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SUBCOMPLEXES OF BOX COMPLEXES OF GRAPHS

By

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Abstract. The box complex $\mathsf{B}(G)$ of a graph G is a simplicial $\mathbf{Z}_2$-complex defined by J. Matoušek and G. M. Ziegler in [4]. They proved that $\chi(G) \geq \mathrm{ind}_{\mathbf{Z}_2}(\|\mathsf{B}(G)\|) + 2$, where $\chi(G)$ is the chromatic number of G and $\mathrm{ind}_{\mathbf{Z}_2}(\|\mathsf{B}(G)\|)$ is the $\mathbf{Z}_2$-index of $\mathsf{B}(G)$. In this paper, to study topology of box complexes, for the union $G \cup H$ of two graphs G and H, we compare $\mathsf{B}(G \cup H)$ with its subcomplex $\mathsf{B}(G) \cup \mathsf{B}(H)$. We give a sufficient condition on G and H so that $\mathsf{B}(G \cup H) = \mathsf{B}(G) \cup \mathsf{B}(H)$ and $\mathsf{B}(G \cap H) = \mathsf{B}(G) \cap \mathsf{B}(H)$ hold. Moreover, under that condition, we show

$$\max\{\chi(G), \chi(H)\} \leq \chi(G \cup H) \leq \max\{\chi(G) + l_H, \chi(H)\},$$

where l_H is the number defined in Definition 3.8. Also we prove

$$\mathrm{ind}_{\mathbf{Z}_2}(\|\mathsf{B}(G \cup H)\|) = \max\{\mathrm{ind}_{\mathbf{Z}_2}(\|\mathsf{B}(G)\|), \mathrm{ind}_{\mathbf{Z}_2}(\|\mathsf{B}(H)\|)\}$$

if $\max\{\mathrm{ind}_{\mathbf{Z}_2}(\|\mathsf{B}(G)\|), \mathrm{ind}_{\mathbf{Z}_2}(\|\mathsf{B}(H)\|)\} \geq 1$.

The complex $\mathsf{B}(G)$ of a graph G contains a 1-dimensional free $\mathbf{Z}_2$-subcomplex $\bar{G}$ of $\mathsf{B}(G)$, defined in [2]. As a supplement to [2], we show that for a connected graph G, $\mathsf{B}(G)$ is disconnected if and only if $\bar{G}$ is disconnected if and only if G contains no cycles of odd length, or equivalently, G is bipartite.

1. Introduction

In this paper, we assume that all graphs are finite, simple, undirected and connected. The box complex $\mathsf{B}(G)$ of a graph G is introduced in [4] by J. Matoušek and G. M. Ziegler as one of applications of topological methods to

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obtain a lower bound for the chromatic number $\chi(G)$ of G. The following theorem, in [4], indicates that a lower bound for $\chi(G)$ is obtained from the topology of the complex $\mathsf{B}(G)$ of G.

Theorem 1.1 ([4], p. 81). *For any graph G, we have*

$$\chi(G) \geq \mathrm{ind}_{\mathbf{Z}_2}(\|\mathsf{B}(G)\|) + 2. \tag{1.1}$$

This motivates us to study the relation between topology of box complexes and combinatorics of graphs. In order to obtain a lower bound for $\chi(G)$ by the inequality (1.1), we need to know the $\mathbf{Z}_2$-index of $\|\mathsf{B}(G)\|$, while it is not easy in general to obtain topological information of $\mathsf{B}(G)$ from the definition except for a few examples: complete graphs, paths and cycles etc.

To study the complex $\|\mathsf{B}(G)\|$, we decompose G into subgraphs $G_1, \dots, G_k$ and compare $\mathsf{B}(G)$ with $\bigcup_{i=1}^k \mathsf{B}(G_i)$. It is easy to see that $\mathsf{B}(G)$ contains $\bigcup_{i=1}^k \mathsf{B}(G_i)$ as a subcomplex. One cannot hope that $\mathsf{B}(G) = \bigcup_{i=1}^k \mathsf{B}(G_i)$ and for $i, j = 1, \dots, k$, $\mathsf{B}(G_i) \cap \mathsf{B}(G_j) = \mathsf{B}(G_i \cap G_j)$ in general. We confine ourselves to the case $k = 2$. For the union $G \cup H$ of two graphs G and H, we give a sufficient condition under which $\mathsf{B}(G \cup H) = \mathsf{B}(G) \cup \mathsf{B}(H)$ and $\mathsf{B}(G) \cap \mathsf{B}(H) = \mathsf{B}(G \cap H)$ hold (see Theorem 3.3). For such a graph $G \cup H$, we obtain the following estimate of the chromatic number $\chi(G \cup H)$ in Theorem 3.9:

$$\max\{\chi(G), \chi(H)\} \leq \chi(G \cup H) \leq \max\{\chi(G) + l_H, \chi(H)\}, \tag{1.2}$$

where l_H is the number defined in Definition 3.8. In view of (1.1) and (1.2), it is natural to seek an estimate of $\mathrm{ind}_{\mathbf{Z}_2}(\|\mathsf{B}(G \cup H)\|)$. We prove

$$\mathrm{ind}_{\mathbf{Z}_2}(\|\mathsf{B}(G \cup H)\|) = \max\{\mathrm{ind}_{\mathbf{Z}_2}(\|\mathsf{B}(G)\|), \mathrm{ind}_{\mathbf{Z}_2}(\|\mathsf{B}(H)\|)\} \tag{1.3}$$

if $\max\{\mathrm{ind}_{\mathbf{Z}_2}(\|\mathsf{B}(G)\|), \mathrm{ind}_{\mathbf{Z}_2}(\|\mathsf{B}(H)\|)\} \geq 1$ (see Theorem 3.10). The inequalities (1.1), (1.2) and the equality (1.3) imply that, for the union $G \cup H$ satisfying the condition of Theorem 3.3, the lower bound $\mathrm{ind}_{\mathbf{Z}_2}(\|\mathsf{B}(G \cup H)\|) + 2$ is not better than the trivial one $\max\{\chi(G), \chi(H)\}$ for $\chi(G \cup H)$.

Appendix is a supplement to section 4 of [2]. In [2], a 1-dimensional free $\mathbf{Z}_2$-complex $\overline{G}$ is defined as a subcomplex of $\mathsf{B}(G)$. It is proved that a graph G contains no 4-cycles if and only if $\|\overline{G}\|$ is a strong $\mathbf{Z}_2$-deformation retract of $\|\mathsf{B}(G)\|$ ([2], Theorem 4.3). This indicates $\mathrm{ind}_{\mathbf{Z}_2}(\|\mathsf{B}(G)\|) = \mathrm{ind}_{\mathbf{Z}_2}(\|\overline{G}\|) \leq 1$ when G contains no 4-cycles. In appendix, we investigate the relation between $\mathsf{B}(G)$ and $\overline{G}$ for a general graph G. It turns out that $\overline{G}$ is a natural double covering of G. We prove that $\mathsf{B}(G)$ is disconnected if and only if $\overline{G}$ is disconnected (see Theorem 4.2) if and only if G contains no cycles of odd length, or equivalently, G is bipartite (see [1], Theorem 1.6.1).

2. Preliminaries

First, we recall some basic notions on graphs, abstract simplicial complexes, and the $\mathbf{Z}_2$-index of a $\mathbf{Z}_2$-space. We follow [1] about the standard notation in graph theory.

A *graph* is a pair $G = (V(G), E(G))$, where $V(G)$ is a finite set and $E(G)$ is a family of 2-element subsets of $V(G)$. Under this definition, every graph is simple, that is, it has no loops and multiple edges. Elements of $V(G)$ are called vertices of G and those of $E(G)$ are called edges of G. Two vertices u and v of G are *adjacent*, if $\{u, v\}$ is an edge of G. An edge $\{u, v\}$ of a graph is simply denoted by uv or vu. A subset A of $V(G)$ is said to be *independent* in G, if no two vertices of A are adjacent in G. A vertex of G which is only adjacent to one vertex of G is called an *endvertex*. For two graphs G and H, the union $G \cup H$ is defined by $V(G \cup H) = V(G) \cup V(H)$ and $E(G \cup H) = E(G) \cup E(H)$. If $V(G) \cap V(H) \neq \phi$, the intersection $G \cap H$ is defined by $V(G \cap H) = V(G) \cap V(H)$ and $E(G \cap H) = E(G) \cap E(H)$. A *k-coloring* of G is a map $c : V(G) \to \{1, \dots, k\}$ such that $c(u) \neq c(v)$ whenever $uv \in E(G)$. The *chromatic number* of G, denoted by $\chi(G)$, is the minimum number k such that there exists a k-coloring of G.

An *abstract simplicial complex* is a pair (V, K), where V is a finite set and K is a family of subsets of V such that if $\sigma \in \mathsf{K}$ and $\tau \subset \sigma$, then $\tau \in \mathsf{K}$. The *polyhedron* of K is denoted by $\|\mathsf{K}\|$. The *nth barycentric subdivision* of K is denoted by $\mathsf{sd}^n\,\mathsf{K}$. For a vertex v of K, the *star* of v in K, denoted by $\mathrm{st}_{\mathsf{K}}(v)$, is the union of all interiors of simplices of K which contain v. The *link* of v in K, denoted by $\mathrm{lk}_{\mathsf{K}}(v)$, is the set $\overline{\mathrm{st}_{\mathsf{K}}(v)} \backslash \mathrm{st}_{\mathsf{K}}(v)$, where $\overline{\mathrm{st}_{\mathsf{K}}(v)}$ is the union of all simplices with v.

A $\mathbf{Z}_2$*-space* (X, ν_X) is a topological space X with a homeomorphism $\nu : X \to X$ such that $\nu^2 = \mathrm{id}_X$, called a $\mathbf{Z}_2$*-action* ν on X. A $\mathbf{Z}_2$-action which has no fixed points is said to be *free* (and a space X with a free $\mathbf{Z}_2$-action is also said to be a *free* $\mathbf{Z}_2$-space).

Example 2.1. The n-dimensional sphere $S^n = \{x \in \mathbf{R}^{n+1} \mid \|x\| = 1\}$ with the antipodal map $x \mapsto -x$ is a free $\mathbf{Z}_2$-space. We always think of S^n as a free $\mathbf{Z}_2$-space with this action.

For two $\mathbf{Z}_2$-spaces (X, ν_X) and (Y, ν_Y), a continuous map $f : X \to Y$ which satisfies $\nu_Y \circ f = f \circ \nu_X$ is called a $\mathbf{Z}_2$*-map* from X to Y. For a $\mathbf{Z}_2$-space (X, ν), the $\mathbf{Z}_2$*-index* of (X, ν) is defined as

$$\mathrm{ind}_{\mathbf{Z}_2}(X, \nu) := \min\{n \mid \text{there is a } \mathbf{Z}_2\text{-map } X \to S^n\}.$$

Next, following [3], we introduce the box complex of a graph. Let G be a graph and A a subset of $V(G)$. A vertex v of G is called a *common neighbor* of A if $va \in E(G)$ for all $a \in A$. The set of all common neighbors of A is denoted by $\mathrm{CN}_G(A)$. For a one point set $\{a\}$, we see $\mathrm{CN}_G(\{a\})$ is the set of all neighbors of a in G. It is simply denoted by $\mathrm{CN}_G(a)$. For convenience, we define $\mathrm{CN}_G(\phi) = V(G)$. The following holds:

$$A \subseteq B \Rightarrow \mathrm{CN}_G(A) \supseteq \mathrm{CN}_G(B). \tag{2.1}$$

For $A_1, A_2 \subseteq V(G)$ such that $A_1 \cap A_2 = \phi$, we define $G[A_1, A_2]$ as the bipartite subgraph of G with

$$V(G[A_1, A_2]) = A_1 \cup A_2 \text{ and } E(G[A_1, A_2]) = \{a_1 a_2 \in E(G) \mid a_1 \in A_1,\ a_2 \in A_2\}.$$

The bipartite subgraph $G[A_1, A_2]$ is said to be *complete* if $a_1 a_2 \in E(G)$ for all $a_1 \in A_1$ and $a_2 \in A_2$. For convenience, $G[\phi, A_2]$ and $G[A_1, \phi]$ are also said to be complete.

Let A_1 and A_2 be subsets of $V(G)$. The subset $A_1 \uplus A_2$ of $V(G) \times \{1, 2\}$ is defined as

$$A_1 \uplus A_2 := (A_1 \times \{1\}) \cup (A_2 \times \{2\}).$$

For vertices $a_1, a_2 \in V(G)$, $\{a_1\} \uplus \phi$, $\phi \uplus \{a_2\}$, and $\{a_1\} \uplus \{a_2\}$ are simply denoted by $a_1 \uplus \phi$, $\phi \uplus a_2$ and $a_1 \uplus a_2$ respectively.

The *box complex* of a graph G is an abstract simplicial complex with the vertex set $V(G) \times \{1, 2\}$ defined by

$$\begin{aligned} \mathsf{B}(G) = \{A_1 \uplus A_2 \mid {} & A_1, A_2 \subseteq V(G), A_1 \cap A_2 = \phi, \\ & G[A_1, A_2] \text{ is complete, } \mathrm{CN}_G(A_1) \neq \phi \neq \mathrm{CN}_G(A_2)\}. \end{aligned}$$

Whenever we consider the polyhedron $\|\mathsf{B}(G)\|$, an abstract simplex $A_1 \uplus A_2$ and its geometric simplex are denoted by the same symbol $A_1 \uplus A_2$. The simplicial map $\nu : V(\mathsf{B}(G)) \to V(\mathsf{B}(G))$ defined by

$$v \uplus \phi \mapsto \phi \uplus v \quad \text{and} \quad \phi \uplus v \mapsto v \uplus \phi \quad \text{for all } v \in V(G)$$

induces a free $\mathbf{Z}_2$-action on $\|\mathsf{B}(G)\|$. We always think of $\|\mathsf{B}(G)\|$ as a free $\mathbf{Z}_2$-space with this action.

3. Decomposition of Box Complexes

In this section, to study the box complex $\mathsf{B}(G)$ of a graph G, first we take a decomposition $G = \bigcup_{i=1}^{k} G_i$ and compare $\mathsf{B}(G)$ with its subcomplex

$\bigcup_{i=1}^{k} \mathsf{B}(G_i)$. In the following theorem, we give a sufficient condition so that $\mathsf{B}(G) = \bigcup_{i=1}^{k} \mathsf{B}(G_i)$.

Theorem 3.1. *Let G be a graph and assume that G is represented by the union $G = \bigcup_{i=1}^{k} G_i$, where $G_1, \dots, G_k$ are the subgraphs of G such that*

> *for each maximal subset $M_1 \uplus M_2 \subseteq V(G) \times \{1,2\}$ with respect to the condition $G[M_1, M_2]$ is complete, there is an $i \in \{1, \dots, k\}$ so that $G_i[M_1, M_2]$ is complete.*

Then we obtain

$$\mathsf{B}(G) = \bigcup_{i=1}^{k} \mathsf{B}(G_i).$$

Before proving this theorem, we prove the following lemma.

Lemma 3.2. *Let $G = \bigcup_{i=1}^{k} G_i$ be a graph and assume that $G_1, \dots, G_k$ satisfy the assumption of Theorem 3.1. Then for any subset $A \subseteq V(G)$ such that $\mathrm{CN}_G(A) \neq \phi$, there is an $i \in \{1, \dots, k\}$ such that $\mathrm{CN}_{G_i}(A) \neq \phi$.*

Proof. For a subset A of $V(G)$ such that $\mathrm{CN}_G(A) \neq \phi$, we notice that $G[A, \mathrm{CN}_G(A)]$ is complete. Let $M_1 \uplus M_2$ be a maximal subset of $V(G) \times \{1,2\}$ with respect to $A \subseteq M_1$, $\mathrm{CN}_G(A) \subseteq M_2$ and the condition $G[M_1, M_2]$ is complete. By the assumption, there is an $i \in \{1, \dots, k\}$ such that $G_i[M_1, M_2]$ is complete. Hence, we see $G_i[A, \mathrm{CN}_G(A)]$ is complete. Thus, we obtain $\mathrm{CN}_{G_i}(A) \supseteq \mathrm{CN}_G(A) \neq \phi$, and hence, $\mathrm{CN}_{G_i}(A) \neq \phi$. $\square$

Proof of Theorem 3.1. It follows from the definition of box complex that $\mathsf{B}(G) \supset \bigcup_{i=1}^{k} \mathsf{B}(G_i)$. To show $\mathsf{B}(G) \subset \bigcup_{i=1}^{k} \mathsf{B}(G_i)$, we prove that each simplex of $\mathsf{B}(G)$ is a simplex of some $\mathsf{B}(G_i)$.

(i) For each simplex of the form $A \uplus \phi, \phi \uplus A \in \mathsf{B}(G)$, where A is nonempty, we have $\mathrm{CN}_G(A) \neq \phi$. By Lemma 3.2, there is an $i \in \{1, \dots, k\}$ such that $\mathrm{CN}_{G_i}(A) \neq \phi$. Thus, $A \uplus \phi, \phi \uplus A \in \mathsf{B}(G_i)$.

(ii) For each simplex of the form $A_1 \uplus A_2 \in \mathsf{B}(G)$, where both A_1 and A_2 are nonempty, let $M_1 \uplus M_2$ be a maximal subset of $V(G) \times \{1,2\}$ with respect to $A_1 \subseteq M_1$, $A_2 \subseteq M_2$ and the condition $G[M_1, M_2]$ is complete. By the assumption of this theorem, there is an $i \in \{1, \dots, k\}$ such that $G_i[M_1, M_2]$ is complete. Then, we see that $G_i[A_1, A_2]$ is complete, and hence, $A_1 \uplus A_2 \in \mathsf{B}(G_i)$.

These prove the desired inclusion $\mathsf{B}(G) \subset \bigcup_{i=1}^{k} \mathsf{B}(G_i)$. $\square$

In what follows, we confine ourselves to the case $k = 2$. Next, we present a sufficient condition on $G \cup H$ such that $\mathsf{B}(G) \cap \mathsf{B}(H) = \mathsf{B}(G \cap H)$ in addition to $\mathsf{B}(G \cup H) = \mathsf{B}(G) \cup \mathsf{B}(H)$.

Theorem 3.3. *Let $G \cup H$ be the union of two graphs G and H, and assume that the intersection $G \cap H$ is of the form*:

$$V(G \cap H) = \{u_1, \dots, u_k, v_1, \dots, v_k\} \quad \textit{and} \quad E(G \cap H) = \{u_i v_i \mid i = 1, \dots, k\}.$$

Further we assume that

(1) *$u_1, \dots, u_k$ are endvertices of H*,

(2) *$v_1, \dots, v_k$ are endvertices of G and*

(3) *the set $\{u_1, \dots, u_k\}$ is independent in G.*

Then, we obtain

$$\mathsf{B}(G \cup H) = \mathsf{B}(G) \cup \mathsf{B}(H) \quad \textit{and} \quad \mathsf{B}(G \cap H) = \mathsf{B}(G) \cap \mathsf{B}(H).$$

Note. *Under the condition of Theorem* 3.3, *we notice $u_i v_j \notin E(G \cup H)$ for $i \neq j$. Indeed, we see $u_i v_j \notin E(H)$ for $i \neq j$ by* (1) *and $u_i v_i \in E(H)$. We obtain $u_i v_j \notin E(G)$ for $i \neq j$ by* (2) *and $u_j v_j \in E(G)$.*

Also we notice that

$$\mathsf{B}(G \cap H) = \{u_i \uplus v_i, v_i \uplus u_i \mid i = 1, \dots, k\},$$

the disjoint union of $2k$ *1-simplices, since the intersection $G \cap H$ consists of disjoint k edges.*

To prove $\mathsf{B}(G \cup H) = \mathsf{B}(G) \cup \mathsf{B}(H)$ for the union $G \cup H$ with the condition given in Theorem 3.3, *we present the following two lemmas.*

Lemma 3.4. *Let $G \cup H$ be the union of two graphs G and H with the intersection $G \cap H$ defined by*

$$V(G \cap H) = \{u_1, \dots, u_k, v_1, \dots, v_k\} \quad \textit{and} \quad E(G \cap H) = \{u_i v_i \mid i = 1, \dots, k\}.$$

We assume (1) *and* (2) *of Theorem* 3.3. *If $(G \cup H)[M_1, M_2]$ is complete, we have*

$$M_1, M_2 \subseteq V(G) \quad \textit{or} \quad M_1, M_2 \subseteq V(H).$$

Proof. We assume $(G \cup H)[M_1, M_2]$ is complete. Suppose that

$$\text{“}M_1 \not\subseteq V(G) \text{ or } M_2 \not\subseteq V(G)\text{”} \quad \text{and} \quad \text{“}M_1 \not\subseteq V(H) \text{ or } M_2 \not\subseteq V(H)\text{”}.$$

Our consideration is divided into four cases.

Case 1. $M_1 \not\subset V(G)$ and $M_1 \not\subset V(H)$. There are two vertices $m_1, m_1' \in M_1$ such that $m_1 \in V(H) \backslash V(G)$ and $m_1' \in V(G) \backslash V(H)$. Then, we show that

$$\text{for any } m_2 \in M_2, \text{ either } m_1 \text{ or } m_1' \text{ is not adjacent to } m_2 \text{ in } G \cup H. \quad (*)$$

If both m_1 and m_1' are adjacent to m_2 in $G \cup H$, we notice $m_1 m_2 \in E(H)$ and $m_1' m_2 \in E(G)$ since $m_1 \notin V(G)$ and $m_1' \notin V(H)$. Then, we see $m_2 \in V(G \cap H) = \{u_1, \dots, u_k, v_1, \dots, v_k\}$. If $m_2 = u_i$, then $m_1 = v_i \in V(G)$ by the assumptions (1) and $u_i v_i, m_1 u_i \in E(H)$. This contradicts the choice of $m_1 \notin V(G)$. If $m_2 = v_j$, then $m_1' = u_j \in V(H)$ by the assumptions (2) and $u_j v_j, m_1' m_2 \in E(G)$. This also contradicts the choice of $m_1' \notin V(H)$.

However, the statement $(*)$ contradicts the assumption that $(G \cup H)[M_1, M_2]$ is complete.

Case 2. $M_2 \not\subset V(G)$ and $M_2 \not\subset V(H)$. We can derive a contradiction from the same argument as above **Case 1**.

Case 3. $M_1 \not\subset V(G)$ and $M_2 \not\subset V(H)$. There are two vertices $m_1 \in M_1$ and $m_2 \in M_2$ such that $m_1 \in V(H) \backslash V(G)$ and $m_2 \in V(G) \backslash V(H)$. Then, m_1 is not adjacent to m_2 in $G \cup H$. This contradicts the assumption that $(G \cup H)[M_1, M_2]$ is complete.

Case 4. $M_2 \not\subset V(G)$ and $M_1 \not\subset V(H)$. We can derive a contradiction from the same argument as above **Case 3**.

In all cases, we derived contradictions, and hence, our statement is proved. □

Lemma 3.5. *Let $G \cup H$ be the union of two graphs G and H with the intersection $G \cap H$ defined by*

$$V(G \cap H) = \{u_1, \dots, u_k, v_1, \dots, v_k\} \quad \textit{and} \quad E(G \cap H) = \{u_i v_i \mid i = 1, \dots, k\}.$$

We assume the condition of Theorem 3.3. *If $(G \cup H)[M_1, M_2]$ is complete, we have*

$$G[M_1, M_2] \text{ is complete} \quad \text{or} \quad H[M_1, M_2] \text{ is complete.}$$

Proof. We assume that $(G \cup H)[M_1, M_2]$ is complete. By Lemma 3.4, we see $M_1, M_2 \subset V(G)$ or $M_1, M_2 \subset V(H)$. Suppose that neither $G[M_1, M_2]$ nor $H[M_1, M_2]$ is complete. Our consideration is divided into two cases.

Case 1. $M_1, M_2 \subset V(G)$. As $G[M_1, M_2]$ is not complete, there are two vertices $m_1 \in M_1$ and $m_2 \in M_2$ such that $m_1 m_2 \in E(H) \backslash E(G)$. Hence, we see

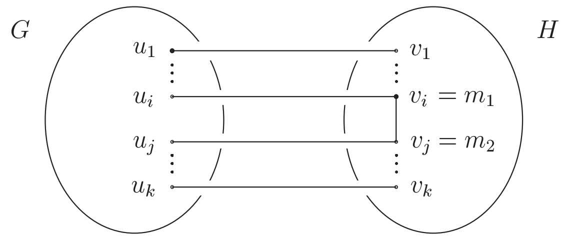

Figure. The union $G \cup H$ of two graphs G and H.

$m_1, m_2 \in V(G \cap H) = \{u_1, \dots, u_k, v_1, \dots, v_k\}$. Since $m_1 m_2 \in E(H) \backslash E(G)$, we notice that both m_1 and m_2 belong to $\{v_1, \dots, v_k\}$ by the assumption (1). Let $m_1 = v_i$ and $m_2 = v_j$ (see Figure).

On the other hand, since $H[M_1, M_2]$ is not complete, there are two vertices $m_1' \in M_1$ and $m_2' \in M_2$ such that $m_1' m_2' \in E(G) \backslash E(H)$. Then, we show that

$$\text{both } m_1' \text{ and } m_2' \text{ belong to } V(H). \tag{$**$}$$

If not, we have $m_1' \in V(G) \backslash V(H)$ or $m_2' \in V(G) \backslash V(H)$. If $m_1' \in V(G) \backslash V(H)$, then we see

$$m_1' v_j = m_1' m_2 \in E(G \cup H) = E(G) \cup E(H),$$

since $(G \cup H)[M_1, M_2]$ is complete. As $m_1' \notin V(H)$, we see that m_1' is adjacent to v_j in G. Then, by the assumptions (2) and $u_j v_j \in E(G)$, we obtain $m_1' = u_j \in V(H)$, which contradicts the choice of $m_1' \notin V(H)$. Similarly, if $m_2' \in V(G) \backslash V(H)$, then we see

$$v_i m_2' = m_1 m_2' \in E(G \cup H) = E(G) \cup E(H).$$

By the same argument as above we obtain $m_2' = u_i \in V(H)$, which contradicts the choice of $m_2' \notin V(H)$. Hence $(**)$ is proved.

By $(**)$ and $m_1' m_2' \in E(G)$, we see $m_1', m_2' \in V(G) \cap V(H) = \{u_1, \dots, u_k, v_1, \dots, v_k\}$. Since m_1' is not adjacent to m_2' in H, we see $\{m_1', m_2'\} \neq \{u_i, v_i\}$ for any $i = 1, \dots, k$. Moreover, we see $\{m_1', m_2'\} \not\subset \{v_1, \dots, v_k\}$ and $\{m_1', m_2'\} \neq \{u_i, v_j\}$ $(i \neq j)$ by the assumption (2). Thus, we conclude that $\{m_1', m_2'\} \subset \{u_1, \dots, u_k\}$. This contradicts the assumption (3).

Case 2. $M_1, M_2 \subset V(H)$. Since $H[M_1, M_2]$ is not complete, there are $m_1 \in M_1$ and $m_2 \in M_2$ such that $m_1 m_2 \in E(G) \backslash E(H)$. Since $m_1, m_2 \in V(H)$ and $m_1 m_2 \in E(G)$, we see $m_1, m_2 \in V(G) \cap V(H) = \{u_1, \dots, u_k, v_1, \dots, v_k\}$. Then, we notice $\{m_1, m_2\} \not\subset \{u_1, \dots, u_k\}$ by the assumption (3). Moreover, we see $\{m_1, m_2\} \not\subset \{v_1, \dots, v_k\}$ and $\{m_1, m_2\} \neq \{u_i, v_j\}$ $(i \neq j)$ by the assumption (2).

Therefore, there is an $i \in \{1, \dots, k\}$ such that $m_1 m_2 = u_i v_i \in E(H)$. This contradicts the condition $m_1 m_2 \notin E(H)$.

These complete the proof of our statement. $\square$

Proof of Theorem 3.3. For any maximal subset $M_1 \uplus M_2 \subseteq V(G) \times \{1, 2\}$ with respect to the condition $(G \cup H)[M_1, M_2]$ is complete, we see that

$$G[M_1, M_2] \text{ is complete} \quad \text{or} \quad H[M_1, M_2] \text{ is complete},$$

by Lemma 3.5. Thus, we obtain $\mathsf{B}(G \cup H) = \mathsf{B}(G) \cup \mathsf{B}(H)$ by Theorem 3.1.

Next, we show that $\mathsf{B}(G \cap H) = \mathsf{B}(G) \cap \mathsf{B}(H)$. It is easy to see that $\mathsf{B}(G \cap H) \subset \mathsf{B}(G) \cap \mathsf{B}(H)$, so we show that $\mathsf{B}(G \cap H) \supset \mathsf{B}(G) \cap \mathsf{B}(H)$. A nonempty set M such that $M \uplus \phi, \phi \uplus M \in \mathsf{B}(G) \cap \mathsf{B}(H)$ is a subset of $V(G) \cap V(H) = \{u_1, \dots, u_k, v_1, \dots, v_k\}$ and it also satisfies $\mathrm{CN}_G(M) \neq \phi$ and $\mathrm{CN}_H(M) \neq \phi$. We see that such a nonempty set M has precisely the following form:

$$M = \{u_i\} \quad \text{or} \quad M = \{v_i\} \quad (i = 1, \dots, k). \tag{4}$$

Indeed, the common neighbors of $\{u_i\}$ and $\{v_i\}$ in G and in H are nonempty. On the other hand, we see that every subset M of $V(G) \cap V(H)$ which is neither $\{u_i\}$ nor $\{v_i\}$ satisfies one of the following three conditions:

(4.1) $M \subseteq \{u_1, \dots, u_k\}$ and $|M| \geq 2$; (4.2) $M \subseteq \{v_1, \dots, v_k\}$ and $|M| \geq 2$;

(4.3) $M \supseteq \{u_i, v_j\}$ $(i, j = 1, \dots, k)$.

For (4.1), we see $\mathrm{CN}_H(M) = \phi$ by the assumptions (1) and $u_i v_i \in E(H)$ for each i. For (4.2), we notice $\mathrm{CN}_G(M) = \phi$ by the assumptions (2) and $u_i v_i \in E(G)$ for each i. For (4.3), we obtain $\mathrm{CN}_G(M) \subseteq \mathrm{CN}_G(\{u_i, v_j\})$ from (2.1). Here we verify $\mathrm{CN}_G(\{u_i, v_j\}) = \phi$. Suppose that $x \in \mathrm{CN}_G(\{u_i, v_j\})$. Then x is adjacent to v_j in G and $x = u_j$ by the assumption (2). Hence, u_i is adjacent to u_j in G. This contradicts the assumption (3).

For any $M \uplus \phi, \phi \uplus M \in \mathsf{B}(G) \cap \mathsf{B}(H)$, we obtain $\mathrm{CN}_{G \cap H}(M), \neq \phi$ by the assumption with respect to the graph $G \cap H$ and (4). Therefore, $M \uplus \phi, \phi \uplus M \in \mathsf{B}(G \cap H)$.

For any $M_1 \uplus M_2 \in \mathsf{B}(G) \cap \mathsf{B}(H)$ such that $M_1 \neq \phi \neq M_2$, we notice that $G[M_1, M_2]$ and $H[M_1, M_2]$ are complete. Hence, we conclude that $(G \cap H)[M_1, M_2]$ is complete, and hence, $M_1 \uplus M_2 \in \mathsf{B}(G \cap H)$. Therefore, we have $\mathsf{B}(G) \cap \mathsf{B}(H) \subset \mathsf{B}(G \cap H)$. $\square$

For the union $G \cup H$ satisfying the condition of Theorem 3.3, an upper bound for its chromatic number is given in the following:

PROPOSITION 3.6. *Let $G \cup H$ be the union of two graphs G and H satisfying the condition of Theorem* 3.3. *Let $l_{c_H} := |\{c_H(u_1), \dots, c_H(u_k)\}|$, where c_H is a $\chi(H)$-coloring of H. Then, there is a* $\max\{\chi(G) + l_{c_H}, \chi(H)\}$*-coloring c of $G \cup H$ such that $c|_{V(H)} = c_H$.*

PROOF. Let $c_H : V(H) \to \{1, \dots, \chi(H)\}$ be a $\chi(H)$-coloring of H. Without loss of generality, we may assume $\{c_H(u_1), \dots, c_H(u_k)\} = \{1, \dots, l_{c_H}\}$. We define a map c on $V(G \cup H)$ as an extension of c_H. First, we define

$$c(v) = c_H(v) \tag{3.1}$$

for all $v \in V(H)$. Next, we define c on $V(G) \backslash V(H)$. Take a $\chi(G)$-coloring c_G of G and let $V_1, \dots, V_{\chi(G)}$ be the color classes of $V(G)$ given by c_G. Then, we define

$$c(v) = l_{c_H} + i \tag{3.2}$$

for $v \in V_i \setminus V(G \cap H)$ and each $i = 1, \dots, \chi(G)$. We notice that $c(V(G) \backslash V(H)) = \{l_{c_H} + 1, \dots, l_{c_H} + \chi(G)\}$. Since $\{u_1, \dots, u_k\}$ is independent in G and $v_1, \dots, v_k$ are endvertices of G, we see that the map c defined by (3.1) and (3.2) is a $\max\{\chi(G) + l_{c_H}, \chi(H)\}$-coloring of $G \cup H$. $\square$

COROLLARY 3.7. *We assume that the union $G \cup H$ of two graphs G and H satisfies the condition of Theorem* 3.3. *Moreover we assume that $\{v_1, \dots, v_k\}$ is independent in H. Then, there is a* $\min\{\max\{\chi(G) + l_{c_H}, \chi(H)\}, \max\{\chi(H) + l_{c_G}, \chi(G)\}\}$*-coloring of $G \cup H$.* $\square$

DEFINITION 3.8. Let H be a graph satisfying the condition of Theorem 3.3. We define

$$l_H := \min\{l_{c_H} \mid c_H \text{ is a } \chi(H)\text{-coloring of } H\}.$$

We remark that $l_H \leq 2$. We take a $\chi(H)$-coloring c_H of H and a number $n \in \{1, \dots, \chi(H)\}$ with $n \neq c_H(v_1)$. Assume that $l_{c_H} = |\{c_H(u_i) \mid i = 1, \dots, k\}| \geq 3$. Then, we can take another $\chi(H)$-coloring c'_H of H defined as follows:

$$c'_H(v) = \begin{cases} c_H(v) & \text{if } v \in V(H) \setminus \{u_1, \dots, u_k\}, \\ c_H(v_1) & \text{if } v = u_i \text{ and } c_H(v_i) \neq c_H(v_1), \\ n & \text{if } v = u_i \text{ and } c_H(v_i) = c_H(v_1). \end{cases}$$

Then, we have $l_H \leq l_{c'_H} = 2$.

As a consequence of Proposition 3.6, we have the following.

THEOREM 3.9. *Let $G \cup H$ be the union of two graphs G and H satisfying the condition of Theorem* 3.3 *and let $k = |E(G \cap H)|$.*

(1) *If* $k \geq 2$, *then we have*

$$\chi(G \cup H) \leq \max\{\chi(G) + l_H, \chi(H)\}.$$

(2) *If* $k = 1$, *we have*

$$\chi(G \cup H) = \max\{\chi(G), \chi(H)\}.$$

PROOF. Our statement (1) follows from Proposition 3.6. We prove (2). If $k = 1$, without loss of generality, we may assume $\chi(G) \geq \chi(H)$. First, take a $\chi(G)$-coloring $c_G : V(G) \to \{1, \dots, \chi(G)\}$ of G and a $\chi(H)$-coloring $c_H : V(H) \to \{1, \dots, \chi(H)\}$ of H. We define a map c on $V(G \cup H)$ as an extension of c_H. First, put $c(v) = c_H(v)$ for $v \in V(H)$. Notice that $c_H(u_1) \in \{1, \dots, \chi(G)\}$. Then, take the transposition $(c_G(V)\, c_H(u_1))$ on $\{1, \dots, \chi(G)\}$, where V is the color class of $V(G)$ given by c_G containing u_1. Then, we define $c(V(G) \setminus V(H)) = ((c_G(V)\, c_H(u_1)) \circ c_G)(V(G) \setminus V(H))$. We see that the map c is a $\chi(G)$-coloring of $G \cup H$. □

In view of (1.1) and Theorem 3.9, it is natural to compute $\mathrm{ind}_{\mathbf{Z}_2}(\|\mathsf{B}(G \cup H)\|)$ for the union $G \cup H$ satisfying the condition of Theorem 3.3. Recall that

$$\mathsf{B}(G) \cap \mathsf{B}(H) = \mathsf{B}(G \cap H) = \{u_i \uplus v_i, v_i \uplus u_i \mid i = 1, \dots, k\},$$

the disjoint union of $2k$ 1-simplices, since the intersection $G \cap H$ consists of disjoint k edges.

THEOREM 3.10. *Let $G \cup H$ be the union of two graphs G and H which satisfies the condition of Theorem* 3.3.

(1) *If* $\max\{\mathrm{ind}_{\mathbf{Z}_2}(\|\mathsf{B}(G)\|), \mathrm{ind}_{\mathbf{Z}_2}(\|\mathsf{B}(H)\|)\} \geq 1$, *we have*

$$\mathrm{ind}_{\mathbf{Z}_2}(\|\mathsf{B}(G \cup H)\|) = \max\{\mathrm{ind}_{\mathbf{Z}_2}(\|\mathsf{B}(G)\|), \mathrm{ind}_{\mathbf{Z}_2}(\|\mathsf{B}(H)\|)\}.$$

(2) *If* $\mathrm{ind}_{\mathbf{Z}_2}(\|\mathsf{B}(G)\|) = \mathrm{ind}_{\mathbf{Z}_2}(\|\mathsf{B}(H)\|) = 0$, *we have*

$$\mathrm{ind}_{\mathbf{Z}_2}(\|\mathsf{B}(G \cup H)\|) \leq 1.$$

PROOF. We use the same notation used in Theorem 3.3. Let $m := \mathrm{ind}_{\mathbf{Z}_2}(\|\mathsf{B}(G)\|)$ and $n := \mathrm{ind}_{\mathbf{Z}_2}(\|\mathsf{B}(H)\|)$. Before we prove (1) and (2), we will define $\mathbf{Z}_2$-maps $\|\mathsf{B}(G)\| \to S^m$ and $\|\mathsf{B}(H)\| \to S^n$ such that each $u_i \uplus v_i$ is mapped to a point. By using these $\mathbf{Z}_2$-maps, we will construct a $\mathbf{Z}_2$-map $\|\mathsf{B}(G \cup H)\| \to S^l$, where $l := \max\{m, n\}$.

First, we construct a $\mathbf{Z}_2$-map from $\|\mathsf{B}(G)\|$ to S^m such that each $u_i \uplus v_i$ is mapped to a point. Let $\mathsf{K} := \mathsf{B}(G\backslash\{v_1, \dots, v_k\})$. We define a simplicial $\mathbf{Z}_2$-map $f_1 : \mathsf{B}(G) \to \mathsf{K}$ as

$$f_1(\phi \uplus v_i) = u_i \uplus \phi, \quad f_1(v_i \uplus \phi) = \phi \uplus u_i$$

and $f_1(v) = v$ for any other vertex v of $\mathsf{B}(G)$. We take a $\mathbf{Z}_2$-map f_2 as the composition

$$\|\mathsf{K}\| \hookrightarrow \|\mathsf{B}(G)\| \to S^m,$$

where the latter map is an arbitrary $\mathbf{Z}_2$-map. Then, the composition $f_2 \circ f_1$ is a desired $\mathbf{Z}_2$-map. Similarly, we can construct a $\mathbf{Z}_2$-map from $\|\mathsf{B}(H)\|$ to S^n such that each $u_i \uplus v_i$ is mapped to a point as follows. Let $\mathsf{L} := \mathsf{B}(H\backslash\{u_1, \dots, u_k\})$. We define a simplicial $\mathbf{Z}_2$-map $g_1 : \mathsf{B}(H) \to \mathsf{L}$ as

$$g_1(\phi \uplus u_i) = v_i \uplus \phi, \quad g_1(u_i \uplus \phi) = \phi \uplus v_i$$

and $g_1(v) = v$ for any other vertex v of $\mathsf{B}(H)$. Let g_2 be the composition $\|\mathsf{L}\| \hookrightarrow \|\mathsf{B}(H)\| \to S^n$, where the latter map is an arbitrary $\mathbf{Z}_2$-map. The composition $g_2 \circ g_1$ is a $\mathbf{Z}_2$-map such that each $u_i \uplus v_i$ is mapped to a point.

Next, to construct a $\mathbf{Z}_2$-map from $\|\mathsf{B}(G \cup H)\|$ to S^l, we need the following claim:

CLAIM. If $m \geq 1$ and $m \geq n$, there exist $\mathbf{Z}_2$-maps $f_3 : \|\mathsf{K}\| \to S^{m+1}$ and $g_3 : \|\mathsf{L}\| \to S^{m+1}$ such that

- $f_3(u_i \uplus \phi) = g_3(\phi \uplus v_i)$ and $f_3(\phi \uplus u_i) = g_3(v_i \uplus \phi)$ for all i,
- the union $\operatorname{im} f_3 \cup \operatorname{im} g_3$ does not contain the north and south poles of S^{m+1}.

We show **Claim**. Let $I : S^n \to S^m$ be the inclusion defined by $I(x) = (x, 0, \dots, 0)$ and $a : S^{m+1} \to S^{m+1}$ the antipodal map. By the continuity of $f_2 : \|\mathsf{K}\| \to S^m$ and $g_2 : \|\mathsf{L}\| \to S^n$, we can take a sufficiently large positive integer $r \geq 1$ so that $f_2(\mathrm{lk}_{\mathsf{sd}^r \mathsf{K}}(u_i \uplus \phi))$ and $g_2(\mathrm{lk}_{\mathsf{sd}^r \mathsf{L}}(\phi \uplus v_i))$ contain no pair of antipodal points for each i. Since $m \geq 1$, the sphere S^m is not covered with the union $a \circ f_2(\mathrm{lk}_{\mathsf{sd}^r \mathsf{K}}(u_i \uplus \phi)) \cup a \circ I \circ g_2(\mathrm{lk}_{\mathsf{sd}^r \mathsf{L}}(\phi \uplus v_i))$. Hence, we see

$$X_i := S^m \backslash (a \circ f_2(\mathrm{lk}_{\mathsf{sd}^r \mathsf{K}}(u_i \uplus \phi)) \cup a \circ I \circ g_2(\mathrm{lk}_{\mathsf{sd}^r \mathsf{L}}(\phi \uplus v_i)))$$

is nonempty. Then, we take a point $w_i \in S^{m+1}$ that belongs to the interior of $\left\{ \frac{x}{\|x\|} \mid x \in p * X_i \right\}$, where p is the north pole of S^{m+1} and $p * X_i$ is the Euclidean cone on X_i with p.

For each i, we modify f_2 on neighborhoods $\mathrm{st}_{\mathsf{sd}^r\mathsf{K}}(u_i \uplus \phi)$ and $\mathrm{st}_{\mathsf{sd}^r\mathsf{K}}(\phi \uplus u_i)$ to obtain a $\mathbf{Z}_2$-map f_3 that maps $u_i \uplus \phi$ to w_i and $\phi \uplus u_i$ to $a(w_i)$. For any $x \in \mathrm{st}_{\mathsf{sd}^r\mathsf{K}}(u_i \uplus \phi) \backslash u_i \uplus \phi$, there exists the unique point $y_x \in \mathrm{lk}_{\mathsf{sd}^r\mathsf{K}}(u_i \uplus \phi)$ such that x is represented by $(1-t)y_x + t(u_i \uplus \phi)$ for some $t \in (0,1)$. Similarly, for $x \in \mathrm{st}_{\mathsf{sd}^r\mathsf{K}}(\phi \uplus u_i) \backslash \phi \uplus u_i$, there exists a unique point $z_x \in \mathrm{lk}_{\mathsf{sd}^r\mathsf{K}}(\phi \uplus u_i)$ such that x is represented by $(1-t)z_x + t(\phi \uplus u_i)$ for some $t \in (0,1)$. Since $r \geq 1$, for $i \neq j$, we see

$$\mathrm{st}_{\mathsf{sd}^r\mathsf{K}}(u_i \uplus \phi) \cap \mathrm{st}_{\mathsf{sd}^r\mathsf{K}}(u_j \uplus \phi) = \phi = \mathrm{st}_{\mathsf{sd}^r\mathsf{K}}(u_i \uplus \phi) \cap \mathrm{st}_{\mathsf{sd}^r\mathsf{K}}(\phi \uplus u_j).$$

We define a $\mathbf{Z}_2$-map $f_3 : \|\mathsf{sd}^r\,\mathsf{K}\| \to S^{m+1}$ as follows:

$$u_i \uplus \phi \mapsto w_i, \qquad \phi \uplus u_i \mapsto a(w_i),$$

$$\begin{aligned}
&x = (1-t)y_x + t(u_i \uplus \phi) \\
&\quad \mapsto \frac{(1-t)(f_2(y_x),0) + tw_i}{\|(1-t)(f_2(y_x),0) + tw_i\|} && \text{if } x \in \mathrm{st}_{\mathsf{sd}^r\mathsf{K}}(u_i \uplus \phi) \backslash u_i \uplus \phi, \\
&x = (1-t)z_x + t(\phi \uplus u_i) \\
&\quad \mapsto \frac{(1-t)(f_2(z_x),0) + t(a(w_i))}{\|(1-t)(f_2(z_x),0) + t(a(w_i))\|} && \text{if } x \in \mathrm{st}_{\mathsf{sd}^r\mathsf{K}}(\phi \uplus u_i) \backslash \phi \uplus u_i, \\
&x \mapsto (f_2(x), 0) && \text{otherwise.}
\end{aligned}$$

Similarly, we can modify $I \circ g_2$ to obtain a $\mathbf{Z}_2$-map $g_3 : \|\mathsf{sd}^r\,\mathsf{L}\| \to S^{m+1}$ such that $g_3(\phi \uplus v_i) = w_i$ and $g_3(v_i \uplus \phi) = a(w_i)$. By the choice of points $\{w_i\}$, we see that the union $\mathrm{im}\, f_3 \cup \mathrm{im}\, g_3$ does not contain the north and south poles of S^{m+1}. This completes the proof of **Claim**.

We prove (1). We assume $m \geq n$. We define a $\mathbf{Z}_2$-map $h : \|\mathsf{B}(G \cup H)\| \to S^{m+1}$ as

$$h(x) = \begin{cases} (f_3 \circ f_1)(x) & \text{if } x \in \|\mathsf{B}(G)\|, \\ (g_3 \circ g_1)(x) & \text{if } x \in \|\mathsf{B}(H)\|, \end{cases}$$

and define a $\mathbf{Z}_2$-map $h' : S^{m+1} \backslash \{p, a(p)\} \to S^m$ as

$$(x_1, \ldots, x_{m+2}) \mapsto \frac{1}{\sqrt{1 - x_{m+2}^2}}(x_1, \ldots, x_{m+1}).$$

We can regard h as a $\mathbf{Z}_2$-map from $\|\mathsf{B}(G \cup H)\|$ to $S^{m+1} \backslash \{p, a(p)\}$. Then, the composition $h' \circ h$ is a $\mathbf{Z}_2$-map from $\|\mathsf{B}(G \cup H)\|$ to S^m, and hence,

$\mathrm{ind}_{\mathbf{Z}_2}(\|\mathsf{B}(G \cup H)\|) \le m$. On the other hand, we see that $\|\mathsf{B}(G)\|$ and $\|\mathsf{B}(H)\|$ are contained in $\|\mathsf{B}(G \cup H)\|$ as $\mathbf{Z}_2$-subcomplexes, and hence, we have $\mathrm{ind}_{\mathbf{Z}_2}(\|\mathsf{B}(G \cup H)\|) \ge m$. Similarly, if $m < n$, we obtain a $\mathbf{Z}_2$-map from $\|\mathsf{B}(G \cup H)\|$ to S^n by the same argument as above. The statement (1) is proved.

We prove (2). If $m = n = 0$, it is not always possible to construct f_3 and g_3 so that they satisfy the latter condition of **Claim**; Example 3.11 is one of such examples. However, we may repeat the argument of **Claim** by taking $\{w_i\}$ as arbitrary points of the upper semicircle of S^1. Then, the map h is a desired $\mathbf{Z}_2$-map from $\|\mathsf{B}(G \cup H)\|$ to S^1. Hence, the statement (2) follows. $\square$

Example 3.11. For a cycle C_5 of length 5, $\|\mathsf{B}(C_5)\|$ is $\mathbf{Z}_2$-homotopy equivalent to S^1, and hence, $\mathrm{ind}_{\mathbf{Z}_2}(\|\mathsf{B}(C_5)\|) = 1$. On the other hand, C_5 is decomposed into P_4 and P_3 such that these satisfy the sufficient condition of Theorem 3.3. Since $\mathrm{ind}_{\mathbf{Z}_2}(\|\mathsf{B}(P)\|) = 0$ for any path P, the inequality of Theorem 3.10 (2) is optimal.

Example 3.12. Let G be the graph defined by

$$V(G) = \{x, u_1, \dots, u_n, v_1, \dots, v_n\} \quad \text{and}$$
$$E(G) = \{xu_i \mid i = 1, \dots, n\} \cup \{u_i v_i \mid i = 1, \dots, n\},$$

where $n \ge 4$. Let H be the graph $K_n + \{u_i v_i \mid i = 1, \dots, n\}$, where $V(K_n) = \{v_1, \dots, v_n\}$. Then, we notice $\mathrm{ind}_{\mathbf{Z}_2}(\|\mathsf{B}(G)\|) = 0$ and $\mathrm{ind}_{\mathbf{Z}_2}(\|\mathsf{B}(H)\|) = n - 2$. By Theorem 3.10 (1), we see $\mathrm{ind}_{\mathbf{Z}_2}(\|\mathsf{B}(G \cup H)\|) = n - 2$. We also have $\chi(G \cup H) \le \max\{4, n\} = n$ by Theorem 3.9 (1). Hence, we see that the inequality of Theorem 3.9 (1) is optimal by the inequality (1.1).

For the union $G \cup H$ satisfying the condition of Theorem 3.3, we obtain

$$\begin{aligned} \mathrm{ind}_{\mathbf{Z}_2}(\|\mathsf{B}(G \cup H)\|) + 2 &= \max\{\mathrm{ind}_{\mathbf{Z}_2}(\|\mathsf{B}(G)\|), \mathrm{ind}_{\mathbf{Z}_2}(\|\mathsf{B}(H)\|)\} + 2 \\ &\overset{(1.1)}{\le} \max\{\chi(G), \chi(H)\} \le \chi(G \cup H) \end{aligned}$$

by Theorem 3.10 (1) and the inequality (1.1), if $\max\{\mathrm{ind}_{\mathbf{Z}_2}(\|\mathsf{B}(G)\|), \mathrm{ind}_{\mathbf{Z}_2}(\|\mathsf{B}(H)\|)\} \ge 1$. The lower bound $\mathrm{ind}_{\mathbf{Z}_2}(\|\mathsf{B}(G \cup H)\|) + 2$ is not better than the trivial one $\max\{\chi(G), \chi(H)\}$ for $\chi(G \cup H)$.

4. Appendix: Addendum to [2]

Here we supplement to section 4 of [2]. For a graph G, let $\overline{G}$ be an abstract simplicial complex with the vertex set $V(\overline{G}) = V(B(G))$ defined by

$$\overline{G} := \{u \uplus \phi, v \uplus \phi, \phi \uplus u, \phi \uplus v, u \uplus v, v \uplus u \mid uv \in E(G)\}.$$

We notice that $\overline{G}$ is a free $\mathbf{Z}_2$-subcomplex of $\mathsf{B}(G)$ with the restriction of the free $\mathbf{Z}_2$-action on $\mathsf{B}(G)$. In [2], the author proved that a graph G contains no 4-cycles if and only if $\|\overline{G}\|$ is a strong $\mathbf{Z}_2$-deformation retract of $\|\mathsf{B}(G)\|$. The $\mathbf{Z}_2$-subcomplex $\overline{G}$ is a natural double covering of G with the map $V(\overline{G}) \to V(G)$ defined by $v \uplus \phi, \phi \uplus v \mapsto v$ for each $v \in V(G)$.

Let T be a spanning tree T of G. Then, the graph G is obtained from T by adding finitely many edges $\{u_i v_i\}_{i=1}^{l}$, where $u_i v_i \in E(G) \setminus E(T)$. Then, we see $\overline{G} = \overline{T} \cup \{u_i \uplus v_i, v_i \uplus u_i\}_{i=1}^{l}$. Since all trees are bipartite, $V(T)$ is the disjoint union of the partite sets A and B. Let $T^1 = T \times \{1\}$ and $T^2 = T \times \{2\}$ be the copies of T with $V(T^1) = A^1 \amalg B^1$ and $V(T^2) = A^2 \amalg B^2$, where $A^1 = A \times \{1\}$, $A^2 = A \times \{2\}$, $B^1 = B \times \{1\}$ and $B^2 = B \times \{2\}$. Then, we notice that $\overline{T}$ is isomorphic to the disjoint union $T^1 \amalg T^2$ of two copies of T by the following correspondence $V(T^1 \amalg T^2) \to V(\overline{T})$:

$$(a,1) \in A^1 \mapsto a \uplus \phi, \quad (b,1) \in B^1 \mapsto \phi \uplus b,$$
$$(a,2) \in A^2 \mapsto \phi \uplus a, \quad (b,2) \in B^2 \mapsto b \uplus \phi.$$

We consider the unique path P in T connecting u_i to v_i for each i. If we add an edge $u_i v_i$ to T so that $T \cup \{u_i v_i\}$ contains a cycle of even length, the path P is of odd length. Then, we notice $u_i \uplus \phi$ and $\phi \uplus v_i$ belong to the same component of $\overline{T}$ and $\phi \uplus u_i$ and $v_i \uplus \phi$ belong to the other component of $\overline{T}$. Hence, $\overline{T} \cup \{u_i \uplus v_i, v_i \uplus u_i\}$ is disconnected. If we add an edge $u_i v_i$ to T so that $T \cup \{u_i v_i\}$ contains a cycle of odd length, the path P is of even length. Then we see $u_i \uplus \phi$ and $v_i \uplus \phi$ belong to the same component of $\overline{T}$ and $\phi \uplus u_i$ and $\phi \uplus v_i$ belong to the other component of $\overline{T}$. Hence, $\overline{T} \cup \{u_i \uplus v_i, v_i \uplus u_i\}$ is connected. Repeating this consideration for edges $u_1 v_1, \dots, u_l v_l$, we see that $\overline{G}$ is disconnected if and only if G contains no cycles of odd length, or equivalently, G is bipartite (see [1], Theorem 1.6.1).

Theorem 4.1. *Let G be a connected graph with k induced cycles of G.*
(1) *If G contains no cycles of odd length, we have $\|\overline{G}\| \simeq \bigvee_k S^1 \amalg \bigvee_k S^1$.*
(2) *If G contains at least one cycle of odd length, we have $\|\overline{G}\| \simeq \bigvee_{2k-1} S^1$.*

Proof. By the preceding argument, the statement (1) holds. Let G be a connected graph which contains at least one cycle of odd length. Then, it follows that $\overline{G}$ is connected. Since the Euler characteristic $\chi_{\overline{G}}$ of $\overline{G}$ is twice as large as that of G, we see

$$\text{rank } H_1(\overline{G}) = 1 - \chi_{\overline{G}} = 1 - 2 \cdot \chi_G = 1 - 2(1-k) = 2k - 1,$$

and hence, the statement (2) follows. □

THEOREM 4.2. *Let G be a connected graph. Then, $\mathsf{B}(G)$ is connected if and only if $\overline{G}$ is connected.*

PROOF. Let G be a connected graph. If $\mathsf{B}(G)$ is disconnected, then $\overline{G}$ is disconnected since $\overline{G}$ is a subcomplex of $\mathsf{B}(G)$ with $V(\overline{G}) = V(\mathsf{B}(G))$.

Conversely, we assume that $\overline{G}$ is disconnected. Then, we see that G contains no cycles of odd length, and hence, $\overline{G}$ is isomorphic to the disjoint union $G \amalg G$. Suppose that $\mathsf{B}(G)$ is connected. Then, for the two vertices $u \uplus \phi$ and $\phi \uplus u$ of $\mathsf{B}(G)$, there exist the vertices $v_0, \dots, v_n$ of $\mathsf{B}(G)$ such that $v_0 = u \uplus \phi$, $v_n = \phi \uplus u$ and each $v_i v_{i+1} \in \mathsf{B}(G)$. Every 1-simplex of $\mathsf{B}(G)$ is one of the following forms: $x \uplus y$, $y \uplus x$, $\{x, y\} \uplus \phi$ and $\phi \uplus \{x, y\}$, in particular, $x \uplus y$ and $y \uplus x$ are simplices of $\overline{G}$. If the 1-simplex $v_i v_{i+1} \in \mathsf{B}(G)$ is the form $\{x, y\} \uplus \phi$, then there is a vertex $z \in V(G)$ such that $z \in \mathrm{CN}_G(\{x, y\})$. The two vertices v_i and v_{i+1} are joined by two simplices $x \uplus z$ and $z \uplus y$ of $\overline{G}$. Similarly, if the 1-simplex $v_i v_{i+1} \in \mathsf{B}(G)$ is the form $\phi \uplus \{x, y\}$, we can join the two vertices v_i and v_{i+1} by simplices of $\overline{G}$. Thus, $u \uplus \phi$ and $\phi \uplus u$ are joined by simplices of $\overline{G}$. This contradicts the fact that $u \uplus \phi$ and $\phi \uplus u$ do not belong to the same component of $\overline{G}$. □

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