International Journal of Algebra, Vol. 7, 2013, no. 14, 673 - 677
HIKARI Ltd, www.m-hikari.com
http://dx.doi.org/10.12988/ija.2013.3434

Subdirect Products and Covering Groups by Subgroups

Mohammad Javad Ataei [1]

Department of Mathematics
Payame Noor University, P.O. Box 19395-3697 Tehran, Iran
ataeymj@pnu.ac.ir

Abstract

In this paper we classification subdirect products of some groups that are $\mathfrak{C}_8$-groups.

Mathematics Subject Classification: 20F99

Keywords: covering groups by subgroups, Subdirect product, maximal iredundant cover, core-free intersection.

1 Introduction and results

Let G be a group. A set $\mathcal{C}$ of proper subgroups of G is called a cover for G if its set-theoretic union is equal to G. If the size of $\mathcal{C}$ is n, we call $\mathcal{C}$ an n-cover for the group G. A cover $\mathcal{C}$ for a group G is called irredundant if no proper subset of $\mathcal{C}$ is a cover for G. A cover $\mathcal{C}$ for a group G is called core-free if the intersection $D = \bigcap_{M\in\mathcal{C}} M$ of $\mathcal{C}$ is core-free in G, i.e. $D_G = \bigcap_{g\in G} g^{-1}Dg$ is the trivial subgroup of G. A cover $\mathcal{C}$ for a group G is called maximal if all the members of $\mathcal{C}$ are maximal subgroups of G. A cover $\mathcal{C}$ for a group G is called a $\mathfrak{C}_n$-cover whenever $\mathcal{C}$ is an irredundant maximal core-free n-cover for G and in this case we say that G is a $\mathfrak{C}_n$-group. A finite group is called semisimple

[1]Department of Mathematics, Payame Noor University (PNU), Isfahan Center, Kohandej Street, Isfahan, Iran

if it has no non-trivial normal abelian subgroups (see p. 86 of [13] for further information on such groups).

Also we use the usual notations ([13]); for example, C_n denotes the cyclic group of order n, $(C_n)^j$ is the direct product of j copies of C_n, the core of a subgroup H of G is denoted by H_G.

In [14], Scorza determined the structure of all groups having an irredundant 3-cover with core-free intersection.

Theorem 1.1 (Scorza [14]) *Let $\{A_i : 1 \leq i \leq 3\}$ be an irredundant cover with core-free intersection D for a group G. Then $D = 1$ and $G \cong C_2 \times C_2$.*

In [11], Greco characterized all groups having an irredundant 4-cover with core-free intersection. Bryce et al.[9], characterized groups with maximal irredundant 5-cover with core-free intersection.

We characterized groups with maximal irredundant 6-cover with core-free intersection in [1]. Abdollahi et al.[3], characterized groups with maximal irredundant 7-cover with core-free intersection.

Also we characterized p-groups with maximal irredundant 8-cover with core-free intersection in [2].

Theorem 1.2 (*See [2]*). *Let G be a $\mathfrak{C}_8$-group. Then G is a p-group for a prime number p if and only if $G \cong (C_3)^4$ or $(C_7)^2$.*

Also we investigated semisimplity condition and covering groups by subgroups in [4]and Covering Semisimple Groups by Subgroups in [6].

Further problems of a similar nature, with slightly different aspects, have been studied by many people (see [5, 7, 8, 15, 16]).

In this paper we investigate all subdirect products of some groups that are not $\mathfrak{C}_8$-groups.

2 Covering groups and subdirect products

To obtain the following results using GAP; we had use several computers mostly for a long time.

Theorem 2.1 *1. Let D denote the intersection of an arbitrary maximal irredundant 8-cover with core-free intersection. Then*

(a) Among all subdirect products of one alternating group Alt_4 and two alternating groups Alt_3, there is only one $\mathfrak{C}_8$-group isomorphic to $C_3 \times C_3 \times A_4$ for which $|D| = 1$.

(b) Among all subdirect products of one symmetric groups Sym_4 and two symmetric group Sym_3, there is only one $\mathfrak{C}_8$-group isomorphic to $(C_3 \times A_4) \rtimes C_2$ for which $|D| = 1$.

(c) Among all subdirect products of two $C_7 \rtimes C_3$ *, there is only one* $\mathfrak{C}_8$*-group isomorphic to* $C_7 \rtimes C_3$ *for which* $|D| = 1$.

(d) All groups of order 14 that isomorphic to $C_7 \rtimes C_2$*, are* $\mathfrak{C}_8$*-groups for which* $|D| = 1$.

(e) All groups of order 21 that isomorphic to $C_7 \rtimes C_3$ *or* $AGL(1, 7)$*, are* $\mathfrak{C}_8$*-groups for which* $|D| = 1$.

(f) Among all subdirect products of two C_2*s and two* Sym_4*s, there is only one* $\mathfrak{C}_8$*-group isomorphic to* $((C_2 \times C_2 \times C_2 \times C_2) \rtimes C_3) \rtimes C_2$ *for which* $D = 1$.

(g) Among all subdirect products of two C_2*s and two* Sym_3*s, there is only one* $\mathfrak{C}_8$*-group isomorphic to* $C_2 \times C_2 \times ((C_3 \times C_3) \rtimes C_2)$ *for which* $D = 1$.

(q) Among all subdirect products of two C_2*s and two primitive groups of degree at most 4, there are two* $\mathfrak{C}_8$*-groups for which* $D = 1$*, one is isomorphic to* $C_2 \times C_2 \times ((C_3 \times C_3) \rtimes C_2)$ *and the other is isomorphic to* $((C_2 \times C_2 \times C_2 \times C_2) \rtimes C_3) \rtimes C_2$ *for which* $D = 1$.

(r) Among all subdirect products of two C_2*s and one* D_{10}*, there is only one* $\mathfrak{C}_8$*-group isomorphic to* $C_2 \times C_2 \times D_{10}$ *for which* $D = 1$.

(s) Among all subdirect products of two C_2*s and one* $AGL(1, 5)$*, there is only one* $\mathfrak{C}_8$*-group isomorphic to* $C_2 \times C_2 \times AGL(1, 5)$ *for which* $D = 1$.

(t) Among all subdirect products of two C_2*s and one primitive group of degree at most 5, there are two* $\mathfrak{C}_8$*-groups isomorphic to* $C_2 \times C_2 \times D_{10}$ *for which* $D = 1$ *or* $C_2 \times C_2 \times AGL(1, 5)$ *for which* $D = 1$.

(u) Among all subdirect products of two Alt_4*, there is only one* $\mathfrak{C}_8$*-group isomorphic to* $(C_2)^4 \rtimes C_3$ *for which* $D = 1$.

(v) Among all subdirect products of two Sym_4*, there is only one* $\mathfrak{C}_8$*-group isomorphic to* $((C_2)^4 \rtimes C_3) \rtimes C_2$ *for which* $D = 1$.

Proof 2.2 *We have used the following function written in* GAP *[10] to show this theorem. The input of the function is a group* G *and the output are all irredundant 8-covers with core-free intersection of* G*, and if there is no such cover for* G*, the output is the empty list.*

```
 f:=function(G) local
S,M,n,C,i,T,Q,R; n:=Size(G); M:=MaximalSubgroups(G);
C:=Combinations(M,8); S:=[]; for i in [1..Size(C)] do if
Size(Union(C[i]))=n then Add(S,C[i]); fi; od; T:=[]; for i in
[1..Size(S)] do if Size(Core(G,Intersection(S[i])))=1 then
Add(T,S[i]); fi;    od; R:=[]; for i in [1..Size(T)] do
Q:=Combinations(T[i],7); if (n in List(Q,i->Size(Union(i))))=false
then Add(R,T[i]); fi; od; return R; end;
```

Acknwoledgements. This research was supported by a grant from Payame Noor University.

References

[1] A. Abdollahi, M.J. Ataei, M. Jafarian, A. Mohammadi Hassanabadi, On groups with maximal irredundant 6-cover, *Comm. Algebra,* **33** (2005), 3225-3238.

[2] A. Abdollahi, M.J. Ataei, A. Mohammadi Hassanabadi, Minimal blocking set in $PG(n, 2)$ and covering groups by subgroups. *Comm. Algebra,* **36** (2008), 365-380.

[3] A. Abdollahi, M. Jafarian, A. Mohammadi Hassanabadi, On groups with an irredundant 7-cover, *Journal of pure and applied algebra,* **209** (2007) 291300.

[4] M.J. Ataei, Semisimplity Condition and Covering Groups by Subgroups. *International Journal of Algebra,* Vol. **4**, (2010), no. 22, 1063 - 1068

[5] M.J. Ataei, $\mathfrak{C}_8$-groups and Nilpotency Condition. *International Journal of Algebra,* Vol. **4**, (2010), no. 22, 1057 - 1062

[6] M.J. Ataei, Covering Semisimple Groups by Subgroups. *International Journal of Algebra,* Vol. **5**, (2011), no. 14, 661 - 665

[7] M.J. Ataei, Characterization of 5-groups with a maximal Irredundant 10-cover. *International Mathematical Forum,* Vol. **6**, (2011), no. 35, 1733 - 1738

[8] M.J. Ataei, Minimal Normal Subgroups and Semisimplity Condition. *International Journal of Algebra,* Vol. **6**, (2012), no. 4, 179 - 183

[9] R.A. Bryce, V. Fedri and L. Serena, Covering groups with subgroups, *Bull. Austral. Math. Soc.* **55** (1997), 469-476.

[10] The GAP Group, *GAP – Groups, Algorithms, and Programming, Version 4.3*; 2002, (`http://www.gap-system.org`).

[11] D. Greco, Sui gruppi che sono somma di quattro o cinque sottogruppi, *Rend. Accad. Sci. Fis. Math. Napoli (4)* **23** (1956), 49-59.

[12] B.H. Neumann, Groups covered by finitely many cosets, *Publ. Math. Debrecen* **3** (1954), 227-242.

[13] D.J.S. Robinson, *A course in the theory of groups,* (Springer-Verlag 1982)

[14] G. Scorza, I gruppi che possono pensarsi come somma di tre loro sottogruppi, *Boll. Un. Mat. Ital.* **5** (1926) 216-218.

[15] M.J. Tomkinson, Groups covered by finitely many cosets or subgroups, *Comm. Algebra* **15** (1987) 845-859.

[16] M.J. Tomkinson, Groups as the union of proper subgroups, *Math. Scand.* **81** (1997), 189-198.

Received: April 15, 2013