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SUBDIRECTLY IRREDUCIBLE AND CONGRUENCE DISTRIBUTIVE Q-LATTICES

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By a *q-lattice* (see [3]) we mean an algebra $A = (A; \vee, \wedge)$ with two binary operations satisfying the following identities:

(associativity):	$a \vee (b \vee c) = (a \vee b) \vee c,$	$a \wedge (b \wedge c) = (a \wedge b) \wedge c,$
(commutativity):	$a \vee b = b \vee a,$	$a \wedge b = b \wedge a,$
(weak absorption):	$a \vee (a \wedge b) = a \vee a,$	$a \wedge (a \vee b) = a \wedge a,$
(weak idempotence):	$a \vee (b \vee b) = a \vee b,$	$a \wedge (b \wedge b) = a \wedge b,$
(equalization):	$a \vee a = a \wedge a.$	

A q-lattice A is called *distributive* if it satisfies the distributive identity

$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$$

for each a, b, c from A. A q-lattice A is *bounded* if there exist elements 0 and 1 of A such that

$$a \wedge 0 = 0 \qquad \text{and} \qquad a \vee 1 = 1$$

for each $a \in A$.

An element a of a q-lattice A is called an *idempotent* if $a \vee a = a$ (and, by equalization, also $a \wedge a = a$). The set of all idempotents of A is called the *skeleton* of A. It is clear that the skeleton of A is a sub-q-lattice of A which is the maximal sublattice contained in A.

A non-singleton subset C of a q-lattice A is called a *cell* of A if $a, b \in C$ implies $a \vee a = b \vee b$ and C is a maximal subset of A with respect to this property.

Evidently, a q-lattice A is a lattice if and only if it has no cell, i.e. if A is equal to its skeleton. Every cell C of A has just one idempotent.

Evidently, every cell D of a q-lattice A is a sub-q-lattice of A. If A is a cell, then the skeleton of A is a singleton.

Distributive and/or bounded q-lattices were investigated in [4]. Let us notice that the distributive identity is equivalent to its dual; on the other hand, the foregoing identities for 0 and 1 do not imply $a \vee 0 = a$ and $1 \wedge a = a$ but only the weaker laws $a \vee 0 = a \vee a$ and $a \wedge 1 = a \wedge a$.

By the foregoing definitions, the class of all distributive q-lattices as well as the class of all bounded distributive lattices form varieties. Therefore, it makes sense to look for SI-members of these varieties. Although q-lattices look rather similar to lattices, these varieties have another number of SI-members than the variety of (bounded) distributive lattices.

Theorem 1. *The variety D of all distributive q-lattices has exactly two non-trivial SI-members, namely those visualized in Fig. 1 as B and C.*

Fig. 1

Before proceeding to proof, let us remark that every q-lattice $A = (A; \vee, \wedge)$ can be viewed as a quasiordered set $(A; Q)$, where the quasiorder Q on A is induced by $\vee$ (or $\wedge$) as follows (see e.g. [3], [4]):

$$\langle a, b \rangle \in Q \quad \text{iff } a \vee b = b \vee b$$

(or, equivalently, $\langle a, b \rangle \in Q$ iff $a \wedge b = a \wedge a$). Henceforth, we can visualize this quasiorder Q in the diagrams of q-lattices by oriented arrows; i. e. $\langle a, b \rangle \in Q$ iff there exists an oriented path from a to b consisting of arrows.

Proof of Theorem 1. Since both B and C are two-element q-lattices, they are subdirectly irreducible. Hence it remains to prove that any other non-trivial distributive q-lattice A different from B, C is subdirectly reducible.

(i) If A has no cell, then A is a lattice. In the case of $A \neq B$, A is subdirectly reducible by [2].

(ii) Let D be a cell of A.

(a) Let A contain an element $a \notin D$. Denote by d the idempotent of D and

$$A_1 = A - (D - \{d\}).$$

Then A_1 and D are sub-q-lattices of A and $\operatorname{card} A_1 > 1$, $\operatorname{card} D > 1$. Introduce a mapping $\alpha\colon A \to A_1 \times D$ by the rule

$$\alpha(x) = \langle x, d\rangle \quad \text{for } x \in A - (D - \{d\}),$$
$$\alpha(x) = \langle d, x\rangle \quad \text{for } x \in D.$$

It is clear that α is an injection and $\mathrm{pr}_1\alpha(A) = A_1$, $\mathrm{pr}_2\alpha(A) = D$. Prove that α is a homomorphism:

if $x \in A_1$, $y \in D$, then

$$\alpha(x \vee y) = \alpha(x \vee d) = \langle x \vee d, d\rangle\,,$$
$$\alpha(x) \vee \alpha(y) = \langle x, d\rangle \vee \langle d, y\rangle = \langle x \vee d, d\rangle\,;$$

if $x, y \in A_1$, then

$$\alpha(x \vee y) = \langle x \vee y, d\rangle = \langle x, d\rangle \vee \langle y, d\rangle = \alpha(x) \vee \alpha(y);$$

if $x, y \in D$, then

$$\alpha(x \vee y) = \alpha(d) = \langle d, d\rangle = \langle d, x\rangle \vee \langle d, y\rangle = \alpha(x) \vee \alpha(y).$$

Dually this can be shown for the meet. Hence A is subdirectly reducible.

(b) Suppose $A = D$. If $A \neq C$, there exist elements a, b of D such that $a \neq b \neq d \neq a$. Put $A_1 = A - \{b\}$ and $A_2 = \{d, b\}$. Thus $\operatorname{card} A_1 > 1$, $\operatorname{card} A_2 > 1$. Introduce a mapping $\alpha\colon A \to A_1 \times A_2$ as follows:

$$\alpha(x) = \langle x, d\rangle \ \text{ for } x \in A_1,$$
$$\alpha(x) = \langle d, x\rangle \ \text{ for } x \in A_2.$$

Evidently, α is an injection and $\mathrm{pr}_1\alpha(A) = A_1$, $\mathrm{pr}_2\alpha(A) = A_2$. Prove that α is a homomorphism:

if $x \in A_1$, $y \in A_2$, then

$$\alpha(x \vee y) = \alpha(d) = \langle d, d\rangle\,,$$
$$\alpha(x) \vee \alpha(y) = \langle x, d\rangle \vee \langle d, y\rangle = \langle d, d\rangle\,;$$

if $x, y \in A_1$, then

$$\alpha(x \vee y) = \alpha(d) = \langle d, d\rangle = \langle x, d\rangle \vee \langle y, d\rangle = \alpha(x) \vee \alpha(y);$$

if $x, y \in A_2$ then

$$\alpha(x \vee y) = \alpha(d) = \langle d, d\rangle = \langle d, x\rangle \vee \langle d, y\rangle = \alpha(x) \vee \alpha(y).$$

Dually this can be done for $\wedge$, i.e. A is a subdirect product of A_1, A_2. □

Theorem 2. *The class of all bounded distributive q-lattices with* $0 \neq 1$ *has exactly three nontrivial SI-members, namely* B *(in Fig. 1),* C_1*,* C_2 *(in Fig. 2).*

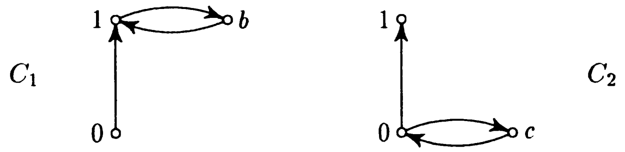

Fig. 2

Proof. As was already mentioned, B is subdirectly irreducible. Since the lattices of congruences not collapsing 0 and 1 of C_1, C_2 are three-element chains, see Fig. 3, also C_1, C_2 are subdirectly irreducible in this class.

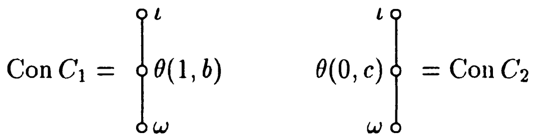

Fig. 3

We have to prove that if A is a bounded distributive q-lattice different from B, C_1, C_2 then A is subdirectly reducible in this class.

(A) If A has no cell than this was done by G. Birkhoff in [2].

(B) If A has at least two cells, say D_1, D_2, then clearly $D_1 \cap D_2 = \emptyset$. Put

$$\Theta_1 = D_1 \times D_1 \cup \omega, \qquad \Theta_2 = D_2 \times D_2 \cup \omega$$

where ω denotes the identity relation on A. It can be easily shown that Θ_1, Θ_2 are congruences on A and $\Theta_1 \cap \Theta_2 = \omega$; thus, by the Birkhoff Theorem [2], A is subdirectly reducible.

(C) It remains to deal with the case when A has just one cell D.

(i) Suppose that the skeleton of A contains just two elements, namely 0 and 1. Let $0 \in D$. Since A is not isomorphic with C_2, it means that D contains at least two non-idempotent elements a, b, i.e. $a \neq 0 \neq b \neq a$. We can put

$$A_1 = \{0, 1, a\}, \qquad A_2 = A - \{a\}.$$

It is easy to see that both A_1, A_2 are bounded distributive q-lattices (moreover, $A_1 \simeq C_2$). Define $\alpha \colon A \to A_1 \times A_2$ as follows:

$$\alpha(0) = \langle 0, 0 \rangle, \quad \alpha(1) = \langle 1, 1 \rangle,$$
$$\alpha(a) = \langle a, 0 \rangle;$$
$$\alpha(x) = \langle 0, x \rangle \text{ for } x \in D, \ x \neq a.$$

We can see that α is an injection and $\mathrm{pr}_1\alpha(A) = A_1$, $\mathrm{pr}_2\alpha(A) = A_2$. It remains to prove that α is a homomorphism. It is almost evident in the case $z, y \in A_1$ that $\alpha(z \vee y) = \alpha(z) \vee \alpha(y)$ and $\alpha(z \wedge y) = \alpha(z) \wedge \alpha(y)$, and analogously for $z, y \in A_2$. Suppose $z \in A_1 - A_2$, $y \in A_2 - A_1$. Then $z = a$ and $y \in D$, $y \neq a$, $y \neq 0$. We have

$$\begin{aligned}
&\alpha(z) \vee \alpha(y) = \alpha(a) \vee \alpha(y) = \langle a, 0\rangle \vee \langle 0, x\rangle = \langle 0, 0\rangle\,,\\
&\alpha(z \vee y) = \alpha(0) = \langle 0, 0\rangle \text{ and}\\
&\alpha(z) \wedge \alpha(y) = \langle a, 0\rangle \wedge \langle 0, x\rangle = \langle 0, 0\rangle = \alpha(0) = \alpha(z \wedge y).
\end{aligned}$$

Consequently, A is isomorphic to a subdirect product of A_1, A_2.

(ii) If the skeleton of A contains just two elements (0 and 1) and $1 \in D$, where D is the unique cell of A, the proof is dual to that of (i).

(iii) Let the skeleton of A have more than two elements. We have three cases:

(a) Suppose there exists an idempotent $d \in A$ with $0 \neq d \neq 1$ and $d \in D$. Put

$$A_1 = \{x; \langle x, d\rangle \in Q\} \text{ and } A_2 = \{x; \langle d, x\rangle \in Q\}$$

for the induced quasiorder Q. Define $\alpha\colon A \to A_1 \times A_2$ as follows:

$$\begin{aligned}
&\alpha(x) = \langle x \wedge d, x \vee d\rangle \text{ for } x \notin D \text{ and}\\
&\alpha(x) = \langle x, x\rangle \text{ for } x \in D.
\end{aligned}$$

Since $x \notin D$ is an idempotent of A (because A has just one cell D), it is easy to verify that α is an injective homomorphism satisfying $\mathrm{pr}_1\alpha(A) = A_1$, $\mathrm{pr}_2\alpha(A) = A_2$, thus A is isomorphic to a subdirect product of A_1, A_2.

(b) Suppose there exists an idempotent $d \in A$ with $0 \neq d \neq 1$ and $0 \in D$. Put

$$A_1 = \{x; \langle x, d\rangle \in Q\}, \quad A_2 = \{x; \langle d, x\rangle \in Q\}$$

and introduce a mapping $\alpha\colon A \to A_1 \times A_2$ by

$$\begin{aligned}
&\alpha(x) = \langle x \wedge d, x \vee d\rangle \text{ for } x \notin D,\\
&\alpha(x) = \langle x, d\rangle \text{ for } x \in D.
\end{aligned}$$

We can easily verify that α is an injective homomorphism with $\mathrm{pr}_i\alpha(A) = A_i$ ($i = 1, 2$), thus A is a subdirect product of A_1, A_2.

(c) The last case with $d \in A$, $0 \neq d \neq 1$, $1 \in D$ is dual to (b), only α is defined for $x \in D$ by $\alpha(x) = \langle d, x\rangle$. □

Corollary. *Every non-trivial distributive q-lattice A is a subdirect product of q-lattices B and C. Every bounded distributive q-lattice A with $0 \neq 1$ is a subdirect product of q-lattices B, C_1, C_2.*

It is well-known than for any lattice L, the congruence lattice $\operatorname{Con} L$ is distributive, see e.g. [1]. We can ask if a similar result is also valid for q-lattices. It is easy to show that the answer is negative in the general case. More precisely, we can state

Lemma. *Let C be a q-lattice which is a cell. Then $\operatorname{Con} C \simeq \Pi_n$, where $n = \operatorname{card} C$ and Π_n is the partition lattice of the set of cardinality n.*

The proof is trivial since every equivalence on C is a congruence.

Theorem 3. *Let A be a q-lattice which has just one n-element cell C, let S be the skeleton of A. Then $\operatorname{Con} A \simeq \Pi_n \times \operatorname{Con} S$.*

Proof. (a) If $\Theta_1 \in \operatorname{Con} S$ and $\Theta_2 \in \operatorname{Con} C \simeq \Pi_n$ and d is the only idempotent of C (i.e. $\{d\} = S \cap C$), then clearly

$$\Theta_1 \cup \Theta_2 \cup \{[d]_{\Theta_1} \cup [d]_{\Theta_2}\}^2 \in \operatorname{Con} A.$$

(b) If $\Theta \in \operatorname{Con} A$, put $\Theta_1 = \Theta \cap S^2$, $\Theta_2 = \Theta \cap C^2$.

Evidently, $\Theta = \Theta_1 \cup \Theta_2 \cup \{[d]_{\Theta_1} \cup [d]_{\Theta_2}\}^2$. Hence each $\Theta \in \operatorname{Con} A$ is of the above mentioned form, i.e. it is uniquely determined by some $\Theta_1 \in \operatorname{Con} S$ and $\Theta_2 \in \operatorname{Con} C$, i.e. the mapping

$$h\colon \Theta \to \langle \Theta_2, \Theta_1 \rangle$$

is a bijection of $\operatorname{Con} A$ onto $\Pi_n \times \operatorname{Con} S$. It is easy to show that h is an isomorphism. □

Theorem 4. *For a q-lattice A, the congruence lattice $\operatorname{Con} A$ is distributive if and only if A contains at most one cell with at most 2 elements. $\operatorname{Con} A$ is modular if and only if A contains at most one cell with at most 3 elements.*

Proof. If A has no cell, then A is a lattice and $\operatorname{Con} A$ is distributive, see [1]. If A contains just one n-element cell then, by Theorem 3, $\operatorname{Con} A \simeq \Pi_n \times \operatorname{Con} S$, where S is the skeleton of A. However, Π_n is distributive if and only if $n \leqslant 2$, Π_n is modular if and only if $n \leqslant 3$ (see e.g. Ex. 5 of Par. 9, Ch. IV in [1]). Since $\operatorname{Con} S$ is distributive, we arrive at the statement.

On the contrary, suppose A has at least two cells C_1, C_2. Let a_i be an idempotent of C_i and $b_1 \in C_1$, $b_1 \neq a_1$, $b_2 \in C_2$, $b_2 \neq a_2$. Then clearly $\langle a_1, a_2 \rangle = \langle b_1 \vee b_2 \rangle \vee \langle b_1 \vee b_2 \rangle$, i.e.

$$\Theta(a_1, a_2) \subseteq \Theta(b_1, b_2).$$

But $\langle b_1, b_2 \rangle \notin \Theta(a_1, a_2)$, i.e. $\Theta(a_1, a_2) \neq \Theta(b_1, b_2)$.

(i) If $a_1 < a_2$ then the congruences

$$\Theta(a_1, b_1),\ \Theta(a_1, a_2),\ \Theta(b_1, b_2),\ \Theta(a_1, b_1) \wedge \Theta(a_1, a_2),\ \Theta(a_1, b_1) \vee \Theta(b_1, b_2)$$

form a sublattice of Con A isomorphic to N_5, see Figs. 4 and 5.

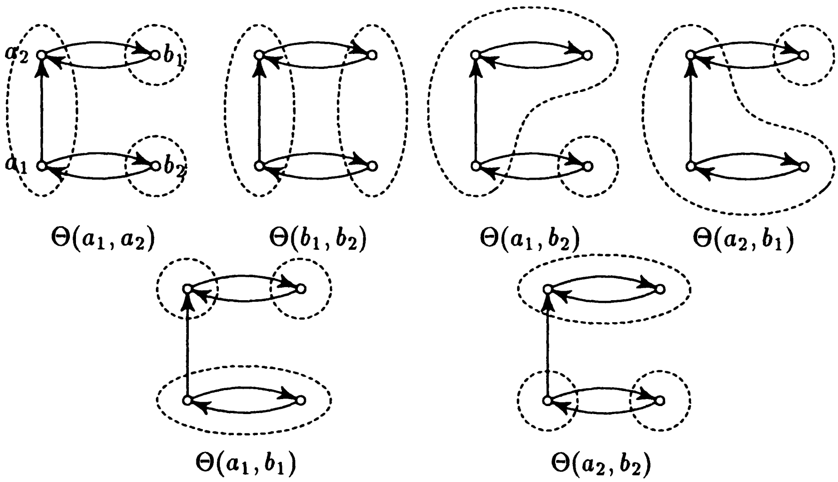

Fig. 4

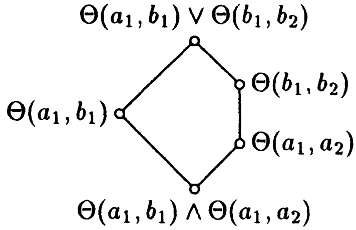

Fig. 5

(ii) If $a_1 \parallel a_2$, then the congruences

$$\Theta(a_1, a_1 \wedge a_2),\ \Theta(a_1, a_2),\ \Theta(b_1, b_2),$$
$$\Psi = \Theta(b_1, a_1 \wedge a_2) \vee \Theta(b_2, a_1 \vee a_2),$$
$$\Phi = \psi \vee \Theta(b_1, b_2)$$

form a sublattice of Con A isomorphic with N_5 again, see Figs. 6 and 7. □

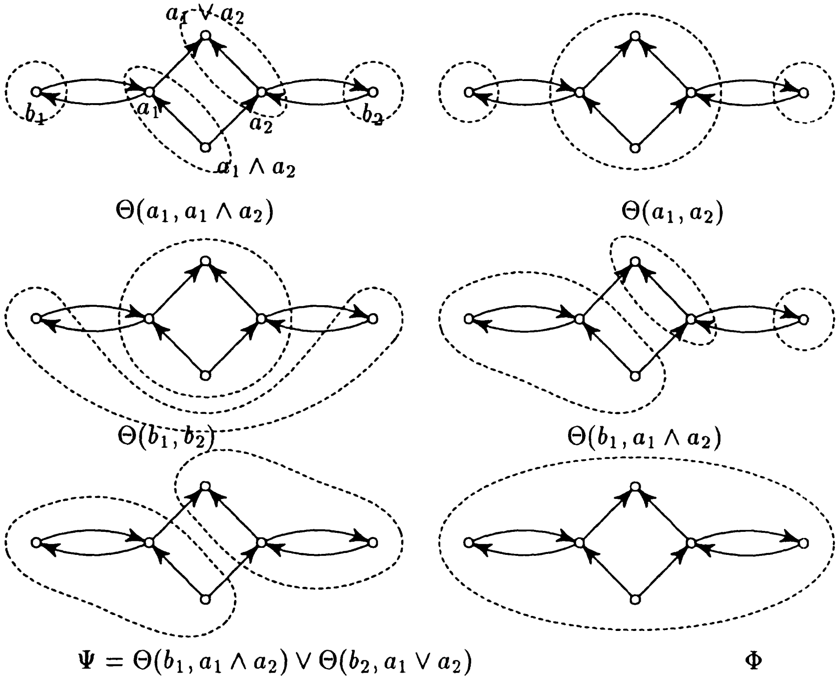

Fig. 6

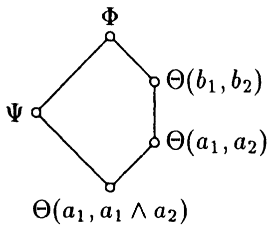

Fig. 7

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