



# Subclasses of bi-univalent functions related to shell-like curves connected with Fibonacci numbers

H. Özlem Güney  
Dicle University, Faculty of Science,  
Department of Mathematics, Turkey  
email: ozlemg@dicle.edu.tr

G. Murugusundaramoorthy  
Department of Mathematics,  
School of Advanced Sciences,  
Vellore Institute of Technology,  
Deemed to be University, India  
email: gmsmoorthy@yahoo.com

J. Sokół  
University of Rzeszów,  
Faculty of Mathematics and  
Natural Sciences, Poland  
email: jsokol@ur.edu.pl

**Abstract.** In this paper, we introduce and investigate new subclasses of bi-univalent functions related to shell-like curves connected with Fibonacci numbers. Furthermore, we find estimates of first two coefficients of functions in these classes. Also, we determine Fekete-Szegő inequalities for these function classes.

## 1 Introduction

Let  $\mathbb{U} = \{z : |z| < 1\}$  denote the unit disc on the complex plane. The class of all analytic functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (1)$$

---

**2010 Mathematics Subject Classification:** 30C45, 30C50

**Key words and phrases:** analytic functions, bi-univalent, shell-like curve, Fibonacci numbers, starlike functions, convex functions

in the open unit disc  $\mathbb{U}$  with normalization  $f(0) = f'(0) - 1 = 0$  is denoted by  $\mathcal{A}$  and the class  $\mathcal{S} \subset \mathcal{A}$  is the class which consists of univalent functions in  $\mathbb{U}$ .

The Koebe one quarter theorem [3] ensures that the image of  $\mathbb{U}$  under every univalent function  $f \in \mathcal{A}$  contains a disk of radius  $\frac{1}{4}$ . Thus every univalent function  $f \in \mathcal{A}$  has an inverse  $f^{-1}$  satisfying

$$f^{-1}(f(z)) = z, \quad (z \in \mathbb{U}) \quad \text{and} \quad f(f^{-1}(w)) = w \left( |w| < r_0(f), \quad r_0(f) \geq \frac{1}{4} \right).$$

A function  $f \in \mathcal{A}$  is said to be bi-univalent in  $\mathbb{U}$  if both  $f$  and  $f^{-1}$  are univalent in  $\mathbb{U}$ . Let  $\Sigma$  denote the class of bi-univalent functions defined in the unit disk  $\mathbb{U}$ . Since  $f \in \Sigma$  has the Maclaurian series given by (1), a computation shows that its inverse  $g = f^{-1}$  has the expansion

$$g(w) = f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3) w^3 + \dots \tag{2}$$

One can see a short history and examples of functions in the class  $\Sigma$  in [12]. Several authors have introduced and investigated subclasses of bi-univalent functions and obtained bounds for the initial coefficients (see [1, 2, 8, 12, 13, 14]).

An analytic function  $f$  is subordinate to an analytic function  $F$  in  $\mathbb{U}$ , written as  $f \prec F$  ( $z \in \mathbb{U}$ ), provided there is an analytic function  $\omega$  defined on  $\mathbb{U}$  with  $\omega(0) = 0$  and  $|\omega(z)| < 1$  satisfying  $f(z) = F(\omega(z))$ . It follows from Schwarz Lemma that

$$f(z) \prec F(z) \iff f(0) = F(0) \quad \text{and} \quad f(\mathbb{U}) \subset F(\mathbb{U}), \quad z \in \mathbb{U}$$

(for details see [3], [7]). We recall important subclasses of  $\mathcal{S}$  in geometric function theory such that if  $f \in \mathcal{A}$  and

$$\frac{zf'(z)}{f(z)} \prec p(z) \quad \text{and} \quad 1 + \frac{zf''(z)}{f'(z)} \prec p(z)$$

where  $p(z) = \frac{1+z}{1-z}$ , then we say that  $f$  is starlike and convex, respectively. These functions form known classes denoted by  $\mathcal{S}^*$  and  $\mathcal{C}$ , respectively. Recently, in [11], Sokół introduced the class  $\mathcal{SL}$  of shell-like functions as the set of functions  $f \in \mathcal{A}$  which is described in the following definition:

**Definition 1** *The function  $f \in \mathcal{A}$  belongs to the class  $\mathcal{SL}$  if it satisfies the condition that*

$$\frac{zf'(z)}{f(z)} \prec \tilde{p}(z)$$

with

$$\tilde{p}(z) = \frac{1 + \tau^2 z^2}{1 - \tau z - \tau^2 z^2},$$

where  $\tau = (1 - \sqrt{5})/2 \approx -0.618$ .

It should be observed  $\mathcal{SL}$  is a subclass of the starlike functions  $\mathcal{S}^*$ .

Later, Dziok et al. in [4] and [5] defined and introduced the class  $\mathcal{KSL}$  and  $\mathcal{SLM}_\alpha$  of convex and  $\alpha$ -convex functions related to a shell-like curve connected with Fibonacci numbers, respectively. These classes can be given in the following definitions.

**Definition 2** The function  $f \in \mathcal{A}$  belongs to the class  $\mathcal{KSL}$  of convex shell-like functions if it satisfies the condition that

$$1 + \frac{zf''(z)}{f'(z)} \prec \tilde{p}(z) = \frac{1 + \tau^2 z^2}{1 - \tau z - \tau^2 z^2},$$

where  $\tau = (1 - \sqrt{5})/2 \approx -0.618$ .

**Definition 3** The function  $f \in \mathcal{A}$  belongs to the class  $\mathcal{SLM}_\alpha$ , ( $0 \leq \alpha \leq 1$ ) if it satisfies the condition that

$$\alpha \left( 1 + \frac{zf''(z)}{f'(z)} \right) + (1 - \alpha) \frac{zf'(z)}{f(z)} \prec \tilde{p}(z) = \frac{1 + \tau^2 z^2}{1 - \tau z - \tau^2 z^2},$$

where  $\tau = (1 - \sqrt{5})/2 \approx -0.618$ .

The class  $\mathcal{SLM}_\alpha$  is related to the class  $\mathcal{KSL}$  only through the function  $\tilde{p}$  and  $\mathcal{SLM}_\alpha \neq \mathcal{KSL}$  for all  $\alpha \neq 1$ . It is easy to see that  $\mathcal{KSL} = \mathcal{SLM}_1$ .

Besides, let's define the class  $\mathcal{SLG}_\gamma$  of so-called gamma-starlike functions related to a shell-like curve connected with Fibonacci numbers as follows.

**Definition 4** The function  $f \in \mathcal{A}$  belongs to the class  $\mathcal{SLG}_\gamma$ , ( $\gamma \geq 0$ ), if it satisfies the condition that

$$\left( \frac{zf'(z)}{f(z)} \right)^\gamma \left( 1 + \frac{zf''(z)}{f'(z)} \right)^{1-\gamma} \prec \tilde{p}(z) = \frac{1 + \tau^2 z^2}{1 - \tau z - \tau^2 z^2},$$

where  $\tau = (1 - \sqrt{5})/2 \approx -0.618$ .

The function  $\tilde{p}$  is not univalent in  $\mathbb{U}$ , but it is univalent in the disc  $|z| < (3 - \sqrt{5})/2 \approx 0.38$ . For example,  $\tilde{p}(0) = \tilde{p}(-1/2\tau) = 1$  and  $\tilde{p}(e^{\mp i \arccos(1/4)}) = \sqrt{5}/5$ , and it may also be noticed that

$$\frac{1}{|\tau|} = \frac{|\tau|}{1 - |\tau|},$$

which shows that the number  $|\tau|$  divides  $[0, 1]$  such that it fulfils the golden section. The image of the unit circle  $|z| = 1$  under  $\tilde{p}$  is a curve described by the equation given by

$$(10x - \sqrt{5})y^2 = (\sqrt{5} - 2x)(\sqrt{5}x - 1)^2,$$

which is translated and revolved trisectrix of Maclaurin. The curve  $\tilde{p}(re^{it})$  is a closed curve without any loops for  $0 < r \leq r_0 = (3 - \sqrt{5})/2 \approx 0.38$ . For  $r_0 < r < 1$ , it has a loop, and for  $r = 1$ , it has a vertical asymptote. Since  $\tau$  satisfies the equation  $\tau^2 = 1 + \tau$ , this expression can be used to obtain higher powers  $\tau^n$  as a linear function of lower powers, which in turn can be decomposed all the way down to a linear combination of  $\tau$  and 1. The resulting recurrence relationships yield Fibonacci numbers  $u_n$ :

$$\tau^n = u_n \tau + u_{n-1}.$$

In [10], taking  $\tau z = t$ , Raina and Sokół showed that

$$\begin{aligned} \tilde{p}(z) &= \frac{1 + \tau^2 z^2}{1 - \tau z - \tau^2 z^2} = \left(t + \frac{1}{t}\right) \frac{t}{1 - t - t^2} \\ &= \frac{1}{\sqrt{5}} \left(t + \frac{1}{t}\right) \left(\frac{1}{1 - (1 - \tau)t} - \frac{1}{1 - \tau t}\right) \\ &= \left(t + \frac{1}{t}\right) \sum_{n=1}^{\infty} \frac{(1 - \tau)^n - \tau^n}{\sqrt{5}} t^n \\ &= \left(t + \frac{1}{t}\right) \sum_{n=1}^{\infty} u_n t^n = 1 + \sum_{n=1}^{\infty} (u_{n-1} + u_{n+1}) \tau^n z^n, \end{aligned} \tag{3}$$

where

$$u_n = \frac{(1 - \tau)^n - \tau^n}{\sqrt{5}}, \tau = \frac{1 - \sqrt{5}}{2} \quad (n = 1, 2, \dots). \tag{4}$$

This shows that the relevant connection of  $\tilde{p}$  with the sequence of Fibonacci numbers  $u_n$ , such that  $u_0 = 0$ ,  $u_1 = 1$ ,  $u_{n+2} = u_n + u_{n+1}$  for  $n = 0, 1, 2, \dots$ .

And they got

$$\begin{aligned} \tilde{p}(z) &= 1 + \sum_{n=1}^{\infty} \tilde{p}_n z^n = 1 + (u_0 + u_2)\tau z + (u_1 + u_3)\tau^2 z^2 \\ &\quad + \sum_{n=3}^{\infty} (u_{n-3} + u_{n-2} + u_{n-1} + u_n)\tau^n z^n \\ &= 1 + \tau z + 3\tau^2 z^2 + 4\tau^3 z^3 + 7\tau^4 z^4 + 11\tau^5 z^5 + \dots \end{aligned} \quad (5)$$

Let  $\mathcal{P}(\beta)$ ,  $0 \leq \beta < 1$ , denote the class of analytic functions  $p$  in  $\mathbb{U}$  with  $p(0) = 1$  and  $\operatorname{Re}\{p(z)\} > \beta$ . Especially, we will use  $\mathcal{P}$  instead of  $\mathcal{P}(0)$ .

**Theorem 1** [5] *The function  $\tilde{p}(z) = \frac{1+\tau^2 z^2}{1-\tau z-\tau^2 z^2}$  belongs to the class  $\mathcal{P}(\beta)$  with  $\beta = \sqrt{5}/10 \approx 0.2236$ .*

Now we give the following lemma which will use in proving.

**Lemma 1** [9] *Let  $p \in \mathcal{P}$  with  $p(z) = 1 + c_1 z + c_2 z^2 + \dots$ , then*

$$|c_n| \leq 2, \quad \text{for } n \geq 1. \quad (6)$$

In this present work, we introduce two subclasses of  $\Sigma$  associated with shell-like functions connected with Fibonacci numbers and obtain the initial Taylor coefficients  $|a_2|$  and  $|a_3|$  for these function classes. Also, we give bounds for the Fekete-Szegő functional  $|a_3 - \mu a_2^2|$  for each subclass.

## 2 Bi-univalent function class $\mathcal{SLM}_{\alpha, \Sigma}(\tilde{p}(z))$

In this section, we introduce a new subclass of  $\Sigma$  associated with shell-like functions connected with Fibonacci numbers and obtain the initial Taylor coefficients  $|a_2|$  and  $|a_3|$  for the function class by subordination.

Firstly, let  $p(z) = 1 + p_1 z + p_2 z^2 + \dots$ , and  $p \prec \tilde{p}$ . Then there exists an analytic function  $u$  such that  $|u(z)| < 1$  in  $\mathbb{U}$  and  $p(z) = \tilde{p}(u(z))$ . Therefore, the function

$$h(z) = \frac{1 + u(z)}{1 - u(z)} = 1 + c_1 z + c_2 z^2 + \dots \quad (7)$$

is in the class  $\mathcal{P}(0)$ . It follows that

$$u(z) = \frac{c_1 z}{2} + \left(c_2 - \frac{c_1^2}{2}\right) \frac{z^2}{2} + \left(c_3 - c_1 c_2 + \frac{c_1^3}{4}\right) \frac{z^3}{2} + \dots \quad (8)$$

and

$$\begin{aligned} \tilde{p}(u(z)) &= 1 + \tilde{p}_1 \left\{ \frac{c_1 z}{2} + \left( c_2 - \frac{c_1^2}{2} \right) \frac{z^2}{2} + \left( c_3 - c_1 c_2 + \frac{c_1^3}{4} \right) \frac{z^3}{2} + \dots \right\} \\ &\quad + \tilde{p}_2 \left\{ \frac{c_1 z}{2} + \left( c_2 - \frac{c_1^2}{2} \right) \frac{z^2}{2} + \left( c_3 - c_1 c_2 + \frac{c_1^3}{4} \right) \frac{z^3}{2} + \dots \right\}^2 \\ &\quad + \tilde{p}_3 \left\{ \frac{c_1 z}{2} + \left( c_2 - \frac{c_1^2}{2} \right) \frac{z^2}{2} + \left( c_3 - c_1 c_2 + \frac{c_1^3}{4} \right) \frac{z^3}{2} + \dots \right\}^3 + \dots \quad (9) \\ &= 1 + \frac{\tilde{p}_1 c_1 z}{2} + \left\{ \frac{1}{2} \left( c_2 - \frac{c_1^2}{2} \right) \tilde{p}_1 + \frac{c_1^2}{4} \tilde{p}_2 \right\} z^2 \\ &\quad + \left\{ \frac{1}{2} \left( c_3 - c_1 c_2 + \frac{c_1^3}{4} \right) \tilde{p}_1 + \frac{1}{2} c_1 \left( c_2 - \frac{c_1^2}{2} \right) \tilde{p}_2 + \frac{c_1^3}{8} \tilde{p}_3 \right\} z^3 + \dots . \end{aligned}$$

And similarly, there exists an analytic function  $v$  such that  $|v(w)| < 1$  in  $\mathbb{U}$  and  $p(w) = \tilde{p}(v(w))$ . Therefore, the function

$$k(w) = \frac{1 + v(w)}{1 - v(w)} = 1 + d_1 w + d_2 w^2 + \dots \quad (10)$$

is in the class  $\mathcal{P}(0)$ . It follows that

$$v(w) = \frac{d_1 w}{2} + \left( d_2 - \frac{d_1^2}{2} \right) \frac{w^2}{2} + \left( d_3 - d_1 d_2 + \frac{d_1^3}{4} \right) \frac{w^3}{2} + \dots \quad (11)$$

and

$$\begin{aligned} \tilde{p}(v(w)) &= 1 + \frac{\tilde{p}_1 d_1 w}{2} + \left\{ \frac{1}{2} \left( d_2 - \frac{d_1^2}{2} \right) \tilde{p}_1 + \frac{d_1^2}{4} \tilde{p}_2 \right\} w^2 \\ &\quad + \left\{ \frac{1}{2} \left( d_3 - d_1 d_2 + \frac{d_1^3}{4} \right) \tilde{p}_1 + \frac{1}{2} d_1 \left( d_2 - \frac{d_1^2}{2} \right) \tilde{p}_2 + \frac{d_1^3}{8} \tilde{p}_3 \right\} w^3 + \dots . \end{aligned} \quad (12)$$

**Definition 5** For  $0 \leq \alpha \leq 1$ , a function  $f \in \Sigma$  of the form (1) is said to be in the class  $\mathcal{SLM}_{\alpha, \Sigma}(\tilde{p}(z))$  if the following subordination hold:

$$\alpha \left( 1 + \frac{zf''(z)}{f'(z)} \right) + (1 - \alpha) \left( \frac{zf'(z)}{f(z)} \right) \prec \tilde{p}(z) = \frac{1 + \tau^2 z^2}{1 - \tau z - \tau^2 z^2} \quad (13)$$

and

$$\alpha \left( 1 + \frac{wg''(w)}{g'(w)} \right) + (1 - \alpha) \left( \frac{wg'(w)}{g(w)} \right) \prec \tilde{p}(w) = \frac{1 + \tau^2 w^2}{1 - \tau w - \tau^2 w^2} \quad (14)$$

where  $\tau = (1 - \sqrt{5})/2 \approx -0.618$  where  $z, w \in \mathbb{U}$  and  $g$  is given by (2).

Specializing the parameter  $\alpha = 0$  and  $\alpha = 1$  we have the following, respectively:

**Definition 6** A function  $f \in \Sigma$  of the form (1) is said to be in the class  $\mathcal{SL}_{\Sigma}(\tilde{p}(z))$  if the following subordination hold:

$$\frac{zf'(z)}{f(z)} \prec \tilde{p}(z) = \frac{1 + \tau^2 z^2}{1 - \tau z - \tau^2 z^2} \quad (15)$$

and

$$\frac{wg'(w)}{g(w)} \prec \tilde{p}(w) = \frac{1 + \tau^2 w^2}{1 - \tau w - \tau^2 w^2} \quad (16)$$

where  $\tau = (1 - \sqrt{5})/2 \approx -0.618$  where  $z, w \in \mathbb{U}$  and  $g$  is given by (2).

**Definition 7** A function  $f \in \Sigma$  of the form (1) is said to be in the class  $\mathcal{KL}_{\Sigma}(\tilde{p}(z))$  if the following subordination hold:

$$1 + \frac{zf''(z)}{f'(z)} \prec \tilde{p}(z) = \frac{1 + \tau^2 z^2}{1 - \tau z - \tau^2 z^2} \quad (17)$$

and

$$1 + \frac{wg''(w)}{g'(w)} \prec \tilde{p}(w) = \frac{1 + \tau^2 w^2}{1 - \tau w - \tau^2 w^2} \quad (18)$$

where  $\tau = (1 - \sqrt{5})/2 \approx -0.618$  where  $z, w \in \mathbb{U}$  and  $g$  is given by (2).

In the following theorem we determine the initial Taylor coefficients  $|a_2|$  and  $|a_3|$  for the function class  $\mathcal{SLM}_{\alpha, \Sigma}(\tilde{p}(z))$ . Later we will reduce these bounds to other classes for special cases.

**Theorem 2** Let  $f$  given by (1) be in the class  $\mathcal{SLM}_{\alpha, \Sigma}(\tilde{p}(z))$ . Then

$$|a_2| \leq \frac{|\tau|}{\sqrt{(1 + \alpha)^2 - (1 + \alpha)(2 + 3\alpha)\tau}} \quad (19)$$

and

$$|a_3| \leq \frac{|\tau| [(1 + \alpha)^2 - (3\alpha^2 + 9\alpha + 4)\tau]}{2(1 + 2\alpha)(1 + \alpha) [(1 + \alpha) - (2 + 3\alpha)\tau]}. \quad (20)$$

**Proof.** Let  $f \in \mathcal{SLM}_{\alpha, \Sigma}(\tilde{p}(z))$  and  $g = f^{-1}$ . Considering (13) and (14), we have

$$\alpha \left( 1 + \frac{zf''(z)}{f'(z)} \right) + (1 - \alpha) \left( \frac{zf'(z)}{f(z)} \right) = \tilde{p}(u(z)) \quad (21)$$

and

$$\alpha \left( 1 + \frac{wg''(w)}{g'(w)} \right) + (1 - \alpha) \left( \frac{wg'(w)}{g(w)} \right) = \tilde{p}(v(w)) \tag{22}$$

where  $\tau = (1 - \sqrt{5})/2 \approx -0.618$  where  $z, w \in \mathbb{U}$  and  $g$  is given by (2). Since

$$\alpha \left( 1 + \frac{zf''(z)}{f'(z)} \right) + (1 - \alpha) \left( \frac{zf'(z)}{f(z)} \right) = 1 + (1 + \alpha)a_2z + (2(1 + 2\alpha)a_3 - (1 + 3\alpha)a_2^2)z^2 + \dots$$

and

$$\alpha \left( 1 + \frac{wg''(w)}{g'(w)} \right) + (1 - \alpha) \left( \frac{wg'(w)}{g(w)} \right) = 1 - (1 + \alpha)a_2w + ((3 + 5\alpha)a_2^2 - 2(1 + 2\alpha)a_3)w^2 + \dots$$

Thus we have

$$\begin{aligned} & 1 + (1 + \alpha)a_2z + (2(1 + 2\alpha)a_3 - (1 + 3\alpha)a_2^2)z^2 + \Delta\Delta\Delta \\ &= 1 + \frac{\tilde{p}_1c_1z}{2} + \left[ \frac{1}{2} \left( c_2 - \frac{c_1^2}{2} \right) \tilde{p}_1 + \frac{c_1^2}{4} \tilde{p}_2 \right] z^2 \\ & \quad + \left[ \frac{1}{2} \left( c_3 - c_1c_2 + \frac{c_1^3}{4} \right) \tilde{p}_1 + \frac{1}{2}c_1 \left( c_2 - \frac{c_1^2}{2} \right) \tilde{p}_2 + \frac{c_1^3}{8} \tilde{p}_3 \right] z^3 + \dots \end{aligned} \tag{23}$$

and

$$\begin{aligned} & 1 - (1 + \alpha)a_2w + ((3 + 5\alpha)a_2^2 - 2(1 + 2\alpha)a_3)w^2 + \Delta\Delta\Delta, \\ &= 1 + \frac{\tilde{p}_1d_1w}{2} + \left[ \frac{1}{2} \left( d_2 - \frac{d_1^2}{2} \right) \tilde{p}_1 + \frac{d_1^2}{4} \tilde{p}_2 \right] w^2 \\ & \quad + \left[ \frac{1}{2} \left( d_3 - d_1d_2 + \frac{d_1^3}{4} \right) \tilde{p}_1 + \frac{1}{2}d_1 \left( d_2 - \frac{d_1^2}{2} \right) \tilde{p}_2 + \frac{d_1^3}{8} \tilde{p}_3 \right] w^3 + \dots \end{aligned} \tag{24}$$

It follows from (23) and (24) that

$$(1 + \alpha)a_2 = \frac{c_1\tau}{2}, \tag{25}$$

$$2(1 + 2\alpha)a_3 - (1 + 3\alpha)a_2^2 = \frac{1}{2} \left( c_2 - \frac{c_1^2}{2} \right) \tau + \frac{c_1^2}{4} 3\tau^2, \tag{26}$$

and

$$-(1 + \alpha)a_2 = \frac{d_1\tau}{2}, \tag{27}$$

$$(3 + 5\alpha)a_2^2 - 2(1 + 2\alpha)a_3 = \frac{1}{2} \left( d_2 - \frac{d_1^2}{2} \right) \tau + \frac{d_1^2}{4} 3\tau^2. \tag{28}$$



From (25) and (27), we have

$$c_1 = -d_1, \quad (29)$$

and

$$2a_2^2 = \frac{(c_1^2 + d_1^2)}{4(1 + \alpha)^2} \tau^2. \quad (30)$$

Now, by summing (26) and (28), we obtain

$$2(1 + \alpha)a_2^2 = \frac{1}{2}(c_2 + d_2)\tau - \frac{1}{4}(c_1^2 + d_1^2)\tau + \frac{3}{4}(c_1^2 + d_1^2)\tau^2. \quad (31)$$

By putting (30) in (31), we have

$$2(1 + \alpha)[(-2 - 3\alpha)\tau + (1 + \alpha)]a_2^2 = \frac{1}{2}(c_2 + d_2)\tau^2. \quad (32)$$

Therefore, using Lemma (1) we obtain

$$|a_2| \leq \frac{|\tau|}{\sqrt{(1 + \alpha)^2 - (1 + \alpha)(2 + 3\alpha)\tau}}. \quad (33)$$

Now, so as to find the bound on  $|a_3|$ , let's subtract from (26) and (28). So, we find

$$4(1 + 2\alpha)a_3 - 4(1 + 2\alpha)a_2^2 = \frac{1}{2}(c_2 - d_2)\tau. \quad (34)$$

Hence, we get

$$4(1 + 2\alpha)|a_3| \leq 2|\tau| + 4(1 + 2\alpha)|a_2|^2. \quad (35)$$

Then, in view of (33), we obtain

$$|a_3| \leq \frac{|\tau| [(1 + \alpha)^2 - (3\alpha^2 + 9\alpha + 4)\tau]}{2(1 + 2\alpha)(1 + \alpha) [(1 + \alpha) - (2 + 3\alpha)\tau]}. \quad (36)$$

□

If we can take the parameter  $\alpha = 0$  and  $\alpha = 1$  in the above theorem, we have the following the initial Taylor coefficients  $|a_2|$  and  $|a_3|$  for the function classes  $\mathcal{SL}_\Sigma(\tilde{p}(z))$  and  $\mathcal{KSL}_\Sigma(\tilde{p}(z))$ , respectively.

**Corollary 1** *Let  $f$  given by (1) be in the class  $\mathcal{SL}_\Sigma(\tilde{p}(z))$ . Then*

$$|a_2| \leq \frac{|\tau|}{\sqrt{1 - 2\tau}} \quad (37)$$

and

$$|a_3| \leq \frac{|\tau|(1 - 4\tau)}{2(1 - 2\tau)}. \quad (38)$$

**Corollary 2** Let  $f$  given by (1) be in the class  $\mathcal{KSL}_\Sigma(\tilde{p}(z))$ . Then

$$|a_2| \leq \frac{|\tau|}{\sqrt{4-10\tau}} \tag{39}$$

and

$$|a_3| \leq \frac{|\tau|(1-4\tau)}{3(2-5\tau)}. \tag{40}$$

### 3 Bi-univalent function class $\mathcal{SLG}_{\gamma,\Sigma}(\tilde{p}(z))$

In this section, we define a new class  $\mathcal{SLG}_{\gamma,\Sigma}(\tilde{p}(z))$  of  $\gamma$ - bi-starlike functions associated with Shell-like domain.

**Definition 8** For  $\gamma \geq 0$ , we let a function  $f \in \Sigma$  given by (1) is said to be in the class  $\mathcal{SLG}_{\gamma,\Sigma}(\tilde{p}(z))$ , if the following conditions are satisfied:

$$\left(\frac{zf'(z)}{f(z)}\right)^\gamma \left(1 + \frac{zf''(z)}{f'(z)}\right)^{1-\gamma} \prec \tilde{p}(z) = \frac{1 + \tau^2 z^2}{1 - \tau z - \tau^2 z^2} \tag{41}$$

and

$$\left(\frac{wg'(w)}{g(w)}\right)^\gamma \left(1 + \frac{wg''(w)}{g'(w)}\right)^{1-\gamma} \prec \tilde{p}(w) = \frac{1 + \tau^2 w^2}{1 - \tau w - \tau^2 w^2}, \tag{42}$$

where  $\tau = (1 - \sqrt{5})/2 \approx -0.618$  where  $z, w \in \mathbb{U}$  and  $g$  is given by (2).

**Remark 1** Taking  $\gamma = 1$ , we get  $\mathcal{SLG}_{1,\Sigma}(\tilde{p}(z)) \equiv \mathcal{SL}_\Sigma(\tilde{p}(z))$  the class as given in Definition 5 satisfying the conditions given in (15) and (16).

**Remark 2** Taking  $\gamma = 0$ , we get  $\mathcal{SLG}_{0,\Sigma}(\tilde{p}(z)) \equiv \mathcal{KL}_\Sigma(\tilde{p}(z))$  the class as given in Definition 6 satisfying the conditions given in (17) and (18).

**Theorem 3** Let  $f$  given by (1) be in the class  $\mathcal{SLG}_{\gamma,\Sigma}(\tilde{p}(z))$ . Then

$$|a_2| \leq \frac{\sqrt{2}|\tau|}{\sqrt{2(2-\gamma)^2 - (5\gamma^2 - 21\gamma + 20)\tau}}$$

and

$$|a_3| \leq \frac{|\tau| [2(2-\gamma)^2 - (5\gamma^2 - 29\gamma + 32)\tau]}{2(3-2\gamma) [2(2-\gamma)^2 - (5\gamma^2 - 21\gamma + 20)\tau]}.$$

**Proof.** Let  $f \in \mathcal{SLG}_{\gamma, \Sigma}(\tilde{p}(z))$  and  $g = f^{-1}$  given by (2) Considering (41) and (42), we have

$$\left(\frac{zf'(z)}{f(z)}\right)^\gamma \left(1 + \frac{zf''(z)}{f'(z)}\right)^{1-\gamma} \prec \tilde{p}(u(z)) \quad (43)$$

and

$$\left(\frac{wg'(w)}{g(w)}\right)^\gamma \left(1 + \frac{wg''(w)}{g'(w)}\right)^{1-\gamma} \prec \tilde{p}(v(w)) \quad (44)$$

where  $\tau = (1 - \sqrt{5})/2 \approx -0.618$  where  $z, w \in \mathbb{U}$  and  $g$  is given by (2). Since,

$$\begin{aligned} \left(\frac{zf'(z)}{f(z)}\right)^\gamma \left(1 + \frac{zf''(z)}{f'(z)}\right)^{1-\gamma} &= 1 + (2 - \gamma)a_2z \\ &+ \left(2(3 - 2\gamma)a_3 + \frac{1}{2}[(\gamma - 2)^2 - 3(4 - 3\gamma)]a_2^2\right)z^2 + \dots \prec \tilde{p}(u(z)) \end{aligned} \quad (45)$$

and

$$\begin{aligned} \left(\frac{wg'(w)}{g(w)}\right)^\gamma \left(1 + \frac{wg''(w)}{g'(w)}\right)^{1-\gamma} &= 1 - (2 - \gamma)a_2w \\ &+ \left([8(1 - \gamma) + \frac{1}{2}\gamma(\gamma + 5)]a_2^2 - 2(3 - 2\gamma)a_3\right)w^2 + \dots \prec \tilde{p}(v(w)). \end{aligned} \quad (46)$$

Equating the coefficients in(45) and (46), with (9) and (12) respectively we get,

$$(2 - \gamma)a_2 = \frac{c_1\tau}{2}, \quad (47)$$

$$2(3 - 2\gamma)a_3 + \frac{1}{2}[(\gamma - 2)^2 - 3(4 - 3\gamma)]a_2^2 = \frac{1}{2} \left(c_2 - \frac{c_1^2}{2}\right)\tau + \frac{c_1^2}{4}3\tau^2, \quad (48)$$

and

$$-(2 - \gamma)a_2 = \frac{d_1\tau}{2}, \quad (49)$$

$$-2(3 - 2\gamma)a_3 + [8(1 - \gamma) + \frac{1}{2}\gamma(\gamma + 5)]a_2^2 = \frac{1}{2} \left(d_2 - \frac{d_1^2}{2}\right)\tau + \frac{d_1^2}{4}3\tau^2. \quad (50)$$

From (47) and (49), we have

$$a_2 = \frac{c_1\tau}{2(2 - \gamma)} = -\frac{d_1\tau}{2(2 - \gamma)}, \quad (51)$$

which implies

$$c_1 = -d_1 \tag{52}$$

and

$$a_2^2 = \frac{(c_1^2 + d_1^2)\tau^2}{8(2 - \gamma)^2}. \tag{53}$$

Now, by summing (48) and (50), we obtain

$$(\gamma^2 - 3\gamma + 4)a_2^2 = \frac{1}{2}(c_2 + d_2)\tau - \frac{1}{4}(c_1^2 + d_1^2)\tau + \frac{3}{4}(c_1^2 + d_1^2)\tau^2. \tag{54}$$

Proceeding similarly as in the earlier proof of Theorem 2, using Lemma (1) we obtain

$$|a_2| \leq \frac{\sqrt{2}|\tau|}{\sqrt{2(2 - \gamma)^2 - (5\gamma^2 - 21\gamma + 20)\tau}}. \tag{55}$$

Now, so as to find the bound on  $|a_3|$ , let's subtract from (48) and (50). So, we find

$$4(3 - 2\gamma)a_3 - 4(3 - 2\gamma)a_2^2 = \frac{1}{2}(c_2 - d_2)\tau. \tag{56}$$

Hence, we get

$$4(1 + 2\gamma)|a_3| \leq 2|\tau| + 4(1 + 2\gamma)|a_2|^2. \tag{57}$$

Then, in view of (55), we obtain

$$|a_3| \leq \frac{|\tau| [2(2 - \gamma)^2 - (5\gamma^2 - 29\gamma + 32)\tau]}{2(3 - 2\gamma) [2(2 - \gamma)^2 - (5\gamma^2 - 21\gamma + 20)\tau]}. \tag{58}$$

□

**Remark 3** *By taking  $\gamma = 1$  and  $\gamma = 0$  in the above theorem, we have the initial Taylor coefficients  $|a_2|$  and  $|a_3|$  for the function classes  $\mathcal{SL}_\Sigma(\tilde{p}(z))$  and  $\mathcal{KSL}_\Sigma(\tilde{p}(z))$ , as stated in Corollary 1 and Corollary 2 respectively.*

## 4 Fekete-Szegö inequalities for the function classes $\mathcal{SLM}_{\alpha,\Sigma}(\tilde{p}(z))$ and $\mathcal{SLG}_{\gamma,\Sigma}(\tilde{p}(z))$

Fekete and Szegö [6] introduced the generalized functional  $|a_3 - \mu a_2^2|$ , where  $\mu$  is some real number. Due to Zaprawa [15], in the following theorem we determine the Fekete-Szegö functional for  $f \in \mathcal{SLM}_{\alpha,\Sigma}(\tilde{p}(z))$ .

**Theorem 4** Let  $f$  given by (1) be in the class  $\mathcal{SLM}_{\alpha,\Sigma}(\tilde{p}(z))$  and  $\mu \in \mathbb{R}$ . Then we have

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{|\tau|}{2(1+2\alpha)}, & |\mu - 1| \leq \frac{(1+\alpha)[(1+\alpha)-(2+3\alpha)\tau]}{2(1+2\alpha)|\tau|} \\ \frac{|1-\mu|\tau^2}{(1+\alpha)[(1+\alpha)-(2+3\alpha)\tau]}, & |\mu - 1| \geq \frac{(1+\alpha)[(1+\alpha)-(2+3\alpha)\tau]}{2(1+2\alpha)|\tau|} \end{cases}.$$

**Proof.** From (32) and (34) we obtain

$$\begin{aligned} a_3 - \mu a_2^2 &= (1 - \mu) \frac{\tau^2(c_2 + d_2)}{4(1 + \alpha)[(1 + \alpha) - (2 + 3\alpha)\tau]} + \frac{\tau(c_2 - d_2)}{8(1 + 2\alpha)} \\ &= \left( \frac{(1 - \mu)\tau^2}{4(1 + \alpha)[(1 + \alpha) - (2 + 3\alpha)\tau]} + \frac{\tau}{8(1 + 2\alpha)} \right) c_2 \\ &\quad + \left( \frac{(1 - \mu)\tau^2}{4(1 + \alpha)[(1 + \alpha) - (2 + 3\alpha)\tau]} - \frac{\tau}{8(1 + 2\alpha)} \right) d_2. \end{aligned} \quad (59)$$

So we have

$$a_3 - \mu a_2^2 = \left( h(\mu) + \frac{|\tau|}{8(1 + 2\alpha)} \right) c_2 + \left( h(\mu) - \frac{|\tau|}{8(1 + 2\alpha)} \right) d_2 \quad (60)$$

where

$$h(\mu) = \frac{(1 - \mu)\tau^2}{4(1 + \alpha)[(1 + \alpha) - (2 + 3\alpha)\tau]}. \quad (61)$$

Then, by taking modulus of (60), we conclude that

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{|\tau|}{2(1+2\alpha)}, & 0 \leq |h(\mu)| \leq \frac{|\tau|}{8(1+2\alpha)} \\ 4|h(\mu)|, & |h(\mu)| \geq \frac{|\tau|}{8(1+2\alpha)} \end{cases}.$$

Taking  $\mu = 1$ , we have the following corollary.

**Corollary 3** If  $f \in \mathcal{SLM}_{\alpha,\Sigma}(\tilde{p}(z))$ , then

$$|a_3 - a_2^2| \leq \frac{|\tau|}{2(1 + 2\alpha)}. \quad (62)$$

If we can take the parameter  $\alpha = 0$  and  $\alpha = 1$  in the above theorem, we have the following the Fekete-Szegő inequalities for the function classes  $\mathcal{SL}_{\Sigma}(\tilde{p}(z))$  and  $\mathcal{KSL}_{\Sigma}(\tilde{p}(z))$ , respectively.

**Corollary 4** Let  $f$  given by (1) be in the class  $\mathcal{SL}_\Sigma(\tilde{p}(z))$  and  $\mu \in \mathbb{R}$ . Then we have

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{|\tau|}{2}, & |\mu - 1| \leq \frac{1-2\tau}{2|\tau|} \\ \frac{|1-\mu|\tau^2}{1-2\tau}, & |\mu - 1| \geq \frac{1-2\tau}{2|\tau|} \end{cases} .$$

**Corollary 5** Let  $f$  given by (1) be in the class  $\mathcal{KSL}_\Sigma(\tilde{p}(z))$  and  $\mu \in \mathbb{R}$ . Then we have

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{|\tau|}{6}, & |\mu - 1| \leq \frac{2-5\tau}{3|\tau|} \\ \frac{|1-\mu|\tau^2}{2(2-5\tau)}, & |\mu - 1| \geq \frac{2-5\tau}{3|\tau|} \end{cases} .$$

In the following theorem, we find the Fekete-Szegő functional for  $f \in \mathcal{SLG}_{\gamma,\Sigma}(\tilde{p}(z))$ .

**Theorem 5** Let  $f$  given by (1) be in the class  $\mathcal{SLG}_{\gamma,\Sigma}(\tilde{p}(z))$  and  $\mu \in \mathbb{R}$ . Then we have

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{|\tau|}{2(3-2\gamma)}, & |\mu - 1| \leq \frac{2(2-\gamma)^2 + (-5\gamma^2 + 21\gamma - 20)\tau}{4(3-2\gamma)|\tau|} \\ \frac{2|1-\mu|\tau^2}{2(2-\gamma)^2 + [(-5\gamma^2 + 21\gamma - 20)\tau]}, & |\mu - 1| \geq \frac{2(2-\gamma)^2 + (-5\gamma^2 + 21\gamma - 20)\tau}{4(3-2\gamma)|\tau|} \end{cases} .$$

Taking  $\mu = 1$ , we have the following corollary.

**Corollary 6** If  $f \in \mathcal{SLG}_{\gamma,\Sigma}(\tilde{p}(z))$ , then

$$|a_3 - a_2^2| \leq \frac{|\tau|}{2(3-2\gamma)}. \tag{63}$$

By taking  $\gamma = 1$  and  $\gamma = 0$  in the above theorem, we have the Fekete-Szegő inequality for the function classes  $\mathcal{SL}_\Sigma(\tilde{p}(z))$  and  $\mathcal{KSL}_\Sigma(\tilde{p}(z))$ , as stated in Corollary 4 and Corollary 5, respectively.

## References

- [1] D. A. Brannan, J. Clunie and W. E. Kirwan, Coefficient estimates for a class of star-like functions, *Canad. J. Math.*, **22** (1970), 476–485.
- [2] D. A. Brannan and T. S. Taha, On some classes of bi-univalent functions, *Stud. Univ. Babeş-Bolyai Math.*, **31** (2) (1986), 70–77.
- [3] P. L. Duren, *Univalent Functions*. In: Grundlehren der Mathematischen Wissenschaften, Band 259, New York, Berlin, Heidelberg and Tokyo, Springer-Verlag, 1983.

- [4] J. Dziok, R. K. Raina, J. Sokół, Certain results for a class of convex functions related to a shell-like curve connected with Fibonacci numbers, *Comp. Math. Appl.*, **61** (2011), 2605–2613.
- [5] J. Dziok, R. K. Raina, J. Sokół, On  $\alpha$ -convex functions related to a shell-like curve connected with Fibonacci numbers, *Appl. Math. Comp.*, **218** (2011), 996–1002.
- [6] M. Fekete, G. Szegő, Eine Bemerkung über ungerade schlichte Functionen, *J. London Math. Soc.*, **8** (1933), 85–89.
- [7] S. S. Miller and P. T. Mocanu Differential Subordinations Theory and Applications, *Series of Monographs and Text Books in Pure and Applied Mathematics*, **225**, Marcel Dekker, New York (2000).
- [8] M. Lewin, On a coefficient problem for bi-univalent functions, *Proc. Amer. Math. Soc.*, **18** (1967), 63–68 .
- [9] Ch. Pommerenke, *Univalent Functions*, Math. Math, Lehrbucher, Vandenhoeck and Ruprecht, Göttingen, 1975.
- [10] R. K. Raina, J. Sokół, Fekete-Szegő problem for some starlike functions related to shell-like curves, *Math. Slovaca*, **66** (2016), 135–140.
- [11] J. Sokół, On starlike functions connected with Fibonacci numbers, *Folia Scient. Univ. Tech. Resoviensis*, **175** (1999), 111–116.
- [12] H. M. Srivastava, A. K. Mishra and P. Gochhayat, Certain subclasses of analytic and bi-univalent functions, *Appl. Math. Lett.*, **23** (10) (2010), 1188–1192.
- [13] Q.-H. Xu, Y.-C. Gui and H. M. Srivastava, Coefficient estimates for a certain subclass of analytic and bi-univalent functions, *Appl. Math. Lett.*, **25** (2012), 990–994.
- [14] X-F. Li and A-P Wang, Two new subclasses of bi-univalent functions, *International Mathematical Forum*, **7** (30) (2012), 1495–1504.
- [15] P. Zaprawa, On the Fekete-Szegő problem for classes of bi-univalent functions, *Bull. Belg. Math. Soc. Simon Stevin*, **21** (1) (2014), 169–178.

*Received: April 29, 2018*