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SUBBUNDLES OF THE TANGENT BUNDLE

BY

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ABSTRACT. This paper studies pairs (M, ξ) where M is a closed manifold and ξ is a k-dimensional subbundle of the tangent bundle of M in terms of cobordism.

1. **Introduction.** The purpose of this note is to analyze pairs (M, ξ) where M is an n-dimensional manifold and ξ is a k-dimensional subbundle of the tangent bundle of M, $k \leqslant n$, in terms of cobordism.

In §2, the cobordism class of M is analyzed and the main result is

PROPOSITION. *A class* $\alpha \in \mathfrak{N}_n$ *is represented by a manifold* M^n *whose tangent bundle has a* k*-dimensional subbundle,* $k \leqslant n$, *if and only if either*

(a) k *is even, or*

(b) k *is odd and* $w_n(\alpha) = 0$.

In section §3, the case $k = 1$, i.e., ξ a line bundle, will be studied more closely. One defines a homomorphism $\theta\colon \mathfrak{N}_n(BO_1) \to Z_2$ as follows. If $\alpha \in \mathfrak{N}_n(BO_1)$, choose a manifold M^n and map $f\colon M^n \to BO_1$ representing α. Let $i \in H^1(BO_1; Z_2)$ be the nonzero class, and let $\theta(\alpha)$ be the characteristic number

$$\{w_n(M) + w_{n-1}(M)f^*(i) + \cdots + w_{n-r}(M)(f^*(i))^r + \cdots + (f^*(i))^n\}[M].$$

Letting γ be the universal line bundle over BO_1, the class α is the class of the pair $(M, f^*(\gamma))$, and interpreting $\mathfrak{N}_n(BO_1)$ as the cobordism classes of n-manifolds with a line bundle, one has

PROPOSITION. *A class* $\alpha \in \mathfrak{N}_n(BO_1)$ *is represented by a pair* (M^n, ξ) *where* ξ *is a sub-line-bundle of the tangent bundle of* M *if and only if* $\theta(\alpha) = 0$.

Note. In order to make this result seem plausible, one should note that the given characteristic number is the nth Stiefel-Whitney number of $\tau_M - f^*(\gamma)$, which is an $(n-1)$-plane bundle if $f^*(\gamma)$ is a subbundle of τ_M.

In §4, the problem is stabilized, and the main result is

PROPOSITION. *A class* $\alpha = [M, f] \in \mathfrak{N}_n(BO_k)$ *is represented by a pair* (M', ξ') *with* $\tau_{M'} \oplus 1 \cong \xi' \oplus \eta' \oplus 1$ *where* η' *is an* $n - k$ *plane bundle if*

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and only if every Stiefel-Whitney number of α *involving a class* $w_i(\tau - f^*(\gamma))$ *for* $i > n - k$ *is zero.*

In §5, the case $k = 2$ is studied more closely.

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2. The cobordism class of M.

LEMMA 2.1. *If* M^n *is a closed n-manifold and* ξ^k *is a subbundle of the tangent bundle of* M *with* k *odd, then* $w_n[M] = 0$; *i.e.,* M *has even Euler characteristic.*

PROOF. If n is odd, $w_n[M] = 0$, so one may assume n even. Let $k = 2p + 1$, $n - k = 2q + 1$ and let η be a complement of ξ in τ, the tangent bundle of M, so that $\xi \oplus \eta = \tau$. Then

$$\begin{aligned}
w_n[M] &= w_n(\tau)[M] \\
&= w_{2p+1}(\xi) \cup w_{2q+1}(\eta)[M], \\
&= (Sq^1 w_{2p}(\xi) + w_1(\xi) \cup w_{2p}(\xi)) \cup w_{2q+1}(\eta)[M] \\
&= \{Sq^1 w_{2p}(\xi) \cup w_{2q+1}(\eta) + (w_1(\tau) + w_1(\eta)) \cup w_{2p}(\xi) \cup w_{2q+1}(\eta)\}[M] \\
&= \{Sq^1 w_{2p}(\xi) \cup w_{2q+1}(\eta) + v_1(\tau) \cup w_{2p}(\xi) \cup w_{2q+1}(\eta) \\
&\qquad + w_{2p}(\xi) \cup Sq^1 w_{2q+1}(\eta)\}[M] \\
&= \{v_1(\tau) \cup + Sq^1\}\{w_{2p}(\xi) w_{2q+1}(\eta)\}[M]
\end{aligned}$$

but cup-product with the Wu class $v_1(\tau) = w_1(\tau)$ gives Sq^1, and so this vanishes. □

In order to prove the converse, one needs some examples of manifolds. For this, one may use the result of [5, 3.4]:

LEMMA 2.2. *Let* $RP(n_1, n_2, \cdots, n_t)$, $t > 1$, *be the bundle of lines in the fibers of* $\lambda_1 \oplus \cdots \oplus \lambda_t$ *over* $RP(n_1) \times \cdots \times RP(n_t)$, *where* λ_i *is the pullback of the canonical bundle over* $RP(n_i)$. *Then* $RP(n_1, \cdots, n_t)$ *is a closed manifold of dimension* $n + t - 1$ *where* $n = n_1 + \cdots + n_t$, *and is indecomposable in* $\mathfrak{N}_*$ *if and only if*

$$\binom{n+t-2}{n_1} + \cdots + \binom{n+t-2}{n_t}$$

is odd.

One now defines manifolds X^n of dimension n for $n \neq 2^s - 1$ and $n \neq 2$ as follows:

(a) if $n = 4s,\ s \geqslant 1$,

$$X^n = RP(\underbrace{1, \cdots, 1}_{2s}, 0),$$

(b) if $n = 4s + 2,\ s \geqslant 1$,

$$X^n = RP(\underbrace{1, \cdots, 1}_{2s}, 0, 0, 0),$$

(c) if $n = 2^p(2q + 1) - 1,\ p,\ q > 0$,

$$X^n = RP(2^p, \underbrace{1, \cdots, 1}_{2^p q - 1}, 0).$$

The above criterion immediately shows that these manifolds are indecomposable in $\mathfrak{N}_*$.

The manifolds X^n have the additional property that, for each integer $k \leqslant n$, the tangent bundle of X^n has a k-dimensional subbundle. In fact, for $n \neq 5$, the tangent bundle of X^n is a Whitney sum of line bundles.

To see this, let λ be the canonical line bundle over $RP(n_1, \cdots, n_t)$ and π: $RP(n_1, \cdots, n_t) \to RP(n_1) \times \cdots \times RP(n_t)$ the projection. Let λ_i denote $\pi^*(\lambda_i)$ and τ_i the pullback of the tangent bundle of $RP(n_i)$. Then

$$\tau_{RP(n_1, \cdots, n_t)} \cong \pi^* \tau_{RP(n_1) \times \cdots \times RP(n_t)} \oplus \mu \cong \tau_1 \oplus \cdots \oplus \tau_t \oplus \mu$$

where μ is the bundle along the fibers. Then

$$\mu \oplus l \cong (\lambda \otimes \lambda_1) \oplus \cdots \oplus (\lambda \otimes \lambda_t) \quad \text{and} \quad \tau_i \oplus l = (n_i + 1)\lambda_i$$

where l is the trivial line bundle. If $n_i = 0$ or 1, τ_i is trivial, since the tangent bundles of $RP(1) = S^1$ and $RP(0) =$ point are trivial. Adding the trivial τ_i with $n_i = 1$ to other τ_j or μ represents them as sums of line bundles.

For $n = 5$, $RP(2, 1, 0)$ has tangent bundle $\tau_1 \oplus l \oplus \mu$ which is a line bundle and two 2-plane bundles, while in all other cases there are at least two l's and the tangent bundle is a sum of line bundles.

One now has

PROPOSITION 2.3. *A class $\alpha \in \mathfrak{N}_n$ is represented by a manifold M^n whose tangent bundle has a k-dimensional subbundle, $k \leqslant n$, if either*:

(a) *k is even, or*

(b) *k is odd and $w_n(\alpha) = 0$.*

PROOF. Every class $\alpha \in \mathfrak{N}_n$ is represented by the disjoint union of manifolds

$$\underbrace{RP(2) \times \cdots \times RP(2)}_{q} \times X^{n_1} \times \cdots \times X^{n_s}$$

with $2q + n_1 + \cdots + n_s = n$. For any integer $k \leqslant n$ of the form $2u + v$ with $u \leqslant q$, $v \leqslant n_1 + \cdots + n_s$, this component has a subbundle of its tangent bundle of dimension k. In particular, every even integer can be put in this form, and every odd integer will be of this form except for the component $[RP(2)]^{n/2}$ which has $w_n \neq 0$. □

This completes the proof of the proposition given in the introduction.

REMARK. If ξ is the line bundle over $RP(1)$ and λ is the line bundle over the Klein bottle $RP(\xi \oplus l)$, then the 5-manifold $RP(\lambda \oplus 3)$ is indecomposable in $\mathfrak{N}_*$ and has tangent bundle a sum of line bundles. This manifold could be used in place of X^5 and so five plays no special role.

3. Line bundles.

LEMMA 3.1. *If M^n is a closed n-manifold, ξ a sub-line-bundle of the tangent bundle of M and $f: M \to BO_1$ classifies ξ, then* $\theta([M, f]) = 0$.

PROOF. Let η be a complement in τ for ξ. Then $w(\eta) = w(\tau)/w(\xi)$, so since η is an $(n-1)$-plane bundle

$$0 = w_n(\eta) = w_n(\tau) + w_{n-1}(\tau)w_1(\xi) + \cdots + (w_1(\xi))^n.$$

Since $w_1(\xi) = f^*(i)$ and $w_i(\tau) = w_i(M)$, this gives $\theta([M, f]) = 0$. □

In order to prove the converse, one needs to analyze the bordism of BO_1. Henceforth, classes of $\mathfrak{N}_*(BO_1)$ will be denoted $[M, \xi]$ where M is a closed manifold and ξ is a line bundle over M. There is a homomorphism of $\mathfrak{N}_*$ modules, called the Smith homomorphism,

$$\Delta: \mathfrak{N}_*(BO_1) \to \mathfrak{N}_*(BO_1)$$

of degree -1 assigning to $[M, \xi]$ the class $[N, \xi|N]$ where $N \subset M$ is the codimension one submanifold of M dual to ξ. Specifically, if $f: M \to BO_1 = RP(\infty)$ classifies ξ, f maps M into some $RP(n)$ and may be homotoped in $RP(n)$ to be transverse regular on $RP(n-1)$, with N then taken to be the inverse image of $RP(n-1)$.

Letting $1 = [\text{point}, l] \in \mathfrak{N}_0(BO_1)$, there are unique classes $x_i = [M^i, \xi^i] \in \mathfrak{N}_i(BO_1)$, $i \geqslant 0$, with

(1) $x_0 = 1$,

(2) $\Delta x_i = x_{i-1}$, and

(3) for $i > 0$, M^i bounds.

These classes form a base for $\mathfrak{N}_*(BO_1)$ as $\mathfrak{N}_*$ module. (A proof of this statement, or more precisely, its complex analogue appears in [2, (5.3)].)

LEMMA 3.2. *For $i > 0$, x_i is the class of the canonical bundle λ over $RP(1, 0, \cdots, 0)$ ($i - 1$ 0's).*

PROOF. In [1, (2.2)], $RP(1, 0, \cdots, 0)$ ($i - 1$ 0's) is denoted $RP(\xi \oplus (i - 1))$, where ξ is the canonical line bundle over $RP(1)$, and is shown to bound. In [4, p. 160] it is shown that for any vector bundle ρ over M, the submanifold dual to λ over $RP(\rho \oplus I)$ is $RP(\rho)$, from which the behaviour of Δ follows. □

For $i > 1$, the tangent bundle of $RP(1, 0, \cdots, 0)$ ($i - 1$ 0's) is $1 \oplus \mu = \lambda \otimes \pi^*(\xi) \oplus (i - 1)\lambda$, which contains a copy of λ, so $\theta(x_i) = 0$ if $i > 1$.

Now if ξ is a line bundle over M, and N is a closed manifold, $\pi_M^*(\xi)$ is a line bundle over $M \times N$, with $[N] \cdot [M, \xi] = [M \times N, \pi_M^*(\xi)]$ giving the module structure of $\mathfrak{N}_*(BO_1)$. If N has dimension n, it is immediate that $\theta([N] \cdot [M, \xi]) = w_n[N] \cdot \theta([M, \xi])$.

Since $\theta(x_0) = \theta(x_1) = 1$, one then has

LEMMA 3.3. $\theta(\Sigma_i [N^{n-i}] x_i) = w_n(N^n) + w_{n-1}(N^{n-1})$.

PROPOSITION 3.4. *If $\alpha \in \mathfrak{N}_n(BO_1)$ with $\theta(\alpha) = 0$, then $\alpha = [M, \xi]$ where ξ is a sub-line-bundle of the tangent bundle of M.*

PROOF. Let $\alpha = \Sigma_{i=1}^n a_i x_i$ with $a_i \in \mathfrak{N}_{n-i}$. Then $w_n(a_0) = 0, w_{n-1}(a_1) = 0$, for if n is odd $w_n(a_0) = 0$ for dimensional reasons while $w_{n-1}(a_1) = \theta(\alpha) = 0$ and if n is even $w_{n-1}(a_1) = 0$ for dimensional reasons while $w_n(a_0) = \theta(\alpha) = 0$. By [1, (4.5)] there are manifolds N^n and N^{n-1} fibered over S^1, with $[N^{n-i}] = a_i$, $i = 0, 1$. Choose manifolds N^{n-i} representing a_i for $i > 1$, and let

$$M^n = N^n \cup (N^{n-1} \times RP(1)) \cup \bigcup_{i>1} (N^{n-i} \times RP(1, \underbrace{0, \cdots, 0}_{i-1}))$$

and let ξ be the line bundle over M whose restriction to N^n is trivial, to $N^{n-1} \times RP(1)$ is the pullback of the canonical bundle over $RP(1)$ and to $N^{n-i} \times RP(1, 0, \cdots, 0)$ is the pullback of λ. Then $\alpha = [M, \xi]$.

Since N^n fibers over S^1, the pullback of τ_{S^1} is a trivial line bundle in τ_{N^n}. Since $N^{n-1} \times RP(1)$ fibers over $S^1 \times S^1$, its tangent bundle contains a trivial 2 plane-bundle, but if ξ' is the canonical bundle over $RP(1)$, $2\xi' = 2$ so the tangent bundle contains two copies of the pullback of ξ'. As noted, λ is a subbundle of the tangent bundle of $RP(1, 0, \cdots, 0)$ ($i - 1$ 0's) if $i > 1$.

Thus ξ is a subbundle of the tangent bundle of M. □

Combining this with Lemma 3.1 gives the second proposition of the introduction.

Now restricting attention to oriented manifolds one has

PROPOSITION 3.5. *A class* $\alpha \in \Omega_n$ *is represented by an oriented manifold* M^n *whose tangent bundle contains a line bundle if and only if the Stiefel-Whitney number* $w_n(\alpha)$ *is zero.*

A class $\alpha \in \Omega_n(RP(\infty))$ is represented by a pair $[M^n, \xi]$ where ξ is a sub-line-bundle of the tangent bundle of the oriented manifold M if and only if the Stiefel-Whitney number $\theta(\alpha)$ is zero.

PROOF. These conditions are clearly necessary. To see that they are sufficient, consider $\alpha \in \Omega_n$ for which $w_n(\alpha) = 0$ and choose a representative M^n for α. Using surgery, one may replace M by the connected sum of its components; i.e., may assume M connected. If n is odd, the tangent bundle has a nonvanishing section, while if n is even, such a section exists if and only if the Euler class of the tangent bundle $X(\tau)$ is zero. Since M is connected, $X(\tau) = \chi(M)\sigma$, where $\chi(M)$ is the Euler characteristic of M and σ is a generator of $H^n(M; Z) \cong Z$. Mod 2, $\chi(M)$ is $w_n(\alpha)$ so $\chi(M)$ is even, and by forming the connected sum of M with copies of $S^p \times S^q$ for suitable $p, q > 0$, one obtains a new M with $\chi(M) = 0$ also in α. [*Note.* If $n = 2, \alpha = 0$ and M may be taken empty or $S^1 \times S^1$ while if $n = 2k$, $k > 1$, the connected sum with $S^2 \times S^{n-2}$ increases χ by 2 while that with $S^1 \times S^{n-1}$ decreases it by 2.] Thus every $\alpha \in \Omega_n$ with $w_n(\alpha) = 0$ is represented by a manifold M^n for which τ_M contains a trivial line bundle.

Now turning to $\Omega_*(RP(\infty))$, one has $\Omega_*(RP(\infty)) \cong \Omega_* \oplus \widetilde{\Omega}_*(RP(\infty))$ and $\widetilde{\Omega}_*(RP(\infty)) \cong \mathfrak{N}_{*-1}$. A class in the Ω_n summand of $\Omega_n(RP(\infty))$ is represented by a manifold M^n with trivial line bundle, and $\theta([M, 1]) = \langle w_n(\tau), [M]\rangle$ so that by the above, a class α in the Ω_* summand is represented by a subbundle if and only if $\theta(\alpha) = 0$. The summand $\mathfrak{N}_{n-1}$ of $\Omega_n(RP(\infty))$ is realized as follows. If $\beta \in \mathfrak{N}_{n-1}$, let N^{n-1} be a manifold in β and let M^n be the real projective space bundle $RP(\xi \oplus 1)$ where ξ is the determinant bundle of the tangent bundle of N and let λ be the canonical line bundle over $RP(\xi \oplus 1)$. Assigning to β the class of $[M, \lambda]$ gives the isomorphism $\mathfrak{N}_{n-1} \cong \widetilde{\Omega}_n(RP(\infty))$. Now $\theta([M, \lambda]) = w_{n-1}(\beta)$, and if $\theta([M, \lambda]) = 0$, β is represented by a manifold N whose tangent bundle has a section and so λ is a subbundle of the tangent bundle of $RP(\xi \oplus 1)$. Noting that θ vanishes on the Ω_* summand if n is odd and on the $\widetilde{\Omega}_*(RP(\infty))$ summand if n is even, one sees that every class in the kernel of θ is realized by a subbundle of the tangent bundle.

4. **Stabilization.** One now considers stabilization of the subbundle problem. This permits the use of homotopy theoretic techniques.

One may consider a manifold M^n together with an isomorphism $\tau_M \oplus$

$j \cong \xi^k \oplus \eta^{n-k} \oplus j$ where j denotes a trivial j plane bundle. By stability the existence of an isomorphism is independent of j if $j \geqslant 1$. The manifold M^n with this structure bounds if $M = \partial V$ where $\tau_V \oplus (j-1) \cong \rho^k \oplus \sigma^{n-k+1} \oplus (j-1)$ is a compatible decomposition; i.e., ρ restricts to ξ and σ to $\eta \oplus 1$. Assuming V has no closed components, V has the homotopy type of an n-dimensional complex, so $\tau_V \cong \rho \oplus \sigma$, but this need not be compatible with the chosen isomorphism along M unless $j > 1$.

Let $\phi_r^k\colon BO_k \times BO_r \to BO$ be a map classifying the complement of the Whitney sum $\gamma_k \oplus \gamma_r$ of the universal bundles (converted to a fibration). The structure on M is precisely a lift of the normal map of M to $BO_k \times BO_{n-k}$, while that of V is a lift to $BO_k \times BO_{n-k+1}$.

The techniques of bordism of manifolds with normal structure [3] give that the bordism group of manifolds M^n of the given type is the image of the stable homotopy homomorphism

$$\pi_n^S(T(BO_k \times BO_{n-k})) \to \pi_n^S(T(BO_k \times BO_{n-k+1}))$$

where $T(BO_k \times BO_r)$ is the Thom spectrum associated with the fibration ϕ_r^k.

Specifically, if one takes the induced fibration

$$\begin{array}{ccc} E & \longrightarrow & BO_k \times BO_r \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \phi_r^k} \\ BO_s & \longrightarrow & BO \end{array}$$

then $\pi_n^S(T(BO_k \times BO_r)) = \lim_{s\to\infty} \pi_{n+s}(T(\pi^*(\gamma_s)))$. One may also describe these groups as

$$\pi_n^S(T(BO_k \times BO_r)) = \lim_{s,t\to\infty} \pi_{n+s+t}(T(\gamma_s \oplus \gamma_t))$$

where γ_s, γ_t are the universal s and t plane bundles over the Grassmann manifolds $G_{k,s}$ and $G_{r,t}$.

One may now consider the homomorphism

$$\begin{array}{ccc} \pi_n^S(T(BO_k \times BO_{n-k})) & \to & \lim_{r\to\infty} \pi_n^S(T(BO_k \times BO_r)) \\ & & \| \\ & & \pi_n^S(T(BO_k \times BO)). \end{array}$$

One has $\pi_1 \times \oplus\colon BO_k \times BO \to BO_k \times BO$, which is a homotopy equivalence, and induces an equivalence $T(BO_k \times BO) \cong BO_k^+ \wedge MO$ and hence

$$\pi_n^S(T(BO_k \times BO)) \cong \mathfrak{N}_n(BO_k).$$

This describes the forgetful homomorphism assigning to M^n with its structure the bordism class of (M, ξ).

One now embarks on a program of analyzing the stable homotopy groups involved.

LEMMA 4.1. *Let γ_s be the universal s plane bundle over $G_{r,s}$, $s > r$, and let p be an odd prime. Then $\widetilde{H}^i(T(\gamma_s); Z_p) = 0$ for $i < r + s$.*

PROOF. One has the inclusion $G_{r,s} \subset G_{r+1,s}$ with $G_{r+1,s}$ obtained by attaching cells of dimension $(r + 1)$ and higher. This induces an inclusion of Thom spaces $T(\gamma_s|G_{r,s}) \subset T(\gamma_s|G_{r+1,s})$ and the cofiber has cells of dimension $r + 1 + s$ and higher. From the exact cohomology sequence

$$\widetilde{H}^i(T(\gamma_s|G_{r,s}); Z_p) \cong \widetilde{H}^i(T(\gamma_s|G_{r+1,s}); Z_p) \quad \text{if} \quad i < r + s.$$

Thus

$$\widetilde{H}^i(T(\gamma_s|G_{r,s}); Z_p) \cong \widetilde{H}^i(T(\gamma_s|G_{r+t,s}); Z_p) \quad \text{if } i < r + s,\ t \geqslant 0,$$

but for t large this is $\widetilde{H}^i(MO_s; Z_p)$ which is zero. □

LEMMA 4.2. *$\pi_i^S(T(BO_k \times BO_r))$ is a 2 group if $i < k + r$.*

PROOF. Let γ_s, γ_t be the universal bundles over $G_{k,s}$ and $G_{r,t}$, s and t large. Then $T(\gamma_s \oplus \gamma_t) = T(\gamma_s) \wedge T(\gamma_t)$ and $\widetilde{H}^i(T(\gamma_s \oplus \gamma_t); Z_p) = 0$ if $i < k + r + s + t$ if p is odd. By the mod C Hurewicz theorem $\pi_i(T(\gamma_s \oplus \gamma_t))$ is a 2 group if $i < k + r + s + t$. □

Thus, for $r \geqslant n - k + 1$, $\pi_n^S(T(BO_k \times BO_r))$ is a 2 group, and the problem is entirely a 2 primary problem.

In order to begin the 2 primary analysis, one analyzes a cofibration of spectra

$$T(BO_k \times BO_r) \to T(BO_k \times BO_{r+1}) \to \mathbf{X}$$

which one realizes by a cofibration $T(\gamma_s \oplus \gamma_t) \to T(\gamma_s \oplus \gamma_t') \to X$ where γ_s, γ_t are universal bundles over $G_{k,s}$, $G_{r,t}$ and γ_t' is the universal bundle over $G_{r+1,t}$, with s and t being large.

First, consider $G_{r+1,t}$ as the space of $r + 1$ planes in R^{r+1+t} with π: $D(\gamma_{r+1}) \to G_{r+1,t}$ the projection of the disc bundle. Letting $S(\gamma_{r+1})$ be the unit sphere bundle, one has a cofibration

$$\underset{\textstyle A}{\underset{\|}{\frac{D(\pi^*(\gamma_t')|S(\gamma_{r+1}))}{S(\pi^*(\gamma_t')|S(\gamma_{r+1}))}}} \to \underset{\textstyle B}{\underset{\|}{\frac{D(\pi^*(\gamma_t'))}{S(\pi^*(\gamma_t'))}}} \to \underset{\textstyle C}{\underset{\|}{\frac{D(\pi^*(\gamma_t'))}{D(\pi^*(\gamma_t')|S(\gamma_{r+1})) \cup S(\pi^*(\gamma_t'))}}}.$$

Since $D(\pi^*(\gamma_t'))$ is identifiable with $D(\gamma_{r+1} \oplus \gamma_t')$, C is the Thom space of the trivial bundle $\gamma_{r+1} \oplus \gamma_t'$, and $C \cong \Sigma^{r+t+1}(G_{r+1,t}^{\;+})$ is the $(r+t+1)$-fold suspension of $G_{r+1,t}$ with a base point adjoined. Since π is a homotopy equivalence, $B \cong T(\gamma_t')$.

Finally, $S(\gamma_{r+1})$ may be considered as pairs (α, x) with α an $(r+1)$-plane in R^{r+1+t} and x a unit vector in α. Assigning to (α, x) the point $x \in S^{r+t}$ defines a fibration p: $S(\gamma_{r+1}) \to S^{r+t}$. The inverse image of $x \in S^{r+t}$ is the space of r planes in R^{r+1+t} orthogonal to x, i.e., $S(\gamma_{r+1})$ is the Grassmann bundle of r planes in the fibers of the tangent bundle of S^{r+t}. The inclusion $G_{r,t} \to G_{r+1,t}$ may then be considered as factoring via the inclusion as a fiber in $S(\gamma_{r+1})$. The inclusion of the fiber $G_{r,t} \to S(\gamma_{r+1})$ induces isomorphisms in homotopy and homology in dimensions less than $r+t-1$, and so the inclusion $T(\gamma_t) \to A$ is a homotopy equivalence (for the prime 2) in dimensions less than $r+2t-1$. Since t is large, one then obtains a cofibration

$$T(\gamma_t) \to T(\gamma_t') \to \Sigma^{r+t+1}(G_{r+1,t}^{\;+}).$$

Smashing with $T(\gamma_s)$ gives a cofibration sequence

$$T(\gamma_s \oplus \gamma_t) \to T(\gamma_s \oplus \gamma_t') \to T(\gamma_s) \wedge \Sigma^{r+t+1}(G_{r+1,t}^{\;+})$$

(i.e., X may be identified with $T(\gamma_s) \wedge \Sigma^{r+t+1}(G_{r+1,t}^{\;+})$ for the prime 2, having isomorphic mod 2 cohomology up to dimension $s+r+2t-1$ induced by a map of spaces).

One now considers $T(\gamma_s) \wedge \Sigma^{r+t+1}(G_{r+1,t}^{\;+})$ as $\Sigma^{r+t+1}\, T(\gamma_s) \wedge G_{r+1,t}^{\;+}$ and analyzes the maps

$$G_{k,s} \to G_{k,s+r+t+1} \to G_{m,s+r+t+1}$$

inducing

$$\Sigma^{r+t+1}\, T(\gamma_s) \to T(\gamma_{s+r+t+1}) \to MO_{s+r+t+1}$$

(m being large). The maps of Grassmannians induce isomorphisms in mod 2 cohomology in dimensions less than or equal to k and hence the Thom spaces have isomorphic mod 2 cohomology in dimensions less than or equal to $k+s+r+t+1$.

Thus X may be identified with $MO_{s+r+t+1} \wedge (G_{r+1,t})^+$ in dimensions less than or equal to $k+s+r+t+1$ (in mod 2 cohomology). In particular, in dimensions less than or equal to $k+s+r+t+1$ $\widetilde{H}^*(X; Z_2)$ is a free module over the Steenrod algebra and

$$\pi_{i+s+t}(X) \cong \pi_{i+s+t}(MO_{s+r+t+1} \wedge (G_{r+1,t})^+) \cong \mathfrak{N}_{i-r-1}(G_{r+1,t})$$

if $i+s+t \leqslant k+s+r+t$, $i \leqslant k+r$ (for 2 primary structure).

Being given a manifold M^i with $\tau_M \oplus j \cong \xi^k \oplus \eta^{r+1} \oplus (i+j-k-r-1)$ representing a class in $\pi_i^S(T(BO_k \times BO_{r+1}))$, $i \leqslant k+r$, the class in $\pi_i^S(\mathrm{X}) \cong \mathfrak{N}_{i-r-1}(BO_{r+1})$ obtained from the cofibration is represented by the submanifold of M^i dual to η^{r+1} with the $(r+1)$-plane bundle obtained by restricting η. The map to X is induced by including $T(\gamma_t')$ in $T(\gamma_t' \oplus \gamma_{r+1})$ and making the maps transverse regular involves finding the submanifold dual to γ_{r+1}, from which one has the given assertion.

On the other hand, a class in $\pi_i^S(T(BO_k \times BO_{r+1}))$, $i \leqslant k+r$, is in the image of $\pi_i^S(T(BO_k \times BO_r))$ if and only if it goes to zero in $\pi_i^S(\mathrm{X})$. Since $\widetilde{H}^*(X; Z_2)$ is a free module over the Steenrod algebra in dimensions up to $k+s+r+t+1$, a homotopy element in $\pi_{i+s+t}(X)$ is detected by mod 2 cohomology. Since $T(\gamma_s \oplus \gamma_t') \to X$ maps $\widetilde{H}^*(X; Z_2)$ isomorphically onto the multiples of $\Phi(w_{r+1})$, the Thom isomorphism image of w_{r+1}, in the $H^*(G_{k,s} \times G_{r+1,t}; Z_2)$ module structure, this asserts that all characteristic numbers involving w_{r+1} should vanish. Thus, one has

LEMMA 4.3. *A manifold M^i with $\tau_M \oplus j \cong \xi^k \oplus \eta^{r+1} \oplus (i+j-k-r-1)$ representing a class in $\pi_i^S(T(BO_k \times BO_{r+1}))$, $i \leqslant k+r$, comes from $\pi_i^S(T(BO_k \times BO_r))$ if and only if all characteristic numbers involving $w_{r+1}(\eta)$ are zero.*

For $r \geqslant n-k$, this determines the image of

$$\pi_n^S(T(BO_k \times BO_r)) \to \pi_n^S(T(BO_k \times BO_{r+1})).$$

For $r \geqslant n-k+1$, this homomorphism is monic, which may be seen as follows.

Consider the homomorphism

$$\pi_{n+1}^S(T(BO_k \times BO_{r+1})) \to \pi_{n+1}^S(\mathrm{X}).$$

Now $\pi_{n+1}^S(\mathrm{X}) \cong \mathfrak{N}_{n-r}(BO_{r+1})$ for $n+1 \leqslant k+r$, and $\mathfrak{N}_{n-r}(BO_{r+1})$ is generated over Z_2 by the manifolds

$$P = M^m \times RP(\lambda_1 \oplus k_1) \times \cdots \times RP(\lambda_s \oplus k_s) \times (\text{point})$$

where λ_i is the nontrivial bundle over $RP(1)$, $k_i \geqslant 0$, with $m + (k_1+1) + \cdots + (k_s+1) = n-r$ with bundle

$$\lambda^{(1)} \oplus \cdots \oplus \lambda^{(s)} \oplus (r+1-s)$$

where $\lambda^{(i)}$ is the canonical bundle over $RP(\lambda_i \oplus k_i)$. To see this, one notes that the $RP(\lambda \oplus k)$, $k \geqslant 0$, and the point generate $\mathfrak{N}_*(BO_1)$, over $\mathfrak{N}_*$ and forming the products of $r+1$ of these gives a $\mathfrak{N}_*$ generating set for $\mathfrak{N}_*(BO_{r+1})$.

One then considers the manifold

$$Q = M^m \times RP(\lambda_1 \oplus k_1 \oplus 1) \times \cdots \times RP(\lambda_s \oplus k_s \oplus 1) \times RP(r + 1 - s)$$

of dimension $m + (k_1 + 2) + \cdots + (k_s + 2) + r + 1 - s = n - r + s + r + 1 - s = n + 1$ and letting λ be the canonical line bundle over $RP(r + 1 - s)$, the submanifold dual to $\lambda^{(1)} \oplus \cdots \oplus \lambda^{(s)} \oplus (r + 1 - s)\lambda = \eta^{r+1}$ is the manifold P given above, with η restricting to the given bundle. Now the tangent bundle of $RP(\lambda_i \oplus k_i \oplus 1)$ is $\lambda^{(i)} \otimes \lambda_i \oplus (k_i + 1)\lambda^{(i)}$ so

$$\begin{aligned}\tau_Q \oplus 1 \cong [\tau_M \oplus (\lambda^{(1)} \oplus \lambda_i \oplus k_i\lambda^{(1)}) \oplus \cdots \\ \oplus (\lambda^{(s)} \otimes \lambda_s \oplus k_s\lambda^{(s)}) \oplus \lambda] \oplus \eta \\ = \xi' \oplus \eta\end{aligned}$$

where ξ' is an $m + (k_1 + 1) + \cdots + (k_s + 1) + 1 = n - r + 1 \leqslant k$ bundle. Thus $\tau_Q \oplus 1 \oplus (k + r - n - 1) \cong [\xi' \oplus (k + r - 1)] \oplus \eta = \xi^k \oplus \eta$ giving a structure on Q mapping to the class of P in $\mathfrak{N}_{n-r}(BO_{r+1})$.

This proves that the forgetful homomorphism $\pi_n^S(T(BO_k \times BO_r)) \to \mathfrak{N}_n(BO_k)$ is monic for $r \geqslant n - k + 1$, and that

$$\operatorname{im}\{\pi_n^S(T(BO_k \times BO_{n-k})) \to \pi_n^S(T(BO_k \times BO_{n-k+1}))\}$$

is mapped monomorphically into $\mathfrak{N}_n(BO_k)$ with image precisely those classes for which all numbers involving $w_i(\tau - f^*(\gamma_k))$ for $i > n - k$ are zero, or one has

PROPOSITION 4.4. *A class $\alpha = [M, f] \in \mathfrak{N}_n(BO_k)$ is represented by a manifold M^n with $\tau_M \oplus 1 \cong f^*(\gamma_k) \oplus \eta^{n-k} \oplus 1$ if and only if all Stiefel-Whitney numbers of α involving $w_i(\tau - f^*(\gamma_k))$ for $i > n - k$ are zero.*

5. **Two plane bundles.** The purpose of this section is to prove

PROPOSITION 5.1. *A class $\alpha = [M, f] \in \mathfrak{N}_n(BO_2)$ is represented by a pair $[M, f]$ with $f^*(\gamma_2)$ a subbundle of the tangent bundle of M if and only if all characteristic numbers of α involving $w_i(\tau - f^*(\gamma_2))$ with $i > n - 2$ are zero.*

To begin the proof, one wants manifolds $M_{i,j}$ of dimension $i + 2j$ for each (i, j) and 2 plane bundles $\lambda_{i,j}$ over $M_{i,j}$ for which

$$w_1^p(\lambda_{i,j})w_2^q(\lambda_{i,j})[M_{i,j}] = \begin{cases} 0 & \text{if } q > j, \quad p + 2q = i + 2j, \\ 1 & \text{if } q = j, \quad p = i \end{cases}$$

Any collection of such manifolds form a base for $\mathfrak{N}_*(BO_2)$ as $\mathfrak{N}_*$ module. The representatives will be chosen so that $\lambda_{i,j}$ is a subbundle of the tangent bundle of $M_{i,j}$ except for $j = 0$ and $i \leqslant 3$.

For $j \geqslant 2$, one lets

$$M_{i,j} = RP(1, \underbrace{0, \cdots, 0}_{j-1}) \times RP(1, \underbrace{0, \cdots, 0}_{i+j-1})$$

and lets $\lambda_{i,j} = \pi_1^*(\lambda) \oplus \pi_2^*(\lambda)$, where λ is the canonical line bundle over $RP(1, 0, \cdots, 0)$.

For $j = 1$, $i \geqslant 3$, one lets $M_{i,j} = RP(1) \times RP(3, 0, \cdots, 0)$ ($i - 2$ 0's) and lets $\lambda_{i,j} = \pi_1^*(\xi) \oplus \pi_2^*(\lambda)$, ξ being the Hopf bundle over $RP(1)$. The tangent bundle of $RP(3)$ is trivial and so the tangent bundle of $M_{i,j}$ is $3 \oplus \pi_2^*(\lambda) \otimes (\pi_2^*(\xi') \oplus (i-2))$, where ξ' is the Hopf bundle over $RP(3)$. Since $2\xi = 2$ and $i \geqslant 3$, $\lambda_{i,j}$ is a subbundle of the tangent bundle.

For $j = 0$, $i \geqslant 4$, one lets $M_{i,j} = RP(3, 0, \cdots, 0)$ ($i - 3$ 0's) and $\lambda_{i,j} = 1 \oplus \lambda$.

For $j = 1$, $i = 0$, one lets $M_{i,j} = RP(2)$, and $\lambda_{i,j} = \tau$, the tangent bundle of $RP(2)$.

For $j = 1$, $i = 1$, one lets $M_{i,j} = RP(1) \times RP(2)$, the tangent bundle being $1 \oplus \pi_2^*(\tau) = 3\pi_2^*(\xi) = (2\pi_1^*(\xi) \otimes \pi_2^*(\xi)) + \pi_2^*(\xi)$ and lets $\lambda_{i,j} = [\pi_1^*(\xi) \otimes \pi_2^*(\xi)] \oplus \pi_2^*(\xi)$.

For $j = 1$, $i = 2$, let $M_{i,j}$ be the bundle of lines in the fibers of $\lambda \oplus 2$ over $RP(1, 0) = RP(\xi \oplus 1)$ where ξ is the Hopf bundle over $RP(1)$, giving projections

$$\pi\colon M_{i,j} \to RP(1, 0), \qquad p\colon RP(1, 0) \to RP(1).$$

Let θ be the bundle along the fibers of p, η the bundle along the fibers of π, and λ' the canonical line bundle over $M_{i,j}$. Then

$$\tau_{M_{i,j}} = \eta \oplus \pi^*(\tau_{RP(1,0)}) = \eta \oplus \pi^*(\theta) \oplus 1 = (\lambda' \otimes \pi^*(\lambda \oplus 2)) \oplus \pi^*(\theta)$$

which contains a copy of $\lambda_{i,j} = \lambda' \oplus \pi^*(\theta)$.

Finally, let $M_{0,0}$ be a point with $\lambda_{0,0}$ trivial, $M_{1,0} = RP(1)$ with $\lambda_{1,0} = \xi \oplus 1$, and $M_{2,0} = RP(1, 0)$, $M_{3,0} = RP(1, 0, 0)$ with $\lambda_{i,0} = \lambda \oplus 1$.

Note that for $M_{i,0}$, $i \leqslant 3$, $\lambda_{i,0}$ is a subbundle of $\tau \oplus 2$. In particular, if $\alpha \in \mathfrak{N}_p$ and $w_p(\alpha) = 0$, α is represented by a manifold M^p fibered over $S^1 \times S^1$ [5, Proposition 6.1] and hence τ_M has 2 sections, so $\lambda_{i,0}$ is a subbundle of the tangent bundle of $M \times M_{i,0}$.

Every class in $\mathfrak{N}_n(BO_2)$ is of the form $\Sigma\, \alpha_{(i,j)}[M_{i,j}, \lambda_{i,j}]$ with $\alpha_{(i,j)} \in \mathfrak{N}_{n-i-2j}$ and every class $\alpha \in \mathfrak{N}_p$ has the form $\beta + aRP(2)^{p/2}$, $a \in Z_2$, $\beta \in \mathfrak{N}_p$ with $w_p(\beta) = 0$. Thus if I is the $\mathfrak{N}_*$ submodule of classes in $\mathfrak{N}_*(BO_2)$

represented by $[M, f]$ with $f^*(\gamma_2)$ a subbundle of the tangent bundle of M, then $\mathfrak{N}_*(BO_2)/I$ is a Z_2 vector space generated by the classes

$$[RP(2)^s] \cdot [M_{i,0}, \lambda_{i,0}] \qquad \text{with } i \leqslant 3.$$

The characteristic numbers $w_n(\tau - f^*(\gamma_2))$ and $w_1(\tau - f^*(\gamma_2)) \cdot w_{n-1}(\tau - f^*(\gamma_2))$ (for $n \geqslant 2$) may be readily seen to give a homomorphism $\mathfrak{N}_n(BO_2)/I \to Z_2 \oplus Z_2$ (or to Z_2 if $n \leqslant 1$) sending the classes $[RP(2)^s] \cdot [M_{i,0}; \lambda_{i,0}]$ of dimension n to linearly independent elements.

This completes the proof of the proposition.

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