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SUBDIFFERENTIALS OF PERFORMANCE FUNCTIONS AND CALCULUS OF CODERIVATIVES OF SET-VALUED MAPPINGS

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Abstract. The paper contains calculus rules for coderivatives of compositions, sums and intersections of set-valued mappings. The types of coderivatives considered correspond to Dini-Hadamard and limiting Dini-Hadamard subdifferentials in Gâteaux differentiable spaces, Fréchet and limiting Fréchet subdifferentials in Asplund spaces and approximate subdifferentials in arbitrary Banach spaces. The key element of the unified approach to obtaining various calculus rules for various types of derivatives presented in the paper are simple formulas for subdifferentials of marginal, or performance functions.

1. Introduction. In a number of recent studies a calculus of coderivatives was developed, first between finite dimensional spaces in [16] and then for coderivatives of various types (approximate [13], limiting Fréchet [17], Fréchet [18]) for set-valued mappings between appropriate Banach spaces (arbitrary in [13] and Asplund in [17, 18]). The primary purpose of this paper is to provide a new insight into the problem and show that these (and certain other, sometimes even more general) results follow from the standard calculus rules for corresponding subdifferentials of functions.

In general the calculi of the three main classes of objects of nonsmooth analysis: subdifferentials, normal cones and coderivatives, are heavily interconnected and every

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result for each of them can, in principle, be obtained from the calculus rule for any other basic operation with any other object, so that the choice of a sequence in which the results are proved or presented is often a matter of taste or personal preferences. A certain justification for giving independent proofs of basic calculus rules for coderivatives of set-valued mappings comes from the fact that coderivatives occupy an intermediate position between subdifferentials of functions and normal cones to sets and calculus rules for the last two classes of objects can easily be deduced from the corresponding rules for coderivatives. But direct proofs for coderivatives are often heavier and more complicated and thr reader may get an impressionthat certain results are independent. For a number of reasons, technical as mentioned above but also substantive, it seems in many cases convenient to consider subdifferential as the primal object and normal cone and coderivative as its derivatives (e.g. if we wish that the normal cone be generated by the subdifferential of the distance function – a useful property known from convex analysis). In addition, proofs for subdifferentials are typically much simpler.

The main vehicle that carries over the rules of subdifferential calculus to corresponding rules for coderivatives are formulae for subdifferentials of performance or, as they are often called, marginal functions[1]. Therefore we consider these formulae (which are very simple and easy to obtain – see also e.g. [7, 12, 21]) in Section 3, immediately after a brief discussion of necessary concepts and preliminary results from nonsmooth analysis. In a short Section 4 we show how marginal functions appear in calculations of coderivatives of the resulting set-valued mappings for two operations: composition and addition. In sections 5 through 7 the main calculus rules for coderivatives are presented. We consider three types of calculus rules: weak fuzzy calculus for so-called elementary coderivatives (e.g. Dini-Hadamard or Fréchet, actually the only elementary coderivatives we consider in the paper), strong fuzzy calculus for Fréchet coderivatives and exact calculus for approximate coderivatives. In the last two cases certain qualification conditions are always needed. In Sections 5–7 we prove calculus rules with very weak "metric qualification conditions" explicitly introduced in [10] but actually considered in certain cases earlier in [7, 13]. In the last section we consider more standard (and much stronger) "subdifferential" qualification conditions involving normal cones and coderivatives.

Some of the results presented here were announced in [10].

We use the following notation:

$X,\ Y,\ Z$ for Banach spaces;

[1]In the proof of the formula for the approximate coderivative of a composition of set-valued mappings we, in certain respects, follow the same pattern as Jourani and Thibault [13]. The basic difference is just that we use the marginal function approach which allows us to prove these and all other results using the same sequence of arguments.

X^*etc.	for their duals;
$\langle x^*, x\rangle$	for the canonical pairing on $X^* \times X$;
$\delta_S(x)$	for the indicator function of S;
$\chi_F(x, y)$	for $\delta_{\mathrm{Graph}F}(x, y)$, if F is a set-valued mapping from X into Y;
$\rho(x, S)$	for the distance from x to S;
B_r	for the closed ball of radius r around the origin;
B	for the unit ball.

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2. Preliminaries: subdifferentials, normal cones and coderivatives. By a *subdifferential* we usually mean a set-valued mapping which associates with every function defined on a Banach space X and any $x \in X$ a set $\partial f(x) \subset X^*$ (possibly empty), called the *subdifferential of f at* x, in such a way that

(a) $\partial f(x) = \emptyset$ if $x \notin \mathrm{dom} f$;

(b) $\partial f(x) = \partial g(x)$ if f and g coincide in a neighborhood of x;

(c) if f is convex, then $\partial f(x)$ is the subdifferential of f in the sense of convex analysis;

(d) $0 \in \partial f(x)$ if f attains a local minimum at x;

(e) if f satisfies the Lipschitz condition near x with constant K, then $\|x^*\| \le K$ if $x^* \in \partial f(x)$;

(f) if $f(x, y) = f_1(x) + f_2(y)$, then $\partial f(x, y) = \partial f_1(x) \times \partial f_2(y)$;

(g) if $f(x) = \lambda g(Ax + v)$, where $A : X \to Y$ is a bounded linear operator *onto* Y and $\lambda > 0$, then $\partial f(x) \subset \lambda A^* \partial g(Ax + v)$.

Certain subdifferentials make sense only for certain classes of functions and spaces, so the corresponding specifications are often necessary. (By "make sense" we mean that they should be able to carry useful information about the function. A minimal requirement for that is that, say, $\partial f(x) \neq \emptyset$ for x of a dense subset of the domain of f for any function of the class.)

Given a subdifferential which makes sense for arbitrary lower semicontinuous functions, we can define the *normal cone (associated with the subdifferential)* to a set S at $x \in S$ as the subdifferential of the indicator of S at x:

$$N(S, x) = \partial \delta_S(x). \tag{2.1}$$

Now if F is a set-valued mapping from X into Y and $y \in F(x)$, then the *coderivative* of F at (x, y) *associated with the given subdifferential* is the set-valued mapping from Y^* into X^* defined as follows:

$$D^*F(x,y)(y^*) = \{x^* : \ (x^*, -y^*) \in N(\text{Graph}F, (x,y))\}. \tag{2.2}$$

In the paper we shall consider the following classes of subdifferentials (which are among the most important both in the theory and applications):

1. *Fréchet subdifferential* of f at x consists of all vectors x^* satisfying

$$\liminf_{\|h\| \to 0} \|h\|^{-1}(f(x+h) - f(x) - \langle x^*, h\rangle) \geq 0. \tag{2.3}$$

Frèchet subdifferential is nonempty on a dense subset of the domain of any lower semicontinuous function on X if and only if X is an Asplund space (see [6] for the "only if" part and [5] for the "if" part of the statement).

2. *Dini-Hadamard subdifferential* of f at x consists of all x^* satisfying

$$\langle x^*, h\rangle \leq \liminf_{(t,u) \to (+0,h)} t^{-1}(f(x+th) - f(x)) = d^- f(x;h), \ \ \forall\, h \in X.$$

The quantity on the right is known as *Hadamard directional derivative* of f at x (along h). If f is Lipschitz near x, it coincides with the Dini directional derivative.

Dini-Hadamard subdifferential is nonempty on a dense subset of the domain of any l.s.c. function on a space having a Gâteaux differentiable renorm (more generally on a space on which there exists a Gâteaux differentiable locally Lipschitz bump function), in particular on any separable space.

The two subdifferentials just defined belong to the group of so called *elementary* subdifferentials. Their definitions are natural modifications of the definitions of the corresponding derivatives: Fréchet and Hadamard. More elementary subdifferentials can be obtained in the same way using other concepts of derivatives. We do not consider here the so called *viscosity* subdifferentials which are very close (see [4]).

3. *Limiting elementary subdifferentials.* Using elementary subdifferentials as the starting point, we can define subdifferentials of a new type called *limiting.* Namely, we say that x^* belongs to the limiting (Fréchet, Dini-Hadamard) subdifferential of f at x if there are sequences $\{x_n\}$ and $\{x_n^*\}$ converging respectively to x in the norm topology and to x^* in the weak* topology and such that every x_n^* belongs to the (Fréchet, Hadamard) subdifferential of f at x_n.

The definition is supported by the fact that the spaces on which the elementary subdifferentials (and hence limiting elementary) are known to make sense have duals with sequentially weak* compact unit balls. Limiting subdifferential (on an appropriate space) is nonempty whenever the function satisfies the Lipschitz condition near the point.

4. *Approximate subdifferential.* This is a subdifferential which makes sense on every Banach space. Approximate subdifferential is first defined for Lipschitz functions. Namely, given a function satisfying the Lipschitz condition near x and a linear subspace $L \subset X$, we set

$$\partial^-_{L\varepsilon} f(x) = \{u^* : \ \langle u^*, h\rangle \le d^- f(u; h) + \varepsilon\|h\|, \ \ \forall h \in L\},$$

omitting the subsript ε if $\varepsilon = 0$ and define the approximate subdifferential of f at x by

$$\partial f(x) = \bigcap_{L\in\mathcal{F}} \limsup_{u\to x} \partial^-_L f(u) \bigcap B_K = \bigcap_{L\in\mathcal{F}} \limsup_{u\to x, \varepsilon\to 0} \partial^-_{L\varepsilon} f(u) \bigcap B_{(K+\varepsilon)}, \tag{2.4}$$

where $\mathcal{F}$ is the collection of finite dimensional subspaces of X and K is any number greater than the Lipschitz constant of f near x. This is always a nonempty set.

The *approximate normal cone* to a set S at $x \in S$ is defined as the cone generated by the subdifferential of the distance function to S:

$$N(S, x) = \bigcup_{\lambda>0} \lambda\partial\rho(x, S). \tag{2.5}$$

Now approximate subdifferential can be defined for an arbitrary lower semicontinuous function through the normal cone to its epigraph as follows:

$$\partial f(x) = \{x^* : \ \ (x^*, -1) \in N(\text{epi} f, (x, f(x))\}. \tag{2.6}$$

These definitions are correct in the sense that they do not depend on the choice of a specific equivalent norm, the definition of the normal cone by (2.1) gives the same object as (2.5) and if the function is Lipschitz near x then the last formula gives the same result as (2.4). (We refer to [7] where a slightly different definitions were introduced.)

In a reflexive (more generally, Asplund and weakly compactly generated) space the limiting Fréchet subdifferential coincides with the approximate subdifferential for any l.s.c. function; in an arbitrary WCG (not necessarily reflexive) space the limiting Hadamard subdifferential coincides with the approximate subdifferential for any locally Lipschitz function [1, 2] (and is never smaller than the approximate subdifferential).

We also mention the connection between the approximate subdifferential and the generalized gradient of Clarke (not considered in this paper for the calculus rules available for it usually require additional convexification). Namely, Clarke's normal cone always coincides with the weak* convex closure of the approximate normal cone at the same point, so that the approximate subdifferential is always a subset of the generalized gradient. If the function is Lipschitz near the point in question, then the latter again is the weak* convex closure of the approximate subdifferential.

We shall next consider the three basic types of calculus rules that subdifferentials may obey. For the sake of brevity, we shall talk here only about rules for sums of functions. As was mentioned in the introduction, knowing them is sufficient to get corresponding rules for other operations and the purpose of the paper is to show how it can be done for various operations with set-valued mappings.

(A) Weak fuzzy calculus. We say that a subdifferential has the *weak fuzzy calculus* on a given space X (or class of spaces) if for any finite collection $\{f_1, \dots, f_k\}$ of lower semicontinuous functions, any x at which all of them are finite, any $\varepsilon > 0$ and any weak* neighborhood U^* of zero in X^*

$$x^* \in \partial(f_1 + \cdots + f_k)(x) \ \Rightarrow\ \exists\, x_i,\ x_i^* : \\ \|x_i - x\| \le \varepsilon,\ |f_i(x_i) - f_i(x)| \le \varepsilon,\ x_i^* \in \partial f_i(x_i),\ i = 1, \dots, k,\ \sum x_i^* \in x^* + U^*.$$

All subdifferentials we have mentioned have the weak fuzzy calculus on the corresponding spaces, namely, Dini-Hadamard and limiting Dini-Hadamard subdifferentials on spaces with Gâteaux differentiable Lipschitz bump functions, Fréchet and limiting Fréchet subdifferentials on Asplund spaces and the approximate subdifferential on every Banach space.

(B) Strong fuzzy calculus. The subdifferential has a *strong fuzzy calculus* if for any finite collection $\{f_1, \dots, f_k\}$ of lower semicontinuous functions, any x at which all of them are finite, any $\varepsilon > 0$

$$x^* \in \partial(f_1 + \cdots + f_k) \ \Rightarrow\ \exists\, x_i, x_i^* : \\ \|x_i - x\| \le \varepsilon,\ |f_i(x_i) - f_i(x)| \le \varepsilon,\ x_i^* \in \partial f_i(x_i),\ i = 1, \dots, k,\ \|\textstyle\sum_i x_i^* - x^*\| \le \varepsilon,$$

provided f_i satisfy the following *general metric qualification condition* (equivalent to the *uniform lower semicontinuity property* of [2, 11] -see [9]):

there is a nonnegative nondecreasing function $\omega(t)$ (generally, extended-real-valued) which is continuous and equal to zero at zero and such that

$$\rho\left((u, \alpha), \mathrm{epi}\left(\sum f_i\right)\right) \le \omega\left(\sum \rho((u, \alpha_i), \mathrm{epi} f_i)\right)$$

for all u sufficiently close to x and all $\alpha,\ \alpha_i$ satisfying $\sum \alpha_i = \alpha$.

Fréchet subdifferential on an Asplund space is the only known example of a subdifferential with strong fuzzy calculus [5, 8].

We further observe that both the weak and the strong fuzzy calculus rules we have formulated are consequences of the following property which we call the *basic fuzzy principle*:

$$0 \in \operatorname{argmin}(f_1 + \cdots + f_k) \ \Rightarrow \ \exists\, x_i, x_i^* : \\ \|x_i - x\| \le \varepsilon,\ |f_i(x_i) - f_i(x)| \le \varepsilon,\ x_i^* \in \partial f_i(x_i),\ i = 1, \ldots, k,\ \|\textstyle\sum_i x_i^*\| \le \varepsilon,$$

provided the general metric qualification condition is satisfied for $f_1, \ldots, f_k$ at x. We shall say, slightly modifying the definition given in [6], that X is a *∂-trustworthy space* if the basic fuzzy principle holds for ∂ on X. All spaces mentioned above are ∂-trustworthy for the corresponding subdifferentials, namely, spaces with Gâteaux differentiable Lipschitz bump functions for the Dini-Hadamard and limiting Dini-Hadamard subdifferentials [6], Asplund spaces for Fréchet and limiting Fréchet subdifferential [5] and all Banach spaces for the approximate subdifferential [9].

(C) Exact calculus. A subdifferential has *exact calculus* if for any finite collection $\{f_1, \ldots, f_k\}$ of lower semicontinuous functions and any x at which all of them are finite

$$\partial(f_1 + \cdots + f_k)(x) \subset \partial f_1(x) + \cdots + \partial f_k(x), \tag{2.7}$$

provided the following *linear-rate metric qualification condition* is satisfied:

there is a $K > 0$ such that the inequality

$$\rho\left((u, \alpha), \operatorname{epi} \sum f_i\right) \le K \sum \rho((u, \alpha_i), \operatorname{epi} f_i)$$

for all u sufficiently close to x and all α, α_i satisfying $\sum \alpha_i = \alpha$ and α_i close to $f_i(x)$ for all i.

The approximate subdifferential on any Banach space and the limiting Fréchet subdifferential on an Asplund space are the two known examples of subdifferentials with exact calculus. (We note that Clarke's generalized gradient and the "moderate" subdifferential of Michel-Penot and the "b-subdifferential" of Treiman do have the property like (2.7) under certain subdifferential qualification condition but it is not known whether they have the exact calculus as stated here, under the linear-rate metric qualification condition.)

The metric qualification conditions stated above are actually very weak. To see this, consider the (closed) positive and negative orthants in l^2 or, even simpler, the cones $\{(x, y, z) \in \mathbb{R}^3 : z \ge (x^2 + y^2)^{1/2}\}$ and $\{(x, y, z) \in \mathbb{R}^3 : z \le -(x^2 + y^2)^{1/2}\}$ the linear-rate metric qualification condition is obviously satisfied for the indicator functions of each of these two pairs of sets at zero whereas even the standard qualification condition of convex analysis (e.g. relative interiors have a nonempty intersection –

see [24]) fails to be valid. Traditionally, stronger qualification conditions in terms of subdifferentials themselves rather than distance estimates have been used in theorems containing calculus rules for subdifferentials etc. We shall discuss this question in the concluding section.

3. Subdifferentials of marginal functions. Consider a function $f(x, y)$ on the product of Banach spaces X and Y and set

$$\varphi(x) = \min_y f(x, y).$$

We shall be interested in connection between subdifferentials of the marginal function φ at a certain x with the corresponding subdifferentials of f at points (x, y) with y in the solution set argmin $f(x, \cdot) = \{y : \ f(x, y) = \varphi(x)\}$. To get desired results for the limiting and approximate subdifferentials we need certain amount of well posedness of the approximation problems involved which can be naturally expressed in terms of lower semicontinuity type properties of the solution maps. So we start with the definitions of the lower semicontinuity properties needed. Given a set-valued mapping F from X into Y and an $S \subset F(x)$, F is called *(sequentially) lower semicontinuous* at (x, S) (see [20]) if for any sequence $\{x_n\}$ converging to x, there exists a subsequence $\{x_{n_k}\}$ and a sequence $\{y_k\}$ converging to a certain $y \in S$ such that $y_k \in F(x_{n_k})$ for each k.

It is an easy matter to see that when S is a singleton $\{y\}$, this definition corresponds to usual lower semicontinuity of F at (x, y). Another extreme case (in which we call F, following [17], *sequentially lower semicompact* at x) is when $S = Y$ (which is equivalent to $S = F(x)$ if Graph$F$ is closed). This is actually a very weak property. Intermediate cases may also be of an interest.

We shall also consider topological versions of the properties. Namely, let us say that F is *topologically lower semicontinuous* at (x, S) if for any net $\{x_\nu\}$ converging to x there is a subnet $\{x_{\nu_\alpha}\}$ and a net $\{y_\alpha\}$ converging to a certain $y \in S$ and such that $y_\alpha \in F(x_{\nu_\alpha})$ for all α. It is clear that for S being a singleton the topological and sequential properties are equivalent. But topological lower semicompactness may be a stronger property if dim $Y = \infty$.

Proposition 3.1. *Let ∂ be either of the elementary subdifferentials (Dini-Hadamard or Fréchet). Assume that $y \in$ argmin $f(x, \cdot)$, that is $\varphi(x) = f(x, y)$, and $x^* \in \partial\varphi(x)$. Then $(x^*, 0) \in \partial f(x, y)$.*

P r o o f. If $x^* \in \partial\varphi(x)$, then by definition

$$\liminf_{t \to 0} t^{-1}(\varphi(x + th) - \varphi(x) - t\langle x^*, h\rangle) \geq 0,$$

either for any h (in case of the Dini-Hadamard subdifferential) or uniformly for h of the unit ball (in case of the Fréchet subdifferential). Therefore, as $\varphi(x) = f(x, y)$,

$$\liminf_{t \to 0} \inf_{\|v\| \le 1} t^{-1}(f(x + th, y + tv) - f(x, y) - t\langle x^*, h\rangle) \ge 0,$$

either for any h or uniformly for $h \in B$. In either case this means that $(x^*, 0)$ belongs to the corresponding subdifferential of f at x.

It is possible to slightly generalize Proposition 3.1 as follows: for any $\varepsilon > 0$, let us denote by $\partial_\varepsilon f(x)$ the subdifferential at x of the function $f_\varepsilon(u) = f(u) + \varepsilon\|x - u\|$. Then the same argument as in the proof of Proposition 3.1 gives

$$x^* \in \partial_\varepsilon^- \varphi(x) \ \&\ y \in \text{argmin } f(x, \cdot) \ \Rightarrow\ (x^*, 0) \in \partial_\varepsilon f(x, y). \qquad \square \tag{3.1}$$

Proposition 3.2. *Let ∂ be a limiting or the approximate subdifferential, and let $f(x, y) = \varphi(x)$. Assume that f is l.s.c. and the solution mapping* argmin $f(x, \cdot)$ *is lower semicontinuous at (x, y) Then*

$$x^* \in \partial\varphi(x) \ \Rightarrow\ (x^*, 0) \in \partial f(x, y).$$

Proposition 3.3. *Assume that f is lower semicontinuous and*

– either ∂ is a limiting subdifferential and the solution mapping $S(x) =$ argmin $f(x, \cdot)$ *is sequentially lower semicompact at x;*

– or ∂ is the approximate subdifferential and the solution mapping is topologically lower semicompact at x.

Then

$$x^* \in \partial\varphi(x) \ \Rightarrow\ (x^*, 0) \in \bigcup_{y \in \text{argmin } f(x, \cdot)} \partial f(x, y).$$

We shall give a joint proof of Propositions 3.2 and 3.3 paying main attention to the case of ∂ being the approximate subdifferential. (For an alternative proof of this see [7].) The proof for limiting subdifferentials is much simpler and is actually a by-product of the arguments of the first part of the proof below. Consider first the case when f satisfies the Lipschitz condition with constant K. Then φ also satisfies the Lipschitz condition with the same constant. Let $x^* \in \partial\varphi(x)$. This means that for any finite dimensional $L \subset X$ there is a sequence $\{x_n\}$ converging to x such that

$$x^* \in \limsup_{n \to \infty} \partial_L^- \varphi(x_n). \tag{3.2}$$

If the solution mapping is l.s.c. at (x, y), we may be sure, taking if necessary a subsequence, that there are $y_n \in S(x_n)$ converging to y. By Proposition 3.1, $(x^*, 0) \in \limsup_{n\to\infty} \partial^-_{L\times Y} f(x_n, y_n)$. This immediately proves Proposition 3.2 in case of a Lipschitz f. (The same argument without restrictions to any finite dimensional subspaces proves the proposition for the limiting subdifferentials of an arbitrary lower semicintinuous f as well as the part of Proposition 3.3 relating to limiting subdifferentials. In the latter case we have to refer to sequential lower semicontinuity at $(x, S(x))$ rather than at (x, y) so that we shall have a certain $y \in S(x)$, not the given one.)

To prove the second part of Proposition 3.3 related to the approximate subdifferential, we choose for any finite dimensional L an $x(L) \in X$ and an $x^*(L) \in \partial^-_L \varphi(x(L))$ such that $\|x(L) - x\| \le (\dim L)^{-1}$ and $x^*(L) \in x^* + L^\perp + (\dim L)^{-1}B$ which is possible by (3.2). Then $x(L) \to x$ and $x^*(L) \to x^*$ (weak-star) as L runs along the net of finite dimensional subspaces of X naturally ordered by inclusion. As the solution mapping is topologically lower semicompact, it follows that there is a net L_α of finite dimensional subspaces of X (such that every finite dimensional subspace belongs to some L_α) and a net $y_\alpha \subset Y$ such that $y_\alpha \in S(x_\alpha)$ for any α (here and below we set $x_\alpha = x(L_\alpha)$ and $x^*_\alpha = x^*(L_\alpha)$). By Proposition 3.1 this implies that $(x^*_\alpha, 0) \in \partial^- \varphi_{L_\alpha}(x_\alpha)$ and, consequently,

$$(x^*, 0) \in \partial^- f_{L_\alpha \times Y}(x_\alpha, y_\alpha) + U(L_\alpha),$$

where $U(L) = (L^\perp + (\dim L)^{-1}B) \times \{0\}$.

We further observe that, as f satisfies the Lipschitz condition,

$$\partial^- f_{M\times Y}(u, v) \subset \partial^- f_{L\times Y}(u, v)$$

if $L \subset M$ and, obviously, $U(M) \subset U(L)$ in this case. Therefore

$$(x^*, 0) \in \partial^- f_{L\times Y}(x_\alpha, y_\alpha) + U(L)$$

whenever $L \subset L_\alpha$. It immediately follows that for any L there is a sequence $\{(x_n, y_n)\}$ converging to (x, y) such that $(x^*, 0) \in \limsup \partial^- f_{L\times Y}(x_n, y_n)$. (Just take $(x_n, y_n) = (x_{\alpha_n}, y_{\alpha_n})$ such that $L \subset L_{\alpha_n}$, $\dim L_{\alpha_n} \to \infty$ and $\|y - y_{\alpha_n}\| \le 1/n$.) Therefore $(x^*, 0) \in \partial f(x, y)$.

This completes the proof of Proposition 3.3 for the case of a Lipschitz f. Passing to the general case, we first note that nothing would change if we had chosen $x^*(L)$ subject to a looser condition $x^*(L) \in \partial^- \varphi_{L\varepsilon(L)}(x(L)$ with, say, $\varepsilon(L) = (\dim L)^{-1}$. On the other hand, it is an easy matter to see that

$$\rho((x, \alpha), \mathrm{epi}\varphi) = \inf_y \rho((x, y, \alpha), \mathrm{epi} f) \tag{3.3}$$

(if, say we take the sum norm in $X \times Y \times \mathbb{R}$) and if $\alpha = \varphi(x) = f(x, y)$, then $0 = \rho((x, \alpha), \mathrm{epi}\varphi) = \rho((x, y, \alpha), \mathrm{epi} f)$. Then $x^* \in \partial\varphi(x)$ means that there is a $\lambda > 0$ such that $(x^*, -1) \in \lambda\partial\rho((x, \varphi(x)), \mathrm{epi}\varphi)$. By Proposition 2.4 of [7] for any finite dimensional $L \subset X$ there are sequences $\{\varepsilon_n\}$ converging to zero and $\{x_n\}$ converging to x with $\varphi(x_n)$ converging to $\varphi(x)$ such that

$$(x^*, -1) \in \limsup_{n\to\infty} \partial^-_{(L\times\mathbb{R})\varepsilon_n} \rho((x_n, \varphi(x_n)), \mathrm{epi}\varphi),$$

so that everything reduces to the above considered case of a Lipschitz function.

Remark. It is not clear whether it is possible to get the result of Proposition 3.3 for the approximate subdifferential under the sequential lower semicompactness assumption. What follows from the proof in this case is that for a Lipschitz function the proposition holds if X is Dini-Hadamard trustworthy (for in this case no restrictions to finite dimensional subspaces is needed).

4. Operations with set-valued mappings. We consider here three operations with set-valued mappings: composition, addition and intersection. Given two set-valued mappings, F from X into Y and G from Y into Z, the composition $G \circ F$ is defined as follows:

$$(G \circ F)(x) = \bigcup_{y \in F(x)} G(y).$$

If $F_1, \dots, F_k$ are set-valued mappings from X into Y, then adding (algebraically) their values at every x we get the sum of F_i:

$$(F_1 + \dots + F_k)(x) = F_1(x) + \dots + F_k(x),$$

and intersection of values gives the intersection of F_i:

$$(\bigcap F_i)(x) = \bigcap F_i(x).$$

The following proposition is elementary.

Proposition 4.1. *(a) For the composition $G \circ F$ of set-valued mappings F from X into Y and G from Y into Z we have*

$$\chi_{G\circ F}(x, z) = \inf_y (\chi_F(x, y) + \chi_G(y, z));$$

(b) for the sum $F = F_1 + \dots + F_k$ of set-valued mappings from X into Y we have

$$\chi_F(x, y) = \inf\{\chi_{F_1}(x, y_1) + \dots + \chi_{F_k}(x, y_k) :\ y_1 + \dots + y_k = y\};$$

(c) for the intersection $F = \bigcap F_i$ of set-valued mappings from X into Y we have

$$\chi_F(x, y) = \chi_{F_1}(x, y) + \dots + \chi_{F_k}(x, y).$$

We see that in each case the indicator function of the resulting operation appears as a result of two subsequent operations: addition and minimization one of which can be absent. Thus, knowing estimates for subdifferentials of sums of functions and marginal functions, we can obtain estimates for coderivatives of composed set-valued mappings.

5. Weak fuzzy calculus of elementary coderivatives. In this section either ∂ is the Dini-Hadamard subdifferential and X, Y, Z, as well as their products[2], are Dini-Hadamard trustworthy spaces, or ∂ is the Fréchet subdifferential and the spaces are Asplund. (In fact, what we need is that ∂ satisfies the weak fuzzy calculus rule.)

Proposition 5.1. *Let $F: X \to Y$ and $G: Y \to Z$ be set-valued mappings with closed graphs. Let $\overline{z} \in (G \circ F)(\overline{x})$ and $\overline{x}^* \in D^*(G \circ F)(\overline{x}, \overline{z})(\overline{z}^*)$. Then for any $\overline{y} \in F(\overline{x}) \bigcap G^{-1}(\overline{z})$, any $\varepsilon > 0$ and any weak* neighborhoods U^*, V^* and W^* of zeros in X^*, Y^* and Z^* respectively there are $x \in X$, y_1, $y_2 \in Y$, $z \in Z$, and $x^* \in X^*$, y_1^*, $y_2^* \in Y^*$, $z^* \in Z^*$ such that*

$$
\left\{
\begin{array}{l}
\|x - \overline{x}\| < \varepsilon,\ \|y_i - \overline{y}\| < \varepsilon,\ \|z - \overline{z}\| < \varepsilon; \\
x^* \in D^*F(x, y_1)(y_1^*),\ y_2^* \in D^*G(y_2, z)(z^*); \\
x^* \in \overline{x}^* + U^*,\ y_1^* - y_2^* \in V^*,\ z^* \in \overline{z}^* + W^*.
\end{array}
\right.
\tag{5.1}
$$

P r o o f. Consider the sets $S_1 = \mathrm{Graph} F \times Z$ and $S_2 = X \times \mathrm{Graph} G$. Then

$$
\inf_y (\chi_F(x, y) + \chi_G(y, z)) = \inf_y (\delta_{S_1}(x, y, z) + \delta_{S_2}(x, y, z)).
\tag{5.2}
$$

Therefore by Propositions 3.1 and 4.1(a)

$$
(\overline{x}^*, 0, -\overline{z}^*) \in \partial(\delta_{S_1} + \delta_{S_2})(\overline{x}, \overline{y}, \overline{z}).
\tag{5.3}
$$

As the sets S_i are closed, the functions δ_{S_i} are lower semicontinuous and we can apply the weak fuzzy calculus rule to conclude that for given $\varepsilon > 0$, U^*, V^*, W^* there are x_i, y_i, z_i and x_i^*, y_i^*, z_i^*, $(i = 1, 2)$, such that

$$
\left\{
\begin{array}{l}
\|x_i - \overline{x}\| < \varepsilon,\ \|y_i - \overline{y}\| < \varepsilon,\ \|z_i - \overline{z}\| < \varepsilon; \\
(x_1^*, -y_1^*, -z_1^*) \in \partial\delta_{S_1}(x_1, y_1, z_1),\ (x_2^*, y_2^*, -z_2^*) \in \partial\delta_{S_2}(x_2, y_2, z_2); \\
x_1^* + x_2^* \in \overline{x}^* + U^*,\ y_1^* - y_2^* \in V^*,\ z_1^* + z_2^* \in \overline{z}^* + W^*.
\end{array}
\right.
\tag{5.4}
$$

[2]It is not known whether a product of two ∂-trustworthy space is ∂-trustworthy. This is certainly the case for the Dini-Hadamard subdifferential if there are Gâteaux differentiable Lipschitz bump functions on the spaces.

It remains to notice that (by the axioms (f) and (c) of subdifferential) $\partial\delta_{S_1}(x,y,z) = \partial\chi_F(x,y)\times\{0\}$ and $\partial\delta_{S_2}(x,y,z) = \{0\}\times\partial\chi_G(y,z)$ and set $x = x_1$, $x^* = x_1^*$, $z = z_2$ and $z^* = z_2^*$. □

Proposition 5.2. *Let $F_i : X \to Y$, $(i = 1,\dots,k)$ be set-valued mappings with closed graphs. Set $F(x) = F_1(x) + \cdots + F_k(x)$ and assume that $\overline{y} \in F(\overline{x})$ and $\overline{x}^* \in D^*F(\overline{x},\overline{y})(\overline{y}^*)$. Assume further that $\overline{y}_i \in F_i(\overline{x})$, $\overline{y}_1 + \cdots + \overline{y}_k = \overline{y}$. Then for any $\varepsilon > 0$ and any weak* neighborhoods U^*, V^* of zeros in X^* and Y^* respectively there are $x_i,\ y_i,\ x_i^*,\ y_i^*\ \ (i = 1,\dots,k)$ such that*

$$\begin{cases} \|x_i - \overline{x}\| < \varepsilon,\ \|y_i - \overline{y}_i\| < \varepsilon, \\ x_i^* \in D^*F_i(x_i,y_i)(y_i^*), \\ x_1^* + \cdots + x_k^* \in \overline{x}^* + U^*,\ y_i^* \in \overline{y}^* + W^*\ (i = 1,\dots,k), \end{cases} \tag{5.5}$$

P r o o f. Consider the following sets and functions on $X \times Y^{k+1}$:

$$S_i = \{(x,y,y_1,\dots,y_k) :\ y_i \in F_i(x)\},\ i = 1,\dots,k;$$
$$S_0 = \{(x,y,y_1,\dots,y_k) :\ y = y_1 + \cdots + y_k\};$$
$$f_i(x,y,y_1,\dots,y_k) = \delta_{S_i}(x,y,y_1,\dots,y_k),\ i = 0,\dots,k.$$

Then by Proposition 4.1(b)

$$\chi_F(x,y) = \inf_{y_1,\dots,y_k} \sum_{i=0}^{k} f_i(x,y,y_1,\dots,y_k).$$

By Proposition 3.1

$$(\overline{x}^*, -\overline{y}^*, 0,\dots,0) \in \partial\left(\sum_{i=0}^{k} f_i\right)(\overline{x},\overline{y},\overline{y}_1,\dots,\overline{y}_k), \tag{5.6}$$

so applying the weak fuzzy calculus rule (as all f_i are l.s.c.) we shall find (for given ε, U^*, V^*) $x_i,\ v_i,\ y_{ij},\ x_i^*,\ v_i^*,\ y_{ij}^*,\ (i,j = 1,\dots,k)$, such that

$$\begin{cases} \|x_i - \overline{x}\| < \varepsilon,\ \|v_i - \overline{y}\| < \varepsilon,\ \|y_{ij} - \overline{y}\| < \varepsilon; \\ (x_i^*, -v_i^*, -y_{i1}^*,\dots,-y_{ik}^*) \in \partial f_i(x_i,v_i,y_{i1},\dots,y_{ik}); \\ \displaystyle\sum_{i=1}^{k} x_i^* \in \overline{x}^* + U^*,\ \sum_{i=0}^{k} v_i^* \in \overline{y}^* + V^*/2,\ \sum_{i=0}^{k} y_{ij}^* \in V^*/2\ (j = 1,\dots,k). \end{cases} \tag{5.7}$$

We have for $i = 1,\dots,k$ (as f_i does not depend on y and y_{ij} for $j \neq i$): $v_i^* = 0$, $y_{ij}^* = 0$ if $i \neq j$. Set $y_i^* = y_{ii}^*$, $y_i = y_{ii}$. Then $(x_i^*, -y_i^*) \in \partial\chi_{F_i}(x_i,y_i)$ which is the same as

$x_i^* \in D^*F_i(x_i, y_i)(y_i^*)$ for $i = 1, \dots, k$. Furthermore (as S_0 is a convex set), $y_{0j}^* = -v_0^*$ for every $j = 1, \dots, k$. So by (5.7) $v_0^* \in \overline{y}^* + V^*/2$ and $-v_0^* + y_i^* \in V^*/2$, that is $y_i^* \in \overline{y}^* + V^*$. This completes the proof. □

Proposition 5.3. *Let F_i be as in Proposition 5.2 and $F(x) = \bigcap F_i(x)$. Assume that $\overline{y} \in F(\overline{x})$ and $\overline{x}^* \in D^*F(\overline{x}, \overline{y})(\overline{y}^*)$. Then for any $\varepsilon > 0$ and any weak* neighborhoods U^*, V^* of zeros in X^* and Y^* respectively there are x_i, y_i, x_i^*, y_i^* $(i = 1, \dots, k)$ such that*

$$\begin{aligned}
&\|x_i - \overline{x}\| < \varepsilon,\ \|y_i - \overline{y}_i\| < \varepsilon,\\
&x_i^* \in D^*F_i(x_i, y_i)(y_i^*),\\
&x_1^* + \cdots + x_k^* \in \overline{x}^* + U^*,\ y_1^* + \cdots + y_k^* \in \overline{y}^* + W^*.
\end{aligned}$$

P r o o f. By Proposition 4.1(c), $\chi_F = \sum \chi_{F_i}$ and the direct application of the weak fuzzy calculus rule gives the desired result. □

6. Strong fuzzy calculus for Fréchet coderivatives. In this section all spaces are Asplund (Fréchet trustworthy) and ∂ is the Fréchet subdifferential.

The strong fuzzy calculus rule requires the general metric qualification condition. So we have to find out first what this condition means for indicator functions.

Proposition 6.1. *Let $S_1, \dots, S_k$ be closed subsets of X, and let $x \in \bigcap S_i$. Then the indicator functions δ_{S_i} satisfy the general metric qualification condition at $\overline{x} \in \bigcap S_i$ if and only if there is a nondecreasing nonnegative function $\omega(t)$ on $\mathbb{R}_+$ which is continuous and equal to zero at zero and such that*

$$\rho(x, \bigcap S_i) \le \omega\left(\sum \rho(x, S_i)\right) \tag{6.1}$$

for all x of a neighborhood of $\overline{x}$.

P r o o f. If we consider the sum norm $\|(x, \alpha)\| = \|x\| + |\alpha|$ in $X \times \mathbb{R}$, then for any set S

$$\rho((x, \alpha), \mathrm{epi}\delta_S) = \rho(x, S) + \alpha^-$$

(where $\alpha^- = \max\{0, -\alpha\}$). Therefore if (6.1) is valid, then setting $f = \delta_{\cap S_i}$, we have for $\alpha = \sum \alpha_i$.

$$\begin{aligned}
\rho((x, \alpha), \mathrm{epi}\delta_{\cap S_i}) &= \rho(x, \bigcap S_i) + \alpha^-\\
&\le \omega\left(\sum \rho(x, S_i)\right) + \alpha^- \le \omega\left(\sum \rho(x, S_i)\right) + \sum \alpha_i^-\\
&\le \omega'\left(\sum (\rho(x, S_i) + \alpha_i^-)\right) = \omega'\left(\sum \rho((x, \alpha_i), \mathrm{epi}\delta_{S_i})\right),
\end{aligned}$$

where $\omega'(t) = \omega(t) + t$.

Conversely, let

$$\rho((x,\alpha),\mathrm{epi}\delta_{\cap S_i}) \le \omega(\sum \rho((x,\alpha_i),\mathrm{epi}\delta_{S_i}).$$

Then taking $\alpha=\alpha_i=0$, we get (6.1).

Thus, if $F_1,\dots,F_k$ are set-valued mappings from X into Y, then the indicator functions of their graphs satisfy the general metric qualification condition at $(\overline{x},\overline{y})$ if and only if for a certain function ω with the properties described above

$$\rho((x,y),\bigcap \mathrm{Graph}F_i) \le \omega\left(\sum_i \rho((x,y),\mathrm{Graph}F_i)\right)$$

for all (x,y) of a neighborhood of $(\overline{x},\overline{y})$. □

Proposition 6.2. *Let $F: X\to Y$ and $G: Y\to Z$ be set-valued mappings with closed graphs. Set $H(x,z)=F(x)\bigcap G^{-1}(z)$, let $\overline{z}\in(G\circ F)(\overline{x})$, $\overline{x}^*\in D^*(G\circ F)(\overline{x},\overline{z})(\overline{z}^*)$ and $\overline{y}\in H(\overline{x},\overline{z})$. Assume that there is a nonnegative nondecreasing function ω on $\mathbb{R}_+$, which is continuous and equal to zero at zero, such that*

$$\rho((x,y,z),\mathrm{Graph}H)\le\omega(\rho((x,y),\mathrm{Graph}F)+\rho((x,y),\mathrm{Graph}G))$$

for all (x,y,z) of a neighborhood of $(\overline{x},\overline{y},\overline{z})$. Then for any $\varepsilon>0$ there are $x\in X$, $y_1,\ y_2\in Y$, $z\in Z$, and $x^*\in X^*$, $y_1^*,\ y_2^*\in Y^*$, $z^*\in Z^*$ such that

$$\begin{cases} \|x-\overline{x}\|<\varepsilon,\ \|y_i-\overline{y}\|<\varepsilon,\ \|z-\overline{z}\|<\varepsilon; \\ x^*\in D^*F(x,y_1)(y_1^*),\ y_2^*\in D^*G(y_2,z)(z^*); \\ \|x^*-\overline{x}^*\|<\varepsilon,\ \|y_1^*-y_2^*\|<\varepsilon,\ \|z^*-\overline{z}^*\|<\varepsilon. \end{cases} \tag{6.2}$$

P r o o f. The same as the proof of Proposition 5.1. We only need to observe that $S_1\bigcap S_2=\mathrm{Graph}H$, $\rho((x,y),\mathrm{Graph}F)=\rho((x,y,z),S_1)$, and $\rho((y,z),\mathrm{Graph}G)=\rho((x,y,z),S_2)$ and apply the strong fuzzy calculus rule to pass from (5.3) (with ∂ being the Fréchet subdifferential) to (6.2). □

Proposition 6.3. *Let $F_i: X\to Y$, $(i=1,\dots,k)$ be set-valued mappings with closed graphs. Set $F(x)=F_1(x)+\dots+F_k(x)$ and assume that $\overline{y}\in F(\overline{x})$ and $\overline{x}^*\in D^*F(\overline{x},\overline{y})(\overline{y}^*)$. Let $\overline{y}_i\in F_i(\overline{x})$, $\overline{y}_1+\dots+\overline{y}_k=\overline{y}$. Assume finally that there is a nonnegative nondecreasing function ω on $\mathbb{R}_+$, which is continuous and equal to zero at zero, such that*

$$\rho((x,y),\mathrm{Graph}F)\le\omega\left(\sum\rho((x,y_i),\mathrm{Graph}F_i)+\|y-\sum y_i\|\right)$$

for all $(y,y_1,\dots,y_k)$ of a neighborhood of $(\overline{y},\overline{y}_1,\dots,\overline{y}_k)$.

Then for any $\varepsilon > 0$ *there are* $x_i,\ y_i,\ x_i^*,\ y_i^*\ \ (i = 1, \dots, k)$ *such that*

$$\left\{\begin{array}{l} \|x_i - \overline{x}\| < \varepsilon,\ \|y_i - \overline{y}_i\| < \varepsilon; \\ x_i^* \in D^*F_i(x_i, y_i)(y_i^*)\ (i = 1, \dots, k); \\ \|x_1^* + \dots + x_k^* - \overline{x}^*\| < \varepsilon,\ \|y_i^* - \overline{y}^*\| < \varepsilon. \end{array}\right. \tag{6.3}$$

P r o o f. The same as the proof of Proposition 5.2. We only need to observe that $\rho((x, y_i), \mathrm{Graph} F_i) = \rho((x, y, y_1, \dots, y_k), S_i)$ for $i = 1, \dots, k$ and $\|y - \sum y_i\|$ is the distance from $(x, y, y_1, \dots, y_k)$ to S_0 and apply the strong fuzzy calculus to get (6.3) from (5.6). □

Proposition 6.4. *Let* F_i *be as in Proposition* 6.3 *and* $F(x) = \bigcap F_i(x)$*. Assume that* $\overline{y} \in F(\overline{x})$ *and* $\overline{x}^* \in D^*F(\overline{x}, \overline{y})(\overline{y}^*)$*. Assume finally that there is a nonnegative nondecreasing function* ω *on* $\mathbb{R}_+$*, which is continuous and equal to zero at zero, such that* (6.1) *holds.*

Then for any $\varepsilon > 0$ *there are* $x_i,\ y_i,\ x_i^*,\ y_i^*\ \ (i = 1, \dots, k)$ *such that*

$$\begin{array}{l} \|x_i - \overline{x}\| < \varepsilon,\ \|y_i - \overline{y}_i\| < \varepsilon, \\ x_i^* \in D^*F_i(x_i, y_i)(y_i^*)\ (i = 1, \dots, k), \\ \|x_1^* + \dots + x_k^* - \overline{x}^*\| < \varepsilon,\ \|y_1^* + \dots + y_k^* - \overline{y}^*\| < \varepsilon. \end{array}$$

7. Exact calculus for approximate subdifferentials. In this section X, Y, Z are arbitrary Banach spaces and ∂ is the approximate subdifferential. Results analogous to those presented in the section are also valid for the limiting Fréchet subdifferential, and no change in proofs is required. Exact calculus for approximate subdifferentials requires the linear-rate metric qualification condition which is a special case of the general metric qualification condition corresponding to $\omega(t) = Kt$. Therefore for such $\omega(\cdot)$ Proposition 6.1 gives the characterization for the linear-rate metric qualification condition for indicator functions as well.

Proposition 7.1. *Let* $F : X \to Y$ *and* $G : Y \to Z$ *be set-valued mappings with closed graphs. Let* $\overline{z} \in (G \circ F)(\overline{x})$*. Set* $H(x, z) = F(x) \bigcap G^{-1}(z)$*.*

(a) Assume that for a certain $\overline{y} \in H(\overline{x}, \overline{z})$ *the set-valued mapping* H *is lower semicontinuous at* $(\overline{x}, \overline{y}, \overline{z})$ *and the inequality*

$$\rho((x, y, z), \mathrm{Graph} H) \le K(\rho((x, y), \mathrm{Graph} F) + \rho((y, z), \mathrm{Graph} G)) \tag{7.1}$$

is satisfied for all (x, y, z) *in a neighborhood of* $(\overline{x}, \overline{y}, \overline{z})$*. Then*

$$D^*(G \circ F)(\overline{x}, \overline{z})(z^*) \subset (D^*F(\overline{x}, \overline{y})) \circ (D^*G(\overline{y}, \overline{z}))(z^*) \tag{7.2}$$

for all z^*.

(b) Assume that H *is topologically lower semicompact at* $(\overline{x},\overline{z})$ *and for any* $\overline{y}\in H(\overline{x},\overline{z})$ *there is a* $K>0$ *such that (7.1) holds for all* (x,y,z) *of a neighborhood of* $(\overline{x},\overline{y},\overline{z})$. *Then*

$$D^*(G\circ F)(\overline{x},\overline{z})(z^*)\subset\bigcup_{\overline{y}\in H(\overline{x},\overline{z})}(D^*F(\overline{x},\overline{y}))\circ(D^*G(\overline{y},\overline{z}))(z^*) \tag{7.3}$$

for all z^*.

P r o o f. Let $\overline{x}^*\in D^*(G\circ F)(\overline{x},\overline{z})(\overline{z}^*)$. If (a) holds then, as in the proof of Proposition 5.1, we get (5.3) (with ∂ being the approximate subdifferential) applying Propositions 3.2 and 4.1(a) to (5.2). Then, as in the proof of Proposition 6.2, we observe that (7.1) implies the linear-rate metric qualification condition for δ_{S_1} and δ_{S_2}. This means that the exact calculus rule can be applied to these functions at $(\overline{x},\overline{y},\overline{z})$ which implies the existence of a $\overline{y}^*$ such that $(\overline{x}^*,-\overline{y}^*,0)\in\partial\delta_{S_1}(\overline{x},\overline{y},\overline{z})$ and $(0,\overline{y}^*,-\overline{z}^*)\in\partial\delta_{S_2}(\overline{x},\overline{y},\overline{z})$, from which we get that $y^*\in D^*G(\overline{y},\overline{z})(z^*)$ and $x^*\in D^*F(\overline{x},\overline{y})(y^*)$ (by specifying the expressions for $\partial\delta_{S_1}$ and $\partial\delta_{S_1}$ with the help of the properties (f) and (c) of subdifferential as it was done in the proof of Proposition 5.1).

This completes the proof of (a).

In case of (b) we have to apply Proposition 3.3 instead of Proposition 3.2 to get, instead of (5.3), the inclusion

$$(\overline{x}^*,0,-\overline{z}^*)\in\bigcup_{\overline{y}\in H(\overline{x},\overline{z})}\partial(\delta_{S_1}+\delta_{S_2})(\overline{x},\overline{y},\overline{z}).$$

The rest of the proof is exactly the same as in (a). □

Proposition 7.2. *Let* $F_i: X\to Y$ $(i=1,\dots,k)$ *be set-valued mappings with closed graphs. Set* $F(x)=F_1(x)+\dots+F_k(x)$. *Let* $(\overline{x},\overline{y})\in \mathrm{Graph}F$. *Define the set-valued mapping* $H:\ X\times Y\to Y^k$:

$$H(x,y)=\{(y_1,\dots,y_k):\ y_i\in F_i(x),\ \sum_i y_i=y\}.$$

(a) Assume that there are $(\overline{y}_1,\dots,\overline{y}_k)\in H(\overline{x},\overline{y})$ *and a* $K>0$ *such that* H *is lower semicontinuous at* $(\overline{x},\overline{y},\overline{y}_1,\dots,\overline{y}_k)$ *and*

$$\rho((x,y),\mathrm{Graph}F)\le K\left(\sum\rho((x,y_i),\mathrm{Graph}F_i)+\|y-\sum y_i\|\right) \tag{7.4}$$

for all $(x,y,y_1,\dots,y_k)$ *of a neighborhood of* $(\overline{x},\overline{y},\overline{y}_1,\dots,\overline{y}_k)$. *Then*

$$D^*F(\overline{x},\overline{y})(y^*)\subset D^*F(\overline{x},\overline{y}_1)(y^*)+\dots+D^*F(\overline{x},\overline{y}_k)(y^*) \tag{7.5}$$

for all y^*.

(b) Assume that H *is topologically lower semicompact at* $(\overline{x},\overline{y})$ *and for any* $(\overline{y}_1,\dots,\overline{y}_k)\in H(\overline{x},\overline{y})$ *there is a* $K>0$ *such that* (7.4) *holds for all* $(x,y,y_1,\dots,y_k)$ *of a neighborhood of* $(\overline{x},\overline{y},\overline{y}_1,\dots,\overline{y}_k)$. *Then*

$$D^*F(\overline{x},\overline{y})(y^*)\subset \bigcup_{(\overline{y}_1,\dots,\overline{y}_k)\in H(\overline{x},\overline{y})} (D^*F_1(\overline{x},\overline{y}_1)(y^*)+\cdots+D^*F_k(\overline{x},\overline{y}_k)(y^*)) \tag{7.6}$$

for all y^*.

Proof. Assume that the condition of (a) holds. Let $\overline{x}^*\in D^*F(\overline{x},\overline{y})(\overline{y}^*)$. Then applying Propositions 3.2 and 4.1(b), we get (5.6) (with ∂ being the approximate subdifferential and the same functions f_i as in the proof of Proposition 5.2). Then, as in the proof of Proposition 6.3, we observe that (7.4) is exactly the linear-rate metric qualification condition for functions f_i. Now applying to (5.6) the exact calculus rule, we find $x_i^*,\ v_i^*,\ y_{ij}^*$ such that $(x_i^*,-v_i^*,-y_{i1}^*,\dots,-y_{ik}^*)\in\partial f_i(\overline{x},\overline{y},\overline{y}_1,\dots,\overline{y}_k)$ such that $\sum_{i=0}^k x_i^*=\overline{x}^*$, $\sum_{i=0}^k v_i^*=y^*$, $\sum_{i=0}^k y_{ij}^*=0$ for every $i=1,\dots,k$. As each f_i $(i=1,\dots,k)$ depends only on x and y_i, we have $v_i^*=0$ and $y_{ij}^*=0$ $(i\neq j)$ for such i (as in the proof of Proposition 5.2) and as f_0 is a convex function not depending on x, we have $x_0^*=0$ and $v_0^*=y_{ii}^*$ for $i=1,\dots,k$. It follows that $y_{ii}^*=v_0^*=\overline{y}^*$ and $\sum_{i=1}^k x_i^*=\overline{x}^*$.

Assuming (b), we have to use Proposition 3.3 instead of 3.2. So instead of (5.6) we shall have

$$(\overline{x}^*,-\overline{y}^*,0,\dots,0)\in \bigcup_{(\overline{y}_1,\dots,\overline{y}_k)\in H(\overline{x},\overline{y})} \partial\left(\sum f_i\right)(\overline{x},\overline{y},\overline{y}_1,\dots,\overline{y}_k)$$

which means that there is a certain $(\overline{y}_1,\dots,\overline{y}_k)\in H(\overline{x},\overline{y})$ for which (5.6) holds. The rest of the proof is as in (a). □

Proposition 7.3. *Let* F *and* F_i *be as in Proposition 6.4. Assume that* $\overline{y}\in F(\overline{x})$ *and*

$$\rho((x,y),\mathrm{Graph}F)\le K\left(\sum\rho((x,y_i),\mathrm{Graph}F_i)\right)$$

for all (x,y) *in a neighborhood of* $(\overline{x},\overline{y})$. *Then, given* x^* *and* y^* *such that* $x^*\in D^*F(\overline{x},\overline{y})(y^*)$, *there are* $y_1^*,\dots,y_k^*$ *such that* $y_1^*+\cdots+y_k^*=y^*$ *and*

$$x^*\in D^*F_1(\overline{x},\overline{y})(y_1^*)+\cdots+D^*F_k(\overline{x},\overline{y})(y_k^*).$$

Proof. This is an immediate consequence of the exact calculus rule and Proposition 6.1. □

8. Subdifferential qualification conditions. By that we mean sufficient criteria for metric qualification conditions stated in terms of subdifferentials (or normal cones, or coderivatives).

Proposition 8.1. *Assume that X is a ∂-trustworthy space as well as its powers. Let S_i, $(i = 1, \dots, k)$ be closed subsets of X, and let $\overline{x} \in \bigcap S_i = S$. Assume finally that there are $\alpha \in (0,1)$ and $\varepsilon > 0$ such that*

$$x_i \in S_i \backslash S,\ \|x_i - \overline{x}\| \le \varepsilon,\ x_i^* \in N(S_i, x_i),\ \|\sum x_i^*\| \le \varepsilon \ \Rightarrow\ \max_i \|x_i^*\| \le \alpha. \tag{8.1}$$

Then there is a $K > 0$ such that

$$\rho(x, S) \le K \sum \rho(x, S_i) \tag{8.2}$$

for all x of a neighborhood of $\overline{x}$.

Moreover, if ∂ has exact calculus on X^{k+1}, then the condition $x_i^* \in N(S_i, x_i)$ in the left part of (8.1) *can be strengthened and replaced by $x_i^* \in (k+1)\partial\rho(x_i, S_i)$.*

(To see how much weaker is the linear-rate metric condition (8.2) in comparison with (8.1), consider again the positive and the negative orthants in $\mathbb{R}^2$ or l^2.)

Proof. A sketch of the proof for two sets can be found in [10]. The complete argument follows (which is basically a slight modification of that given in the proof of Theorem 5.1 of [7]).

Assuming the contrary, we shall find a sequence $\{u_n\}$ converging to $\overline{x}$ such that $\varepsilon_n = \rho(u_n, \bigcap S_i) > 2n \sum \rho(u_n, S_i)$. This means that there are $u_{in} \in S_i$ such that

$$\sum_i \|u_n - u_{in}\| < \varepsilon_n / 2n. \tag{8.3}$$

(Observe that $\rho(u_{in}, \bigcap S_i) \ge (1 - (1/2n))\varepsilon_n$ in this case.) Applying Ekeland's variational principle to the function $\sum \|x - x_i\| + \sum \delta_{S_i}(x_i)$, we shall find, taking (8.3) into account, a v_n and $v_{1n} \in S_1, \dots, v_{kn} \in S_k$ such that

$$\|u_n - v_n\| + \sum \|u_{in} - v_{in}\| \le \varepsilon_n / 2 \tag{8.4}$$

and the function

$$(x, x_1, \dots, x_k) \mapsto \sum \|x - x_i\| + n^{-1}(\|x - v_n\| + \sum \|x_i - v_{in}\|)$$

attains minimum on $X \times S_1 \times \dots \times S_k$ at $(v_n, v_{1n}, \dots, v_{kn})$.

Observe that, as follows from (8.3), (8.4), neither v_n nor any of v_{in} can belong to $\bigcap S_i$. This means that at least for one i we have $v_n \ne v_{in}$.

Consider first the case when ∂ has exact calculus. As the function above satisfies the Lipschitz condition with constant not exceeding $k+1$, the function

$$f(x, x_1, \dots, x_k) = \sum \|x - x_i\| + n^{-1}\left(\|x - v_n\| + \sum \|x_i - v_{in}\|\right) + (k+1)\sum \rho(x_i, S_i)$$

attains an unconditional minimum at $(v_n, v_{1n}, \dots, v_{kn})$.

This means that $(0, 0, \dots, 0) \in \partial f(v_n, v_{1n}, \dots, v_{kn})$. As f is a sum of Lipschitz functions, hence satisfying the linear metric qualification condition, we can apply the exact fuzzy calculus rule and find $u^*_{in} \in \partial\|\cdot\|(v_n - v_{in})$ and $v^*_{in} \in (k+1)\partial\rho(v_{in}, S_i)$ such that

$$\|\sum v^*_{in}\| \le n^{-1},\ \| - u^*_{in} + v^*_{in}\| \le n^{-1},\ i = 1, \dots, k.$$

As there is an index i for which $v_n \ne v_{in}$, we have $\|u^*_{in}\| = 1$ at least for one i. Thus, we have sequences $\{v_{in}\} \subset S_i \backslash S$, $i = 1, \dots, k$, converging to $\overline{x}$ and $\{v^*_{in}\}$ such that $v^*_{in} \in (k+1)\partial\rho(v_{in}, S_i)$ and $\|\sum v^*_{in}\| \to 0$, and on the other hand $\max_i \|v^*_{in}\| \to 1$. Setting $x_n = v_n$ we see that the result contradicts the assumptions.

In the general case when ∂ may not have exact calculus, we consider instead of f the function

$$g(x, x_1, \dots, x_k) = \sum \|x - x_i\| + n^{-1}\left(\|x - v_n\| + \sum \|x_i - v_{in}\|\right) + \sum \delta_{S_i}(x_i)$$

Clearly, this function attains its minimum at $(v_n, v_{1n}, \dots, v_{kn})$ so that we have as above $(0, 0, \dots, 0) \in \partial g(v_n, v_{1n}, \dots, v_{kn})$.

We can consider g as the sum of a Lipschitz (actually convex continuous) function

$$g_1(x, x_1, \dots, x_k) = \sum \|x - x_i\| + n^{-1}\left(\|x - v_n\| + \|x_i - v_{in}\|\right)$$

and a lower semicontinuous function

$$g_2(x, x_1, \dots, x_k) = \sum \delta_{S_i}(x_i).$$

Therefore g satisfies the general metric qualification condition and we can apply the basic fuzzy principle and find $x_n, x_{1n}, \dots, x_{kn}, w_{1n}, \dots, w_{kn}$ such that $\|x_n - v_n\| \to 0$, $\|x_{in} - v_{in}\| \to 0$, $\|w_{in} - v_{in}\| \to 0$ as $n \to \infty$ with x_n and w_{in} so close to v_n and v_{in} respectively that $x_n \ne w_{in}$ for at least one i and, on the other hand, $(x^*_n, w^*_{1n}, \dots, w^*_{kn}) \in \partial g_1(x_n, w_{1n}, \dots, w_{kn})$ and $(w^*_n, x^*_{1n}, \dots, x^*_{kn}) \in \partial g_2(x_{1n}, \dots, x_{kn})$ such that $\|x^*_n - w^*_n\| < n^{-1}$, $\|x^*_{in} - w^*_{in}\| < n^{-1}$, $i = 1, \dots, k$.

As all terms in g_1 are convex continuous, for some $u^*_{in} \in \partial\|\cdot\|(x_n - w_{in})$ we have by standard rules of convex analysis: $\|x^*_n - \sum u^*_{in}\| \le n^{-1}$ and $\|u^*_{in} - w^*_{in}\| \le n^{-1}$. On the other hand, by the property (f) of subdifferentials, $x^*_{in} \in \partial\delta_{S_i}(x_{in}) = N(S_i, x_{in})$ and $w^*_{in} = 0$. As in the first part of the proof, $\|u^*_{jn}\| = 1$ for at least one index j and therefore the norm of the corresponding x^*_{in} goes to one as $n \to \infty$ and we again see that (8.1) is satisfied for x^*_{in} and therefore come to a contradiction proving the claim.

As an immediate corollary of the proposition, we get the following result.

Corollary 8.2. *Let ∂ and X be as in Proposition* 8.1*, and let the spaces Y and Z belong to the same class as X (as well as powers and products of the three spaces). Let F, G and H be as in Proposition* 6.2*, and let $\overline{y} \in H(\overline{x}, \overline{z})$. Assume finally that there are $\varepsilon > 0$ and $\alpha \in (0,1)$ such that*

$$\left.\begin{array}{l} \|x-\overline{x}\| < \varepsilon,\ \|y-\overline{y}\| < \varepsilon,\ \|z-\overline{z}\| < \varepsilon; \\ x^* \in D^*F(x,y_1)(y_1^*),\ y_2^* \in D^*G(y_2,z)(z^*); \\ \|x^*\| < \varepsilon,\ \|y_1^*-y_2^*\| < \varepsilon, \|z^*\| < \varepsilon \end{array}\right\} \Rightarrow \max\{\|y_1^*\|, \|y_2^*\|\} \le \alpha.$$

Then there is a $K > 0$ such that

$$\rho((x,y,z), \mathrm{Graph}H) \le K(\rho((x,y), \mathrm{Graph}F) + \rho((y,z), \mathrm{Graph}G)) \tag{8.5}$$

for all (x,y,z) sufficiently close to $(\overline{x}, \overline{y}, \overline{z})$.

Proof. Apply Proposition 8.1 to $S_1 = \mathrm{Graph}F \times Z$ and $S_2 = X \times \mathrm{Graph}G$. □

Corollary 8.3. *Let ∂, X, Y be as in Corollary* 8.2*. Let F and F_i be as in Proposition* 6.3*, and let $\overline{y}_i \in F_i(\overline{x})$, $\sum \overline{y}_i = \overline{y}$. Assume that there are $\varepsilon > 0$ and $\alpha \in (0,1)$ such that*

$$\left.\begin{array}{l} \|x_i-\overline{x}\| + \sum \|y_i-\overline{y}\| < \varepsilon,\ y_i \in F_i(x_i); \\ \|y_i^*-y^*\| < \varepsilon, \|y^*\| < \varepsilon, x_i^* \in D^*F_i(x_i,y_i)(y_i^*), \|\sum x_i^*\| < \varepsilon \end{array}\right\} \Rightarrow \max_i \|x_i^*\| \le \alpha. \tag{8.6}$$

Then there is a $K > 0$ such that

$$\rho((x,y), \mathrm{Graph}F) \le K \sum \rho((x,y_i), \mathrm{Graph}G) \tag{8.7}$$

for all $(x,y,y_1,\ldots,y_k)$ sufficiently close to $(\overline{x}, \overline{y}, \overline{y}_1, \ldots, \overline{y}_k)$ and such that $y_1 + \cdots + y_k = y$.

Proof. Let S_i, $i = 0, \ldots, k$ be as in the proofs of Propositions 5.2 and 6.3. Then (8.6) reduces precisely to (8.1): we have seen that for such S_i

$$(x^*, y^*, y_1^*, \ldots, y_k^*) \in N((x,y,y_1,\ldots,y_k), S_i),$$

implies $y^* = 0$, $y_j^* = 0$ if $j \ne i$ for $i = 1, \ldots, k$, and $x^* = 0$, $y^* = y_1^* = \cdots = y_k^*$ for $i = 0$. Therefore (8.2) holds which in our case is precisely (8.7).

In the context of Asplund spaces, the condition (8.6) was recently introduced in [18] under the name "fuzzy qualification condition" and used to obtain the strong fuzzy calculus rule. We see that this condition is strictly stronger that the linear-rate metric qualification condition, and all the more than the "uniform lower semicontinuity condition" of [2, 11] known to be sufficient for the strong fuzzy calculus in Asplund spaces.

The three concluding results are corollaries from Proposition 8.1 for approximate subdifferentials.

Corollary 8.4. *Let X be a Banach space, and let ∂ stand for approximate subdifferential. Let S_i, $i = 1, \dots, k$ be closed subsets of X and $\overline{x} \in \bigcap S_i$. Suppose further that all S_i with a possible exception of one of them are compactly epi-Lipschitz*[3] *at $\overline{x}$. Then the condition*

$$x_i^* \in N(\overline{x}, S_i),\ \sum x_i^* = 0 \ \Rightarrow\ x_1^* = \cdots = x_k^* = 0 \tag{8.8}$$

is sufficient for the existence of a $K > 0$ such that (8.2) *holds for all x sufficiently close to $\overline{x}$.*

As stated, the result was proved in [15, 12]; earlier it was proved in [7] under a stronger assumption that the sets are epi-Lipschitz at $\overline{x}$.

For approximate coderivatives of set-valued mappings a corresponding calculus rule can be obtained under a weaker (than the compactly epi-Lipschitz property of the graph) condition, as was first observed in [19] for limiting Fréchet subdifferentials in Asplund spaces.

We shall say following [9] that a set-valued mapping F is *sequentially codirectionally compact* at $(\overline{x}, \overline{y}) \in \mathrm{Graph} F$ if for any sequence $\{x_n, y_n, x_n^*, y_n^*\}$ such that $y_n \in F(x_n)$, $x_n^* \in D^*F(x_n, y_n)(y_n^*)$, $(x_n, y_n) \to (\overline{x}, \overline{y})$, $\|x_n^*\| \to 0$ zero belongs to the norm closure of $\{y_n^*\}$, provided it belongs to the weak* closure of the sequence. This is a sequential version of the property introdused in [23] (for arbitrary cone-valued mappings, not necessarily coderivatives). Both versions are satisfied when the graph of F is compactly epi-Lipschitz at $(\overline{x}, \overline{y})$.

Corollary 8.5. *Let X, Y, Z be Banach spaces, let F and G be set-valued mappings from X into Y and from Y into Z respectively, and let ∂ stand for the approximate subdifferential. Set as above $H(x, z) = F(x) \bigcap G^{-1}(z)$ and assume that $(\overline{x}, \overline{y}, \overline{z}) \in \mathrm{Graph} H$. Suppose finally that the following two conditions are satisfied:*

(a) either G is sequentially codirectionally compact at $(\overline{y}, \overline{z})$ or F^{-1} is sequentially codirectionally compact at $(\overline{y}, \overline{x})$;

[3] S is compactly epi-Lipschitz at x (see [3]) if there are $\varepsilon > 0$ and a compact set $P \subset X$ such that for any $t \in [0, \varepsilon]$

$$(x + \varepsilon B) \cap S + tB \subset S + tP.$$

(b) $0 \in D^*F(\overline{x},\overline{y})(y^*)$ & $y^* \in D^*G(\overline{y},\overline{z})(0) \Rightarrow y^* = 0.$
Then there is a $K > 0$ *such that* (8.5) *holds for all* (x, y, z) *of a neighborhood of* $(\overline{x},\overline{y},\overline{z})$.

Proof. Again we consider the sets $S_1 = \text{Graph}F \times Z$ and $S_2 = X \times \text{Graph}G$. All we need to verify is that the conditions of Proposition 8.1 are satisfied for these sets at $(\overline{x},\overline{y},\overline{z})$. Assuming the contrary, we find sequences of $(x_{in}, y_{in}, z_{in}) \in S_i$ converging to $(\overline{x},\overline{y},\overline{z})$ and $(x^*_{in}, y^*_{in}, -z^*_{in}) \in 3\partial\rho((x_{in}, y_{in}, z_{in}), S_i)$, $i = 1, 2,$ such that

$$\|x^*_{1n} + x^*_{2n}\| \to 0, \ \|y^*_{1n} + y^*_{2n}\| \to 0, \ \|z^*_{1n} + z^*_{2n}\| \to 0 \tag{8.9}$$

and for any n the maximum of the norms of the six functionals is not smaller than one. We then observe (as in the proofs of Propositions 5.1, 6.2 and 7.1) that $x^*_{2n} = 0$ and $z^*_{1n} = 0$ which implies that $\max_i \|y^*_{in}\| \geq 1$ and consequently by (8.9) that

$$\liminf_{n\to\infty} \|y^*_{in}\| \geq 1, \ i = 1, 2. \tag{8.10}$$

Setting $x_n = x_{1n}$, $x^*_n = x^*_{1n}$, $z_n = z_{2n}$, $z^*_n = z^*_{2n}$, we have

$$(x^*_n, y^*_{1n}) \in 3\partial\rho((x_n, y_{1n}), \text{Graph}F), \ \ (y^*_{2n}, -z^*_n) \in 3\partial\rho((y_{2n}, z_n), \text{Graph}G).$$

Let y^* belong to the weak* closure of $\{y^*_{2n}\}$ (note that this is a bounded sequence). Then by (8.9) (and in view of the upper semicontinuity of approximate subdifferentials of a Lipschitz function),

$$(0, -y^*) \in 2\partial\rho((\overline{x},\overline{y}), \text{Graph}F), \ \ (y^*, -z^*) \in 2\partial\rho((\overline{y},\overline{z}), \text{Graph}G).$$

By (b) this implies that $y^* = 0$ and by (a) zero must belong to the norm closure of either of $\{y^*_{in}\}$, in contradiction with (8.10). □

Corollary 8.6. *Let* X, Y *be Banach spaces, let* F_i, $i = 1, \dots, k$, *be set-valued mappings from* X *into* Y *with closed graphs, and let* ∂ *stand for the approximate subdifferential. Let* $F = F_1 + \cdots + F_k$, *let* $\overline{y} \in F(\overline{x})$, *and let* $\overline{y}_i \in F_i(\overline{x})$ *be such that* $\sum \overline{y}_i = \overline{y}$. *Suppose finally that the following two conditions are satisfied:*

(a) the mappings F_i^{-1} *with possible exception of one of them are sequentially codirectionally compact at* $(\overline{y}_i, \overline{x})$ *respectively;*

(b) $x^*_i \in D^*F_i(\overline{x},\overline{y}_i)(0)$, $\sum x^*_i = 0 \Rightarrow x^*_1 = \cdots = x^*_k = 0.$
Then there is a $K > 0$ *such that* (8.7) *holds for any* $(x, y, y_1, \dots, y_k)$ *of a neighborhood of* $(\overline{x},\overline{y},\overline{y}_1,\dots,\overline{y}_k)$ *satisfying* $\sum y_i = y$.

Proof. The general scheme of the proof is exactly as in the previous case. Take the same S_i as in the proofs of Propositions 5.2 and 6.3: $S_i = \{(x, y, y_1, \dots, y_k) \in X \times Y^{k+1} : \ y_i \in F_i(x)\}$ for $i = 1, \dots, k$ and $S_0 = \{(x, y, y_1, \dots, y_k) \in X \times Y^{k+1} :$

$\sum y_i = y\}$. Assuming that the conditions of Proposition 8.1 are not satisfied for S_i at $(\overline{x}, \overline{y}, \overline{y}_1, \dots, \overline{y}_k)$, we shall find sequences of $(x_{in}, y_{in}, y_{1in}, \dots, y_{kin}) \in S_i$ and $(x^*_{in}, v^*_{in}, y^*_{1in}, \dots, y^*_{kin}) \in (k+1)\partial\rho((x_{in}, y_{in}, y_{1in}, \dots, y_{kin}), S_i)$ such that

$$(8.11) \qquad \sum_i \|x^*_{in}\| \to 0, \ \sum_i \|v^*_{in}\| \to 0, \ \sum_i \|y^*_{jin}\| \to 0, \quad j = 1, \dots, k$$

and for any n maximum of the norms of the functionals is not smaller than one. As in the proofs of Propositions 5.2, 6.3 and 7.2, we notice that $x^*_{0n} = 0$, $y^*_{jin} = 0$ if $i \geq 1$, $j \neq i$, $v^*_{in} = 0$ if $i \geq 1$ and $y^*_{j0n} = v^*_{0n}$ for $i = 1, \dots, k$. So setting $y_{in} = y_{iin}$ and $y^*_{in} = y^*_{iin}$ for $i = 1, \dots, k$, $y^*_{0n} = v^*_{on}$, we conclude that $\|y^*_{in}\| \to 0$ for all $i = 0, \dots, k$,

$$(8.12) \qquad (x^*_{in}, -y^*_{in}) \in (k+1)\partial\rho((x_{in}, y_{in}), \mathrm{Graph} F_i),$$

and $\max_i \|x^*_{in}\| \geq 1$, so that, taking if necessary a subsequence, we conclude from (8.11)

$$(8.13) \qquad \liminf_{n\to\infty} \|x^*_{in}\| \geq 1$$

at least for two indices i. As $\{x^*_{in}\}$ are bounded sequences, we can find points x^*_i in their respective weak* closures such that $(x^*_1, \dots, x^*_k)$ belongs to the weak-star closure of $\{(x^*_{1n}, \dots, x^*_{kn})\}$ in $(X^*)^k$ and $\sum x^*_i = 0$ (by (8.11)). By (8.12), $x^*_i \in D^*F_i(\overline{x}, \overline{y}_i)(0)$ which by (b) implies that all x^*_i are zeros. By (a) this means that the sequences x^*_{in} save at most one of them contain subsequences norm converging to zero. But this contradicts the established fact that (8.13) must be valid at least for two sequences. This completes the proof of the corollary. □

We leave to the reader the elementary task of combining Corollaries 8.2, 8.3, 8.5 and 8.6 with the corresponding propositions of §§ 5 – 7 to obtain calculus rules under subdifferential qualification conditions.

We also note that the last three corollaries are valid also for the limiting Fréchet subdifferential in Asplund spaces [17]. Actually, the proofs of analogues of the last two corollaries in this case are even simpler: one only needs to pass to limits in Corollaries 8.2 and 8.3 applied to the Fréchet subdifferential on corresponding spaces.

Remark (added in proof). In [17], [19] a property similar to sequential coderivative compactness (SCC) was called "partial sequential normal compactness". To eliminate the confusion we have to note that in the original version of [19] another property was defined in which sequences $\{(x_n, y_n)\}$ sufficiently close to $(\overline{x}, \overline{y})$, not only those converging to $(\overline{x}, \overline{y})$, were considered while the name "partial normal compactness" (PNS) wes used for a closely connected but differentproperty, and no sequential property was used in the original version of [17]. And it is in the development of (PNS) that the mentioned net predecessor of (SCC) was introduced in [23] and also in [14].

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