Subdiffusive transport of charged particles perpendicular to the large scale magnetic field

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[1] Transport of charged particles across the mean magnetic field due to broad band, powerlaw-distributed magnetic fluctuations is examined by direct computation of a large number of charged test particle trajectories. Fluctuations consist of a one dimensional slab spectrum with an additional 0.01 percent level of two dimensional fluctuations; the net field has no ignorable directions. We find that, at long times (t), the mean square displacements subdiffusively scale as $\sim t^{1/2}$. This confirms that perpendicular diffusion is suppressed when turbulence lacks strong three dimensional structure, in agreement with theories of Urch and of Kóta and Jokipii. *INDEX TERMS:* 7859 Space Plasma Physics: Transport processes; 2104 Interplanetary Physics: Cosmic rays; 2114 Interplanetary Physics: Energetic particles, heliospheric (7514); 7514 Solar Physics, Astrophysics, and Astronomy: Energetic particles (2114)

1. Introduction

[2] The collisionless spatial transport of energetic charged particles is fundamental in space and astrophysical plasma physics, entering in an essential way into cosmic ray modulation, solar energetic particle transport, shock acceleration, and wave particle interactions. It is also related to the issue of energy confinement in laboratory plasmas. The large scale mean magnetic field, say, $\mathbf{B}_0$, sets a preferred direction for transport, and an additional random or "turbulent" magnetic field component induces transport. Unperturbed orbits, computed in terms of $\mathbf{B}_0$ alone, are the starting point for quasilinear theory (QLT) [*Jokipii*, 1966], which has dominated the conceptualization of transport for more than 30 years. There are distinct regimes of diffusive transport parallel and perpendicular to $\mathbf{B}_0$, and these can be computed in QLT independently of one another. Perpendicular transport remains especially challenging, and even puzzling, in that numerical studies have not confirmed any theory to be accurate for both low and high energy charged particles [*Giacalone and Jokipii*, 1999; *Mace et al.*, 2000]. Among possible resolutions of these discrepancies are possibilities that nonquasilinear or nonlinear effects can alter transport drastically. In particular, *Urch* [1977] [see also *Lingenfelter et al.*, 1971; *Forman and Jokipii*, 1978] suggested that strong parallel diffusion can suppress or eliminate perpendicular diffusion. This issue has been recently reexamined by *Kóta and Jokipii* [2000]. Here we examine this issue of "compound" transport using numerically computed test particle trajectories. We focus upon trajectories in magnetostatic turbulence models that are close to one dimensional (1D), but which nonetheless have no ignorable coordinate. We show that the evolution of the ensemble towards a classical perpendicular diffusion regime is thwarted by parallel diffusion. The perpendicular trajectories settle into a stable subdiffusive mode of transport which is quantitatively described by the formulas of *Urch* [1977] and *Kóta and Jokipii* [2000].

2. Approach and Methods

[3] Let us consider a total magnetic field $\mathbf{B}(\mathbf{x}) = \mathbf{B}_0 + \mathbf{b}(\mathbf{x})$ consisting of a uniform mean magnetic field $\mathbf{B}_0$ and zero-average fluctuation $\mathbf{b}(\mathbf{x})$, here taken to be time independent (magnetostatic) and spatially homogeneous over the scales of interest in the three dimensional position $\mathbf{x} = (x, y, z)$. We choose the axes so that the mean field is in the $\hat{z}$ direction, $\mathbf{B}_0 = B_0\hat{z}$. Presently we select a specific model for the fluctuations. We are interested in a large number N_p of time (t) dependent trajectories of charged test particles under the influence of $\mathbf{B}$, designated by $\{\mathbf{X}^\sigma(t)\}$, $\sigma = 1, 2 \ldots N_p$ where $\mathbf{X}^\sigma = (X^\sigma, Y^\sigma, Z^\sigma)$ are the components of the particle trajectories. Below we will suppress the labels σ, although in practice averaging over a set of trajectories substitutes for an ensemble average $\langle \ldots \rangle$. Using the standard Fokker Planck definitions, we consider the perpendicular diffusion, associated with a diffusion coefficient $\kappa_\perp = \kappa_{xx} = \langle(\Delta X)^2\rangle/2\Delta t$, where Δt is an increment in time and ΔX and its components are the associated spatial increments during that time period. The diffusion coefficient is understood to be associated with an appropriate large Δt limit, and we will restrict ourselves here to the axisymmetric case in which $\kappa_\perp = \kappa_{xx} = \kappa_{yy}$. The parallel diffusion coefficient is defined similarly as $\kappa_\parallel = \langle(\Delta Z)^2\rangle/2\Delta t$. Each charged test particle with rest mass m, Lorentz factor γ, and velocity $\mathbf{v} = d\mathbf{X}/dt$ obeys the Newton-Lorentz equation

$$\gamma m \frac{d\mathbf{v}(t)}{dt} = \frac{q\mathbf{v}(t)}{c} \times \mathbf{B}(\mathbf{X}). \tag{1}$$

Here we assume a static $\mathbf{B}(\mathbf{X})$ and that the laboratory frame electric field is zero.

[4] The choice of fluctuation properties is central to the problem. Here we take $\mathbf{b}(\mathbf{r})$ to be a realization of magnetostatic broad band turbulence, constructed by Fourier analysis in a large rectangular box with square cross section relative to $\mathbf{B}_0$. Most of the energy (99.99%) is put in "slab" modes having a one dimensional spatial dependence

$$\mathbf{b}_{slab} = \sum_{k_z} \hat{\mathbf{b}}(k_z) \exp\, ik_z z \tag{2}$$

We choose two physical length scales for the slab modes: a parallel outer scale (correlation scale) λ_c, and a maximum excited wave number k_z^{max}. A discrete Fast Fourier Transform method is employed. The spatial resolution is selected according to the number of parallel grid points N_z, typically $2^{22} = 4194304$. The k-values represented are $k_z(j) = j/L_z$, ($j = 0, 1 \ldots N_z/2 - 1$). Each k_z is associated with a complex amplitude determined by

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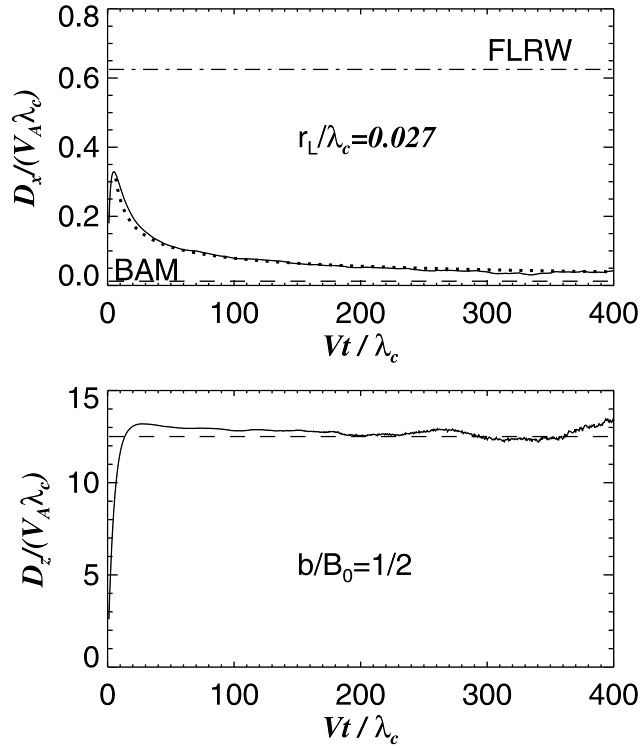

Figure 1. Results of a test particle simulation with $r_L/\lambda_c = 0.027$ and $\delta b/B_0 = 0.5$. (Bottom panel) Solid line is running parallel diffusion coefficient vs. time. For reference, dashed line is approximate quasilinear theory result. (Top panel) Solid line is running perpendicular diffusion coefficient. Dotted line corresponds to subdiffusion. FLRW and BAM theoretical results (see text) are also shown.

selecting real and imaginary parts using a Gaussian random number generator and a spectral shape function. The latter is

$$S^{slab}(k_z) = \frac{C^{slab}}{\left(1 + (k_z\lambda)^2\right)^{5/6}}, \quad (3)$$

where $\lambda \approx \lambda_c$. For the simulations shown, $k_{max}\lambda = 1318$ ensures a realistic inertial (powerlaw) range spanning more than 3 decades. To avoid periodicity effects, the periodic box is very long, $L_z/\lambda = 10000$, and each particle trajectory is limited to a small fraction of the full length, i.e., the time of integration t is such that $Vt/L_z < 0.04$ for the simulations shown.

[5] In addition to $\mathbf{b}^{slab}$ we include a small amplitude two dimensional (2D) component of the fluctuations, making up 0.01% of the total energy and defined in a way analogous to the slab part, except that the populated wave vectors have $k_z = 0$ and are axisymmetrically distributed in the (x, y) plane. The 2D part $\mathbf{b}^{2D}$ is associated with the Fourier amplitudes $\hat{\mathbf{b}}^{2D}(k_x, k_y)$ which are represented in a square cross section box with dimension $L_x = L_y$ and resolved modes denoted by $(k_x, k_y) = (i, j)/L_x$, (i, j = 0, 1, 2 ... $N_x/2 + 1$). In the present runs the perpendicular spatial resolution is $N_x \times N_y = 4096 \times 4096$. The presence of the 2D component may avoid problems that might be incurred due to ignorable coordinates [*Jokipii et al.*, 1993]. On the other hand there is substantial evidence [*Bieber et al.*, 1996] that the solar wind on average is more distant from a pure slab symmetry than the case we examine. However, there are circumstances that favor production of one dimensional waves, and a 99.99% slab model may be resonable in some cases, e.g., near regions of strong wave particle interactions.

[6] Fields are calculated by transforming (once) to the equally spaced real space points and then employing linear interpolation to the particle positions.

[7] Our results are based upon a series of runs with the above fluctuation properties and grid characteristics, carried out for particle number $N_p = 1000$. Initial conditions have particle positions and gyrophases selected randomly. The Newton Lorentz equations are integrated using a fourth order Runge-Kutta scheme with adaptive time stepping regulated by a fifth order error estimate step. Typically we set the error parameter to 10^{-9}.

3. Numerical Simulation Results

[8] It is convenient to investigate the approach to diffusive behavior by computing the running diffusion coefficients, $\tilde{\kappa}_{xx} = d\left\langle(\Delta x)^2\right\rangle \big/ 2dt$ and $\tilde{\kappa}_{zz} = d\left\langle(\Delta z)^2\right\rangle \big/ 2dt$, where the time derivative is computed using a first order finite difference, and the brackets $\langle\ldots\rangle$ indicate ensemble average, or average over many test particles. When the mean square displacements are diffusive ($\propto t$), the running diffusion coefficient is identical to the usual one.

[9] Figure 1 shows the parallel and perpendicular running diffusion coefficient for the "near slab" fluctuations described above, for a numerical experiment consisting of particles having $r_L/\lambda_c = 0.027$, where r_L is the particle Larmor radius. Referring to the top panel, the solid line shows the computed $\tilde{\kappa}_{xx}$. For reference we show two theoretical predictions for perpendicular diffusion: The horizontal dot-dashed line is the so called field line random walk (FLRW) limit of perpendicular diffusion, in which the perpendicular spread of particles occurs at a rate proportional

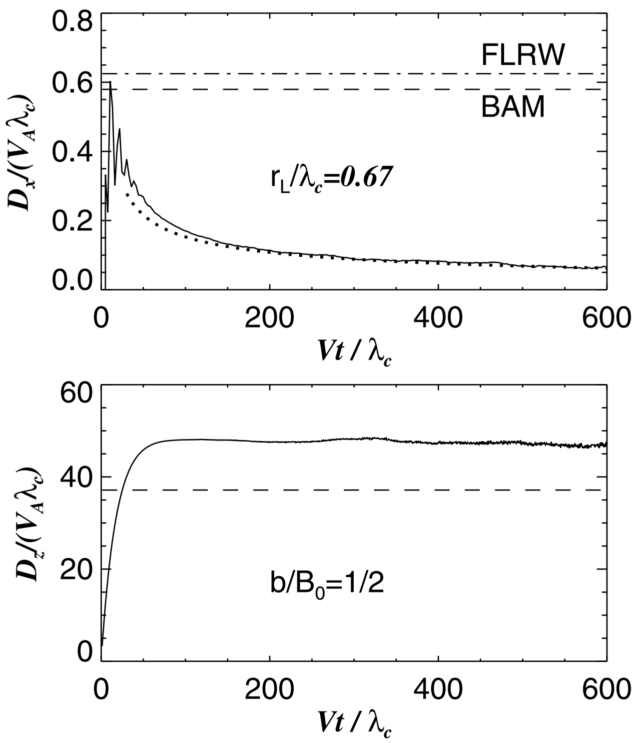

Figure 2. Results of another test particle simulation with $r_L/\lambda_c = 0.67$, and $\delta b/B_0 = 0.5$ in same format as Figure 1. (Bottom panel) Running parallel diffusion coefficient vs. time. (Top panel) Running perpendicular diffusion coefficient.

to the spatial rate of spreading of the field lines themselves. In this limit, $\kappa_{xx} = (v/2)D_\perp$ where v is particle speed and $D_\perp = \langle(\Delta x)^2/2\Delta z$ is the Fokker Planck coefficient for diffusive spread of field lines. On fairly general grounds, the FLRW limit might be expected [*Jokipii*, 1966; *Jokipii and Parker*, 1968] for slab turbulence and when the quasilinear approximation is valid. The other horizontal reference line in the top panel of Figure 1 is a theory [*Bieber and Matthaeus*, 1997; "BAM"] based upon the Green-Kubo-Taylor [e.g., *Kubo*, 1957] approach, in which one models the particle velocity autocorrelation function in order to compute κ_{xx}. Recent numerical evidence [*Giacalone and Jokipii*, 1999; *Mace et al.*, 2000] does not affirm the accuracy of either FLRW or BAM theories at lower energies ($r_L/\lambda_c \ll 1$).

[10] Besides the reference lines, Figure 1 (top) also shows the computed $\tilde{\kappa}_{xx}$ as a solid line. We can see that at very early times the running diffusion coefficient increases rapidly with time ($\langle(\Delta x)^2\rangle \propto t^2$ is expected in this "free streaming" regime.) At around Vt/λ = 10 or so, the running perpendicular diffusion coefficient flattens out, but instead of attaining the expected (horizontal) diffusive limit, it reaches a maximum and then begins decreasing. The subsequent behavior, in which the running diffusion coefficient behaves very nearly as $\tilde{\kappa}_{xx} \propto 1/t^{1/2}$ is the focus of the present paper and is further discussed in the next section.

[11] The bottom panel of Figure 1 shows the parallel running diffusion coefficient $\tilde{\kappa}_{zz}$ as a function of time, scaled to the same units as in the top panel. It is noteworthy that $\tilde{\kappa}_{zz}$ settles into a constant value, corresponding to parallel diffusion, after about vt/λ_c > 20 - about the same time frame in which $\tilde{\kappa}_{xx}$ reaches its maximum. This also corresponds to a time at which most particles have traveled a parallel mean free path. For reference a theoretical value for κ_{zz} is shown in the bottom panel as a solid horizontal line. This corresponds to a quasilinear parallel scattering theory for powerlaw inertial range extending to infinite wavenumber [*Jokipii*, 1966; *Bieber et al.*, 1995].

[12] A very similar picture is seen for other physical parameters provided that we remain within the confines of the 0.9999 slab case. A second example in the same format as Figure 1 is shown in Figure 2, for test particles at higher energy, r_L/λ_c = 0.67. Again, the rate of increase of perpendicular displacements increases in time towards the QLT value at times prior to establishment of parallel diffusive behavior. Once parallel diffusion sets in, the running perpendicular diffusion coefficient begins to decrease in time. In this regime, the perpendicular mean square displacements are increasing as $\sim t^{1/2}$, which is slower than the $\sim t$ scaling characteristic of true diffusion. [Simple least square fits to the logarithm of the running perpendicular diffusion data gives $t^{-0.59 \pm 0.04}$ in the range vt/λ_c: [20,400] for the data in Figure 1, and $t^{-0.56 \pm 0.09}$ in the range vt/λ_c: [50,600] for the data in Figure 2]. We have seen similar behavior repeatedly in simulations in which the fluctuations are of the slab (not shown) or nearly slab types. Such fluctuations give rise to subdiffusive, and not diffusive time evolution of displacements perpendicular to the large scale mean magnetic field. We turn now to a theoretical discussion of this numerical result.

4. Compound Subdiffusion

[13] The suggestion that parallel transport can suppress perpendicular diffusion was made by *Urch* [1977], [see also, *Lingenfelter et al.*, 1971]. The standard familiar description of perpendicular diffusion is that the gyrocenters of charged particles follow magnetic field lines, and thus their diffusive spread perpendicular to the mean magnetic field is governed by the diffusive spread of field lines due to field line random walk. Thus, if the spread of field lines is at a (Fokker-Plank) rate of $D_\perp = \langle(\Delta x)^2\rangle/2\Delta z$ and the particle speed is v, then the diffusion of particles occurs at a diffusive rate

$$\kappa_{xx} = \frac{\left\langle (\Delta x)^2 \right\rangle}{2\Delta t} \tag{4}$$

$$= \frac{\Delta z}{\Delta t} \times \frac{\left\langle (\Delta x)^2 \right\rangle}{2\Delta z} \tag{5}$$

$$\sim v \times D_\perp \tag{6}$$

where $\Delta z/\Delta t$ denotes the mean rate of progression of particles down the field lines without regard to direction. Urch's point is that, while particles (or, their gyrocenters) attempt to follow field lines, their motions are not free streaming, but rather represent a parallel random walk. Particles that back scatter through 90° return along the same field line (or, a very similar field line - this is a key issue), and therefore experience a tendency towards a decreased perpendicular displacement. This possibility has become known as "compound diffusion." It is however readily seen that it is not a type of diffusion at all, and we suggest it would likely be better known as subdiffusion.

[14] We note that the particle subdiffusion reported here is unrelated to the field line subdiffusion discussed by *Ragot* [1999]. That field line effect depends on specific forms of the low wavenumber portion of the spectrum, which is different from the form we assume. Our turbulence model has a normal, finite correlation length, and the field lines do indeed spread diffusively. The particle subdiffusion reported here results not only from field line properties, but also from the interplay between parallel and perpendicular particle transport.

[15] The issue of compound diffusion has been debated in the literature [*Forman and Jokipii*, 1978] and is not without controversy. Recently, *Kóta and Jokipii* [2000] revisited the issue of compound subdiffusion in the context of the Green-Kubo-Taylor formulation ["TGK", e.g., *Kubo*, 1957] which expresses diffusion coefficients in terms of integrals over the two-time particle velocity autocorrelation [see also, *Forman*, 1977 and *Bieber and Matthaeus*, 1997]. Kóta and Jokipii found that the TGK approach can be reconciled with compound (sub)diffusion provided that one retains a term ordinarily discarded as small. In particular, they concur with Urch's conclusions regarding the temporal behavior of the mean square perpendicular displacement in the presence of strong parallel diffusion.

[16] The algebraic structure of compound subdiffusion is simple. In the above standard reasoning one employed the free streaming estimate $\Delta z/\Delta t \sim v$. This is incorrect when there is parallel diffusion. Instead, $\Delta z \approx (2\kappa_{zz}\Delta t)^{1/2}$ and thus $\Delta z/\Delta t \sim (2\kappa_{zz}/t)^{1/2}$. For large Δt the estimate of $\kappa_{xx} \to 0$ and there is no diffusion. More carefully, one finds [*Kóta and Jokipii*, 2000] that $\left\langle (\Delta x)^2 \right\rangle = 4D_\perp\sqrt{\kappa_{zz}t/\pi}$. Consequently in the presence of strong parallel diffusion the running perpendicular diffusion coefficient becomes

$$\tilde{\kappa}_{xx} = D_\perp\sqrt{\frac{\kappa_{zz}}{\pi t}}. \tag{7}$$

In both Figures 1 and 2 this functional form of running perpendicular diffusion is plotted (dashed lines) for appropriate values of the parameters alongside the empirically determined $\tilde{\kappa}_{xx}$. The theoretical value of the field line diffusion coefficient is used [*Jokipii*, 1966; *Matthaeus et al.*, 1995]. However, the actual fitted values of the particle parallel diffusion coefficient κ_{zz} are employed. The theoretical and empirically determined running perpendicular diffusion coefficients agree quite well.

5. Conclusions

[17] Our results are twofold. First, classical diffusive transport across the mean magnetic field is not maintained in nearly slab magnetostatic fluctuations. Parallel diffusion is the culprit. Once it sets in, i.e., after particles travel about a parallel scattering mean free path, perpendicular random walk is suppressed to a subdiffusive level. Our second result is that the subdiffusive perpendicular transport implied by the numerically computed test particle trajectories is very well described by the quantitative treatments of compound (sub)diffusion given by *Urch* [1977] and *Kóta and Jokipii* [2000].

[18] The above conclusions are closely tied to the structure of the magnetic fluctuations, and in particular the fact that they are nearly slab-like, varying only in a single cartesian spatial coordinate. In realistic system a 0.9999 slab energy may be unreasonable, and a magnetic field such as the solar wind may be much more 3-dimensional [see e.g., *Bieber et al.*, 1996].

[19] The rationale for subdiffusive increase of the displacements rests crucially on the similarity of nearby field lines. Pitch angle scattering eventually causes scattering through 90°. When particles reverse the sign of their velocity along the mean magnetic field, they begin to retrace their path. The transverse displacement is thus reduced provided that the field line going back is the same or nearly the same as the field line going forward. On the other hand if the magnetic field depends strongly upon the transverse coordinate, the particle will in general not closely retrace its earlier motion, and we would not expect a reduction to subdiffusive transport.

[20] Although we have investigated only a limited class of magnetic field models, it seems unavoidable that similar reasoning applies more broadly. It is clear that it is the degree of dissimilarity of field lines lying within a gyro-orbit that determines whether parallel scattering can suppress perpendicular displacements. Clearly if nearby lines belong to an equivalence class [*Jones et al.*, 1998] then the conditions should be met for subdiffusion. For sets of field lines having weak transverse variation (“near-equivalence classes”) such as the type employed above, we have seen that subdiffusion emerges. What is not clear at present is what quantitative condition determines whether the dissimilarity of nearby field lines is sufficient to restore diffusive perpendicular transport. It would appear that this question is analogous to one that arises in quasilinear calculation of field line random walk. [*Jokipii*, 1966; *Jokipii and Parker*, 1968; *Matthaeus et al.*, 1996; *Isichenko*, 1991]. In particular, the condition for validity of neglect of transverse displacements in the field line random walk problem is that the Kubo number $K = (\delta b/B_0)(\lambda_{c\|}/\lambda_{c\perp})$ is small. In view of the above discussion the hypothesis arises that the condition for perpendicular subdiffusion is also that perpendicular displacements are negligible, and therefore that the Kubo number is small. If this is so, then the curious condition would emerge that the condition for validity of quasilinear field line random walk and the condition for quasilinear perpendicular particle diffusion are mutually exclusive, or at least complementary. It would also seem reasonable that a small ratio of Larmor radius to correlation scale $r_L/\lambda_{c\perp}$ would favor perpendicular subdiffusion. So far our numerical results have only confirmed that particle motion in slab and near slab fluctuations exhibits compound subdiffusion. Answers to the broader questions that have arisen will await more comprehensive numerical investigations, which are currently in progress.

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