$. But a spherical d-tuple cannot be the zero d-tuple except when it is the nil d-tuple, and thus we have proved that $\bar{T}$ is a pure d-contraction if and only if $\bar{Z}$ is nil. □

The two extreme cases of Theorem 8.5 in which $n=0$ and $n=1$ are noteworthy. From the case $n=0$ we deduce the following result of Athavale [11], which was established by entirely different methods.

COROLLARY 1. *Let $T_1, ..., T_d$ be a set of commuting operators on a Hilbert space H such that $T_1^* T_1 + ... + T_d^* T_d = \mathbf{1}$. Then $(T_1, ..., T_d)$ is a subnormal d-tuple.*

Proof. Let $A_k = T_k^*$. $(A_1, ..., A_d)$ is a d-contraction for which

$$n = \operatorname{rank}(\mathbf{1} - A_1 A_1^* - \ldots - A_d A_d^*) = 0.$$

Theorem 8.5 implies that there is a spherical operator $\bar{Z} = (Z_1, ..., Z_d)$ acting on a Hilbert space $\widetilde{H} \supseteq H$ such that $Z_k^* H \subset H$ and A_k is the compression of Z_k to H, $k = 1, ..., d$. Hence $T_k = A_k^* = Z_k^* \restriction_H$ for every k, so that $(Z_1^*, ..., Z_d^*)$ is a normal d-tuple which extends $(T_1, ..., T_d)$ to a larger Hilbert space. □

From the case $n=1$ we have the following description of all d-contractions that can be obtained by compressing the d-shift to a co-invariant subspace.

COROLLARY 2. *Every nonzero co-invariant subspace $K\subseteq H^2$ for the d-shift $\bar{S}=(S_1,\dots,S_d)$ is full, and the compression of $\bar{S}$ to K,*

$$T_k=P_K S_k\restriction_K,\quad k=1,\dots,d,$$

defines a pure d-contraction $\bar{T}=(T_1,\dots,T_d)$ for which

$$\operatorname{rank}(\mathbf{1}-T_1T_1^*-\dots-T_dT_d^*)=1. \tag{8.12}$$

If K' is another co-invariant subspace for $\bar{S}$ which gives rise to $\bar{T}'$, then $\bar{T}$ and $\bar{T}'$ are unitarily equivalent if and only if $K=K'$.

Every pure d-contraction $(T_1,\dots,T_d)$ satisfying (8.12) *is unitarily equivalent to one obtained by compressing $(S_1,\dots,S_d)$ to a co-invariant subspace of H^2.*

Proof. Let $\{0\}\neq K\subseteq H^2$ be a a co-invariant subspace for the set of operators $\{S_1,\dots,S_d\}$. Since $\mathcal{T}_d$ is an irreducible C^*-algebra it follows that K satisfies condition (8.4.2), hence it is full. Let T_j be the compression of S_j to K, $j=1,\dots,d$. The canonical triple associated with $\bar{T}=(T_1,\dots,T_d)$ is therefore $(1,\mathrm{nil},K)$, and the third paragraph of Theorem 8.5 implies that $\bar{T}$ is a pure d-contraction satisfying (8.12).

If K' is another co-invariant subspace of H^2 giving rise to a d-contraction $\bar{T}'$ which is unitarily equivalent to $\bar{T}$ then Theorem 8.5 implies that there is a unitary operator V which commutes with $\mathcal{S}=\{S_1,\dots,S_d\}$ such that $VK=K'$. Because V is unitary it must commute with $\mathcal{S}^*$ as well, and hence with the Toeplitz algebra $\mathcal{T}_d$. The latter is irreducible, hence V must be a scalar multiple of the identity operator, hence $K'=K$.

Finally, if $\bar{T}=(T_1,\dots,T_d)$ is any pure d-contraction then Theorem 8.5 implies that the spherical summand $\bar{Z}$ of its dilation must be the nil d-tuple, and if in addition

$$\operatorname{rank}(\mathbf{1}-T_1T_1^*-\dots-T_dT_d^*)=1,$$

then the canonical triple associated with $\bar{T}$ is $(1,\mathrm{nil},K)$ for some subspace K of H^2 which is co-invariant under the d-shift. □

Lemma 7.13 asserts that the identity representation of the Toeplitz C^*-algebra is a boundary representation for the unital operator space generated by the d-shift. This fact has a number of significant consequences, and we conclude with a brief discussion of two of them. Rudin posed the following function-theoretic problem in the sixties: Do there exist nonconstant inner functions in $H^\infty(B_d)$ [37]? This problem was finally solved (affirmatively) in 1982 by B. A. Aleksandrov [38]. The following proposition implies that the answer to the analogue of Rudin's question for the multiplier algebra $\mathcal{M}$ is the opposite: there are no nontrivial isometries in $\mathcal{B}(H^2)$ which commute with $\{S_1,\dots,S_d\}$ when $d\geqslant 2$. Indeed, we have the following more general assertion.

PROPOSITION 8.13. *Let $T_1, T_2, \dots$ be a finite or infinite sequence of operators on H^2_d, $d \geqslant 2$, which commute with the d-shift and which satisfy*

$$T_1^* T_1 + T_2^* T_2 + \dots = \mathbf{1}. \tag{8.14}$$

Then each T_j is a scalar multiple of the identity operator.

Proof. Consider the completely positive linear map ϕ defined on $\mathcal{B}(H^2)$ by

$$\phi(A) = T_1^* A T_1 + T_2^* A T_2 + \dots .$$

The sum converges strongly for every operator A because by (8.14) we have

$$\|T_1 \xi\|^2 + \|T_2 \xi\|^2 + \dots = \|\xi\|^2 < \infty, \quad \xi \in H^2.$$

Moreover, since each T_k commutes with each S_j we have $T_k^* S_j T_k = T_k^* T_k S_j$, and thus from (8.14) we conclude that $\phi(A) = A$ for every A in $\mathcal{S} = \operatorname{span}\{\mathbf{1}, S_1, \dots, S_d\}$. Since the identity representation of $\mathcal{T}_d$ is a boundary representation for $\mathcal{S}$ it follows that $\phi(A) = A$ for every A in the Toeplitz C^*-algebra $\mathcal{T}_d$.

Let n be the number of operators in the sequence $T_1, T_2, \dots$ and let V be the linear map of H^2 to $n \cdot H^2$ defined by

$$V\xi = (T_1 \xi, T_2 \xi, \dots).$$

Because of (8.14), V is an isometry. Letting π be the representation of $\mathcal{B}(H^2)$ on $n \cdot H^2$ defined by

$$\pi(A) = A \oplus A \oplus \dots,$$

we find that (V, π) is a Stinespring pair for ϕ,

$$\phi(A) = V^* \pi(A) V, \quad A \in \mathcal{B}(H^2).$$

Since

$$(VA - \pi(A)V)^*(VA - \pi(A)V) = A^* \phi(\mathbf{1}) A - \phi(A)^* A - A^* \phi(A) + \phi(A^* A) = 0,$$

we conclude that $VA - \pi(A)V = 0$. By examining the components of this operator equation one sees that $T_k A = A T_k$ for every k and every $A \in \mathcal{T}_d$. Since $\mathcal{T}_d$ is an irreducible C^*-algebra it follows that each T_k must be a scalar multiple of the identity operator. □

Finally, we offer a few remarks about C^*-envelopes, that is to say, noncommutative Silov boundaries (see the discussion preceding Lemma 7.13).

THEOREM 8.15. *The Toeplitz algebra $\mathcal{T}_d$ is the C^*-envelope of the commutative algebra $\mathcal{A}$ of all polynomials in the d-shift $(S_1, ..., S_d)$. Moreover, every irreducible representation of $\mathcal{T}_d$ is a boundary representation for $\mathcal{A}$.*

Proof. Lemma 7.13 implies that the intersection of the kernels of all boundary representations for $\mathcal{A}$ is $\{0\}$, and the first assertion follows.

The irreducible representations of $\mathcal{T}_d$ are easily identified using Proposition 8.3. In addition to the identity representation (and other members of its unitary equivalence class) there are the one-dimensional representations corresponding to points of the boundary ∂B_d. One may verify directly that the latter are boundary representations. □

Remarks. $\mathcal{T}_d$ is generated as a C^*-algebra by two other natural abelian subalgebras, namely the algebra of all multiplications by polynomials in the Hardy space of the boundary $H^2(\partial B_d)$, or by the corresponding algebra acting on the Bergman space $H^2(B_d)$ of the interior. However, in both of the latter cases the C^*-envelopes are not $\mathcal{T}_d$ but rather its commutative quotient C^*-algebra

$$\mathcal{T}_d/\mathcal{K} = C(\partial B_d).$$

Appendix A. Trace estimates

Fix $d=1, 2, ...$, let E_d be a d-dimensional Hilbert space, and let

$$\mathcal{F}_+(E_d) = \mathbf{C} \oplus E_d \oplus E_d^2 \oplus \ldots$$

be the symmetric Fock space over E_d. The number operator is the unbounded self-adjoint diagonal operator N satisfying $N\xi = n\xi$, $\xi \in E_d^n$, $n=0, 1, \ldots$. Let P_n be the projection on E_d^n. Then for every $p>0$, $(\mathbf{1}+N)^{-p}$ is a positive compact operator,

$$(\mathbf{1}+N)^{-p} = \sum_{n=0}^{\infty} (n+1)^{-p} P_n,$$

whose trace is given by

$$\operatorname{trace}(\mathbf{1}+N)^{-p} = \sum_{n=0}^{\infty} \frac{\dim E_d^n}{(n+1)^p}. \tag{A.1}$$

Thus $(\mathbf{1}+N)^{-1}$ belongs to the Schatten class $\mathcal{L}^p(\mathcal{F}_+(E_d))$ if and only if the infinite series (A.1) converges. In this appendix we show that that is the case if and only if $p>d$. Notice that the function of a complex variable defined for $\operatorname{Re} z > d$ by

$$\zeta_d(z) = \operatorname{trace}(\mathbf{1}+N)^{-z}$$

is a d-dimensional variant of the Riemann zeta function, since for $d=1$ we have $\dim E_d^n=1$ for all n and hence

$$\zeta_1(z)=\sum_{n=1}^{\infty}\frac{1}{n^z}.$$

We calculate the generating function for the coefficients $\dim E_d^n$.

LEMMA A.2. *The numbers $a_{n,d}=\dim E_d^n$ are the coefficients of the series expansion*

$$(1-z)^{-d}=\sum_{n=0}^{\infty}a_{n,d}z^n, \quad |z|<1.$$

Proof. Note that the numbers $a_{n,d}$ satisfy the recurrence relation

$$a_{n,d+1}=a_{0,d}+a_{1,d}+\ldots+a_{n,d}, \quad n=0,1,\ldots,\ d=1,2,\ldots\,. \tag{A.3}$$

Indeed, if we choose a basis $e_1,\ldots,e_d$ for E_d then the set of symmetric products

$$\{e_{i_1}e_{i_2}\ldots e_{i_n}:1\leqslant i_1\leqslant\ldots\leqslant i_n\leqslant d\}$$

forms a basis for the vector space E_d^n, and hence $a_{n,d}$ is the cardinality of the set

$$S_{n,d}=\{(i_1,\ldots,i_n)\in\{1,\ldots,d\}^n:1\leqslant i_1\leqslant\ldots\leqslant i_n\leqslant d\}.$$

Since $S_{n,d+1}$ decomposes into a disjoint union

$$S_{n,d+1}=\bigsqcup_{k=0}^{n}\{(i_1,\ldots,i_n)\in S_{n,d+1}:i_k\leqslant d,\ i_{k+1}=\ldots=i_n=d+1\},$$

and since the kth set on the right has the same cardinality $a_{k,d}$ as $S_{k,d}$, (A.3) follows.

From (A.3) we find that $a_{n,d+1}-a_{n-1,d+1}=a_{n,d}$. Thus if we let f_d be the formal power series

$$f_d(z)=\sum_{n=0}^{\infty}a_{n,d}z^n \tag{A.4}$$

then $f_{d+1}(z)-zf_{d+1}(z)=f_d(z)$, and hence

$$f_{d+1}(z)=\frac{f_d(z)}{1-z}.$$

Lemma A.2 follows after noting that $f_1(z)=1+z+z^2+\ldots=(1-z)^{-1}$. $\square$

Remark. Notice that the power series of Lemma A.2 converges absolutely to the generating function $(1-z)^{-d}$ throughout the open unit disk $|z|<1$.

By evaluating successive derivatives of the generating function at the origin, we find that

$$\dim E_d^n = \frac{(n+d-1)(n+d-2)\dots d}{n!} = \frac{(n+d-1)!}{n!\,(d-1)!}. \tag{A.5}$$

A straightforward application of Stirling's formula [36, p. 194]

$$N! \sim \sqrt{2\pi}\, N^{N+1/2} e^{-N}$$

leads to

$$\lim_{n\to\infty} (n+1)^{-d+1} \frac{(n+d-1)!}{n!} = \frac{1}{(d-1)!},$$

and hence

$$\dim E_d^n \sim \frac{(n+1)^{d-1}}{(d-1)!}. \tag{A.6}$$

We now prove the assertion of (5.2).

THEOREM. *For* $p>0$ *we have* $\operatorname{trace}(\mathbf{1}+N)^{-p}<\infty$ *if and only if* $p>d$.

Proof. By (A.6), the infinite series

$$\operatorname{trace}(\mathbf{1}+N)^{-p} = \sum_{n=0}^{\infty} \frac{\dim E_d^n}{(n+1)^p}$$

converges if and only if the series

$$\sum_{n=0}^{\infty} \frac{1}{(n+1)^{p-d+1}}$$

converges; i.e., if and only if $p>d$. □

Appendix B. Quasinilpotent operator spaces

In this appendix we prove that if $\mathcal{S}$ is a finite-dimensional operator space generated by commuting quasinilpotent operators then the energy sequence is itself quasinilpotent in the sense that

$$\lim_{n\to\infty} E_n(\mathcal{S})^{1/n} = 0. \tag{B.1}$$

In particular, for such an operator space we must have $\lim_{n\to\infty} E_n(\mathcal{S})=0$.

We first show that if $\mathcal{S}$ is an arbitrary operator space of finite dimension d, then the energy sequence can be defined in terms of d-tuples (rather than arbitrarily long sequences in $\operatorname{seq}(\mathcal{S})$). Indeed, for every $n=1,2,\dots$ we claim that

$$E_n(\mathcal{S}) = \sup \|\bar{T}^n\|_{\mathrm{col}}^2 \tag{B.2}$$

where the sup on the right is taken over all d-tuples $\bar{T}=(T_1, ..., T_d)$ with components in $\mathcal{S}$ which satisfy $\|\bar{T}\|_{\text{row}} \leqslant 1$. In view of the description of $E_n(\mathcal{S})$ in terms of completely positive maps (see 7.4), the formula (B.2) is an immediate consequence of the following observation. We remark that the relationship between completely positive maps of $\mathcal{B}(H)$ and the theory of operator spaces is developed more fully in [8].

LEMMA B.3. *Let $\mathcal{S}$ be a finite-dimensional operator space, let $T_1, T_2, ... T_m \in \mathcal{S}$ be a finite sequence of elements of $\mathcal{S}$ and let ϕ be the completely positive map of $\mathcal{B}(H)$ defined by $\phi(X)=T_1XT_1^*+...+T_mXT_m^*$. Then there is a linearly independent set $T_1', ..., T_r'$ in $\mathcal{S}$, $r \leqslant \dim(\mathcal{S})$, such that*

$$\phi(X)=\sum_{k=1}^{r} T_d' X T_k'^*, \quad X \in \mathcal{B}(H).$$

Proof. Let $m \cdot H$ denote the direct sum of m copies of the underlying Hilbert space H, and define an operator $V \in \mathcal{B}(H, m \cdot H)$ by

$$V\xi=(T_1^*\xi, T_2^*\xi, ..., T_m^*\xi).$$

If $\tilde{\pi}(X)=X \oplus ... \oplus X$ is the natural representation of $\mathcal{B}(H)$ on $m \cdot H$, then we have

$$\phi(X)=V^*\tilde{\pi}(X)V, \quad X \in \mathcal{B}(H).$$

Let π be the subrepresentation of $\tilde{\pi}$ defined by restricting it to the invariant subspace

$$K=[\tilde{\pi}(X)\xi : X \in \mathcal{B}(H), \xi \in H].$$

Then $\phi(X)=V^*\pi(X)V$ is a minimal Stinespring representation of the completely positive map ϕ.

π is a normal representation of $\mathcal{B}(H)$, and therefore the projection onto K can be decomposed into an orthogonal sum

$$P_K=E_1+E_2+...$$

of minimal projections E_j in the commutant of $\tilde{\pi}(\mathcal{B}(H))$. For each j let $U_j: H \to K$ be an isometry satisfying $U_jU_j^*=E_j$ and $U_jX=\tilde{\pi}(X)U_j$ for $X \in \mathcal{B}(H)$.

$\{U_1, U_2, ...\}$ is of course a linearly independent set of operators. Set $T_j'=V^*U_j$. We claim that $\{T_1', T_2', ...\}$ is a linearly independent set of operators in $\mathcal{S}$ for which

$$\phi(X)=T_1'XT_1'^*+T_2'XT_2'^*+..., \quad X \in \mathcal{B}(H).$$

Indeed, since $U_jX=\tilde{\pi}(X)U_j$ for all $X \in \mathcal{B}(H)$, U_j must have the form

$$U_j\xi=(\lambda_j^1\xi, \lambda_j^2\xi, ...)$$

for some sequence of scalars $(\lambda_j^1, \lambda_j^2, ...)$. Hence $T_j' = \sum_k \lambda_j^k T_k \in \mathcal{S}$. To see that the $\{T_j\}$ are linearly independent, choose $c_1, ..., c_s \in \mathbf{C}$ such that $c_1 T_1' + ... + c_s T_s' = 0$. Then for every $\xi \in H$ and every $X \in \mathcal{B}(H)$ we have

$$V^* \tilde{\pi}(X) \sum_j c_j U_j \xi = \sum_j c_j V^* \tilde{\pi}(X) U_j \xi = \sum_j c_j V^* U_j X \xi = \sum_j c_j T_j' X \xi = 0.$$

By taking the inner product with a vector of the form $V\zeta$ for $\zeta \in H$ we find that $(\sum c_j U_j)\xi$ is orthogonal to all vectors in $m \cdot H$ of the form $\tilde{\pi}(X^*)V\zeta$. Since the latter vectors span K and since $(\sum c_j U_j)\xi$ belongs to K, it follows that $\sum_j c_j U_j = 0$, and hence $c_1 = ... = c_s = 0$.

In particular, there are at most $d = \dim(\mathcal{S})$ elements in the set $\{T_1', T_2', ...\}$. Finally,

$$\sum_k T_k' X T_k'^* = \sum_k V^* U_k X U_k^* V = \sum_k V^* \pi(X) E_k V = V^* \pi(X) V = \phi(X),$$

because $\sum_k E_k = P_K$ and $P_K V = V$. □

Turning now to the proof of (B.1), let $A_1, ..., A_d$ be a linearly independent commuting set of quasinilpotent operators and consider the operator space

$$\mathcal{S} = \{a_1 A_1 + ... + a_d A_d : a_1, ..., a_d \in \mathbf{C}\}.$$

Formula (B.2) implies that, in order to estimate $E_n(\mathcal{S})$, we may confine attention to sequences $T_1, ..., T_d \in \mathcal{S}$ of length d which satisfy

$$\left\| \sum_{k=1}^d T_k T_k^* \right\| \leqslant 1, \tag{B.4}$$

and for such a sequence we must find appropriate estimates of the norms

$$\left\| \sum_{i_1, ..., i_n = 1}^d T_{i_1}^* ... T_{i_n}^* T_{i_n} ... T_{i_1} \right\|, \quad n = 1, 2, ...,$$

independently of the particular choice of $T_1, ..., T_d$ satisfying (B.4).

This is done as follows. Since $A_1, ..., A_d$ are linearly independent, we can define a positive constant K by

$$K = \sup\{|a_1| + ... + |a_d| : \|a_1 A_1 + ... + a_d A_d\| \leqslant 1\}.$$

Choose a sequence $T_1, ..., T_d \in \mathcal{S}$ satisfying (B.4). Then there is a $(d \times d)$-matrix (a_{ij}) such that

$$T_i = \sum_{j=1}^d a_{ij} A_j.$$

Since $T_1, ..., T_d$ satisfy (B.4) we have

$$\|T_i\|^2 = \|T_i T_i^*\| \leqslant \|T_1 T_1^* + ... + T_d T_d^*\| \leqslant 1$$

for every $i=1, 2, ..., d$, and hence

$$\sum_{j=1}^{d} |a_{ij}| \leqslant K, \quad i = 1, 2, ..., d.$$

It follows that for every i, j we have

$$\|T_i T_j\| \leqslant \sum_{p,q=1}^{d} |a_{ip}| \cdot |a_{jq}| \cdot \|A_p A_q\| \leqslant K^2 \max_{1 \leqslant p,q \leqslant d} \|A_p A_q\|,$$

and similarly for every choice of $i_1, i_2, ..., i_n \in \{1, 2, ..., d\}$ we have

$$\|T_{i_1} ... T_{i_n}\| \leqslant K^n \max_{1 \leqslant j_1, ..., j_n \leqslant d} \|A_{j_1} ... A_{j_n}\| = K^n \alpha_n,$$

where

$$\alpha_n = \max_{1 \leqslant j_1, ..., j_n \leqslant d} \|A_{j_1} ... A_{j_n}\|$$

is the largest norm of any n-fold product of elements drawn from $\{A_1, ..., A_d\}$. Thus for every $n=1, 2, ...$ we have

$$\begin{aligned} \Big\| \sum_{i_1, ..., i_n = 1}^{d} T_{i_1}^* ... T_{i_n}^* T_{i_n} ... T_{i_1} \Big\| &\leqslant \sum_{i_1, ..., i_n = 1}^{d} \|T_{i_1} ... T_{i_n}\|^2 \\ &\leqslant \sum_{i_1, ..., i_n = 1}^{d} K^{2n} \alpha_n^2 = d^n K^{2n} \alpha_n^2, \end{aligned}$$

which implies the following upper bound on the energy sequence:

$$E_n(\mathcal{S}) \leqslant (dK^2)^n \alpha_n^2, \quad n = 1, 2, ... \tag{B.5}$$

Note that we have not used commutativity in establishing (B.5).

To complete the proof we estimate α_n as follows. Choose $\varepsilon > 0$. For every $j_1, ..., j_n \in \{1, 2, ..., d\}$ we use commutativity to write $A_{j_1} ... A_{j_n}$ in the form

$$A_{j_1} ... A_{j_n} = A_1^{p_1} ... A_d^{p_d}$$

where $p_1, ..., p_d$ are nonnegative integers summing to n. Since each of the operators A_j is quasinilpotent there is a constant $C > 0$ (depending on ε) such that

$$\|A_j^p\| \leqslant C \varepsilon^p$$

for every $p=0,1,2,\ldots$ and every $j=1,2,\ldots,d$. Hence

$$\|A_1^{p_1}\ldots A_d^{p_d}\| \leqslant C^d\varepsilon^{p_1+\ldots+p_d} = C^d\varepsilon^n.$$

We may conclude that

$$\alpha_n = \max_{1\leqslant j_1,\ldots,j_n\leqslant d} \|A_{j_1}\ldots A_{j_n}\| \leqslant C^d\varepsilon^n$$

for every $n=1,2,\ldots$. From (B.5) it follows that

$$E_n(\mathcal{S}) \leqslant C^{2d}(\varepsilon^2 dK^2)^n, \quad n=1,2,\ldots .$$

The preceding inequality implies that

$$\limsup_{n\to\infty} E_n(\mathcal{S})^{1/n} \leqslant \varepsilon^2 dK^2,$$

and since ε is arbritrarily small, (B.1) follows.

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