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SUBDIVISION SCHEMES, BIORTHOGONAL WAVELETS AND IMAGE COMPRESSION

by

BIN HAN ©

A thesis submitted to the Faculty of Graduate Studies and Research in partial fulfilment of the requirements for the degree of DOCTOR OF PHILOSOPHY

in

MATHEMATICS

Department of Mathematical Sciences

Edmonton, Alberta

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ABSTRACT

In this thesis, we study subdivision schemes, biorthogonal wavelets and wavelet-based image compression. Subdivision schemes are important in computer aided geometric design to generate curves and high dimensional surfaces. First, We shall characterize the L_p convergence of any subdivision scheme with a finitely supported refinement mask in multivariate case. Then we shall study the error behaviour of any subdivision scheme if there is a round-off of its refinement mask. Next, we study a special kind of subdivision schemes – interpolatory subdivision schemes. We shall analyze the optimal properties, such as sum rules of an interpolatory refinement mask and the smoothness of its associated refinable function, of any interpolatory subdivision scheme. A general construction of optimal interpolatory subdivision schemes is presented. Next, we shall study biorthogonal multivariate wavelets since there is a well known close relation between interpolatory subdivision schemes and biorthogonal wavelets. We shall study the optimal approximation and smoothness properties of any biorthogonal wavelet. More importantly, a general and easy way (CBC algorithm) is presented to construct multivariate biorthogonal wavelets. As an example, a family of optimal bivariate biorthogonal wavelets is given. Finally, we try to apply our results in image compression and a 2-D wavelet transform C++ program is established which can use a library of bivariate biorthogonal wavelet filters.

DEDICATION

To my family: Shuang Liao and Rachel Irene Han
my parents to whom I owe much
and all of my teachers and friends

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Chapter 1

L_p Convergence of Subdivision Schemes

1.1 Introduction

A **refinable function** ϕ is a function satisfying the following so-called **refinement equation**:

$$\phi(x) = \sum_{\alpha \in \mathbb{Z}^s} a(\alpha)\phi(2x - \alpha), \qquad x \in \mathbb{R}^s, \tag{1.1.1}$$

where a is a sequence on $\mathbb{Z}^s$ called the **refinement mask**. If a is a finitely supported sequence with $\sum_{\alpha \in \mathbb{Z}^s} a(\alpha) = 2^s$, then it is known that (see [6]) there exists a unique compactly supported distribution ϕ satisfying the refinement equation (1.1.1) subject to the condition $\widehat{\phi}(0) = 1$. This distribution is said to be the **normalized solution** of the refinement equation (1.1.1) with the refinement mask a and will be denoted by ϕ_a throughout this thesis. The Fourier transform of a function f in $L_1(\mathbb{R}^s)$ is defined to be

$$\widehat{f}(\xi) := \int_{\mathbb{R}^s} f(x)e^{-ix\cdot\xi}\, dx, \qquad \xi \in \mathbb{R}^s, \tag{1.1.2}$$

where $x \cdot \xi$ denotes the inner product of two vectors x and ξ in $\mathbb{R}^s$. The domain of the Fourier transform can be naturally extended to include compactly supported distributions.

In order to solve the refinement equation (1.1.1), we start with the initial function

$$\phi_0(x) = \prod_{i=1}^{s} h(x_i), \qquad x = (x_1, \cdots, x_s) \in \mathbb{R}^s, \tag{1.1.3}$$

where h is the hat function defined by

$$h(x) := \begin{cases} 1+x & \text{for } x \in [-1, 0), \\ 1-x & \text{for } x \in [0, 1], \\ 0 & \text{for } x \in \mathbb{R}\backslash[-1, 1]. \end{cases} \tag{1.1.4}$$

Then we employ the iteration scheme $Q_a^n \phi_0$, $n = 0, 1, 2, \cdots$, where Q_a is the bounded linear operator on $L_p(\mathbb{R}^s)$ $(1 \le p \le \infty)$ given by

$$Q_a f := \sum_{\beta \in \mathbb{Z}^s} a(\beta) f(2 \cdot -\beta), \qquad f \in L_p(\mathbb{R}^s). \tag{1.1.5}$$

This iteration scheme is called a **subdivision scheme** (see [6]) or a **cascade algorithm** (see [28]) associated with the mask a. We say that the **subdivision scheme associated with a converges** in the L_p norm if there exists a function $f \in L_p(\mathbb{R}^s)$ such that $\lim_{n \to \infty} \|Q_a^n \phi_0 - f\|_p = 0$. If this is the case, then it is necessary that f must be the normalized solution ϕ_a of the refinement equation (1.1.1) with the refinement mask a.

Refinable functions are encountered in computer aided geometric design where subdivision schemes are used to construct smooth curves and surfaces. The reader is referred to [6, 30, 31, 38, 39, 40, 51, 52, 59, 62, 84, 85, 89] and the references cited there for detailed discussion on subdivision schemes and their applications to generate curves and surfaces. They are also known as scaling functions in the wavelet theory, for example, see [10, 13, 15, 16, 18, 26, 27, 62, 67, 79, 102]. For more detail about the applications of subdivision schemes on computer graphics, please see Chapter 3.

The structure of this chapter is as follows. In Section 1.2, we will introduce the concepts of subdivision operators and ℓ_p-norm joint spectral radius, then we shall study the relation between the subdivision operator and the ℓ_p-norm joint spectral radius. In Section 1.3, based on the work Jia [59], by employing ℓ_p-norm joint spectral radius, we characterize the L_p convergence of a multivariate subdivision scheme. In Section 1.4, we demonstrate that when $p = 2$, the ℓ_2-norm joint spectral radius can be easily computed by calculating the eigenvalues of a transition operator on a finite dimensional linear space. Examples will be given to illustrate the general theory. Finally in Section 1.5, the subdivision schemes associated with any general dilation matrix are discussed. All the results in this chapter are joint work with my supervisor Professor Rong-Qing Jia.

1.2 ℓ_p-Norm Joint Spectral Radius

This section is devoted to a study of joint spectral radii of a finite collection of linear operators associated to a refinement equation.

Let $\mathcal{A}$ be a finite collection of linear operators on a *finite dimensional* vector space V. A vector norm $\|\cdot\|$ on V induces a norm on the linear operators on V as follows. For a linear operator A on V, define

$$\|A\| := \max_{\|v\|=1}\{ \|Av\| \}.$$

For a positive integer n we denote by $\mathcal{A}^n$ the Cartesian power of $\mathcal{A}$:

$$\mathcal{A}^n := \{(A_1, \ldots, A_n) : A_1, \ldots, A_n \in \mathcal{A}\}.$$

When $n = 0$, we interpret $\mathcal{A}^0$ as the set $\{I\}$, where I is the identity mapping on V. Let

$$\|\mathcal{A}^n\|_\infty := \max\{\|A_1 \cdots A_n\| : (A_1, \ldots, A_n) \in \mathcal{A}^n\}.$$

Then the uniform joint spectral radius of $\mathcal{A}$ is defined to be

$$\rho_\infty(\mathcal{A}) := \lim_{n\to\infty} \|\mathcal{A}^n\|_\infty^{1/n}.$$

For $1 \le p < \infty$, we define

$$\|\mathcal{A}^n\|_p := \left(\sum_{(A_1,\dots,A_n)\in\mathcal{A}^n} \|A_1 \cdots A_n\|^p \right)^{1/p}.$$

For $1 \le p \le \infty$, the **ℓ_p-norm joint spectral radius** of $\mathcal{A}$ is defined to be

$$\rho_p(\mathcal{A}) := \lim_{n\to\infty} \|\mathcal{A}^n\|_p^{1/n}.$$

It is easily seen that this limit indeed exists, and

$$\lim_{n\to\infty} \|\mathcal{A}^n\|_p^{1/n} = \inf_{n\ge 1} \|\mathcal{A}^n\|_p^{1/n}.$$

Clearly, $\rho_p(\mathcal{A})$ is independent of the choice of the vector norm on V.

The uniform joint spectral radius was first introduced by Rota and Strang in [93] and the mean spectral radius was studied by Wang in [101]. The general ℓ_p-norm joint spectral radius of a finite collection of linear operators was introduced by Jia in [59].

By some basic properties of ℓ_p spaces we have that, for $1 \le p \le r \le \infty$,

$$(\#\mathcal{A})^{1/r-1/p}\rho_p(\mathcal{A}) \le \rho_r(\mathcal{A}) \le \rho_p(\mathcal{A}),$$

where $\#\mathcal{A}$ denotes the number of elements in $\mathcal{A}$. Furthermore, it is easily seen from the definition of the joint spectral radius that $\rho(A) \le \rho_\infty(\mathcal{A})$ for any element A in $\mathcal{A}$, where $\rho(A)$ is the spectral radius of A.

The subdivision operator is important in the study of convergence of a subdivision scheme. Let a be a sequence on $\mathbb{Z}^s$. The **subdivision operator** associated with a is defined by

$$S_a\lambda(\alpha) := \sum_{\beta\in\mathbb{Z}^s} a(\alpha - 2\beta)\lambda(\beta), \qquad \alpha \in \mathbb{Z}^s, \tag{1.2.1}$$

where $\lambda \in \ell_0(\mathbb{Z}^s)$ and $\ell_0(\mathbb{Z}^s)$ denotes the linear space of sequences on $\mathbb{Z}^s$ with finite support. Note that $\ell_0(\mathbb{Z}^s)$ is a subspace of $\ell_p(\mathbb{Z}^s)$. The ℓ_p norm of an element $\lambda \in \ell_p(\mathbb{Z}^s)$ is denoted by $\|\lambda\|_p$.

In order to study convergence of the subdivision scheme, we need to analyze the sequences $S_a^n\delta$, $n = 1, 2, \ldots$. For this purpose, we introduce the biinfinite matrices A_ε ($\varepsilon \in \mathbb{Z}^s$) as follows:

$$A_\varepsilon(\alpha, \beta) := a(\varepsilon + 2\alpha - \beta), \qquad \alpha, \beta \in \mathbb{Z}^s. \tag{1.2.2}$$

Lemma 1.1 *Suppose $\alpha = \varepsilon_1 + 2\varepsilon_2 + \cdots + 2^{n-1}\varepsilon_n + 2^n\gamma$, where $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n, \gamma \in \mathbb{Z}^s$. Then for any $\beta \in \mathbb{Z}^s$,*

$$S_a^n\delta(\alpha - \beta) = A_{\varepsilon_n} \cdots A_{\varepsilon_1}(\gamma, \beta).$$

Proof: The proof proceeds by induction on n. For $n = 1$ and $\alpha = \varepsilon_1 + 2\gamma$, we have

$$S_a\delta(\alpha - \beta) = a(\varepsilon_1 + 2\gamma - \beta) = A_{\varepsilon_1}(\gamma, \beta).$$

Suppose $n > 1$ and the lemma has been verified for $n - 1$. For $\alpha = \varepsilon_1 + 2\alpha_1$, where $\alpha_1, \varepsilon \in \mathbb{Z}^s$, we have

$$S_a^n\delta(\alpha - \beta) = \sum_{\eta \in \mathbb{Z}^s} a(\alpha - \beta - 2\eta) S_a^{n-1}\delta(\eta) = \sum_{\eta \in \mathbb{Z}^s} a(\varepsilon_1 + 2\eta - \beta) S_a^{n-1}\delta(\alpha_1 - \eta). \tag{1.2.3}$$

Suppose that $\alpha_1 = \varepsilon_2 + \cdots + 2^{n-2}\varepsilon_n + 2^{n-1}\gamma$. Then by the induction hypothesis we have

$$S_a^{n-1}\delta(\alpha_1 - \eta) = A_{\varepsilon_n} \cdots A_{\varepsilon_2}(\gamma, \eta).$$

This in connection with (1.2.3) gives

$$S_a^n\delta(\alpha - \beta) = \sum_{\eta \in \mathbb{Z}^s} A_{\varepsilon_n} \cdots A_{\varepsilon_2}(\gamma, \eta) A_{\varepsilon_1}(\eta, \beta) = A_{\varepsilon_n} \cdots A_{\varepsilon_2} A_{\varepsilon_1}(\gamma, \beta),$$

thereby completing the induction procedure. ∎

The biinfinite matrices A_ε ($\varepsilon \in \mathbb{Z}^s$) defined in (1.2.2) may be viewed as the linear operators given by

$$A_\varepsilon v(\alpha) = \sum_{\beta \in \mathbb{Z}^s} a(\varepsilon + 2\alpha - \beta) v(\beta), \qquad v \in \ell_0(\mathbb{Z}^s),\ \alpha \in \mathbb{Z}^s. \tag{1.2.4}$$

Now let $\mathcal{A}$ be a finite collection of linear operators on a vector space V, which is not necessarily finite dimensional. A subspace W of V is said to be $\mathcal{A}$-invariant, if it is invariant under every operator A in $\mathcal{A}$. Let U be a subset of V. The intersection of all $\mathcal{A}$-invariant subspaces of V containing U is $\mathcal{A}$-invariant, and we call it the **minimal $\mathcal{A}$-invariant subspace** generated by U, or the **minimal common invariant subspace** of the operators A in $\mathcal{A}$ generated by U. This subspace is spanned by the set

$$\{A_1 \cdots A_j u \ :\ u \in U,\ (A_1, \ldots, A_j) \in \mathcal{A}^j,\ j = 0, 1, \ldots\}.$$

If, in addition, V is finite dimensional, then there exists a positive integer k such that the set

$$\{A_1 \cdots A_j u \ :\ u \in U,\ (A_1, \ldots, A_j) \in \mathcal{A}^j,\ j = 0, 1, \ldots, k\}$$

already spans the minimal $\mathcal{A}$-invariant subspace generated by U.

We define, for $1 \le p < \infty$,

$$\|\mathcal{A}^n v\|_p := \left(\sum_{(A_1, \ldots, A_n) \in \mathcal{A}^n} \|A_1 \cdots A_n v\|^p \right)^{1/p},$$

and for $p = \infty$,

$$\|\mathcal{A}^n v\|_\infty := \max\{\|A_1 \cdots A_n v\| \ :\ (A_1, \ldots, A_n) \in \mathcal{A}^n\}.$$

The s-torus $\mathbb{T}^s$ is defined by

$$\mathbb{T}^s := \{(z_1, \ldots, z_s) \in \mathbb{C}^s \ :\ |z_1| = \cdots = |z_s| = 1\}.$$

The **symbol** of a sequence $a \in \ell_0(\mathbb{Z}^s)$ is the Laurent polynomial $\tilde{a}(z)$ given by

$$\tilde{a}(z) := \sum_{\alpha \in \mathbb{Z}^s} a(\alpha) z^\alpha, \qquad z \in \mathbb{T}^s, \tag{1.2.5}$$

where $z^\alpha := z_1^{\alpha_1} \cdots z_s^{\alpha_s}$ for $z = (z_1, \ldots, z_s) \in \mathbb{T}^s$ and $\alpha = (\alpha_1, \ldots, \alpha_s) \in \mathbb{Z}^s$.

For any $\alpha \in \mathbb{Z}^s$, by δ_α we denote the sequence on $\mathbb{Z}^s$ such that $\delta_\alpha(\alpha) = 1$ and $\delta_\alpha(\beta) = 0$ for $\beta \in \mathbb{Z}^s \backslash \{\alpha\}$. In particular, we use δ to denote δ_0.

For $\beta \in \mathbb{Z}^s$ we denote by τ^β the shift operator on $\ell_0(\mathbb{Z}^s)$ given by

$$\tau^\beta \lambda := \lambda(\cdot - \beta), \qquad \lambda \in \ell_0(\mathbb{Z}^s).$$

Let ν be an element of $\ell_0(\mathbb{Z}^s)$. Then its symbol $\tilde{\nu}(z)$ is a Laurent polynomial, which induces the difference operator $\tilde{\nu}(\tau) := \sum_{\beta \in \mathbb{Z}^s} \nu(\beta)\tau^\beta$. Note that $\tilde{\nu}(\tau)\delta = \nu$.

Let Ω be the set of vertices of the unit cube $[0,1]^s$. Thus, each element $\alpha \in \mathbb{Z}^s$ can be uniquely represented as $\varepsilon + 2\gamma$, where $\varepsilon \in \Omega$ and $\gamma \in \mathbb{Z}^s$.

As usual, for $1 \le p \le \infty$, $\ell_p(\mathbb{Z}^s)$ denotes the Banach space of all sequences on $\mathbb{Z}^s$ such that $\|a\|_p < \infty$, where

$$\|a\|_p := \left(\sum\nolimits_{\beta \in \mathbb{Z}^s} |a(\beta)|^p\right)^{1/p} \qquad \text{for } 1 \le p < \infty,$$

and $\|a\|_\infty$ is the supremum of a on $\mathbb{Z}^s$. In the following lemma, the underlying vector norm on $\ell_0(\mathbb{Z}^s)$ is chosen to be the ℓ_p-norm.

Lemma 1.2 *Let S_a be the subdivision operator associated with a refinement mask a. Let $\mathcal{A} := \{A_\varepsilon : \varepsilon \in \Omega\}$, where A_ε are the linear operators on $\ell_0(\mathbb{Z}^s)$ given by* (1.2.4). *Then for $1 \le p \le \infty$ and $\nu \in \ell_0(\mathbb{Z}^s)$,*

$$\|\tilde{\nu}(\tau) S_a^n \delta\|_p = \|\mathcal{A}^n \nu\|_p, \qquad n = 1, 2, \ldots. \tag{1.2.6}$$

Proof: Suppose that $\alpha = \varepsilon_1 + 2\varepsilon_2 + \cdots + 2^{n-1}\varepsilon_n + 2^n\gamma$, where $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n \in \Omega$ and $\gamma \in \mathbb{Z}^s$. Then by Lemma 1.1 we have

$$\begin{aligned}
\tilde{\nu}(\tau) S_a^n \delta(\alpha) &= \sum_{\beta \in \mathbb{Z}^s} \nu(\beta) S_a^n \delta(\alpha - \beta) \\
&= \sum_{\beta \in \mathbb{Z}^s} A_{\varepsilon_n} \cdots A_{\varepsilon_1}(\gamma, \beta)\nu(\beta) = A_{\varepsilon_n} \cdots A_{\varepsilon_1}\nu(\gamma).
\end{aligned}$$

Hence, (1.2.6) is true for $p = \infty$. When $1 \le p < \infty$ we have

$$\sum_{\alpha \in \mathbb{Z}^s} \left|\tilde{\nu}(\tau) S_a^n \delta(\alpha)\right|^p = \sum_{(\varepsilon_1, \ldots, \varepsilon_n) \in \Omega^n} \sum_{\gamma \in \mathbb{Z}^s} \left|A_{\varepsilon_n} \cdots A_{\varepsilon_1} \nu(\gamma)\right|^p.$$

This verifies (1.2.6) for $1 \le p < \infty$. ∎

Let $\mathcal{A} := \{A_\varepsilon : \varepsilon \in \Omega\}$. We shall demonstrate that, for each ν in $\ell_0(\mathbb{Z}^s)$, the minimal $\mathcal{A}$-invariant subspace generated by ν is finite dimensional. To establish this result, we shall introduce the concept of admissible sets. For a finite subset K of $\mathbb{R}^s$, by $\ell(K)$ we denote the linear subspace of $\ell_0(\mathbb{Z}^s)$ consisting of all sequences supported on $K \cap \mathbb{Z}^s$. Let A be a linear operator on $\ell_0(\mathbb{Z}^s)$. A finite subset K of $\mathbb{Z}^s$ is said to be **admissible** for A if $\ell(K)$ is invariant under A. See [44] for the related notion of *good* sets. The following lemma shows that there exists a finite subset K of $\mathbb{Z}^s$ such that K contains the support of ν and is admissible for all A_ε, $\varepsilon \in \Omega$.

Lemma 1.3 *Suppose that a is a sequence on $\mathbb{Z}^s$ with its support $E := \operatorname{supp} a$ where $\operatorname{supp} a := \{\alpha \in \mathbb{Z}^s : a(\alpha) \neq 0\}$ being finite. Let A_ε ($\varepsilon \in \Omega$) be the linear operators on $\ell_0(\mathbb{Z}^s)$ given by* (1.2.4). *Then a finite subset K of $\mathbb{Z}^s$ is admissible for $A := A_0$ if and only if*

$$2^{-1}(E + K) \cap \mathbb{Z}^s \subseteq K. \tag{1.2.7}$$

Consequently, for any finite subset G of $\mathbb{Z}^s$, there exists a finite subset K of $\mathbb{Z}^s$ such that K contains G and K is admissible for all A_ε, $\varepsilon \in \Omega$.

Proof: Suppose that K is admissible for A. Let $\alpha \in 2^{-1}(E + K) \cap \mathbb{Z}^s$. Then $2\alpha = \gamma + \beta$ for some $\gamma \in E$ and $\beta \in K$. It follows that $A\delta_\beta(\alpha) = a(2\alpha - \beta) = a(\gamma) \neq 0$. Since K is admissible for A, we have $A\delta_\beta \in \ell(K)$, and therefore $\alpha \in K$. This shows that (1.2.7) is true.

Conversely, suppose that (1.2.7) is true. Let $v \in \ell(K)$ and $\alpha \in \mathbb{Z}^s$. Then

$$Av(\alpha) = \sum_{\beta \in \mathbb{Z}^s} a(2\alpha - \beta) v(\beta) \neq 0$$

implies that $2\alpha - \beta \in E$ for some $\beta \in K$. It follows that $2\alpha \in E + K$. Therefore $\alpha \in 2^{-1}(E+K) \cap \mathbb{Z}^s$, and so $\alpha \in K$ by (1.2.7). This shows that A maps $\ell(K)$ to $\ell(K)$. In other words, K is admissible for A.

From the above proof we see that a finite subset K of $\mathbb{Z}^s$ is admissible for A_ε if and only if

$$2^{-1}(E - \varepsilon + K) \cap \mathbb{Z}^s \subseteq K. \tag{1.2.8}$$

The set $E - \Omega$ consists of all the points $x - \varepsilon$, where $x \in E$ and $\varepsilon \in \Omega$.

Now suppose that G is a finite subset of $\mathbb{Z}^s$. Let $H := 2G \cup (E - \Omega) \cup \{0\}$, and let

$$K := \left(\sum_{n=1}^{\infty} 2^{-n} H\right) \cap \mathbb{Z}^s.$$

In other words, an element $\alpha \in \mathbb{Z}^s$ belongs to K if and only if $\alpha = \sum_{n=1}^{\infty} 2^{-n} h_n$ for some sequence of elements $h_n \in H$. Since $0 \in H$ and $2^{-1}H \supseteq G$, we have

$$K \supseteq \mathbb{Z}^s \cap 2^{-1}H \supseteq \mathbb{Z}^s \cap G = G.$$

Moreover,

$$\begin{aligned} 2^{-1}(E - \varepsilon + K) \cap \mathbb{Z}^s &\subseteq 2^{-1}(H + K) \cap \mathbb{Z}^s \\ &= (2^{-1}H + 2^{-1}K) \cap \mathbb{Z}^s \\ &\subseteq (2^{-1}H + 2^{-2}H + \cdots) \cap \mathbb{Z}^s = K. \end{aligned} \tag{1.2.9}$$

Thus, K satisfies (1.2.8). Hence K is admissible for all A_ε, $\varepsilon \in \Omega$. ∎

Lemma 1.4 *Let $\mathcal{A}$ be a finite collection of linear operators on a vector space V. Let ν be a vector in V, and let $V(\nu)$ be the minimal $\mathcal{A}$-invariant subspace generated by ν. If $V(\nu)$ is finite dimensional, then there exist two positive constants C_1 and C_2 such that*

$$C_1 \|\mathcal{A}^n|_{V(\nu)}\|_p \le \|\mathcal{A}^n \nu\|_p \le C_2 \|\mathcal{A}^n|_{V(\nu)}\|_p \qquad \forall n \in \mathbb{N} \tag{1.2.10}$$

and therefore,

$$\lim_{n\to\infty} \|\mathcal{A}^n\nu\|_p^{1/n} = \rho_p(\mathcal{A}|_{V(\nu)}). \tag{1.2.11}$$

Proof: Let $\|\cdot\|$ be a vector norm on $V(\nu)$. Since $V(\nu)$ is finite dimensional, there exists a positive integer k such that $V(\nu)$ is spanned by the set

$$Y := \{A_1 \cdots A_j\nu : (A_1, \dots, A_j) \in \mathcal{A}^j,\ j = 0, 1, \dots, k\}.$$

Thus, there exists a positive constant C_1 such that $\|\mathcal{A}^n y\|_p \le C_1\|\mathcal{A}^n\nu\|_p$ for all $y \in Y$ and all $n = 1, 2, \dots$. Moreover, there exists a positive constant C_2 such that

$$\|\mathcal{A}^n|_{V(\nu)}\|_p \le C_2 \max_{y\in Y} \|\mathcal{A}^n y\|_p, \qquad n = 1, 2, \dots .$$

Therefore, there exists a positive constant C such that for all $n = 1, 2, \dots,$

$$\|\mathcal{A}^n|_{V(\nu)}\|_p \le C\|\mathcal{A}^n\nu\|_p .$$

Hence, (1.2.10) holds true. But $\|\mathcal{A}^n\nu\|_p \le \|\mathcal{A}^n|_{V(\nu)}\|_p\|\nu\|$. This proves the desired relation (1.2.11). ∎

Theorem 1.5 *Let S_a be the subdivision operator associated with a refinement mask a. Let $\mathcal{A} := \{A_\varepsilon : \varepsilon \in \Omega\}$, where Ω is the set of vertices of the unit cube $[0,1]^s$, and A_ε are the linear operators on $\ell_0(\mathbb{Z}^s)$ given by (1.2.4). Then for any $\nu \in \ell_0(\mathbb{Z}^s)$, there exist two positive constants C_1 and C_2 such that for all $n \in \mathbb{N}$,*

$$C_1\|\{A_\varepsilon|_{V(\nu)} : \varepsilon \in \Omega\}^n\|_p \le \|\tilde{\nu}(\tau)S_a^n\delta\|_p \le C_2\|\{A_\varepsilon|_{V(\nu)} : \varepsilon \in \Omega\}^n\|_p \tag{1.2.12}$$

and therefore,

$$\lim_{n\to\infty} \|\tilde{\nu}(\tau)S_a^n\delta\|_p^{1/n} = \rho_p(\{A_\varepsilon|_{V(\nu)} : \varepsilon \in \Omega\}), \tag{1.2.13}$$

where $V(\nu)$ is the minimal $\mathcal{A}$-invariant subspace generated by ν. Moreover, if W is the minimal $\mathcal{A}$-invariant subspace generated by a finite set Y, then there exist two positive constants C_1 and C_2 such that for all $n \in \mathbb{N}$,

$$C_1\|\{A_\varepsilon|_W : \varepsilon \in \Omega\}^n\|_p \le \max_{\nu\in Y}\{\|\tilde{\nu}(\tau)S_a^n\delta\|_p\} \le C_2\|\{A_\varepsilon|_W : \varepsilon \in \Omega\}^n\|_p \tag{1.2.14}$$

and therefore,

$$\rho_p(\{A_\varepsilon|_W : \varepsilon \in \Omega\}) = \max_{\nu \in Y}\{\lim_{n\to\infty} \|\tilde{\nu}(\tau)S_a^n\delta\|_p^{1/n}\}. \qquad (1.2.15)$$

Proof: By Lemma 1.3, the linear space $V(\nu)$ is finite dimensional, and so the relevant joint spectral radius in (1.2.13) is well defined. By Lemma 1.2 we have

$$\|\tilde{\nu}(\tau)S_a^n\delta\|_p = \|\mathcal{A}^n\nu\|_p, \qquad 1 \le p \le \infty,\ n = 1, 2, \dots .$$

Applying Lemma 1.4 to the present situation, we obtain (1.2.12) and (1.2.13).

For the second part of the theorem, we let W be the minimal $\mathcal{A}$-invariant subspace generated by a finite set Y, and observe that W is a finite sum of the linear subspaces $V(\nu)$, $\nu \in Y$. Hence

$$\rho_p(\{A_\varepsilon|_W : \varepsilon \in \Omega\}) = \max_{\nu \in Y}\{\rho_p(\{A_\varepsilon|_{V(\nu)} : \varepsilon \in \Omega\})\}.$$

This together with (1.2.12) and (1.2.13) verifies (1.2.14) and (1.2.15). ∎

1.3 L_p Convergence of Subdivision Schemes

In this section, we characterize the L_p-convergence ($1 \le p \le \infty$) of a subdivision scheme in terms of the corresponding refinement mask.

Let f be the normalized solution of the refinement equation (1.1.1) with a mask a. Taking the Fourier transform of the functions on both sides of (1.1.1), we obtain

$$\hat{f}(\xi) = 2^{-s}\tilde{a}(e^{-i\xi/2})\hat{f}(\xi/2), \qquad \xi \in \mathbb{R}^s. \qquad (1.3.1)$$

It is evident that $\tilde{a}(e^{-i2\pi\beta}) = 2^s$ for all $\beta \in \mathbb{Z}^s$ since $\tilde{a}(1) = \sum_{\beta\in\mathbb{Z}^s} a(\beta) = 2^s$. Thus, for all positive integers k and all $\beta \in \mathbb{Z}^s$, it follows from (1.3.1) that

$$\hat{f}(2\pi 2^k\beta) = \hat{f}(2\pi\beta).$$

If, in addition, f lies in $L_1(\mathbb{R}^s)$, then by the Riemann-Lebesgue lemma we have

$$\hat{f}(2\pi\beta) = \lim_{k\to\infty} \hat{f}(2\pi 2^k \beta) = 0 \qquad \forall\, \beta \in \mathbb{Z}^s \setminus \{0\}.$$

A function f is said to satisfy the **moment conditions** of order 1, if $\hat{f}(0) = 1$ and $\hat{f}(2\pi\beta) = 0$ for all $\beta \in \mathbb{Z}^s \setminus \{0\}$. Thus, if the normalized solution f of the refinement equation (1.1.1) lies in $L_1(\mathbb{R}^s)$, then f satisfies the moment conditions of order 1. If $Q_a^n f$ converges to the normalized solution of the refinement equation (1.1.1), then f must satisfy the moment conditions of order 1.

In our study of convergence, the concept of stability plays an important role. The shifts of a function ϕ in $L_p(\mathbb{R}^s)$ are said to be **stable** if there are two positive constants C_1 and C_2 such that

$$C_1\|\lambda\|_p \le \left\| \sum_{\alpha\in\mathbb{Z}^s} \lambda(\alpha)\phi(\cdot - \alpha) \right\|_p \le C_2\|\lambda\|_p \qquad \forall\, \lambda \in \ell_0(\mathbb{Z}^s). \tag{1.3.2}$$

It was proved by Jia and Micchelli in [68] that a compactly supported function ϕ in $L_p(\mathbb{R}^s)$ satisfies the L_p-stability condition in (1.3.2) if and only if, for any $\xi \in \mathbb{R}^s$, there exists an element $\beta \in \mathbb{Z}^s$ such that

$$\hat{\phi}(\xi + 2\pi\beta) \ne 0.$$

Note that the shifts of the hat function h given in (1.1.4) are stable. It is easily seen that the shifts of the function ϕ_0 given in (1.1.3) are stable.

For any vector $y \in \mathbb{Z}^s$, the difference operator ∇_y on $\ell(\mathbb{Z}^s)$ is defined to be

$$\nabla_y \lambda = \lambda - \lambda(\cdot - y), \qquad \lambda \in \ell(\mathbb{Z}^s),$$

where $\ell(\mathbb{Z}^s)$ denotes the linear space of all sequences on $\mathbb{Z}^s$. In particular, ∇_i denotes the difference operator ∇_{e_i} where e_i is the ith coordinate unit vector.

First, we give a necessary condition for the subdivision scheme to converge.

Theorem 1.6 *Let a be an element in $\ell_0(\mathbb{Z}^s)$ with $\sum_{\alpha\in\mathbb{Z}^s} a(\alpha) = 2^s$, and S_a be the subdivision operator associated with a given in* (1.2.1). *If the subdivision scheme* *associated with the mask a converges in the L_p-norm ($1 \le p \le \infty$), then for any vector $y \in \mathbb{Z}^s$,*

$$\lim_{n\to\infty} 2^{-ns/p}\|\nabla_y S_a^n \delta\|_p = 0. \tag{1.3.3}$$

Consequently, if the subdivision scheme associated with the mask a converges in the L_p-norm, then

$$\sum_{\beta\in\mathbb{Z}^s} a(\alpha+2\beta) = 1 \qquad \forall\,\alpha \in \mathbb{Z}^s. \tag{1.3.4}$$

Proof: Suppose that ϕ is a compactly supported function in $L_p(\mathbb{R}^s)$, ϕ satisfies the moment conditions of order 1, and the shifts of ϕ are stable. For $n = 0, 1, 2, \ldots$, let $a_n := S_a^n\delta$ and $f_n := Q_a^n\phi$, where Q_a is the operator given in (1.1.5). Then by (1.1.5) we have

$$f_n = \sum_{\alpha\in\mathbb{Z}^s} a_n(\alpha)\phi(2^n\cdot - \alpha).$$

Hence, for $y \in \mathbb{Z}^s$ we have

$$\begin{aligned} f_n - f_n(\cdot - 2^{-n}y) &= \sum_{\alpha\in\mathbb{Z}^s} \big[a_n(\alpha) - a_n(\alpha - y)\big]\phi(2^n\cdot - \alpha) \\ &= \sum_{\alpha\in\mathbb{Z}^s} \nabla_y a_n(\alpha)\phi(2^n\cdot - \alpha). \end{aligned}$$

Since the shifts of ϕ are stable, there exists a constant $C > 0$ such that

$$2^{-ns/p}\|\nabla_y a_n\|_p \le C\|f_n - f_n(\cdot - 2^{-n}y)\|_p. \tag{1.3.5}$$

In particular, the above estimate is valid for $f_n = Q_a^n\phi_0$, where ϕ_0 is the function given in (1.1.3). If the subdivision scheme converges in the L_p-norm, then there exists a compactly supported function f in $L_p(\mathbb{R}^s)$ ($f \in C(\mathbb{R}^s)$ in the case $p = \infty$) such that $\|f_n - f\|_p \to 0$ as $n \to \infty$. Moreover, by the triangle inequality, we have

$$\|f_n - f_n(\cdot - 2^{-n}y)\|_p \le \|f - f(\cdot - 2^{-n}y)\|_p + 2\|f - f_n\|_p.$$

Hence, $\|f_n - f_n(\cdot - 2^{-n}y)\|_p \to 0$ as $n \to \infty$. This together with (1.3.5) verifies (1.3.3).

For the second part of the theorem, we observe that if the subdivision scheme converges in the L_p-norm for some p with $1 \le p \le \infty$, then it also converges in the L_1-norm. Thus, we only have to deal with the case $p = 1$.

Let Ω be the set of vertices of $[0,1]^s$. Then $\#\Omega = 2^s$, and $\mathbb{Z}^s$ is the disjoint union of $\alpha + 2\mathbb{Z}^s$, $\alpha \in \Omega$. Since $\sum_{\alpha\in\mathbf{Z}^s} a(\alpha) = 2^s$, we have

$$\sum_{\alpha\in\Omega}\sum_{\beta\in\mathbf{Z}^s} a(\alpha - 2\beta) = 2^s.$$

Thus, (1.3.4) will be proved if we can show

$$\sum_{\beta\in\mathbf{Z}^s} a(\alpha - 2\beta) = \sum_{\beta\in\mathbf{Z}^s} a(-2\beta) \qquad \forall\, \alpha \in \Omega. \tag{1.3.6}$$

To this end, we deduce from $a_n = S_a a_{n-1}$ that

$$\sum_{\alpha\in\mathbf{Z}^s} a_n(\alpha) = \sum_{\alpha\in\mathbf{Z}^s}\sum_{\beta\in\mathbf{Z}^s} a(\alpha - 2\beta)a_{n-1}(\beta) = 2^s \sum_{\beta\in\mathbf{Z}^s} a_{n-1}(\beta).$$

An induction argument gives $\sum_{\alpha\in\mathbf{Z}^s} a_n(\alpha) = 2^{ns}$. Moreover,

$$\begin{aligned}\sum_{\beta\in\mathbf{Z}^s} a_n(\alpha - 2\beta) &= \sum_{\beta\in\mathbf{Z}^s}\sum_{\gamma\in\mathbf{Z}^s} a(\alpha - 2\beta - 2\gamma)a_{n-1}(\gamma)\\ &= \sum_{\beta\in\mathbf{Z}^s} a(\alpha - 2\beta)\sum_{\gamma\in\mathbf{Z}^s} a_{n-1}(\gamma - \beta) = 2^{s(n-1)}\sum_{\beta\in\mathbf{Z}^s} a(\alpha - 2\beta).\end{aligned}$$

Thus, we have

$$\sum_{\beta\in\mathbf{Z}^s}\big[a(\alpha - 2\beta) - a(-2\beta)\big] = 2^{-s(n-1)}\sum_{\beta\in\mathbf{Z}^s}\big[a_n(\alpha - 2\beta) - a_n(-2\beta)\big].$$

It follows that

$$\left|\sum_{\beta\in\mathbf{Z}^s}\big[a(\alpha - 2\beta) - a(-2\beta)\big]\right| \le 2^{-(n-1)s}\|\nabla_\alpha a_n\|_1. \tag{1.3.7}$$

If the subdivision scheme is L_1-convergent, then by the first part of the theorem we have $2^{-(n-1)s}\|\nabla_\alpha a_n\|_1 \to 0$ as $n \to \infty$. This together with (1.3.7) implies (1.3.6), as desired. ∎

It was proved by Cavaretta, Dahmen, and Micchelli [6] that the condition in (1.3.4) is necessary if the subdivision scheme converges in the L_∞-norm.

The next theorem gives us a characterization of convergence of a multivariate subdivision scheme.

Theorem 1.7 *Let a be an element in $\ell_0(\mathbb{Z}^s)$ such that $\sum_{\alpha\in\mathbb{Z}^s} a(\alpha) = 2^s$, and S_a the corresponding subdivision operator. Then the subdivision scheme associated with the mask a converges in the L_p-norm ($1 \le p \le \infty$) if and only if*

$$\lim_{n\to\infty} \|\nabla_j S_a^n \delta\|_p^{1/n} < 2^{s/p} \qquad \text{for } j = 1, \dots, s. \tag{1.3.8}$$

Proof: Let A_ε be the linear operators on $\ell_0(\mathbb{Z}^s)$ given by (1.2.4), and let V be the minimal common invariant subspace of A_ε ($\varepsilon \in \Omega$) generated by $\nabla_j\delta$, $j = 1, \dots, s$. Then V is finite dimensional and by Theorem 1.5 we have

$$\rho_p := \rho_p(\{A_\varepsilon|_V : \varepsilon \in \Omega\}) = \max_{1\le j\le s}\{\lim_{n\to\infty} \|\nabla_j S_a^n \delta\|_p^{1/n}\}.$$

Thus, (1.3.8) is equivalent to $\rho_p(\{A_\varepsilon|_V : \varepsilon \in \Omega\}) < 2^{s/p}$.

Let $\mathcal{A} := \{A_\varepsilon|_V : \varepsilon \in \Omega\}$. If $\rho_p(\{A_\varepsilon|_V : \varepsilon \in \Omega\}) \ge 2^{s/p}$, then we have

$$\inf_{n\ge1} \|\mathcal{A}^n\|_p^{1/n} = \lim_{n\to\infty} \|\mathcal{A}^n\|_p^{1/n} \ge 2^{s/p}.$$

It follows that

$$2^{-ns/p}\|\mathcal{A}^n\|_p \ge 1 \qquad \forall n \in \mathbb{N}.$$

By Lemma 1.4, there exists a constant $C > 0$ such that $\|\mathcal{A}^n\|_p \le C \max_{1\le j\le s} \|\mathcal{A}^n \nabla_j \delta\|_p$ for all n. By Lemma 1.2, we have $\|\mathcal{A}^n \nabla_j \delta\|_p = \|\nabla_j S_a^n \delta\|_p$. Hence

$$\rho_p \ge 2^{s/p} \quad \Longrightarrow \quad \max_{1\le j\le s}\{2^{-ns/p}\|\nabla_j S_a^n \delta\|_p\} \ge 1/C.$$

Thus, the subdivision scheme associated with a is not L_p-convergent, by Theorem 1.6. This shows that (1.3.8) is necessary for the subdivision scheme to converge in the L_p-norm.

In the following we prove the sufficiency part of the theorem. By (1.3.8), there exist positive constants r and C such that $0 < r < 1$ and

$$2^{-ns/p}\|\nabla_j S_a^n \delta\|_p \le C r^n \qquad \forall n \in \mathbb{N}, j = 1, \cdots, s. \tag{1.3.9}$$

By induction, we observe that

$$f_{n+1} = \sum_{\alpha \in \mathbb{Z}^s} S_a^{n+1}\delta(\alpha)\phi_0(2^{n+1}\cdot - \alpha), \tag{1.3.10}$$

where ϕ_0 is defined in (1.1.3). Note that ϕ_0 satisfies the following refinement equation

$$\phi_0 = \sum_{\alpha \in \mathbb{Z}^s} b(\alpha)\phi_0(2\cdot - \alpha),$$

where $\widetilde{b}(z) := 2^{-s}\prod_{j=1}^{s}(z_j^{-1} + 2 + z_j), z = (z_1, \cdots, z_s) \in \mathbb{T}^s$. Therefore,

$$f_n = \sum_{\alpha \in \mathbb{Z}^s} S_a^n \delta(\alpha)\phi_0(2^n \cdot - \alpha) = \sum_{\alpha \in \mathbb{Z}^s} S_b S_a^n \delta(\alpha)\phi_0(2^{n+1}\cdot - \alpha). \tag{1.3.11}$$

From (1.3.10) and (1.3.11), we deduce

$$f_{n+1} - f_n = \sum_{\alpha \in \mathbb{Z}^s} (S_a - S_b) S_a^n \delta(\alpha)\phi_0(2^{n+1}\cdot - \alpha) = \sum_{\alpha \in \mathbb{Z}^s} S_{a-b} S_a^n \delta(\alpha)\phi_0(2^{n+1}\cdot - \alpha).$$

Since ϕ_0 is compactly supported, there exists a positive constant C_1 such that

$$\|f_{n+1} - f_n\|_p \le C_1 2^{-ns/p}\|S_{a-b} S_a^n \delta\|_p. \tag{1.3.12}$$

By Theorem 1.6, we have

$$\sum_{\beta \in \mathbb{Z}^s} \big(a(\alpha + 2\beta) - b(\alpha + 2\beta)\big) = 0 \qquad \forall\, \alpha \in \mathbb{Z}^s.$$

Thus, by Lemma 2.5 in Chapter 2, it follows that there exist sequences $c_j, j = 1, \cdots, s$ in $\ell_0(\mathbb{Z}^s)$ such that

$$\widetilde{a}(z) - \widetilde{b}(z) = \sum_{j=1}^{s}(1 - z_j^2)\widetilde{c_j}(z), \qquad z = (z_1, \cdots, z_s) \in \mathbb{T}^s.$$

Hence, $\widetilde{S_{a-b}S_a^n\delta}(z) = \sum_{j=1}^s \widetilde{c_j}(z)\widetilde{\nabla_j S_a^n\delta}(z^2), z = (z_1, \cdots, z_s) \in \mathbb{T}^s$. From (1.3.12) and (1.3.9), we deduce that

$$\|f_{n+1} - f_n\|_p \le C_1 2^{-ns/p} \sum_{j=1}^{s} \|c_j\|_1 \|\nabla_j S_a^n \delta\|_p \le \left(CC_1 \sum_{j=1}^{s} \|c_j\|_1 \right) r^n \qquad \forall\, n \in \mathbb{N}.$$

Therefore, the subdivision scheme associated with a converges in the L_p norm. ∎

Suppose that K is an admissible set for every A_ε, $\varepsilon \in \Omega$, and $\ell(K)$ contains $\nabla_j\delta$ for $j = 1, \ldots, s$. Let

$$V := \left\{ v \in \ell(K) \ : \ \sum_{\alpha \in \mathbb{Z}^s} v(\alpha) = 0 \right\}. \tag{1.3.13}$$

If $\sum_{\beta \in \mathbb{Z}^s} a(\alpha - 2\beta) = 1$ for all $\alpha \in \mathbb{Z}^s$, then V is invariant under every A_ε, $\varepsilon \in \Omega$. Thus, we may restate Theorem 1.7 as follows.

Theorem 1.8 *Under the conditions of Theorem* 1.7, *the subdivision scheme associated with a converges in the L_p-norm* ($1 \le p \le \infty$) *if and only if the following two conditions are satisfied:*

(*a*) $\sum_{\beta \in \mathbb{Z}^s} a(\alpha - 2\beta) = 1$ *for all* $\alpha \in \mathbb{Z}^s$;

(*b*) $\rho_p(\{A_\varepsilon|_V : \varepsilon \in \Omega\}) < 2^{s/p}$ *where the linear space V is defined in* (1.3.13).

Proof: For $j = 1, \ldots, s$, $\nabla_j\delta \in V$. Conversely, V is spanned by vectors of the form $\tau^\beta \nabla_j \delta$, where $\beta \in \mathbb{Z}^s$, $j = 1, \ldots, s$. Let $\mathcal{A} := \{A_\varepsilon|_V : \varepsilon \in \Omega\}$. By Lemma 1.2 we have

$$\|\mathcal{A}^n \tau^\beta \nabla_j \delta\|_p = \|\tau^\beta \nabla_j S_a^n \delta\|_p = \|\nabla_j S_a^n \delta\|_p.$$

This shows that

$$\rho_p(\{A_\varepsilon|_V : \varepsilon \in \Omega\}) = \max_{1 \le j \le s} \{ \lim_{n \to \infty} \|\nabla_j S_a^n \delta\|_p^{1/n} \}.$$

Thus, Theorem 1.8 follows from Theorem 1.7 at once. ∎

After a closer examination of the proof of Theorems 1.6 and 1.7, we obtain the following result.

Theorem 1.9 *Let a be a finitely supported sequence on $\mathbb{Z}^s$ with $\sum_{\alpha\in\mathbb{Z}^s} a(\alpha) = 2^s$, and let Q_a be the linear operator given by* (1.1.5). *Suppose u is a compactly supported function in $L_p(\mathbb{R}^s)$ $(1 \le p \le \infty)$, u satisfies the moment conditions of order* 1, *and the shifts of u are stable. If there exists a function $f \in L_p(\mathbb{R}^s)$ (a continuous function f in the case $p = \infty$) such that*

$$\lim_{n\to\infty} \|Q_a^n u - f\|_p = 0, \tag{1.3.14}$$

then for any compactly supported function $v \in L_p(\mathbb{R}^s)$ satisfying the moment conditions of order 1 *we also have*

$$\lim_{n\to\infty} \|Q_a^n v - f\|_p = 0. \tag{1.3.15}$$

Consequently, if the normalized solution f of (1.1.1) *lies in $L_p(\mathbb{R}^s)$ (f is a continuous function in the case $p = \infty$), and if the shifts of f are stable, then the subdivision scheme associated with mask a converges to f in the L_p-norm.*

Proof: Suppose that (1.3.14) is true for a function u that satisfies the moment conditions of order 1 and has stable shifts. Then the proof of Theorem 1.6 tells us that (1.3.3) is valid for every vector $y \in \mathbb{Z}^s$. Therefore, from the proof of Theorem 1.7 we see that (1.3.8) holds true. Since both u and v are compactly supported and satisfy the moment conditions of order 1, we have $u - v = \sum_{i=1}^s (g_i - g_i(\cdot - e_i))$ for some compactly supported functions $g_i \in L_p(\mathbb{R}^s)$. Hence,

$$Q_a^n u - Q_a^n v = \sum_{i=1}^{s} \sum_{\alpha\in\mathbb{Z}^s} \nabla_i S_a^n \delta(\alpha) g_i(2^n \cdot - \alpha).$$

Since $g_i \in L_p(\mathbb{R}^s)$ are compactly supported, there exists a positive constant C such that

$$\|Q_a^n u - Q_a^n v\|_p \le C 2^{-ns/p} \max\{\ \|\nabla_i S_a^n \delta\|_p \ :\ i = 1, \cdots, s\}.$$

By (1.3.9) and (1.3.14), we have (1.3.15). In particular, if f itself has stable shifts, then we may choose u to be f in (1.3.14). Thus, in such a case, the subdivision scheme converges in the L_p-norm. ∎

Theorem 1.9 implies that for any function u in $L_p(\mathbb{R}^s)$ such that u satisfies the moment conditions of order 1 and the shifts of u are stable, then the function u can serve as an initial function in any subdivision scheme. It is evident that the particular choice ϕ_0 in (1.1.3) has such properties.

Example 1.10 Consider the following refinement equation

$$f = \sum_{\alpha \in \mathbb{Z}^2} a(\alpha) f(2\cdot - \alpha), \tag{1.3.16}$$

where the mask a is given by its symbol

$$\tilde{a}(z) = 1/4\, z_1^{-1} + 1 + 3/4\, z_1 + 3/4\, z_1^{-1} z_2 + z_2 + 1/4\, z_1 z_2, \qquad z = (z_1, z_2) \in \mathbb{T}^2.$$

We claim that the subdivision scheme associated with a is convergent in the L_p-norm for $1 \le p < \infty$, but it is not L_∞-convergent.

For $\varepsilon \in \Omega := \{(0,0), (1,0), (0,1), (1,1)\}$, let A_ε be the operator on $\ell_0(\mathbb{Z}^2)$ given by

$$A_\varepsilon v(\alpha) = \sum_{\beta \in \mathbb{Z}^2} a(\varepsilon + 2\alpha - \beta) v(\beta), \qquad \alpha \in \mathbb{Z}^2,\ v \in \ell_0(\mathbb{Z}^2).$$

Let K be the set consisting of the points $(-1,0), (0,0), (1,0), (-1,1), (0,1), (1,1)$. Then K is admissible for A_ε for all $\varepsilon \in \Omega$. Let $V := \{v \in \ell(K) : \sum_{\alpha \in \mathbb{Z}^2} v(\alpha) = 0\}$. Then V is the minimal common invariant space of A_ε ($\varepsilon \in \Omega$) generated by $\nabla_j \delta$, $j = 1, 2$. The dimension of V is 5. We choose a basis for V as follows:

$$v_1 = \delta - \delta_{(1,0)},\ v_2 = \delta - \delta_{(-1,0)},\ v_3 = \delta_{(0,1)} - \delta_{(1,1)},$$
$$v_4 = \delta_{(0,1)} - \delta_{(-1,1)}, \quad \text{and} \quad v_5 = r(\delta - \delta_{(0,1)}),$$

where r is a number such that $0 < r < (3/2)^{1/p} - 1$ for $1 \le p < \infty$ and $r = 1$ for $p = \infty$.

By computation, the matrix representations of $A_\varepsilon|_V$ ($\varepsilon \in \Omega$) under this basis are given by

$$A_{(0,0)}|_V = \begin{bmatrix} 3/4 & 0 & 0 & 0 & 0 \\ 0 & 1/4 & 0 & 0 & 0 \\ 0 & 0 & 1/4 & 0 & 0 \\ 0 & 0 & 0 & 3/4 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \qquad A_{(1,0)}|_V = \begin{bmatrix} 0 & -1/4 & 0 & 0 & 0 \\ 0 & 3/4 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3/4 & 0 \\ 0 & 0 & 0 & 1/4 & 0 \\ 0 & -r/4 & 0 & 3r/4 & 1 \end{bmatrix},$$

and

$$A_{(0,1)}|_V = \begin{bmatrix} 1/4 & 0 & 0 & 0 & 0 \\ 0 & 3/4 & 0 & 0 & 0 \\ 3/4 & 0 & 0 & 0 & 0 \\ 0 & 1/4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \qquad A_{(1,1)}|_V = \begin{bmatrix} 0 & -3/4 & 0 & 0 & 0 \\ 0 & 1/4 & 0 & 0 & 0 \\ 0 & -1/4 & 0 & 0 & 0 \\ 0 & 3/4 & 0 & 0 & 0 \\ 0 & -r/2 & 0 & 0 & 0 \end{bmatrix}.$$

Since $A_{(0,0)}|_V$ has an eigenvalue 1, we have

$$\rho_\infty(\{A_\varepsilon|_V : \varepsilon \in \Omega\}) \geq 1.$$

Therefore, the subdivision scheme is not L_∞-convergent.

For the case $1 \leq p < \infty$, we choose the maximum row sum norm as the matrix norm. Since $0 < r < (3/2)^{1/p} - 1$, we have

$$\sum_{\varepsilon \in \Omega} \|A_\varepsilon|_V\|^p \leq 1 + (1+r)^p + (3/4)^p + (3/4)^p < 4.$$

This shows that

$$\rho_p(\{A_\varepsilon|_V : \varepsilon \in \Omega\}) < 4^{1/p}.$$

By Theorem 1.8, the subdivision scheme is L_p-convergent for $1 \leq p < \infty$.

1.4 L_2 Convergence of Subdivision Schemes

In general, the ℓ_p-norm joint spectral radius is difficult to compute. However, in this section we demonstrate that the ℓ_2-norm joint spectral radius can be easily computed by calculating the spectral radius of a certain finite matrix.

Given $a \in \ell_0(\mathbb{Z}^s)$, the symbol $\tilde{a}(z)$ is well defined on the s-torus

$$\mathbb{T}^s := \{(z_1, \dots, z_s) \in \mathbb{C}^s : |z_1| = \cdots = |z_s| = 1\}.$$

For $a, b \in \ell_0(\mathbb{Z}^s)$, the discrete convolution of a and b, denoted $a*b$, is given by

$$a*b(\alpha) := \sum_{\beta \in \mathbb{Z}^s} a(\alpha - \beta) b(\beta), \qquad \alpha \in \mathbb{Z}^s.$$

It is easily seen that

$$\widetilde{a*b}(z) = \tilde{a}(z)\tilde{b}(z), \qquad z \in \mathbb{T}^s.$$

For $z \in \mathbb{C}$, we use $\overline{z}$ to denote the complex conjugate of z. Note that for $z \in \mathbb{T}^s$ and $\alpha \in \mathbb{Z}^s$, we have $\overline{z^\alpha} = z^{-\alpha}$. For $a \in \ell_0(\mathbb{Z}^s)$, we denote by a^* the sequence given by $a^*(\alpha) := \overline{a(-\alpha)}$, $\alpha \in \mathbb{Z}^s$. Then for $z \in \mathbb{T}^s$ we have

$$\widetilde{a^*}(z) = \sum_{\alpha \in \mathbb{Z}^s} \overline{a(-\alpha)} z^\alpha = \sum_{\alpha \in \mathbb{Z}^s} \overline{a(-\alpha)\, z^{-\alpha}} = \overline{\tilde{a}(z)}.$$

If $b = a*a^*$, then we have

$$\tilde{b}(z) = \tilde{a}(z)\widetilde{a^*}(z) = |\tilde{a}(z)|^2, \qquad \text{for } z \in \mathbb{T}^s.$$

Theorem 1.11 *For $a \in \ell_0(\mathbb{Z}^s)$, let $b := a*a^*$ and denote by S_a and S_b the subdivision operators associated with a and b, respectively. Then for any $\nu \in \ell_0(\mathbb{Z}^s)$,*

$$\lim_{n\to\infty} \|\tilde{\nu}(\tau) S_a^n \delta\|_2^{1/n} = \sqrt{\rho(T_b|_W)}$$

and

$$\lim_{n\to\infty} \|\tilde{\mu}(\tau) S_b^n \delta\|_\infty^{1/n} = \rho(T_b|_W)$$

where $\mu := \nu*\nu^*$, T_b is the transition operator associated with b on $\ell_0(\mathbb{Z}^s)$ given by

$$T_b\lambda(\alpha) = \sum\nolimits_{\beta \in \mathbb{Z}^s} b(2\alpha - \beta)\lambda(\beta), \qquad \alpha \in \mathbb{Z}^s,\ \lambda \in \ell_0(\mathbb{Z}^s), \tag{1.4.1}$$

and W is the minimal T_b-invariant subspace generated by μ.

Proof: For $n = 1, 2, \ldots$, write a_n for $S_a^n \delta$ and b_n for $S_b^n \delta$. Note that the symbol of $\tilde{\nu}(\tau)a_n$ is $\tilde{\nu}(z)\tilde{a}_n(z)$, and the symbol of $\tilde{\mu}(\tau)b_n$ is $\tilde{\mu}(z)\tilde{b}_n(z)$. By the Parseval identity we have

$$\begin{aligned}\|\tilde{\nu}(\tau)a_n\|_2^2 &= \sum_{\alpha\in\mathbf{Z}^s} \left|\tilde{\nu}(\tau)a_n(\alpha)\right|^2 \\ &= \frac{1}{(2\pi)^s}\int_{[0,2\pi)^s} \left|\tilde{\nu}(e^{i\xi})\tilde{a}_n(e^{i\xi})\right|^2 d\xi = \frac{1}{(2\pi)^s}\int_{[0,2\pi)^s} \tilde{\mu}(e^{i\xi})\tilde{b}_n(e^{i\xi})\, d\xi.\end{aligned}$$

Since $\tilde{\mu}(e^{i\xi})\tilde{b}_n(e^{i\xi}) \geq 0$ for all $\xi \in \mathbb{R}^s$, it follows that

$$\begin{aligned}\tilde{\mu}(\tau)b_n(0) \leq \|\tilde{\mu}(\tau)b_n\|_\infty &\leq \frac{1}{(2\pi)^s}\int_{[0,2\pi)^s} \left|\tilde{\mu}(e^{i\xi})\tilde{b}_n(e^{i\xi})\right| d\xi \\ &= \frac{1}{(2\pi)^s}\int_{[0,2\pi)^s} \tilde{\mu}(e^{i\xi})\tilde{b}_n(e^{i\xi})\, d\xi = \tilde{\mu}(\tau)b_n(0).\end{aligned}$$

On the other hand, by induction, we observe that

$$\begin{aligned}\tilde{\mu}(\tau)b_n(0) &= \sum_{\beta\in\mathbf{Z}^s} \mu(\beta)S_b^n\delta(-\beta) = \sum_{\beta\in\mathbf{Z}^s}\sum_{\gamma\in\mathbf{Z}^s} \mu(\beta)b(-\beta+2\gamma)S_b^{n-1}\delta(-\gamma) \\ &= \sum_{\gamma\in\mathbf{Z}^s} T_b\mu(\gamma)S_b^{n-1}\delta(-\gamma) = T_b^n\mu(0).\end{aligned}$$

Hence,

$$\|\tilde{\nu}(\tau)S_a^n\delta\|_2^2 = \|\tilde{\mu}(\tau)S_b^n\delta\|_\infty = T_b^n\mu(0).$$

It follows that

$$\lim_{n\to\infty} \|\tilde{\mu}(\tau)S_b^n\delta\|_\infty^{1/n} \leq \lim_{n\to\infty} |T_b^n\mu(0)|^{1/n} \leq \rho(T_b|_W).$$

Moreover, since W is the minimal T_b-invariant subspace generated by μ, we have

$$\rho(T_b|_W) \leq \lim_{n\to\infty} \|\tilde{\mu}(\tau)S_b^n\delta\|_\infty^{1/n}.$$

This completes the proof. ■

We remark that Goodman, Micchelli, and Ward in [44] established a result similar to Theorem 1.11 for the special case $\nu = \delta$.

The following theorem discusses the relationship among the spectra of T_a when it is restricted to different invariant subspaces.

Theorem 1.12 *For an element $a \in \ell_0(\mathbb{Z}^s)$, let T_a be the linear operator on $\ell_0(\mathbb{Z}^s)$ given by*

$$T_a\lambda(\alpha) = \sum_{\beta\in\mathbb{Z}^s} a(2\alpha-\beta)\lambda(\beta), \qquad \alpha\in\mathbb{Z}^s, \lambda\in\ell_0(\mathbb{Z}^s).$$

Suppose that E is the support of a. Then the set K_0 given by

$$K_0 := \left(\sum\nolimits_{n=1}^{\infty} 2^{-n}E\right)\cap\mathbb{Z}^s \tag{1.4.2}$$

is admissible for T_a. Moreover, if W is a finite dimensional T_a-invariant subspace, then the eigenvalues of $T_a|_{W\cap\ell(K_0)}$ are also eigenvalues of $T_a|_W$, and all the other eigenvalues of $T_a|_W$ are 0.

Proof: Let K_0 be the set given in (1.4.2). Then

$$2^{-1}(E+K_0)\subseteq \sum\nolimits_{n=1}^{\infty} 2^{-n}E.$$

This shows that K_0 satisfies (1.2.7). Hence K_0 is an admissible set for T_a, by Lemma 1.3. Since $\ell(K_0)$ is an invariant subspace of T_a, the eigenvalues of $T_a|_{W\cap\ell(K_0)}$ are also eigenvalues of $T_a|_W$.

Let K be an admissible set for T_a such that $\ell(K)\supseteq W$. In order to prove that all the other eigenvalues of $T_a|_W$ are 0, it suffices to show that there exists a positive integer N such that

$$T_a^N\lambda\in\ell(K_0) \qquad \forall\,\lambda\in\ell(K). \tag{1.4.3}$$

Indeed, if (1.4.3) is true, and if σ is an eigenvalue of $T_a|_W$ with an eigenvector λ in $W\setminus\ell(K_0)$, then by (1.4.3) we have $\sigma^N\lambda = T_a^N\lambda\in\ell(K_0)$. But $\lambda\notin\ell(K_0)$. Hence this happens only if $\sigma=0$. Thus, it remains to prove (1.4.3). For this purpose, it suffices to prove that, for each $\beta\in K\setminus K_0$, there exists a positive integer N such that $T_a^N\delta_\beta$ lies in $\ell(K_0)$.

Let j be a positive integer. For $\lambda\in\ell(K)$, we have

$$T_a^j\lambda(\alpha) = \sum_{\gamma\in\mathbb{Z}^s} b(2\alpha-\gamma)T_a^{j-1}\lambda(\gamma).$$

Hence $T_a^j\lambda(\alpha) \neq 0$ only if $2\alpha - \gamma \in E$ for some $\gamma \in \mathbb{Z}^s$ with $T_a^{j-1}\lambda(\gamma) \neq 0$. Let n be a positive integer, and let $\alpha, \beta \in \mathbb{Z}^s$. Then $T_a^n\delta_\beta(\alpha) \neq 0$ holds true only if there exist $\alpha_0, \alpha_1, \ldots, \alpha_n \in \mathbb{Z}^s$ such that $\alpha_0 = \beta$, $\alpha_n = \alpha$, and

$$2\alpha_j - \alpha_{j-1} \in E, \qquad \text{for } j = 1, \ldots, n.$$

Hence $T_a^n\delta_\beta(\alpha) \neq 0$ implies

$$\alpha \in 2^{-1}E + 2^{-2}E + \cdots + 2^{-n}E + 2^{-n}K =: \Gamma_n.$$

Let $\Gamma := \sum_{n=1}^{\infty} 2^{-n}E$. Then $K_0 = \mathbb{Z}^s \cap \Gamma$, and $(\mathbb{Z}^s \setminus K_0) \cap \Gamma = \emptyset$. We shall show that Γ is a compact set. Let H be an infinite subset of Γ. Note that E is a finite set. By induction on n we can find a sequence of elements $\omega_n \in E$ $(n = 1, 2, \ldots)$ such that

$$(2^{-1}\omega_1 + \cdots + 2^{-n}\omega_n + 2^{-n-1}E) \cap H$$

is an infinite set. Then the element $\gamma := \sum_{n=1}^{\infty} 2^{-n}\omega_n$ is a limit point of H.

Since $\mathbb{Z}^s \setminus K_0$ is closed and Γ is compact, hence $\eta := \operatorname{dist}(\mathbb{Z}^s \setminus K_0, \Gamma)$, the distance between two sets $\mathbb{Z}^s \setminus K_0$ and Γ, is positive. Note that $2^{-n} \to 0$ as $n \to \infty$. Thus, there exists a positive integer N such that

$$T_a^N\delta_\beta(\alpha) \neq 0 \quad \Longrightarrow \quad \operatorname{dist}(\alpha, \Gamma) < \eta.$$

From $\operatorname{dist}(\alpha, \Gamma) < \eta$ and $\alpha \in \mathbb{Z}^s$ we deduce that $\alpha \in K_0$. This shows that $T_a^N\delta_\beta$ lies in $\ell(K_0)$, as desired. ∎

The L_2-convergence of a subdivision scheme can be determined by using Theorems 1.7 and 1.11. The following theorem gives another form of characterization for the L_2-convergence.

Theorem 1.13 *For $a \in \ell_0(\mathbb{Z}^s)$, let $b := a * a^*$ and let T_b be the transition operator associated with b on $\ell_0(\mathbb{Z}^s)$ given by*

$$T_b\lambda(\alpha) = \sum_{\beta \in \mathbb{Z}^s} b(2\alpha - \beta)\lambda(\beta), \qquad \alpha \in \mathbb{Z}^s,\ \lambda \in \ell_0(\mathbb{Z}^s).$$

Denote by K_0 *the set* $\mathbb{Z}^s \cap \sum_{n=1}^{\infty} 2^{-n}E$*, where* E *is the support of* b*. Let* V *be the linear space*

$$\left\{\lambda \in \ell(K_0) \ : \ \sum_{\alpha\in\mathbb{Z}^s} \lambda(\alpha) = 0\right\}.$$

Then the subdivision scheme associated with a *converges in the* L_2*-norm if and only if the following two conditions are satisfied:*

(a) $\sum_{\beta\in\mathbb{Z}^s} a(\alpha - 2\beta) = 1$ *for all* $\alpha \in \mathbb{Z}^s$*;*

(b) $\rho(T_b|_V) < 2^s$.

Proof: First, assuming that conditions (a) and (b) are satisfied, we shall prove that the subdivision scheme associated with a converges in the L_2-norm. Let W be the minimal T_b-invariant subspace generated by $-\delta_{-e_j} + 2\delta - \delta_{e_j}$, $j = 1, \ldots, s$. By Theorem 1.12,

$$\rho(T_b|_W) = \rho(T_b|_{W\cap\ell(K_0)}).$$

It follows from condition (a) that $\sum_{\beta\in\mathbb{Z}^s} b(\alpha - 2\beta) = 2^s$ for all $\alpha \in \mathbb{Z}^s$. Consequently, if w is an element in $\ell_0(\mathbb{Z}^s)$ such that $\sum_{\alpha\in\mathbb{Z}^s} w(\alpha) = 0$, then $\sum_{\alpha\in\mathbb{Z}^s} T_b w(\alpha) = 0$. This shows $W \cap \ell(K_0) \subseteq V$. Hence $\rho(T_b|_W) \le \rho(T_b|_V)$. By Theorem 1.11 we have that, for $j = 1, \ldots, s$,

$$\lim_{n\to\infty} \|\nabla_j S_a^n \delta\|_2^{1/n} \le \sqrt{\rho(T_b|_W)} \le \sqrt{\rho(T_b|_V)} < 2^{s/2}.$$

By Theorem 1.7 we conclude that the subdivision scheme associated with a converges in the L_2-norm.

Next, suppose the subdivision scheme associated with a converges in the L_2-norm. By Theorem 1.6, condition (a) is satisfied. It remains to prove $\rho(T_b|_V) < 2^s$.

Let ϕ_0 be the function given in (1.1.3), and let $f_n := Q_a^n \phi_0$, where Q_a is given in (1.1.5). Then there exists a function $f \in L_2(\mathbb{R}^s)$ such that $\|f_n - f\|_2 \to 0$ as $n \to \infty$. For a function f defined on $\mathbb{R}^s$, let f^* be the function given by $f^*(x) = \overline{f(-x)}$ for $x \in \mathbb{R}^s$. Let $\phi := \phi_0 * \phi_0^*$ be the convolution of ϕ_0 and ϕ_0^*. Similarly, let $g_n := f_n * f_n^*$

and $g = f*f^*$. It is easily seen that $g_n = Q^n_{2^{-s}b}\phi$, where $Q_{2^{-s}b}$ is the operator given by $Q_{2^{-s}b}\phi = \sum_{\alpha\in\mathbb{Z}^s} 2^{-s}b(\alpha)\phi(2\cdot-\alpha)$. Then we have

$$\begin{aligned}\|g_n - g\|_\infty &= \|f_n*f_n^* - f*f^*\|_\infty \\ &\le \|f_n*(f_n^* - f^*)\|_\infty + \|(f_n - f)*f^*\|_\infty \\ &\le (\|f_n\|_2 + \|f\|_2)\|f_n - f\|_2.\end{aligned}$$

Note that ϕ is a continuous function, ϕ satisfies the moment conditions of order 1, and the shifts of ϕ are stable. Thus, by Theorem 1.9, the subdivision scheme associated with $2^{-s}b$ converges in the L_∞-norm. By Theorem 1.8, we conclude that $\rho(T_{2^{-s}b}|_V) < 1$. Therefore, $\rho(T_b|_V) < 2^s$ which completes the proof of the theorem. ∎

In the case $s = 1$ and $M = (2)$, Theorem 1.13 was established by Jia [59]. In the multivariate case, Theorem 1.13 was also obtained independently by W. Lawton, S. L. Lee, and Z. W. Shen [80]. More recently, it was demonstrated by Zhou in [104] that for any p such that p is an even integer, the ℓ_p-norm joint spectral radius can be obtained by calculating the spectral radius of a finite matrix though when $p > 2$ the size of such matrix is too large to apply the method of [104] in practice.

Example 1.14 Consider the refinement equation (1.3.16) with the mask a given by its symbol

$$\tilde{a}(z) = 1+(1/2+t)(z_1+z_2+z_1z_2)+(1/2-t)(z_1^{-1}+z_2^{-1}+z_1^{-1}z_2^{-1}), \qquad z = (z_1, z_2) \in \mathbb{T}^2$$

with t being a real number. The normalized solution of the refinement equation is the standard linear element if $t = 0$, and is the characteristic function of the unit square $[0,1)^2$ if $t = 1/2$. Let $b := a*a^*$. By computation we find that

$$\begin{aligned}\tilde{b}(z) = |\tilde{a}(z)|^2 =&(5/2 + 6t^2) + (3/2 + 2t^2)(z_1 + z_2 + z_1^{-1} + z_2^{-1}) \\ &+ (3/2 - 2t^2)(z_1z_2 + z_1^{-1}z_2^{-1}) + (1/2 + 2t^2)(z_1z_2^{-1} + z_1^{-1}z_2) \\ &+ (1/4 - t^2)(z_1^2 + z_1^2z_2^2 + z_2^2 + z_1^{-2} + z_1^{-2}z_2^{-2} + z_2^{-2}) \\ &+ (1/2 - 2t^2)(z_1^2z_2 + z_1z_2^2 + z_1^{-2}z_2^{-1} + z_1^{-1}z_2^{-2}).\end{aligned}$$

The transition operator T_b associated with b is given by

$$T_b\lambda(\alpha) = \sum_{\beta\in\mathbb{Z}^2} b(2\alpha-\beta)\lambda(\beta), \qquad \lambda \in \ell_0(\mathbb{Z}^2),\ \alpha \in \mathbb{Z}^2.$$

For $j = 1, 2$, let $\nu_j = \delta - \delta_{e_j}$ and $\mu_j = \nu_j * \nu_j^* = -\delta_{e_j} + 2\delta - \delta_{-e_j}$. Then

$$T_b\mu_j = T_b(-\delta_{e_j} + 2\delta - \delta_{-e_j}) = (1+4t^2)(-\delta_{e_j} + 2\delta - \delta_{-e_j}) = (1+4t^2)\mu_j.$$

Thus, the minimal T_b-invariant subspace W_j generated by μ_j is the one-dimensional subspace spanned by μ_j. By Theorem 1.11 we conclude that

$$\lim_{n\to\infty} \|\nabla_j S_a^n \delta\|_2^{1/n} = \sqrt{\rho(T_b|_{W_j})} = \sqrt{1+4t^2}.$$

By Theorem 1.7, the subdivision scheme associated with a converges in the L_2-norm if and only if $\sqrt{1+4t^2} < \sqrt{4}$, that is, $|t| < \sqrt{3}/2$.

In order to apply Theorem 1.13 to the present example, we first find the support E of b as follows:

$$E = \{(\alpha_1, \alpha_2) \in \mathbb{Z}^2 : |\alpha_1| \le 2,\ |\alpha_2| \le 2,\ |\alpha_1 - \alpha_2| \le 2\}.$$

The set E consists of 19 points. Evidently, $K_0 := 2^{-1}E + 2^{-2}E + \cdots$ equals E. Let V be the linear space

$$\Big\{\lambda \in \ell(K_0) : \sum\nolimits_{\alpha\in\mathbb{Z}^2} \lambda(\alpha) = 0\Big\}.$$

We shall find all the eigenvalues of $B|_V$.

Let $K := \mathbb{Z}^2 \cap (K_0/2)$. We observe that $\ell(K) \cap V$ is T_b-invariant, and the vectors $\delta_{e_1} - \delta$, $\delta_{e_2} - \delta$, $\delta_{e_1+e_2} - \delta$, $\delta_{-e_1} - \delta$, $\delta_{-e_2} - \delta$, and $\delta_{-e_1-e_2} - \delta$ form a basis for $\ell(K) \cap V$. Let $W = W_1 + W_2$ where the linear spaces W_j are defined in the first part of this example. Then W is a B-invariant subspace of $\ell(K) \cap V$. Moreover,

$$B(-\delta_{-e_1-e_2} + 2\delta - \delta_{e_1+e_2}) = -\delta_{-e_1-e_2} + 2\delta - \delta_{e_1+e_2} + w,$$

where w is a vector in W. Hence 1 is an eigenvalue of $B|_{\ell(K)\cap V}$. Let U be the linear space spanned by W and $-\delta_{-e_1-e_2} + 2\delta - \delta_{e_1+e_2}$. Then U is a 3-dimensional B-invariant subspace of $B|_{\ell(K)\cap V}$. Furthermore, with $v_1 := \delta - \delta_{e_1}$, $v_2 = \delta - \delta_{e_2}$, and $v_3 = \delta - \delta_{e_1+e_2}$, we have

$$T_b \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 3/2 + 2t^2 & -1/2 + 2t^2 & 1/2 - 2t^2 \\ -1/2 + 2t^2 & 3/2 + 2t^2 & 1/2 - 2t^2 \\ 1/2 + 2t^2 & 1/2 + 2t^2 & 3/2 - 2t^2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} + \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix},$$

where u_1, u_2, u_3 are some vectors in U. By computation we find that the above matrix has an eigenvalue 2 of multiplicity 2 and an eigenvalue $1/2 + 2t^2$ of multiplicity 1. The remaining eigenvalues of $B|_V$ can be easily found. To summarize, the linear operator $T_b|_V$ has an eigenvalue 2 of multiplicity 2, an eigenvalue 1 of multiplicity 1, an eigenvalue $1 + 4t^2$ of multiplicity 2, an eigenvalue $1/2 + 2t^2$ of multiplicity 3, an eigenvalue $1/4 - t^2$ of multiplicity 6, and an eigenvalue $1/2 - 2t^2$ of multiplicity 4. Thus,

$$\rho(T_b|_V) = \max\{1 + 4t^2, 2\}.$$

By Theorem 1.13, the subdivision scheme associated with a converges in the L_2-norm if and only if $\rho(T_b|_V) < 4$, that is, $|t| < \sqrt{3}/2$.

This example demonstrates that, in order to check the L_2-convergence, it is often more involved to use Theorem 1.13 than to use Theorem 1.11.

1.5 L_p Convergence in the General Case

In this section, we state shortly the results on L_p convergence of subdivision schemes associated with general dilation matrices. The reader is referred to [51] for proofs and detail.

A general refinable function satisfies the following general refinement equation:

$$f = \sum_{\alpha \in \mathbb{Z}^s} a(\alpha) f(M \cdot - \alpha). \tag{1.5.1}$$

where f is the unknown function defined on the s-dimensional Euclidean space $\mathbb{R}^s$, a is a finitely supported sequence on $\mathbb{Z}^s$, and M is an $s \times s$ integer matrix such that $\lim_{n\to\infty} M^{-n} = 0$. Such matrix M is called a **dilation matrix.**

If the mask a satisfies

$$\sum_{\alpha \in \mathbb{Z}^s} a(\alpha) = m := |\det M|,$$

then it is known that there exists a unique compactly supported distribution f satisfying the refinement equation (1.5.1) subject to the condition $\hat{f}(0) = 1$. This distribution is said to be **the normalized solution** to the refinement equation with mask a. This fact was essentially proved by Cavaretta, Dahmen, and Micchelli in [6] for the case in which the dilation matrix is 2 times the $s \times s$ identity matrix I. The same proof applies to the general refinement equation (1.5.1).

Similarly, to solve the refinement equation (1.5.1), we start with the initial function ϕ_0 defined in (1.1.3), and use the iteration scheme $f_n := Q_a^n \phi_0$, $n = 0, 1, 2, \ldots$, where Q_a is the bounded linear operator on $L_p(\mathbb{R}^s)$ given by

$$Q_a \phi := \sum_{\alpha \in \mathbb{Z}^s} a(\alpha) \phi(M \cdot - \alpha), \qquad \phi \in L_p(\mathbb{R}^s).$$

We say that the subdivision scheme associated with mask a **converges in the L_p-norm**, if there is a function $f \in L_p(\mathbb{R}^s)$ such that

$$\lim_{n\to\infty} \|Q_a^n \phi_0 - f\|_p = 0.$$

To describe our results, we introduce the linear operators A_ε ($\varepsilon \in \mathbb{Z}^s$) on $\ell_0(\mathbb{Z}^s)$ as follows:

$$A_\varepsilon v(\alpha) = \sum_{\beta \in \mathbb{Z}^s} a(\varepsilon + M\alpha - \beta) v(\beta), \qquad v \in \ell_0(\mathbb{Z}^s),\ \alpha \in \mathbb{Z}^s.$$

We observe that the set $\mathbb{Z}^s$ is an abelian group under addition, and $M\mathbb{Z}^s$ is a subgroup of $\mathbb{Z}^s$. Let Ω be a complete set of representatives of the distinct cosets of the quotient group $\mathbb{Z}^s/M\mathbb{Z}^s$. For a finite subset K of $\mathbb{Z}^s$, we denote by $\ell(K)$ the linear subspace of $\ell_0(\mathbb{Z}^s)$ consisting of all sequences supported on K. Then there exists a finite subset K of $\mathbb{Z}^s$ such that $\ell(K)$ contains $\nabla_j\delta$ for $j = 1, \dots, s$ and is invariant under every A_ε ($\varepsilon \in \Omega$). Let

$$V := \left\{\lambda \in \ell(K) : \sum\nolimits_{\alpha\in\mathbb{Z}^s} \lambda(\alpha) = 0\right\}.$$

It is easily seen that V is a common invariant subspace of A_ε ($\varepsilon \in \Omega$) if and only if $\sum_{\beta\in\mathbb{Z}^s} a(\alpha - M\beta) = 1$ for all $\alpha \in \mathbb{Z}^s$.

Theorem 1.15 (see [51], Theorems 3.2 and 3.3) *Let M be a dilation matrix with $m := |\det M|$, a an element in $\ell_0(\mathbb{Z}^s)$ such that $\sum_{\alpha\in\mathbb{Z}^s} a(\alpha) = m$. The subdivision scheme associated with mask a and the dilation matrix M converges in the L_p-norm ($1 \le p \le \infty$) if and only if the following two conditions are satisfied:*

(a) $\sum_{\beta\in\mathbb{Z}^s} a(\alpha - M\beta) = 1$ *for all* $\alpha \in \mathbb{Z}^s$;

(b) $\rho_p(\{A_\varepsilon|_V : \varepsilon \in \Omega\}) < m^{1/p}$, *where* $\rho_p(\{A_\varepsilon|_V : \varepsilon \in \Omega\})$ *denotes the ℓ_p-norm joint spectral radius of the linear operators $A_\varepsilon|_V$, $\varepsilon \in \Omega$.*

Or equivalently, the subdivision scheme associated with the mask a and the dilation matrix M converges in the L_p-norm if and only if

$$\lim_{n\to\infty} \|\nabla_j S_a^n \delta\|_p^{1/n} < m^{1/p} \qquad \text{for } j = 1, \dots, s,$$

where S_a is the general subdivision operator associated with the mask a given by

$$S_a\lambda(\alpha) := \sum_{\beta\in\mathbb{Z}^s} a(\alpha - M\beta)\lambda(\beta), \qquad \alpha \in \mathbb{Z}^s, \lambda \in \ell_0(\mathbb{Z}^s).$$

In particular, when $p = 2$, we have the following result:

Theorem 1.16 (see [51], Theorem 4.3) *Let M be an $s \times s$ dilation matrix and let $m := |\det M|$. For $a \in \ell_0(\mathbb{Z}^s)$, let $b := a*a^*$ and let T_b be the transition operator on $\ell_0(\mathbb{Z}^s)$ given by*

$$T_b\lambda(\alpha) = \sum_{\beta\in\mathbb{Z}^s} b(M\alpha - \beta)\lambda(\beta), \qquad \alpha \in \mathbb{Z}^s,\ \lambda \in \ell_0(\mathbb{Z}^s).$$

Denote by K_0 the set $\mathbb{Z}^s \cap \sum_{n=1}^{\infty} M^{-n}E$, where E is the support of b. Let V be the linear space

$$\Big\{\lambda \in \ell(K_0)\ :\ \sum_{\alpha\in\mathbb{Z}^s} \lambda(\alpha) = 0\Big\}.$$

Then the subdivision scheme associated with a converges in the L_2-norm if and only if the following two conditions are satisfied:

(a) $\sum_{\beta\in\mathbb{Z}^s} a(\alpha - M\beta) = 1$ *for all* $\alpha \in \mathbb{Z}^s$;

(b) $\rho(T_b|_V) < m$.

Chapter 2

Error Estimate of Subdivision Schemes

2.1 Introduction

In general, for any compactly supported function in $L_p(\mathbb{R}^s)$ (when $p = \infty$ it is a continuous function with compact support), if it satisfies the moment conditions of order 1 and its shifts are stable, then it can serve as an initial function in any subdivision scheme. For simplicity, throughout this chapter, we shall use ϕ_0, defined in (1.1.3), as our initial function in any subdivision scheme. The proofs in this chapter are the same for any general initial function.

Subdivision schemes play an important role in computer graphics and wavelet analysis. See [38, 39, 40, 85] for their applications to computer aided geometric design, and see [15, 27, 28, 102] for their applications to wavelet decompositions.

Before proceeding further, we recall some notation. By $\ell(\mathbb{Z}^s)$ we denote the linear space of all sequences on $\mathbb{Z}^s$, and by $\ell_0(\mathbb{Z}^s)$ the linear space of all finitely supported sequences on $\mathbb{Z}^s$. By δ we denote the element in $\ell_0(\mathbb{Z}^s)$ given by $\delta(0) = 1$ and $\delta(\beta) = 0$

for all $\beta \in \mathbb{Z}^s\backslash\{0\}$. For $j = 1, \cdots, s$, let e_j be the jth coordinate unit vector. The difference operator ∇_j is defined by $\nabla_j\lambda := \lambda - \lambda(\cdot - e_j)$, $\lambda \in \ell(\mathbb{Z}^s)$.

The subdivision operator is important in the study of convergence of a subdivision scheme. Let a be a sequence on $\mathbb{Z}^s$. Recall that the **subdivision operator** associated with a is defined by

$$S_a\lambda(\alpha) := \sum_{\beta\in\mathbb{Z}^s} a(\alpha - 2\beta)\lambda(\beta), \qquad \alpha \in \mathbb{Z}^s, \tag{2.1.1}$$

where $\lambda \in \ell_0(\mathbb{Z}^s)$. Note that $\ell_0(\mathbb{Z}^s)$ is a subspace of $\ell_p(\mathbb{Z}^s)$. The ℓ_p norm of an element $\lambda \in \ell_p(\mathbb{Z}^s)$ is denoted by $\|\lambda\|_p$. It was proved in Chapter 1 that for any finitely supported refinement mask a, the subdivision scheme associated with a converges in the L_p norm if and only if

$$\lim_{n\to\infty} \max\{\ \|\nabla_j S_a^n \delta\|_p^{1/n}\ :\ j = 1, \cdots, s\ \} < 2^{s/p}.$$

Moreover, it was also proved in Chapter 1 that if the subdivision scheme associated with a converges in the L_p norm to the normalized solution ϕ_a of (1.1.1) with the refinement mask a, then for any

$$2^{-s/p} \lim_{n\to\infty} \max\{\ \|\nabla_j S_a^n \delta\|_p^{1/n}\ :\ j = 1, \cdots, s\ \} < r < 1,$$

there exists a positive constant C such that $\|Q_a^n\phi_0 - \phi_a\|_p \le Cr^n$ for all $n \in \mathbb{N}$ where the operator Q_a is defined in (1.1.5).

Let Ω be the set of vertices of the unit cube $[0,1]^s$. If the subdivision scheme associated with a converges in the L_p norm, then by Theorem 1.6, it is necessary that a should satisfy the **sum rules of order** 1, i.e.,

$$\sum_{\alpha\in\mathbb{Z}^s} a(2\alpha + \varepsilon) = 1 \qquad \forall \varepsilon \in \Omega. \tag{2.1.2}$$

Hence, if we truncate a refinement mask, then it is necessary for the truncated mask to satisfy the sum rules of order 1 so that the subdivision scheme associated with the truncated mask may still converge in the L_p norm.

Refinable functions are encountered in computer graphics where interpolatory subdivision schemes are used to construct smooth curves and surfaces. The reader is referred to [6, 31, 38, 39, 40, 52, 84, 85, 89] for detailed discussion on interpolatory subdivision schemes and their applications to generate curves and surfaces. They are also known as scaling functions in the wavelet theory, for example, see [13, 15, 18, 26, 27, 79, 102]. Here we are concerned with the behaviour of a refinable function when there is a small perturbation of its refinement mask. In applications, under many situations, we need to truncate the refinement masks even though they have finite support. For example, the coefficients in the refinement masks of Daubechies' orthonormal scaling functions given in [26] are often irrational. Although in most cases the coefficients in biorthogonal wavelets are rational, the biorthogonal wavelets given in [1], with one refinement mask of tap 7 and the other of tap 9, have irrational coefficients in their masks. This biorthogonal wavelets are known to have an overall good performance on image compression [1, 76, 94, 95].

Given $N \in \mathbb{N}$, let $\ell([-N,N]^s)$ denote the linear space of sequences on $\mathbb{Z}^s$ which are supported on $[-N,N]^s \cap \mathbb{Z}^s$. For a finitely supported sequence b on $\mathbb{Z}^s$, throughout this chapter, its ℓ_p norm is defined to be $\|b\|_p := \left(\sum_{\alpha\in\mathbb{Z}^s} |b(\alpha)|^p\right)^{1/p}$. The main result in this chapter is as follows:

Theorem 2.1 *Let a be a refinement mask supported on $[-N,N]^s \cap \mathbb{Z}^s$ for some $N \in \mathbb{N}$. Suppose the subdivision scheme with mask a converges in the L_p norm. Then there exist $\eta > 0$ and $C > 0$ such that for any sequence $b \in \ell([-N,N]^s)$ satisfying $\sum_{\alpha\in\mathbb{Z}^s} b(2\alpha+\varepsilon) = 1$ for all $\varepsilon \in \Omega$ and $\|a-b\|_1 < \eta$, the subdivision scheme associated with b converges in the L_p norm and*

$$\|Q_a^n \phi_0 - Q_b^n \phi_0\|_p \le C\|a-b\|_1 \qquad \forall n \in \mathbb{N}, \tag{2.1.3}$$

where the initial function ϕ_0 is defined in (1.1.3) *and the linear operator Q_a is defined in* (1.1.5). *Moreover, let ϕ_a and ϕ_b be the normalized solutions of the refinement* *equations* (1.1.1) *with the refinement masks a and b respectively, then*

$$\|\phi_a - \phi_b\|_p \le C\|a-b\|_1. \tag{2.1.4}$$

The above estimate is sharp in the sense that under the conditions as in Theorem 2.1, we shall give examples to demonstrate that for such examples, there exists a positive constant C such that for any $\eta > 0$, there exists $b \in \ell([-N,N]^s)$ satisfying $\sum_{\alpha\in\mathbb{Z}^s} b(2\alpha+\varepsilon) = 1$ for all $\varepsilon \in \Omega$ and $\|a-b\|_1 < \eta$, but $\|Q_a^n\phi_0 - Q_b^n\phi_0\|_p \ge C\|a-b\|_1$ for all $n \in \mathbb{N}$.

There are few papers in the literature discussing about the effect of truncation of a refinement mask. As we know, for $s = 1$, Bonami, Durand and Weiss in [3] considered the connectivity of the set of orthonormal scaling functions in the L_2 norm. Daubechies and Huang in [29] first investigated the problem how the truncation of a refinement mask will affect its subdivision scheme and its normalized solution. In this chapter, we shall extend and improve their result. Daubechies and Huang [29] considered a special case of Theorem 2.1 for $s = 1$ and $p = \infty$. In Theorem 3.2 [29], they demonstrated that if the subdivision schemes associated with a and b are convergent in the L_p norm, then for any $0 < \lambda < 1$, there exists a positive constant C_λ depending on λ, a and b such that

$$\|Q_a^n\phi_0 - Q_b^n\phi_0\|_\infty \le C_\lambda\|a-b\|_1^\lambda.$$

But their method does not work for $\lambda = 1$. It is not clear in [29] whether the subdivision scheme associated with b converges in the L_p norm if the subdivision scheme associated with a converges in the L_p norm and the mask b is close enough to a. Hence, even in this special case, our result here is stronger than theirs. In this chapter, we discuss the case when the refinement mask has finite support. In passing, we mention that the results in this chapter can be easily generalized to vector subdivision schemes.

Our approach here is different from the approach taken by Daubechies and Huang in [29]. We deal with this problem from the time domain by using the subdivision operator, while they did it from the frequency domain by using Fourier transform.

2.2 Auxiliary Results

The proof of Theorem 2.1 is based on our results in Chapter 1 on convergence of subdivision schemes. The reader is referred to Chapter 1 for detailed discussion on convergence of subdivision schemes and the relation between the subdivision operator and the ℓ_p-norm joint spectral radius.

The concept of ℓ_p-norm joint spectral radius which was introduced in Chapter 1 will be used in our study of the effect of a refinable function caused by truncation of a refinement mask. The reader if referred to Section 1.2 for definition and results of ℓ_p-norm joint spectral radius.

Recall that S_a is the subdivision operator given in (2.1.1). To relate the subdivision operator to the ℓ_p-norm joint spectral radius, we introduce the linear operator A_ε ($\varepsilon \in \Omega$) on $\ell_0(\mathbb{Z}^s)$ as follows:

$$A_\varepsilon \lambda(\alpha) := \sum_{\beta \in \mathbb{Z}^s} a(2\alpha - \beta + \varepsilon)\lambda(\beta), \qquad \alpha \in \mathbb{Z}^s,\ \lambda \in \ell_0(\mathbb{Z}^s). \tag{2.2.1}$$

Before proceeding further, we will need some results in Chapter 1. For the reader's convenience, we cite some results here. The reader is referred to Lemma 1.5 and Theorem 1.7 in Chapter 1 for proofs of such results.

Theorem 2.2 *Let a be a sequence supported on $[-N, N]^s \cap \mathbb{Z}^s$ for some $N \in \mathbb{N}$. Then the subdivision scheme associated with a converges in the L_p norm ($1 \le p \le \infty$) if and only if*

$$\lim_{n\to\infty} \max\{\ \|\nabla_j S_a^n \delta\|_p^{1/n}\ :\ 1 \le j \le s\ \} < 2^{s/p}. \tag{2.2.2}$$

Moreover, if a satisfies the sum rules of order 1, *then*

$$\lim_{n\to\infty} \max\{ \|\nabla_j S_a^n\delta\|_p^{1/n} \ : \ 1 \le j \le s \} = \rho_p(\mathcal{A}),$$

and there exists a positive constant C depending only on N and p such that

$$\max\{ \|\nabla_j S_a^n\delta\|_p \ : \ 1 \le j \le s \} \le C\|\mathcal{A}^n\|_p \qquad \forall n \in \mathbb{N},$$

where $\mathcal{A} := \{A_\varepsilon |_V : \varepsilon \in \Omega\}$ and $V := \{\lambda \in \ell([-N,N]^s) \ : \ \sum_{\alpha\in\mathbb{Z}^s}\lambda(\alpha) = 0\}$, and A_ε is given in (2.2.1).

Based on the above result, we can prove that if the subdivision scheme associated with a converges in the L_p norm and a mask b is close enough to the mask a, then the subdivision scheme associated with b also converges in the L_p norm.

Lemma 2.3 *Let a be a sequence supported on $[-N,N]^s \cap \mathbb{Z}^s$ for some $N \in \mathbb{N}$. Suppose that the subdivision scheme associated with the mask a converges in the L_p norm ($1 \le p \le \infty$). Then there exist $\eta > 0, \nu > 0$ and $C > 0$ such that for any $b \in \ell([-N,N]^s)$, if $\sum_{\alpha\in\mathbb{Z}^s} b(2\alpha+\varepsilon) = 1$ for all $\varepsilon \in \Omega$ and $\|a-b\|_1 < \eta$, then*

$$\|\nabla_j S_b^n\delta\|_p \le C2^{(s/p-\nu)n} \qquad \forall n \in \mathbb{N},\ j = 1,\cdots,s,$$

and therefore, the subdivision scheme associated with b converges in the L_p norm.

Proof: Since the subdivision scheme associated with a converges in the L_p norm, by (2.2.2), there exists a positive real number ν such that

$$\lim_{n\to\infty} \max\{ \|\nabla_j S_a^n\delta\|_p^{1/n} \ : \ 1 \le j \le s \} < 2^{s/p-2\nu}.$$

Note that by Theorem 2.2,

$$\lim_{n\to\infty} \max\{ \|\nabla_j S_a^n\delta\|_p^{1/n} \ : \ j = 1,\cdots,s \} = \lim_{n\to\infty} \|\mathcal{A}^n\|_p^{1/n} = \inf_{n>0} \|\mathcal{A}^n\|_p^{1/n},$$

where $\mathcal{A}$ is given in Theorem 2.2. Hence, there exists a positive integer m such that $\|\mathcal{A}^m\|_p^{1/m} < 2^{s/p-2\nu}$. Therefore, there exists a positive number η such that for any $b \in \ell([-N,N]^s)$, if $\sum_{\alpha\in\mathbb{Z}^s} b(2\alpha+\varepsilon) = 1$ for all $\varepsilon \in \Omega$ and $\|a-b\|_1 < \eta$, then

$$\|\mathcal{B}_b^m\|_p^{1/m} \le 2^{s/p-\nu}, \tag{2.2.3}$$

where $\mathcal{B}_b := \{B_{b,\varepsilon}|_V : \varepsilon \in \Omega\}$, and

$$B_{b,\varepsilon}\lambda(\alpha) = \sum_{\beta\in\mathbb{Z}^s} b(2\alpha-\beta+\varepsilon)\lambda(\beta)$$

and V is given in Theorem 2.2. Since $\sum_{\alpha\in\mathbb{Z}^s} b(2\alpha+\varepsilon) = 1$ for all $\varepsilon \in \Omega$, it is easy to verify that V is also invariant under $B_{b,\varepsilon}$, $\varepsilon \in \Omega$. The above argument in proving (2.2.3) lies in that we can fix a basis of V which is independent of b, then we view each $B_{b,\varepsilon}$ as a matrix under this basis.

Since for any $l,n \in \mathbb{N}$, $\|\mathcal{B}_b^{l+n}\|_p \le \|\mathcal{B}_b^l\|_p \cdot \|\mathcal{B}_b^n\|_p$. Thus for any $n = km+r$ and $0 \le r < m$, we have

$$\|\mathcal{B}_b^n\|_p = \|\mathcal{B}_b^{km+r}\|_p \le \|\mathcal{B}_b^m\|_p^k \cdot \|\mathcal{B}_b^r\|_p \le C_1\|\mathcal{B}_b^m\|_p^k,$$

where C_1 is a positive constant depending only on a, η, m and p. Hence, for $n = km+r$ and $0 \le r < m$, it follows from (2.2.3) that

$$\|\mathcal{B}_b^n\|_p \le C_1\|\mathcal{B}_b^m\|_p^k \le C_1 2^{(s/p-\nu)mk} \le C_1 2^{|s/p-\nu|m} 2^{(s/p-\nu)n} \le C_2 2^{(s/p-\nu)n},$$

where $C_2 = C_1 2^{|s/p-\nu|m}$. Therefore, by Theorem 2.2, there exists a positive constant C_3 depending only on N and p such that

$$\max\{\, \|\nabla_j S_b^n\delta\|_p \,:\, j = 1,\cdots,s \,\} \le C_3\|\mathcal{B}_b^n\|_p \le C_2C_3 2^{(s/p-\nu)n} \qquad \forall n \in \mathbb{N}.$$

Thus we complete the proof. ∎

Recall that for any sequence c on $\mathbb{Z}^s$, the **symbol** of c is defined to be

$$\tilde{c}(z) = \sum_{\alpha\in\mathbb{Z}^s} c(\alpha)z^\alpha, \qquad z \in \mathbb{T}^s,$$

where $z = (z_1, \cdots, z_s)$ and $z^\alpha = z_1^{\alpha_1} \cdots z_s^{\alpha_s}$ for $\alpha = (\alpha_1, \cdots, \alpha_s)$. For simplicity, throughout this chapter, we shall use the following notation:

$$\|\widetilde{c}(z)\|_p := \|c\|_p. \tag{2.2.4}$$

To facilitate our discussion, we establish the following inequality which is crucial in our proof of Theorem 2.1.

Lemma 2.4 *Let a be a sequence supported on $[-N, N]^s \cap \mathbb{Z}^s$ for some $N \in \mathbb{N}$. Then for any sequence $b \in \ell_p(\mathbb{Z}^s)$, we have the following inequality*

$$\|\widetilde{S_a^n \delta}(z)\widetilde{b}(z^{2^n})\|_p = \|S_a^n b\|_p \le (2N+1)^{s/q}\|S_a^n \delta\|_p \cdot \|b\|_p \qquad \forall n \in \mathbb{N},$$

where $1/p + 1/q = 1$.

Proof: Note that $\widetilde{S_a^n b}(z) = \widetilde{S_a^n \delta}(z)\widetilde{b}(z^{2^n})$ and $\widetilde{S_a^n \delta}(z) = \prod_{j=0}^{n-1} \widetilde{a}(z^{2^j})$. Since a is supported on $[-N, N]^s$, it is easy to verify that the sequence $S_a^n \delta$ is supported on $[-(2^n-1)N, (2^n-1)N]^s$. Observe that

$$[-(2^n-1)N, (2^n-1)N]^s \subseteq \bigcup_{\gamma \in [-N,N]^s \cap \mathbb{Z}^s} \Big(2^n\gamma + [0, 2^n)^s\Big).$$

Therefore, the sequence $S_a^n \delta$ can be uniquely decomposed as

$$S_a^n \delta = \sum_{\gamma \in [-N,N]^s \cap \mathbb{Z}^s} a_\gamma$$

with each sequence a_γ given by $a_\gamma(\beta) = a(\beta)$ for $\beta \in (2^n\gamma + [0, 2^n)^s) \cap \mathbb{Z}^s$, and $a_\gamma(\beta) = 0$ otherwise. Hence,

$$\begin{aligned} \|\widetilde{S_a^n \delta}(z)\widetilde{b}(z^{2^n})\|_p &\le \sum_{\gamma \in [-N,N]^s \cap \mathbb{Z}^s} \|\widetilde{a}_\gamma(z)\widetilde{b}(z^{2^n})\|_p \\ &= \sum_{\gamma \in [-N,N]^s \cap \mathbb{Z}^s} \|\widetilde{a}_\gamma(z)\|_p \cdot \|\widetilde{b}(z^{2^n})\|_p \\ &= \sum_{\gamma \in [-N,N]^s \cap \mathbb{Z}^s} \|a_\gamma\|_p \cdot \|b\|_p. \end{aligned} \tag{2.2.5}$$

Note that $\|S_a^n \delta\|_p^p = \sum_{\gamma\in[-N,N]^s\cap\mathbb{Z}^s} \|a_\gamma\|_p^p$. By Hölder inequality, we have

$$\begin{aligned}\sum_{\gamma\in\in[-N,N]^s\cap\mathbb{Z}^s} \|a_\gamma\|_p &\le (2N+1)^{s/q}\Bigl(\sum_{\gamma\in[-N,N]^s\cap\mathbb{Z}^s} \|a_\gamma\|_p^p\Bigr)^{1/p} \\ &= (2N+1)^{s/q}\|S_a^n\delta\|_p,\end{aligned} \tag{2.2.6}$$

where $1/p + 1/q = 1$. Hence

$$\|\widetilde{S_a^n\delta}(z)\widetilde{b}(z^{2^n})\|_p \le (2N+1)^{s/q}\|S_a^n\delta\|_p \cdot \|b\|_p.$$

This completes the proof. ∎

A direct consequence of the above result is that for any refinement mask a supported on $[-N,N]^s$ for some positive integer N,

$$\lim_{n\to\infty} \|S_a^n\delta\|_p^{1/n} = \inf_{n>0}\left((2N+1)^{s/q}\|S_a^n\delta\|_p\right)^{1/n},$$

where $1/p+1/q=1$. In particular, for any sequence $a \in \ell_1(\mathbb{Z}^s)$, we have

$$\lim_{n\to\infty} \|S_a^n\delta\|_1^{1/n} = \inf_{n>0} \|S_a^n\delta\|_1^{1/n}.$$

To prove Theorem 2.1, the following two lemmas will be needed.

Lemma 2.5 *Let $c \in \ell([-N,N]^s)$ for some $N \in \mathbb{N}$ and Ω be the set of vertices of the unit cube $[0,1]^s$. Suppose $\sum_{\alpha\in\mathbb{Z}^s} c(2\alpha+\varepsilon) = 0$ for all $\varepsilon \in \Omega$. Then there exist sequences c_j in $\ell([-N,N]^s)$, $j = 1,\cdots,s$ such that*

$$\widetilde{c}(z) = \sum_{j=1}^{s}(1-z_j^2)\widetilde{c}_j(z), \qquad z = (z_1,\cdots,z_s) \in (\mathbb{C}\backslash\{0\})^s, \tag{2.2.7}$$

and

$$\sum_{j=1}^{s} \|c_j\|_1 \le C_N\|c\|_1, \tag{2.2.8}$$

where C_N is a positive constant depending only on N.

Proof: Define a subspace W of $\ell([0,2N]^s)$ as follows:

$$W := \{ b \in \ell([0,2N]^s) \ : \ \sum_{\alpha \in \mathbb{Z}^s} b(2\alpha + \varepsilon) = 0 \quad \forall \varepsilon \in \Omega \}.$$

We first prove that for any $b \in W$, there exist sequences $b_j \in \ell([0,2N]^s)$ for all $j = 1, \cdots, s$ such that

$$\widetilde{b}(z) = \sum_{j=1}^{s} (1 - z_j^2) \widetilde{b}_j(z), \qquad z \in \mathbb{C}^s. \tag{2.2.9}$$

We will use the idea of the division algorithm on multivariate polynomials to prove (2.2.9). For discussion on the division algorithm, the reader is referred to page 63 in [23]. For any monomial z^α such that $\alpha = (\alpha_1, \cdots, \alpha_s) \in [0,2N]^s \cap \mathbb{Z}^s$, if $\alpha_j \geq 2$ for some $1 \leq j \leq s$, then we observe that $z^\alpha = z^{\alpha - 2e_j} - (1 - z_j^2) z^{\alpha - 2e_j}$, where e_j is the jth coordinate unit vector. Note that $\alpha - 2e_j \in [0,2N]^s \cap \mathbb{Z}^s$ and the total degree of $|\alpha| := \alpha_1 + \cdots + \alpha_s > |\alpha - 2e_j|$. Thus by induction and the division algorithm, it is not difficult to see that for any $b \in W$, we can find sequences $b_j \in \ell([0,2N]^s)$, $j = 1, \cdots, s$ and a remainder sequence $r \in \ell([0,1]^s)$ such that

$$\widetilde{b}(z) = \sum_{j=1}^{s} (1 - z_j^2) \widetilde{b}_j(z) + \widetilde{r}(z), \qquad z \in \mathbb{C}^s.$$

Let a denote the sequence in $\ell([0,2N]^s)$ such that $\widetilde{a}(z) = \sum_{j=1}^{s} (1 - z_j^2) \widetilde{b}_j(z)$. Note that $a \in W$ is equivalent to that $a \in \ell([0,2N]^s)$ and

$$\widetilde{a}(z) = 0 \qquad \forall\, z = (z_1, \cdots, z_s) \ \text{ with } \ z_j \in \{-1, 1\} \ \forall 1 \leq j \leq s.$$

Thus it is easy to see that $a \in W$ which implies $r = b - a \in W$. By $r \in \ell([0,1]^s)$ and $\sum_{\alpha \in \mathbb{Z}^s} r(2\alpha + \varepsilon) = 0$ for all $\varepsilon \in \Omega$, it is easily seen that $r(\alpha) = 0$ for all $\alpha \in \mathbb{Z}^s$. Hence (2.2.9) is verified.

Next we define a norm on W as follows. For any $b \in W$, define

$$\|b\|_W = \inf \left\{ \sum_{j=1}^{s} \|b_j\|_1 \ : \ \widetilde{b}(z) = \sum_{j=1}^{s} (1 - z_j^2) \widetilde{b}_j(z) \quad \text{with all} \quad b_j \in \ell([0,2N]^s) \right\}.$$

By the preceding discussion, $\|\cdot\|_W$ is well-defined and it is not difficult to verify that $\|\cdot\|_W$ is a norm on W. Since W is finite dimensional and $\|\cdot\|_1$ is also a norm on W, there exists a constant C_N depending only on N such that

$$\|b\|_W \le \frac{1}{2} C_N \|b\|_1 \qquad \forall\, b \in W.$$

By the definition of $\|\cdot\|_W$, it follows that there exist sequences $b_j \in \ell([0,2N]^s)$ for any $j = 1, \cdots, s$ such that

$$\widetilde{b}(z) = \sum_{j=1}^{s} (1 - z_j^2) \widetilde{b}_j(z) \quad \text{and} \quad \sum_{j=1}^{s} \|b_j\|_1 \le 2\|b\|_W \le C_N \|b\|_1. \tag{2.2.10}$$

To complete the proof of this lemma, we note that $c \in \ell([-N,N]^s)$ and

$$\sum_{\alpha \in \mathbb{Z}^s} c(2\alpha + \varepsilon) = 0 \qquad \forall\, \varepsilon \in \Omega.$$

Let $b(\alpha) = c(\alpha - (N, N, \cdots, N))$, $\alpha \in \mathbb{Z}^s$. Then it is easy to see that $b \in W$. The claim of this lemma follows directly from (2.2.10). ∎

Lemma 2.6 *Let a be a sequence supported on $\mathbb{Z}^s$. Suppose the subdivision scheme associated with a converges in the L_p norm $(1 \le p \le \infty)$. Then there exists a positive constant C such that*

$$\|S_a^n \delta\|_p \le C 2^{sn/p} \qquad \forall\, n \in \mathbb{N}. \tag{2.2.11}$$

Proof: Let $f_n := Q_a^n \phi_0$. By induction, we deduce that

$$f_n = \sum_{k \in \mathbb{Z}} S_a^n \delta(k) \phi_0(2^n \cdot - k).$$

Since the shifts of ϕ_0 are stable, by (1.3.2), from the above equation, there exists a positive constant C_1 such that $\|S_a^n \delta\|_p \le C_1 2^{sn/p} \|f_n\|_p$ for all $n \in \mathbb{N}$. Note that the series $(f_n, n \in \mathbb{N})$ converges in the L_p norm. Thus there exists a constant C_2 such that $\|f_n\|_p \le C_2$ for all $n \in \mathbb{N}$ thereby completing the proof. ∎

It is easily seen that inequality (2.2.11) is equivalent to the fact that the series $(Q_a^n\phi_0, n \in \mathbb{N})$ is bounded in the L_p norm. Therefore, for the case $p = 2$, the inequality (2.2.11) is equivalent to that $(Q_a^n\phi_0, n \in \mathbb{N})$ weakly converges to ϕ_a in the L_2 norm since it is well known that $\widehat{Q_a^n\phi_0}(\xi)$ converges to $\widehat{\phi}_a(\xi)$ as n goes to ∞ for all $\xi \in \mathbb{R}^s$.

2.3 Proof of Theorem 2.1

Proof: Note that by induction

$$Q_a^n\phi_0(x) = \sum_{\alpha\in\mathbb{Z}^s} S_a^n\,\delta(\alpha)\phi_0(2^n x - \alpha).$$

Take η as in Lemma 2.3. Let $b \in \ell([-N,N]^s)$ satisfy $\sum_{\alpha\in\mathbb{Z}^s} b(2\alpha+\varepsilon) = 1$ for all $\varepsilon \in \Omega$ and $\|a-b\|_1 < \eta$. To estimate $Q_a^n\phi_0 - Q_b^n\phi_0$, we use the following equality

$$Q_a^n\phi_0(x) - Q_b^n\phi_0(x) = \sum_{\alpha\in\mathbb{Z}^s} \big(S_a^n\,\delta(\alpha) - S_b^n\,\delta(\alpha)\big)\phi_0(2^n x - \alpha).$$

Since the shifts of ϕ_0 are stable, from (1.3.2), there exist two positive constants C_1 and C_2 depending only on ϕ_0 such that

$$C_1 2^{-ns/p}\|S_a^n\,\delta - S_b^n\,\delta\|_p \le \|Q_a^n\phi_0 - Q_b^n\phi_0\|_p \le C_2 2^{-ns/p}\|S_a^n\,\delta - S_b^n\,\delta\|_p. \qquad (2.3.1)$$

Hence, to estimate $\|Q_a^n\phi_0 - Q_b^n\phi_0\|_p$, it suffices to deal with $\|S_a^n\,\delta - S_b^n\,\delta\|_p$. Write $S_a^n\,\delta - S_b^n\,\delta$ in the following form:

$$S_a^n\,\delta - S_b^n\,\delta = \sum_{k=1}^{n} S_a^{k-1}(S_a - S_b)S_b^{n-k}\delta = \sum_{k=1}^{n} S_a^{k-1}S_{a-b}S_b^{n-k}\delta. \qquad (2.3.2)$$

Since both a and b satisfy the sum rules of order 1, we have

$$\sum_{\alpha\in\mathbb{Z}^s}\Big(a(2\alpha+\varepsilon) - b(2\alpha+\varepsilon)\Big) = 0 \qquad \forall\varepsilon\in\Omega.$$

Hence, by Lemma 2.5, there exist sequences $c_j \in \ell([-N,N]^s)$, $j = 1,\cdots,s$ such that

$$\widetilde{a}(z) - \widetilde{b}(z) = \sum_{j=1}^{s}(1 - z_j^2)\widetilde{c}_j(z) \quad \text{and} \quad \sum_{j=1}^{s}\|c_j\|_1 \le C_N\|a-b\|_1$$

for some constant C_N depending only on N.

Let $G(z)$ denote the symbol of $S_a^{k-1}S_{a-b}S_b^{n-k}\delta$. Note that $\widetilde{S_a\lambda}(z) = \widetilde{a}(z)\widetilde{\lambda}(z^2)$ for any $\lambda \in \ell_0(\mathbb{Z}^s)$. This gives us

$$G(z) = \widetilde{a}(z^{2^0})\widetilde{a}(z^{2^1})\cdots\widetilde{a}(z^{2^{k-2}})\left(\widetilde{a}(z^{2^{k-1}}) - \widetilde{b}(z^{2^{k-1}})\right)\widetilde{b}(z^{2^k})\widetilde{b}(z^{2^{k+1}})\cdots\widetilde{b}(z^{2^{n-1}}).$$

It follows that

$$G(z) = \sum_{j=1}^{s}\widetilde{c}_j(z^{2^{k-1}})\widetilde{a}(z^{2^0})\widetilde{a}(z^{2^1})\cdots\widetilde{a}(z^{2^{k-2}})(1 - z_j^{2^k})\widetilde{b}(z^{2^k})\cdots\widetilde{b}(z^{2^{n-1}}).$$

From the fact that $\widetilde{\nabla_j S_b^n}\delta(z) = (1 - z_j)\prod_{l=0}^{n-1}\widetilde{b}(z^{2^l})$, we have

$$G(z) = \sum_{j=1}^{s}\widetilde{c}_j(z^{2^{k-1}})\widetilde{S_a^{k-1}\delta}(z)\widetilde{\nabla_j S_b^{n-k}\delta}(z^{2^k}). \tag{2.3.3}$$

Note that for any sequence $\lambda \in \ell_0(\mathbb{Z}^s)$, $\|\widetilde{\lambda}(z)\|_p := \|\lambda\|_p$. By Young's inequality and Lemma 2.4, from (2.3.3), we have

$$\begin{aligned}
\|S_a^{k-1}S_{a-b}S_b^{n-k}\delta\|_p &= \|G(z)\|_p \\
&\le \sum_{j=1}^{s}\|\widetilde{c}_j(z^{2^{k-1}})\|_1 \cdot \|\widetilde{S_a^{k-1}\delta}(z)\widetilde{\nabla_j S_b^{n-k}}\,\delta(z^{2^k})\|_p \\
&\le \left(\sum_{j=1}^{s}\|c_j\|_1\right)\max_{1\le j\le s}\|\widetilde{S_a^{k-1}\delta}(z)\widetilde{\nabla_j\, S_b^{n-k}}\,\delta(z^{2^k})\|_p \qquad (2.3.4)\\
&\le C_N\|a-b\|_1\max_{1\le j\le s}\|\widetilde{S_a^{k-1}\delta}(z)\widetilde{\nabla_j\, S_b^{n-k}}\,\delta(z^{2^k})\|_p \\
&\le C_N(2N+1)^{s/q}\|a-b\|_1\max_{1\le j\le s}\|S_a^{k-1}\delta\|_p\|\nabla_j S_b^{n-k}\delta\|_p,
\end{aligned}$$

where $1/p + 1/q = 1$. By Lemma 2.3 and Lemma 2.6, it follows from the above inequality that there exists a positive constant C depending only on a, N and p such that

$$\begin{aligned}
\|S_a^{k-1}\,S_{a-b}S_b^{n-k}\delta\|_p &\le C\|a-b\|_1 2^{s(k-1)/p}2^{(s/p-\nu)(n-k)} \\
&= C\|a-b\|_1 2^{s(n-1)/p}2^{-\nu(n-k)},
\end{aligned} \tag{2.3.5}$$

where $\nu > 0$ is given in Lemma 2.3. Therefore, by (2.3.2), for all $n \in \mathbb{N}$,

$$\begin{aligned} \|S_a^n \delta - S_b^n \delta\|_p &\le \sum_{k=1}^{n} \|S_a^{k-1} S_{a-b} S_b^{n-k} \delta\|_p \\ &\le C 2^{s(n-1)/p} \|a-b\|_1 \sum_{k=1}^{n} 2^{-\nu(n-k)} \\ &\le C_3 2^{sn/p} \|a-b\|_1, \end{aligned} \tag{2.3.6}$$

where $C_3 = C 2^{-s/p} \sum_{k=0}^{\infty} 2^{-\nu k}$. Thus, by (2.3.1), we infer

$$\|Q_a^n \phi_0 - Q_b^n \phi_0\|_p \le C_2 2^{-sn/p} \|S_a^n \delta - S_b^n \delta\|_p \le C_2 C_3 \|a-b\|_1 \qquad \forall n \in \mathbb{N}.$$

Hence, (2.1.3) is verified. Note that by Lemma 2.3 and Theorem 2.2 the subdivision scheme associated with b converges in the L_p norm. Now (2.1.4) comes directly from (2.1.3) by $\lim_{n\to\infty} \|Q_a^n \phi_0 - \phi_a\|_p = 0$ and $\lim_{n\to\infty} \|Q_b^n \phi_0 - \phi_b\|_p = 0$ which completes the proof. ∎

2.4 Sharpness of Theorem 2.1

In this section, we will give examples to illustrate that the estimate in Theorem 2.1 is optimal. More precisely, we will give examples to demonstrate that the conditions in Theorem 2.1 are satisfied for a sequence $a \in \ell([-N,N]^s)$, but there exists a positive constant C such that for any sufficiently small $\eta > 0$, we can find $b \in \ell([-N,N]^s)$ satisfying $\|a-b\|_1 < \eta$, and $\sum_{\alpha \in \mathbb{Z}^s} b(2\alpha + \varepsilon) = 1$ for all $\varepsilon \in \Omega$ and

$$\|Q_a^n \phi_0 - Q_b^n \phi_0\|_p \ge C \|a-b\|_1 \qquad \forall n \in \mathbb{N}. \tag{2.4.1}$$

Although all the following examples are given in the univariate case, it is easy to obtain such examples in the multivariate case by using the tensor product. To avoid complicated calculation, here we only consider the following simple examples.

Example 2.7 Let a be a sequence given by its symbol $\tilde{a}(z) = (1+z)^2/2$. It is evident that the subdivision scheme associated with a converges in the L_∞ norm since $\lim_{n\to\infty} \|\nabla S_a^n \delta\|_\infty^{1/n} = 1/2$. Take $\widetilde{b}(z) = (1+z)((1-\eta)+(1+\eta)z)/2$ for sufficiently small $\eta > 0$. Then an easy calculation gives us $\|a-b\|_1 = \eta$. Note that for any $n \in \mathbb{N}$,

$$\widetilde{S_a^n\delta}(z) = 2^{-n}\prod_{k=0}^{n-1}(1+z^{2^k})^2 = 2^{-n}\left(\sum_{k=0}^{2^n-1} z^k\right)^2,$$

and

$$\begin{aligned}
\widetilde{S_b^n\delta}(z) &= 2^{-n}\prod_{k=0}^{n-1}(1+z^{2^k})\prod_{k=0}^{n-1}\left(1-\eta+(1+\eta)z^{2^k}\right) \\
&= 2^{-n}\left(\sum_{k=0}^{2^n-1} z^k\right)\prod_{k=0}^{n-1}\left(1-\eta+(1+\eta)z^{2^k}\right) \qquad (2.4.2)\\
&= \left(\frac{1-\eta}{2^n}+\frac{1+\eta}{2^n}z^{2^{n-1}}\right)\left(\sum_{k=0}^{2^n-1} z^k\right)\prod_{k=0}^{n-2}\left(1-\eta+(1+\eta)z^{2^k}\right).
\end{aligned}$$

Note that $\deg\prod_{k=0}^{n-2}(1-\eta+(1+\eta)z^{2^k}) = 2^{n-1}-1$. It is not difficult to see that $S_a^n\delta(2^{n-1}-1) = 1/2$ and $S_b^n\delta(2^{n-1}-1) = (1-\eta)/2$. Hence, for all $n \in \mathbb{N}$,

$$\|S_a^n\delta - S_b^n\delta\|_\infty \geq \left|S_a^n\delta(2^{n-1}-1) - S_b^n\delta(2^{n-1}-1)\right| \geq \eta/2 = \|a-b\|_1/2.$$

Therefore, by (2.3.1), there exists a positive constant C depending only on ϕ_0 such that for any sufficiently small $\eta > 0$,

$$\|Q_a^n\phi_0 - Q_b^n\phi_0\|_\infty \geq 2C\|S_a^n\delta - S_b^n\delta\|_\infty \geq C\|a-b\|_1 \qquad \forall n \in \mathbb{N}.$$

Thus the estimate in Theorem 2.1 is optimal for $p = \infty$.

To show that the estimate in Theorem 2.1 is optimal for $1 \leq p < \infty$, it suffices to give an example such that (4.1) holds true for $p = 1$.

Example 2.8 Let a be a sequence given by its symbol $\widetilde{a}(z) = 1+z$. It is easy to see that the subdivision scheme associated with a converges in the L_p norm for $1 \leq p < \infty$ since $\lim_{n\to\infty} \|\nabla S_a^n\delta\|_p^{1/n} = 1$. Take $\widetilde{b}(z) = (1+z)(1-\eta+\eta z)$ for sufficiently small

$\eta > 0$. It is evident that $\|a-b\|_1 = 2\eta$. By calculation, we have $\widetilde{S_a^n\delta}(z) = \sum_{k=0}^{2^n-1} z^k$ and $\widetilde{S_b^n\delta}(z) = (\sum_{k=0}^{2^n-1} z^k)\prod_{k=0}^{n-1}(1-\eta+\eta z^{2^k})$. Let

$$g(z) := \left(\sum_{k=2^{n-1}}^{2^n-1} z^k\right) \eta z^{2^{n-1}} \prod_{k=0}^{n-2}\left(1-\eta+\eta z^{2^k}\right).$$

Since η is small and positive, if we expand $\widetilde{S_b^n\delta}(z) = (\sum_{k=0}^{2^n-1} z^k)\prod_{k=0}^{n-1}(1-\eta+\eta z^{2^k})$, then each term has a nonnegative coefficient. Note that $\deg \widetilde{S_a^n\delta}(z) = 2^n - 1$. Hence, it is straightforward to see that

$$\|S_a^n\delta - S_b^n\delta\|_1 \geq \sum_{k=2^n}^{\infty} S_b^n(k) \geq g(1) = 2^{n-1}\eta.$$

Hence, by (2.3.1), there exists a positive constant C depending only on ϕ_0 such that for any sufficiently small $\eta > 0$,

$$\|Q_a^n\phi_0 - Q_b^n\phi_0\|_1 \geq C2^{-n+2}\|S_a^n\delta - S_b^n\delta\|_1 \geq 2C\eta = C\|a-b\|_1 \qquad \forall n \in \mathbb{N}.$$

Thus, the estimate in Theorem 2.1 is optimal for $p = 1$. Note that both $Q_a^n\phi_0$ and $Q_b^n\phi_0$ are supported on $[-1, 3]$. It is easy to see that for any $1 \leq p < \infty$,

$$\|Q_a^n\phi_0 - Q_b^n\phi_0\|_p \geq 4^{-1/q}\|Q_a^n\phi_0 - Q_b^n\phi_0\|_1 \geq 4^{-1/q}C\|a-b\|_1 \qquad \forall n \in \mathbb{N},$$

where $1/p + 1/q = 1$. Therefore, the estimate in Theorem 2.1 is also optimal for any $1 \leq p < \infty$.

Chapter 3

Interpolatory Subdivision Schemes

3.1 Introduction

In this chapter we are interested in a special kind of refinable functions — fundamental and refinable functions with compact support.

A function ϕ is said to be **fundamental** if ϕ is continuous, $\phi(0) = 1$, and $\phi(\alpha) = 0$ for all $\alpha \in \mathbb{Z}^s \backslash \{0\}$. A fundamental refinable function is a fundamental function satisfying the following refinement equation

$$\phi = \sum_{\alpha \in \mathbb{Z}^s} a(\alpha)\phi(2 \cdot - \alpha), \tag{3.1.1}$$

where a is a **refinement mask** on $\mathbb{Z}^s$.

As in Chapter 1, throughout this thesis, we assume that a mask a satisfies

$$\sum_{\alpha \in \mathbb{Z}^s} a(\alpha) = 2^s. \tag{3.1.2}$$

The normalized solution of the refinement equation (3.1.1) with a mask a will be denoted by ϕ_a.

If a compactly supported function ϕ is fundamental and satisfies the refinement equation (3.1.1) with a finitely supported refinement mask a, then it is necessary that

$$a(0) = 1 \quad \text{and} \quad a(2\alpha) = 0 \qquad \forall\, \alpha \in \mathbb{Z}^s \backslash \{0\}. \tag{3.1.3}$$

A finitely supported sequence a on $\mathbb{Z}^s$ is called an **interpolatory refinement mask** if it satisfies the above condition (3.1.3).

Before proceeding further, we recall some notation. By $\ell(\mathbb{Z}^s)$ we denote the linear space of all sequences on $\mathbb{Z}^s$, and by $\ell_0(\mathbb{Z}^s)$ the linear space of all finitely supported sequences on $\mathbb{Z}^s$. The **support** of a sequence $a \in \ell_0(\mathbb{Z}^s)$ is denoted by $\operatorname{supp} a$, which is the finite set $\{\alpha \in \mathbb{Z}^s : a(\alpha) \neq 0\}$. For $\alpha \in \mathbb{Z}^s$, we denote by δ_α the element in $\ell_0(\mathbb{Z}^s)$ given by $\delta_\alpha(\alpha) = 1$ and $\delta_\alpha(\beta) = 0$ for all $\beta \in \mathbb{Z}^s \backslash \{\alpha\}$. In particular, we write δ for δ_0. For $j = 1, \cdots, s$, let e_j be the jth coordinate unit vector. The difference operator ∇_j is defined by $\nabla_j \lambda := \lambda - \lambda(\cdot - e_j)$, $\lambda \in \ell(\mathbb{Z}^s)$.

If ϕ is a fundamental and refinable function with a mask a, then for any given sequence $b \in \ell(\mathbb{Z}^s)$, we can construct a function

$$f_b(x) = \sum_{\beta \in \mathbb{Z}^s} b(\beta)\phi(x - \beta), \qquad x \in \mathbb{R}^s,$$

which interpolates the sequence b with $f_b(\beta) = b(\beta)$ for all $\beta \in \mathbb{Z}^s$. Evidently, the smoother the function ϕ is, the smoother f_b is. Moreover, the value of f_b at any dyadic rational number can be easily computed by the following iterative subdivision scheme formula:

$$f_b(2^{-n}\beta) = S_a^n b(\beta) \qquad \forall\, n \in \mathbb{N}, \beta \in \mathbb{Z}^s,$$

where S_a is the **subdivision operator** on $\ell(\mathbb{Z}^s)$ associated with the mask a and is defined by

$$S_a \lambda(\alpha) := \sum_{\beta \in \mathbb{Z}^s} a(\alpha - 2\beta)\lambda(\beta), \qquad \alpha \in \mathbb{Z}^s, \lambda \in \ell(\mathbb{Z}^s). \tag{3.1.4}$$

Subdivision schemes are very useful in computer graphics and wavelet analysis. See [38, 39] and [85] for their applications to computer aided geometric design, and see [15] and [102] for their applications to wavelet decompositions.

Let a be an interpolatory refinement mask. Then the normalized solution of (3.1.1) with the mask a is fundamental if and only if the subdivision scheme associated with a converges in the L_∞ norm. The reason is the following. If ϕ_a is a fundamental refinable function with a refinement mask a, then the shifts of ϕ_a are stable for $p = \infty$. Thus by Theorem 1.9 in Chapter 1, the subdivision scheme associated with a converges in the L_∞ norm. Conversely, if the subdivision scheme associated with a converges in the L_∞ norm, then ϕ_a is a continuous function and

$$\lim_{n\to\infty} \|Q_a^n \phi_0 - \phi_a\|_\infty = 0,$$

where the operator Q_a is defined in (1.1.5) and the initial function ϕ_0 is given in (1.1.3). Since the mask a is an interpolatory mask and the initial function ϕ_0 is a fundamental function, it is easily seen that each function $Q_a^n\phi_0$ is also a fundamental function which implies ϕ_a is also a fundamental function. A subdivision scheme is said to be a C^k interpolatory subdivision scheme if it converges to a function in $C^k(\mathbb{R}^s)$.

The first C^1 interpolatory subdivision scheme on $\mathbb{R}$ was constructed by Dubuc in [33]. His mask is given by

$$a(0) = 1, \quad a(1) = a(-1) = 9/16, \quad a(3) = a(-3) = -1/16,$$

and $a(\alpha) = 0$ for $\alpha \in \mathbb{Z} \setminus \{-3, -1, 0, 1, 3\}$. See Figure 3.1 for the graph of its fundamental refinable function. In [30], Deslauriers and Dubuc proposed a general method to construct symmetric interpolatory subdivision schemes. The L_2 smoothness analysis of their schemes was conducted by Eirola in [42]. In [84] Micchelli discussed connections of their schemes with the Daubechies orthogonal wavelets (see [26]).

For the multivariate case, Dyn, Gregory and Levin [39] constructed the so-called butterfly scheme which is a C^1 bivariate interpolatory subdivision scheme, while Deslauriers, Dubois and Dubuc [31] obtained several continuous bivariate refinable and fundamental functions. Mongeau and Deslauriers [87] obtained several C^1 bivariate refinable and fundamental functions. Recently, using convolutions of box splines with distributions, Riemenschneider and Shen [89] constructed a family of bivariate interpolatory subdivision schemes with symmetry. More recently, Han and Jia [52] constructed a family of bivariate optimal interpolatory subdivision schemes with many desired properties.

The purpose of this chapter is to give a general construction of interpolatory refinement masks such that the corresponding refinable functions possess the optimal approximation and L_2 smoothness properties. Let us discuss these two properties in detail.

For a compactly supported function ϕ in $L_p(\mathbb{R}^s)$, $1 \le p \le \infty$, we define

$$S(\phi) := \left\{ \sum_{\alpha \in \mathbb{Z}^s} \phi(\cdot - \alpha)\lambda(\alpha) \ : \ \lambda \in \ell(\mathbb{Z}^s) \right\}$$

and call it the shift-invariant space generated by ϕ. For $h > 0$, the scaled space S^h is defined by $S^h := \{g(\cdot/h) \ : \ g \in S(\phi)\}$. For a positive integer k, we say that $S(\phi)$ provides **approximation order** k if, for each sufficiently smooth function f in $L_p(\mathbb{R}^s)$, there exists a positive constant C such that

$$\inf_{g \in S^h} \|f - g\|_p \le Ch^k \qquad \forall\, h > 0.$$

Under the assumption $\widehat{\phi}(0) \ne 0$, it was proved by Jia in [58] that $S(\phi)$ provides approximation order k if and only if $S(\phi)$ contains Π_{k-1}, where Π_{k-1} denotes the set of all polynomials of (total) degree at most $k - 1$.

The concept of stability plays an important role in wavelet analysis. Let us recall the definition of stability from Chapter 1. Let ϕ be a compactly supported function in $L_p(\mathbb{R}^s)$ $(1 \le p \le \infty)$. We say that the shifts of ϕ are **stable** if there are two positive constants C_1 and C_2 such that

$$C_1\|\lambda\|_p \le \left\| \sum_{\alpha \in \mathbb{Z}^s} \lambda(\alpha)\phi(\cdot - \alpha) \right\|_p \le C_2\|\lambda\|_p \qquad \forall\, \lambda \in \ell_0(\mathbb{Z}^s). \tag{3.1.5}$$

It was proved by Jia and Micchelli in [68] that a compactly supported function $\phi \in L_p(\mathbb{R}^s)$ satisfies the above L_p-stability condition if and only if, for any $\xi \in \mathbb{R}^s$, there exists an element $\beta \in \mathbb{Z}^s$ such that

$$\hat{\phi}(\xi + 2\pi\beta) \neq 0.$$

Note that a fundamental function has stable shifts.

Now suppose ϕ is the normalized solution of the refinement equation (3.1.1) with a mask a which satisfies (3.1.2). By Ω we denote the set of the vertices of the unit cube $[0,1]^s$. For a positive integer k, we say that a satisfies the **sum rules** of order k if

$$\sum_{\beta \in \mathbb{Z}^s} a(\varepsilon + 2\beta)p(\varepsilon + 2\beta) = \sum_{\beta \in \mathbb{Z}^s} a(2\beta)p(2\beta) \qquad \forall \varepsilon \in \Omega,\ p \in \Pi_{k-1}. \tag{3.1.6}$$

It was proved by Jia in [61] and [62] that a satisfies the sum rules of order k implies $S(\phi)$ contains Π_{k-1}. If, in addition, the shifts of ϕ are stable, then the converse holds true. Thus, in particular, if ϕ is a fundamental and refinable function with mask a, then $S(\phi)$ provides approximation order k if and only if a satisfies the sum rules of order k.

Here is an outline of this chapter. Section 3.2 is devoted to a study of interpolatory refinement masks which satisfy the optimal order of sum rules. For a positive integer r, let a be an interpolatory mask supported on the cube $[1-2r, 2r-1]^s$. We will demonstrate that $2r$ is the optimal order of sum rules that a satisfies. In the univariate case ($s = 1$), there is a unique interpolatory mask supported on $[1-2r, 2r-1]$ and satisfying the sum rules of order $2r$. This is the same interpolatory mask as given by Deslauriers and Dubuc in [30], and will be denoted by b_r. In the multivariate case ($s > 1$), such interpolatory masks are not unique. Let t_r be the sequence on $\mathbb{Z}^s$ given by

$$t_r(\alpha_1, \ldots, \alpha_s) := b_r(\alpha_1) \cdots b_r(\alpha_s), \qquad (\alpha_1, \ldots, \alpha_s) \in \mathbb{Z}^s. \tag{3.1.7}$$

Then t_r is an interpolatory refinement mask supported on $[1-2r, 2r-1]^s$ and it satisfies the sum rules of order $2r$. We denote by ϕ_{t_r} the normalized solution of the refinement equation (3.1.1) with the mask t_r. In Section 3.3, we shall provide a general characterization of L_p ($1 \le p \le \infty$) smoothness order of a refinable function. In Section 3.4 we will give an analysis of smoothness of fundamental functions arising from interpolatory subdivision schemes. It will be demonstrated that ϕ_{t_r} achieves the optimal smoothness in the following sense. If a is an interpolatory mask supported on $[1-2r, 2r-1]^s$ and satisfying the sum rules of order $2r$, and if ϕ_a is the corresponding refinable function, then $\nu_p(\phi_a) \le \nu_p(\phi_{t_r})$ for any $1 \le p \le \infty$ where the L_p critical exponent $\nu_p(\phi)$ will be defined in Section 3.3. Thus, an interpolatory mask supported on $[1-2r, 2r-1]^s$ is said to be *optimal* in the L_2 norm sense if it satisfies the sum rules of order $2r$ and the corresponding refinable function ϕ satisfies $\nu_2(\phi) = \nu_2(\phi_{t_r})$.

Now it is clear that an optimal refinement mask should be chosen in such a way that the size of its support and the number of nonzero coefficients are minimal. The size of a mask a is defined to be the volume of the convex hull of supp a. In Section 3.5, we will give a general construction of two-dimensional optimal interpolatory masks g_r ($r \in \mathbb{N}$). The size of g_r is $8r^2 + O(r)$ and the number of nonzero coefficients of g_r is $2r^2 + O(r)$. In comparison, the size of t_r is $16r^2 + O(r)$ and the number of nonzero coefficients of t_r is $4r^2 + O(r)$. Let RS_r denote the interpolatory mask supported on $[1-2r, 2r-1]^2$ constructed by Riemenschneider and Shen in [89]. The size of RS_r is $12r^2 + O(r)$, but the number of its nonzero coefficients is $9r^2 + O(r)$, which is about twice the number of nonzero coefficients of t_r. The masks RS_r ($r = 2, 3, \dots$) are symmetric about the origin and the line $x_1 = x_2$. Our masks g_r enjoy better symmetric properties. They are symmetric about the origin, the x_1-axis, the x_2-axis, and the lines $x_1 = x_2$ and $x_1 = -x_2$.

Finally, in Section 3.6, we will give several examples, including a 16-point bivariate C^1 interpolatory subdivision scheme and a 30-point bivariate C^2 scheme. Furthermore, we will demonstrate that the refinable functions associated with our masks g_r ($r = 1, 2, \dots, 12$) attain the optimal L_2 smoothness order. Comparison results and graphs of several fundamental refinable functions with the masks g_r are provided at the end of this chapter. Finally, comparison results by applying such interpolatory refinement masks to generate surfaces are illustrated.

3.2 Optimal Sum Rules

In this section, we shall investigate the approximation properties of fundamental and refinable functions. Let ϕ be a fundamental and refinable function. Then the corresponding refinement mask a is interpolatory. In this case, the shift-invariant space generated by ϕ provides approximation order k if and only if a satisfies the sum rules of order k. Thus, the problem reduces to a study of sum rules.

The following theorem gives an upper bound for the order of sum rules that an interpolatory mask satisfies in terms of the support of the mask.

Theorem 3.1 *Let a in $\ell_0(\mathbb{Z})$ be an interpolatory refinement mask satisfying the sum rules of order k. If a is supported on an interval $[-L, H]$ with L and H being nonnegative integers, then $k \le \lfloor \frac{L+1}{2} \rfloor + \lfloor \frac{H+1}{2} \rfloor$, where $\lfloor \cdot \rfloor$ denotes the floor function. Moreover, there exists a unique interpolatory refinement mask such that it is supported on $[-L, H]$ and satisfies the sum rules of order $\lfloor \frac{L+1}{2} \rfloor + \lfloor \frac{H+1}{2} \rfloor$.*

Proof: Since a is an interpolatory refinement mask, we have $a(0) = 1$ and $a(2j) = 0$ for all $j \in \mathbb{Z} \setminus \{0\}$. Set $l := \lfloor \frac{L-1}{2} \rfloor$ and $h := \lfloor \frac{H+1}{2} \rfloor$. Then a satisfies the sum rules of order k if and only if

$$\sum_{j=-l}^{h} a(2j-1) = 1 \quad \text{and} \quad \sum_{j=-l}^{h} a(2j-1)(2j-1)^m = 0, \qquad 0 < m \le k-1.$$

The above equations can be rewritten in the following matrix form:

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ (-2l-1) & (-2l+1) & \cdots & (2h-1) \\ \vdots & \vdots & \ddots & \vdots \\ (-2l-1)^{k-1} & (-2l+1)^{k-1} & \cdots & (2h-1)^{k-1} \end{bmatrix} \begin{bmatrix} a(-2l-1) \\ a(-2l+1) \\ \vdots \\ a(2h-1) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \tag{3.2.1}$$

Note that the matrix $\big((2j-1)^m\big)_{-l\le j\le h, 0\le m\le l+h}$ is a Vandermonde matrix; hence the matrix $\big((2j-1)^m\big)_{-l\le j\le h, 1\le m\le l+h+1}$ is nonsingular. If

$$k > \lfloor \frac{L+1}{2} \rfloor + \lfloor \frac{H+1}{2} \rfloor = l+h+1,$$

then it follows from (3.2.1) that $a(-2l-1) = \cdots = a(2h-1) = 0$, which contradicts the condition $\sum_{j=-l}^{h} a(2j-1) = 1$. This shows $k \le l+h+1$, as desired.

Suppose $k = l+h+1$. Since $\big((2j-1)^m\big)_{-l\le j\le h, 0\le m\le l+h}$ is a Vandermonde matrix, the equation (3.2.1) has a unique solution for $[a(-2l-1), a(-2l+1), \dots, a(2h-1)]$, which can be easily found as follows:

$$a(2j-1) = (-1)^{j+l} \frac{\prod_{k=-l}^{h}(2k-1)}{2^{l+h}(2j-1)\cdot(j+l)!(h-j)!}, \qquad -l \le j \le h. \tag{3.2.2}$$

The proof of the theorem is complete. ∎

Recall that for a sequence $a \in \ell_0(\mathbb{Z}^s)$, the **symbol** of a is defined to be

$$\widetilde{a}(z) := \sum_{\alpha\in\mathbb{Z}^s} a(\alpha) z^\alpha, \qquad z \in \mathbb{T}^s.$$

As an example, we consider the case where $L = 3$ and $H = 5$. In this case the interpolatory refinement mask a obtained from (3.2.2) in Theorem 3.1 is given by its symbol

$$\begin{aligned} \widetilde{a}(z) &= -\frac{5}{128}z^{-3} + \frac{15}{32}z^{-1} + 1 + \frac{45}{64}z - \frac{5}{32}z^3 + \frac{3}{128}z^5 \\ &= \frac{1}{128}(1+z)^5(-5\,z^{-3} + 25\,z^{-2} - 15\,z^{-1} + 3). \end{aligned}$$

It satisfies the sum rules of order 5 and gives rise to a C^2 interpolatory subdivision scheme.

When $L = H = 2r-1$, $r \in \mathbb{N}$, the interpolatory refinement mask given in (3.2.2) is exactly the symmetric interpolatory refinement mask constructed by Deslauriers and Dubuc in [30]. Recall that this mask is denoted by b_r. Correspondingly, ϕ_{b_r} will be used to denote the normalized solution of (3.1.1) with the refinement mask b_r.

For the reader's convenience, we list some b_r $(r = 2, 3, 4, 5)$ in the following.

$$\begin{aligned}
\widetilde{b}_2(z) =& -\frac{1}{16}z^{-3} + \frac{9}{16}z^{-1} + 1 + \frac{9}{16}z - \frac{1}{16}z^3 \\
=& \frac{1}{16}(1+z)^4 z^{-2}(-z^{-1} + 4 - z), \\
\widetilde{b}_3(z) =& \frac{3}{256}z^{-5} - \frac{25}{256}z^{-3} + \frac{75}{128}z^{-1} + 1 + \frac{75}{128}z - \frac{25}{256}z^3 + \frac{3}{256}z^5 \\
=& \frac{1}{256}(1+z)^6 z^{-3}(3z^{-2} - 18z^{-1} + 38 - 18z + 3z^2), \\
\widetilde{b}_4(z) =& \frac{1}{2048}(1+z)^8 z^{-4}(-5z^{-3} + 40z^{-2} - 131z^{-1} + 208 - 131z + 40z^2 - 5z^3), \\
\widetilde{b}_5(z) =& \frac{1}{65536}(1+z)^{10} z^{-5}(35z^{-4} - 350z^{-3} + 1520z^{-2} - 3650z^{-1} + 5018 \\
& - 3650z + 1520z^2 - 350z^3 + 35z^4).
\end{aligned}$$

See Figure 3.1 for the graphs of the functions $\phi_{b_r}, r = 2, 3, 4, 5$.

Next we extend the results of Theorem 3.1 to the multidimensional case.

Theorem 3.2 *Suppose a is an interpolatory refinement mask supported on the closed cell $\Pi_{j=1}^s[-L_j, H_j]$ for some nonnegative integers L_j and H_j. If a satisfies the sum rules of order k, then*

$$k \le \min_{1\le j\le s}\left(\lfloor\frac{L_j+1}{2}\rfloor + \lfloor\frac{H_j+1}{2}\rfloor\right).$$

Proof: For a fixed integer j between 1 and s, let b be the sequence on $\mathbb{Z}$ defined by

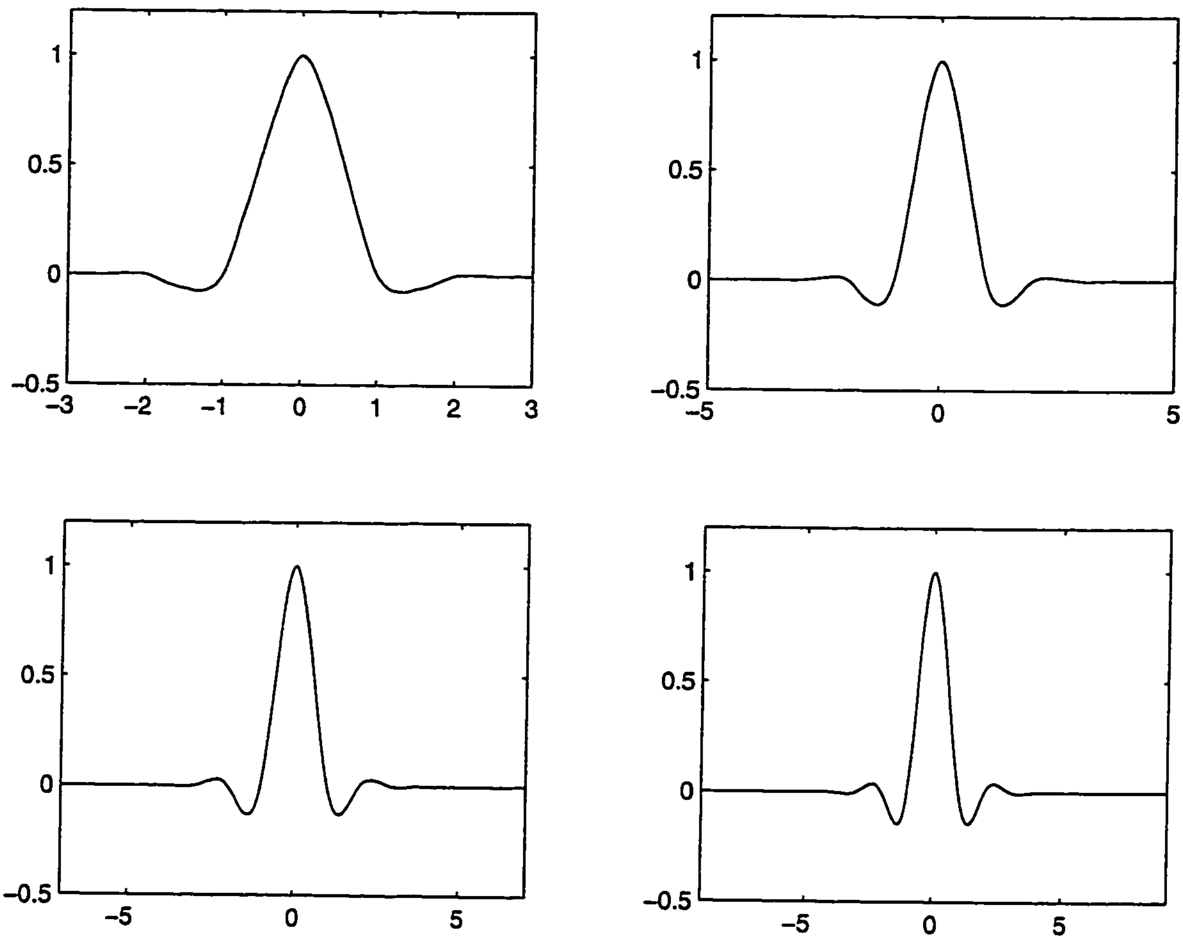

Figure 3.1: The graphs of the Deslauriers and Dubuc's fundamental functions $\phi_{b_r}, r = 2, 3, 4, 5$.

$b(0) = 1$, $b(2\beta) = 0$ for $\beta \in \mathbb{Z}\backslash\{0\}$, and

$$b(2\beta+1) := \frac{1}{2^{s-1}} \sum_{\alpha_1,\cdots,\alpha_{j-1},\alpha_{j+1},\cdots,\alpha_s \in \mathbb{Z}} a(\alpha_1,\cdots,\alpha_{j-1},2\beta+1,\alpha_{j+1},\cdots,\alpha_s), \qquad \beta \in \mathbb{Z}.$$

It is evident that b is an interpolatory refinement mask. Suppose a satisfies the sum rules of order k. We show that b satisfies the sum rules of order at least k. Recall that Ω is the set of all vertices of the cube $[0,1]^s$. Let

$$\Omega_j := \{(\varepsilon_1,\cdots,\varepsilon_s) \in \Omega \ : \ \varepsilon_j = 1\}.$$

For a nonnegative integer m, by the definition of b, we obtain

$$\sum_{\beta \in \mathbb{Z}} (2\beta+1)^m b(2\beta+1) = \frac{1}{2^{s-1}} \sum_{(\varepsilon_1,\ldots,\varepsilon_s)\in\Omega_j} c_m(\varepsilon_1,\ldots,\varepsilon_s),$$

where

$$c_m(\varepsilon_1,\ldots,\varepsilon_s) := \sum_{(\beta_1,\cdots,\beta_s)\in\mathbb{Z}^s} (2\beta_j+\varepsilon_j)^m a(2\beta_1+\varepsilon_1,\cdots,2\beta_s+\varepsilon_s).$$

Since a satisfies the sum rules of order k, we have

$$c_m(\varepsilon_1,\ldots,\varepsilon_s) = \delta_{m0} \qquad \forall\,(\varepsilon_1,\cdots,\varepsilon_s) \in \Omega_j,\ 0 \le m < k,$$

where $\delta_{00} = 1$ and $\delta_{m0} = 0$ for $m \neq 0$. It follows that

$$\sum_{\beta\in\mathbb{Z}} (2\beta+1)^m b(2\beta+1) = \delta_{m0}, \qquad 0 \le m < k.$$

That is, b satisfies the sum rules of order k. But b is supported on $[-L_j, H_j]$. By Theorem 3.1, we conclude that $k \le \lfloor\frac{L_j+1}{2}\rfloor + \lfloor\frac{H_j+1}{2}\rfloor$, thereby completing the proof. ∎

By using tensor product and Theorem 3.1, it is easy to see that there exists an interpolatory refinement mask which is supported on $\Pi_{j=1}^s[-L_j, H_j]$ and satisfies the sum rules of the optimal order $\min_{1\le j\le s}\left(\lfloor\frac{L_j+1}{2}\rfloor + \lfloor\frac{H_j+1}{2}\rfloor\right)$. In general, when $s > 1$, such interpolatory refinement masks are not unique. If a is an interpolatory refinement mask supported on $[1-2r, 2r-1]^s$, $r \in \mathbb{N}$, and satisfies the sum rules of order k, then Theorem 3.2 tells us that $k \leq 2r$. Let t_r be the mask given by (3.1.7). Then t_r satisfies the sum rules of order $2r$. In Section 3.5, we will give a general construction of interpolatory refinement masks g_r ($r \in \mathbb{N}$) on $\mathbb{Z}^2$. Each g_r is supported on the square $[1-2r, 2r-1]^2$ and satisfies the sum rules of order $2r$. But the size of the support of g_r is much smaller than that of t_r.

3.3 L_p Smoothness of Multivariate Refinable Functions

In this section, we will study the smoothness of a refinable function in the multivariate setting. Many results on the analysis of L_2 smoothness of a refinable function both in the univariate case and multivariate case have already been obtained; see, for example [20, 28, 42, 60, 89, 92, 102] and references therein. For $s = 1$, the characterization of L_p smoothness was given by Villemoes in [102]. In [60], Jia gave a complete characterization of L_2 smoothness of a refinable function with any general isotropic dilation matrix. The results in this section were essentially known to Jia. In this section, based on a result of Ditzian [34, 35], we present a simple proof to characterize the L_p smoothness of a multivariate refinable function. Jia will discuss the L_p smoothness of a refinable function with an arbitrary dilation matrix in a forthcoming paper [65].

We shall use the generalized Lipschitz space to measure smoothness of a given function. For any vector y in $\mathbb{R}^s$, the **difference operator** ∇_y on $L_p(\mathbb{R}^s)$ is defined to be

$$\nabla_y f = f - f(\cdot - y), \qquad f \in L_p(\mathbb{R}^s).$$

Let k be a positive integer. The k-th **modulus of smoothness** of a function f in $L_p(\mathbb{R}^s)$ is defined by

$$\omega_k(f,h)_p := \sup_{|y|\leq h} \|\nabla_y^k f\|_p, \qquad h > 0.$$

For $\nu > 0$, let k be an integer greater than ν. The **generalized Lipschitz space** $Lip^*(\nu, L_p(\mathbb{R}^s))$ consists of those functions f in $L_p(\mathbb{R}^s)$ for which

$$\omega_k(f,h)_p \le Ch^\nu \qquad \forall\, h > 0, \tag{3.3.1}$$

where C is a constant independent of h, or in other words, $\omega_k(f,h)_p = O(h^\nu)$. The reader is referred to the books [32, 36, 98] for more detail.

The L_p smoothness of a function $f \in L_p(\mathbb{R}^s)$ in the L_p norm sense is described by its L_p **critical exponent** $\nu_p(f)$ defined by

$$\nu_p(f) := \sup\left\{ \nu \ :\ f \in Lip^*(\nu, L_p(\mathbb{R}^s)) \right\}. \tag{3.3.2}$$

For any $\nu > 0$, the Sobolev space $W_2^\nu(\mathbb{R}^s)$ contains all the functions $f \in L_2(\mathbb{R}^s)$ for which

$$\int_{\mathbb{R}^s} |\widehat{f}(\xi)|^2 (1 + |\xi|^\nu)^2\, d\xi < \infty.$$

It is well known that, for $\nu > \eta > 0$, the inclusion relations

$$W_2^\nu(\mathbb{R}^s) \subseteq Lip^*(\nu, L_2(\mathbb{R}^s)) \subseteq W_2^{\nu-\eta}(\mathbb{R}^s)$$

hold true (see [98]). Therefore,

$$\nu_2(f) = \sup\left\{ \nu \ :\ f \in Lip^*(\nu, L_2(\mathbb{R}^s)) \right\} = \sup\left\{ \nu \ :\ f \in W_2^\nu(\mathbb{R}^s) \right\}.$$

In the following, we will characterize the L_p ($1 \le p \le \infty$) smoothness of a refinable function in multidimensional spaces. To do this, we need the following result on moduli of smoothness, which is based on a result of Ditzian in [34, 35].

Theorem 3.3 *Let f be a function in $L_p(\mathbb{R}^s)$ and ν be a positive real number. Then f belongs to the space $Lip^*(\nu, L_p(\mathbb{R}^s))$ if and only if for an integer k greater than ν, there exists a positive constant C such that*

$$\max\{ \|\nabla^k_{2^{-n}e_i} f\|_p \ :\ i = 1, \cdots, s \} \le C 2^{-n\nu} \qquad \forall\, n \in \mathbb{N}, \tag{3.3.3}$$

where e_i is the i-th coordinate unit vector.

Proof: Necessity: If f belongs to $Lip^*(\nu, L_p(\mathbb{R}^s))$, then by the definition of the Lipschitz space $Lip^*(\nu, L_p(\mathbb{R}^s))$, there exists a positive constant C such that

$$\|\nabla^k_{2^{-n}e_i} f\|_p \le \omega_k(f, 2^{-n}) \le C2^{-n\nu} \qquad \forall\, 1 \le i \le s, n \in \mathbb{N}.$$

Hence, inequality (3.3.3) holds true.

Sufficiency: If inequality (3.3.3) holds true, then we can demonstrate that there exists a positive constant C_1 such that

$$\|\nabla^k_{he_i} f\|_p \le C_1 h^\nu \qquad \forall\, 1 \le i \le s, h > 0. \tag{3.3.4}$$

Let g be a simple function such that $\|g\|_q = 1$ where $1/p + 1/q = 1$. Define

$$F(x) := f * g(x) = \int_{\mathbb{R}^s} f(x-t)g(t)\,dt, \qquad x \in \mathbb{R}^s.$$

Then the function F is continuous and bounded. Note that the inequality (3.3.3) implies that for any $i = 1, \cdots, s$,

$$\|\nabla^k_{2^{-n}e_i} F\|_\infty = \|(\nabla^k_{2^{-n}e_i} f) * g\|_\infty \le \|\nabla^k_{2^{-n}e_i} f\|_p \|g\|_q \le C2^{-n\nu} \qquad \forall\, n \in \mathbb{N}.$$

Therefore, in particular, we have

$$|\nabla^k_{2^{-n}e_i} F(te_i)| \le C2^{-n\nu} \qquad \forall\, t \in \mathbb{R}, n \in \mathbb{N}.$$

By a result of Boman (see Theorem 1 in [2]) and Ditzian [35], there exists a positive constant C_1 depending only on k and C (independent of g) such that

$$|\nabla^k_{he_i} F(te_i)| \le C_1 h^\nu \qquad \forall\, t \in \mathbb{R}, h > 0. \tag{3.3.5}$$

Note that $\nabla^k_{he_i} F(0) = (\nabla^k_{he_i} f) * g(0)$. It follows from the above inequality (3.3.5) that for any simple function g with $\|g\|_q = 1$, we have that for any $i = 1, \cdots, s$,

$$\left| \int_{\mathbb{R}^s} (\nabla^k_{he_i} f)(-x) g(x)\,dx \right| = |(\nabla^k_{he_i} f) * g(0)| = |\nabla^k_{he_i} F(0)| \le C_1 h^\nu \quad \forall\, h > 0.$$

This yields

$$\|\nabla^k_{he_i} f\|_p = \sup_{\|g\|_q = 1} \left| \int_{\mathbb{R}^s} (\nabla^k_{he_i} f)(-x) g(x)\,dx \right| \le C_1 h^\nu \qquad \forall\, 1 \le i \le s, h > 0.$$

Therefore, inequality (3.3.4) is verified. By inequality (3.3.4) and a result of Ditzian (see Corollary 5.2 and also cf. Theorem 5.1 in [34]), it is straightforward to see that the function f belongs to the function space $Lip^*(\nu, L_p(\mathbb{R}^s))$. ∎

Remark 3.4 In fact, the result in Corollary 5.2 of Ditzian [34] is a Marchaud-type inequality which says that to characterize the k-th modulus of smoothness of a function in $L_p(\mathbb{R}^s)$ in the L_p norm sense, the information of the k-th modulus of smoothness in s independent directions is enough. More precisely, for any vector y in $\mathbb{R}^s$, we denote $\omega_k(f, h, y)_p := \sup_{|t|\le h} \|\nabla_{ty}^k f\|_p$, $h > 0$. Let y_i $(i = 1, \cdots, s)$ be s independent vectors in $\mathbb{R}^s$. Then for any $\nu > 0$ and an integer $k > \nu$, $\omega_k(f, h)_p = O(h^\nu)$ if and only if $\omega_k(f, h, y_i)_p = O(h^\nu)$ for all $i = 1, \cdots, s$. Therefore, in Theorem 3.3, the vectors e_i $(i = 1, \cdots, s)$ can be replaced by vectors y_i $(i = 1, \cdots, s)$ provided that y_i $(i = 1, \cdots, s)$ are linearly independent vectors in $\mathbb{R}^s$. For more detail of the above result, the reader is referred to the work of Boman [2], Ditzian [34, 35], and Ditzian and Totik [36].

Based on the above result, the following theorem gives us a characterization of the critical exponent $\nu_p(\phi)$ of a refinable function ϕ in $L_p(\mathbb{R}^s)$ in terms of its mask provided that the shifts of the refinable function ϕ are stable.

Theorem 3.5 *Let a function ϕ in $L_p(\mathbb{R}^s)$ $(1 \le p \le \infty)$ be the normalized solution of the refinement equation* (3.1.1) *with a finitely supported refinement mask a on $\mathbb{Z}^s$ such that $\sum_{\beta\in\mathbb{Z}^s} a(\beta) = 2^s$. For any nonnegative integer k, let*

$$\sigma_{k,p}(a) := \lim_{n\to\infty} \max\{ \|\nabla_i^k S_a^n \delta\|_p^{1/n} \ : \ i = 1, \cdots, s \}.$$

Then

$$\min\{ k, \ \nu_p(\phi) \} \ge s/p - \log_2 \sigma_{k,p}(a). \tag{3.3.6}$$

In addition, if the shifts of ϕ are stable, then

$$\min\{ k, \nu_p(\phi) \} = s/p - \log_2 \sigma_{k,p}(a). \tag{3.3.7}$$

More generally, let $Y := \{ y_i \in \mathbb{Z}^s : i = 1, \cdots, s \}$ be a set of s independent vectors. Define

$$\sigma_{k,p,Y}(a) := \lim_{n\to\infty} \max\{ \|\nabla_{y_i}^k S_a^n \delta\|_p^{1/n} : i = 1, \cdots, s \}.$$

Then the above results still hold true if $\sigma_{k,p}(a)$ is replaced with $\sigma_{k,p,Y}(a)$.

Proof: By the definition of $\sigma_{k,p}(a)$, for any real number r such that $r > \sigma_{k,p}(a)$, there exists a positive constant C_r such that

$$\max\{ \|\nabla_i^k S_a^n \delta\|_p : 1 \le i \le s \} \le C_r r^n \qquad \forall\, n \in \mathbb{N}. \tag{3.3.8}$$

By induction and the definition of the subdivision operator defined in (3.1.4), we observe

$$\nabla_{2^{-n}e_i}^k \phi = \sum_{\beta\in\mathbb{Z}^s} \nabla_i^k S_a^n \delta(\beta) \phi(2^n \cdot -\beta), \qquad i = 1, \cdots, s. \tag{3.3.9}$$

Since the function ϕ in $L_p(\mathbb{R}^s)$ is compactly supported, from Equation (3.3.9), there exists a positive constant C_1 depending only on ϕ such that

$$\|\nabla_{2^{-n}e_i}^k \phi\|_p \le C_1 2^{-ns/p} \|\nabla_i^k S_a^n \delta\|_p \qquad \forall\, 1 \le i \le s,\ n \in \mathbb{N}.$$

Therefore, it follows from inequality (3.3.8) that

$$\|\nabla_{2^{-n}e_i}^k \phi\|_p \le C_1 C_r 2^{-ns/p} r^n \qquad \forall\, n \in \mathbb{N}. \tag{3.3.10}$$

On the other hand, by induction, we observe $\sigma_{k,p}(a) \ge 2^{s/p-k}$ since $\sum_{\beta\in\mathbb{Z}^s} a(\beta) = 2^s$. Therefore, the inequality $k \ge s/p - \log_2 \sigma_{k,p}(a)$ holds true for any nonnegative integer k. Since $r > \sigma_{k,p}(a)$, we deduce that $k \ge s/p - \log_2 \sigma_{k,p}(a) > s/p - \log_2 r$. By Theorem 3.3, it follows from inequality (3.3.10) that $\phi \in Lip^*(s/p - \log_2 r, L_p(\mathbb{R}^s))$ for any r such that $r > \sigma_{k,p}(a)$. So in conclusion, we have

$$\min\{ k, \nu_p(\phi) \} \ge s/p - \log_2 \sigma_{k,p}(a).$$

If the shifts of the function ϕ are stable, to prove Equation (3.3.7), it suffices to prove that $\min\{k, \nu_p(\phi)\} \le s/p - \log_2 \sigma_{k,p}(a)$, equivalently, it suffices to prove that

$$\sigma_{k,p}(a) \le 2^{s/p-\nu} \qquad \text{for all} \quad 0 < \nu < \min\{\, k,\ \nu_p(\phi)\,\}.$$

Since the shifts of the function ϕ are stable and ϕ lies in $L_p(\mathbb{R}^s)$, from (3.3.9), there exists a positive constant C_2 depending only on the function ϕ such that

$$\|\nabla_i^k S_a^n \delta\|_p \le C_2 2^{ns/p} \|\nabla_{2^{-n}e_i}^k \phi\|_p \qquad \forall\, 1 \le i \le s,\ n \in \mathbb{N}.$$

Since $\phi \in Lip^*(\nu, L_p(\mathbb{R}^s))$ and $k > \nu$, by Theorem 3.1, we have

$$\max_{1\le i\le s}\{\, \|\nabla_i^k S_a^n \delta\|_p \,\} \le C_2 2^{ns/p} \max_{1\le i\le s}\{\, \|\nabla_{2^{-n}e_i}^k \phi\|_p \,\} \le C_2 C 2^{n(s/p-\nu)} \qquad \forall\, n \in \mathbb{N}.$$

Therefore, the inequality $\sigma_{k,p}(a) \le 2^{s/p-\nu}$ holds true, as desired. The last assertion of this theorem comes directly from Remark 3.4. ∎

Remark 3.6 If the shifts of the function ϕ are stable and its mask a satisfies the sum rules of order k but not $k+1$, then $\nu_p(\phi) \le k$ (see [61, 62]) and therefore, by Theorem 3.5, $\nu_p(\phi) = s/p - \log_2 \sigma_{k,p}(a)$. Another remark about the above theorem is that by carefully choosing the set Y, the equality in (3.3.6) may hold true even when the shifts of the function are not stable. For example, let

$$\phi(x) = \max\{1 - |x|/2,\, 0\}, \qquad x \in \mathbb{R}.$$

Then the function ϕ is a refinable function with its mask a given by its symbol $\widetilde{a}(z) := 1 + (z^{-2} + z^2)/2$. It is a known fact that the shifts of ϕ are *not* stable and $\nu_p(\phi) = 1 + 1/p$ for any p such that $1 \le p \le \infty$. On the other hand, choose $y = 2$. It is not difficult to verify that $\sigma_{2,p,y}(a) := \lim_{n\to\infty} \|\nabla_y^2 S_a^n \delta\|_p^{1/n} = 1/2$. Therefore, we still have $\nu_p(\phi) = 1/p - \log_2 \sigma_{2,p,y}(a) = 1/p + 1$ for any p such that $1 \le p \le \infty$.

In Chapter 1, we demonstrated that $\sigma_{k,2}(a)$ can be easily computed by calculating the spectral radius of a certain finite matrix. Let b be the sequence given by

$$b(\alpha) := \sum_{\beta\in\mathbb{Z}^s} a(\alpha+\beta)\overline{a(\beta)}, \qquad \alpha \in \mathbb{Z}^s.$$

Recall that the **transition operator** T_b associated with b is defined by

$$T_b\lambda(\alpha) = \sum_{\beta\in\mathbb{Z}^s} b(2\alpha-\beta)\lambda(\beta), \qquad \alpha\in\mathbb{Z}^s,\ \lambda\in\ell_0(\mathbb{Z}^s).$$

From Theorem 1.11 in Chapter 1, we have

$$\lim_{n\to\infty} \|\nabla_j^k S_a^n\delta\|_2^{1/n} = \sqrt{\rho(T_b\,|_W)},$$

where $\rho(T_b\,|_W)$ is the spectral radius of the operator T_b restricted to W, and W is the minimal invariant subspace of T_b generated by $\Delta_j^k\delta$, where

$$\Delta_j\lambda(\alpha) := -\lambda(\alpha-e_j)+2\lambda(\alpha)-\lambda(\alpha+e_j), \qquad \alpha\in\mathbb{Z}^s,\ \lambda\in\ell_0(\mathbb{Z}^s).$$

When the symbol of the mask is reducible, the following result often simplifies the computation of the smoothness order of a refinable function in terms of its mask.

Theorem 3.7 *Let a be a finitely supported sequence on $\mathbb{Z}^s$. Suppose for some positive integer l, $\widetilde{a}(z) = (1+z_i)^l\widetilde{b}(z)$, $z=(z_1,\cdots,z_s)$, where $1\le i\le s$ and b is a finitely supported sequence on $\mathbb{Z}^s$. Then the following relation is valid for $k\ge l$:*

$$\lim_{n\to\infty} \|\nabla_i^k S_a^n\delta\|_p^{1/n} = \lim_{n\to\infty} \|\nabla_i^{k-l} S_b^n\delta\|_p^{1/n}.$$

Proof: We observe that

$$\widetilde{\nabla_i^k S_a^n\delta}(z) = (1-z_i)^k\prod_{j=0}^{n-1}\widetilde{a}(z^{2^j}) = (1-z_i)^k\prod_{j=0}^{n-1}(1+z_i^{2^j})^l\prod_{j=0}^{n-1}\widetilde{b}(z^{2^j})$$

$$= (1-z_i^{2^n})^l(1-z_i)^{k-l}\prod_{j=0}^{n-1}\widetilde{b}(z^{2^j}) = (1-z_i^{2^n})^l\widetilde{\nabla_i^{k-l} S_b^n\delta}(z).$$

Therefore,

$$\|\nabla_i^k S_a^n\delta\|_p \le 2^l\|\nabla_i^{k-l} S_b^n\delta\|_p \qquad \forall n\in\mathbb{N}. \tag{3.3.11}$$

On the other hand, without loss of generality, we may assume that $l=1$ and b is supported on $[0,N]^s$ with $N\ge k-1$. It follows that $\nabla_i^{k-1} S_b^n\delta$ is supported on

$[0, 2^n N]^s$. By what has been proved we see that the following relation is valid for z near 0:

$$\widetilde{\nabla_i^{k-1} S_b^n} \delta(z) = \frac{1}{1 - z_i^{2^n}} \widetilde{\nabla_i^k S_a^n} \delta(z) = \sum_{j=0}^{N} z_i^{2^n j} \widetilde{\nabla_i^k S_a^n} \delta(z) + O(z_i^{(N+1)2^n}).$$

Since $\nabla_i^{k-1} S_b^n \delta$ is supported on $[0, 2^n N]^s$, it follows that

$$\|\nabla_i^{k-1} S_b^n \delta\|_p \leq (N+1) \|\nabla_i^k S_a^n \delta\|_p \qquad \forall n \in \mathbb{N}. \tag{3.3.12}$$

Combining (3.3.11) and (3.3.12) together, we obtain the desired result. ■

Combining Theorem 1.5 in Chapter 1 and Theorem 3.5, we have the following result.

Theorem 3.8 *Let a function ϕ be the normalized solution of the refinement equation (3.1.1) with a mask a. Suppose the shifts of ϕ are stable and $\nu_\infty(\phi) = r$ is a positive integer. Then $\phi \notin C^r(\mathbb{R}^s)$.*

Proof: Let $k = r + 1$. By Theorem 3.5 and $\nu_\infty(\phi) = r$, we have

$$\lim_{n \to \infty} \max\{\|\nabla_i^k S_a^n \delta\|_\infty^{1/n} \ : \ 1 \leq i \leq s\} = 2^{-r}.$$

Let W be the minimal A_ε ($\varepsilon \in \Omega$) invariant space generated by $\{\nabla_i^k \delta \ : \ 1 \leq i \leq s\}$ where the operators A_ε are defined in (1.2.4). By Theorem 1.5, there exist two positive constants c_1 and c_2 such that

$$c_1 \|\mathcal{A}^n\|_\infty \leq \max\{\|\nabla_i^k S_a^n \delta\|_\infty \ : \ 1 \leq i \leq s\} \leq c_2 \|\mathcal{A}^n\|_\infty \qquad \forall n \in \mathbb{N}, \tag{3.3.13}$$

where $\mathcal{A} := \{A_\varepsilon|_W \ : \ \varepsilon \in \Omega\}$.

From the above inequalities, it follows that

$$\inf_{n>0} \|\mathcal{A}^n\|_\infty^{1/n} = \lim_{n \to \infty} \|\mathcal{A}^n\|_\infty^{1/n} = \lim_{n \to \infty} \max\{\|\nabla_i^k S_a^n \delta\|_\infty^{1/n} \ : \ 1 \leq i \leq s\} = 2^{-r},$$

Thus, we have $2^{rn}\|\mathcal{A}^n\|_\infty \geq 1$ for all $n \in \mathbb{N}$. Hence (3.3.13) yields that

$$\max\{2^{rn}\|\nabla_i^k S_a^n \delta\|_\infty \ : \ 1 \leq i \leq s\} \geq c_1 > 0 \qquad \forall n \in \mathbb{N}.$$

Combining the above inequality with (3.1.5), we have

$$\begin{aligned} C_1 \max\{\, 2^{rn}\|\nabla_i^k S_a^n \delta\|_\infty \ : \ 1 \leq i \leq s\} &\leq \max\{2^{rn}\|\nabla_{2^{-n}e_i}^k \phi\|_\infty \ : \ 1 \leq i \leq s\} \\ &\leq 2^{rn}\omega_k(\phi, 2^{-n})_\infty \qquad \forall n \in \mathbb{N}. \end{aligned}$$

Hence $2^{rn}\omega_k(\phi, 2^{-n})_\infty \geq c_1 C_1 > 0$ for all $n \in \mathbb{N}$ which implies that $\phi \notin C^r(\mathbb{R}^s)$ since if $\phi \in C^r(\mathbb{R}^s)$, then $\lim_{h\to 0} h^{-r}\omega_k(\phi, h)_\infty = 0$ for any $k > r$. ∎

Finally, in this section, we prove the following result.

Theorem 3.9 *Suppose a function ϕ is a fundamental real-valued function on the real line and ϕ satisfies the refinement equation* (3.1.1) *with an interpolatory refinement mask a supported on $[-3, 3]$. Then $\nu_\infty(\phi) \leq 2$ and therefore, $\phi \notin C^2(\mathbb{R})$.*

Proof: Suppose $\nu_\infty(\phi) > 2$. Then a must satisfy the sum rules of order at least 3 (see [6, 61]). By a simple calculation, it is not difficult to see that the symbol $\widetilde{a}(z)$ can be written as

$$\widetilde{a}(z) = z^{-3}\,(1+z)^3\,\widetilde{c}(z) \quad \text{with} \quad \widetilde{c}(z) := t - 3\,t\,z + (3/8 + 3t)\,z^2 - (1/8 + t)\,z^3,$$

for some $t \in \mathbb{R}$.

By Theorem 3.7 and Theorem 1.5 in Chapter 1, we observe that

$$\sigma_{3,\infty}(a) = \sigma_{0,\infty}(c) = \lim_{n\to\infty} \max\left\{\|B_1 \cdots B_n\|^{1/n} \ : \ B_1, \cdots, B_n \in \{A_0, A_1\}\right\},$$

where A_0 and A_1 are matrices given by

$$A_0 := \begin{bmatrix} t & 3/8 + 3\,t & 0 \\ 0 & -3\,t & -1/8 - t \\ 0 & t & 3/8 + 3\,t \end{bmatrix},$$

and

$$A_1 := \begin{bmatrix} -3t & -1/8-t & 0 \\ t & 3/8+3t & 0 \\ 0 & -3t & -1/8-t \end{bmatrix}.$$

Therefore, it is evident that

$$\sigma_{3,\infty}(a) = \sigma_{0,\infty}(c) \geq \lim_{n\to\infty} \|A_0^n\|^{1/n} =: \rho(A_0),$$

where $\rho(A_0)$ is the spectral radius of A_0. Note that $\lambda = 3/16 + \sqrt{(3/8)^2 + 4(t+8t^2)}/2$ is an eigenvalue of A_0 and

$$\lambda = 3/16 + \sqrt{1/64 + 32(t+1/8)^2}/2 \geq 1/4 \qquad \forall\, t \in \mathbb{R}.$$

This yields

$$\sigma_{3,\infty}(a) = \sigma_{0,\infty}(c) \geq \rho(A_0) \geq 1/4.$$

Since the function ϕ is a fundamental function, the shifts of ϕ are stable. By Theorem 3.5, we have

$$\min\{\,3,\ \nu_\infty(\phi)\,\} = -\log_2 \sigma_{3,\infty}(a) \leq -\log_2(1/4) = 2.$$

This is a contradiction to our assumption $\nu_\infty(\phi) > 2$. Hence, the inequality $\nu_\infty(\phi) \leq 2$ holds true. We are done. ∎

3.4 Optimal Fundamental Refinable Functions

Before proceeding further, we need the following two lemmas.

Lemma 3.10 *Let a be an interpolatory mask on $\mathbb{Z}^s$ supported on $[1-2r, 2r-1]^s$ for some positive integer r. Define a new sequence a_1 on $\mathbb{Z}$ as follows:*

$$a_1(k) = 2^{1-s} \sum_{\alpha_2\in\mathbb{Z}} \cdots \sum_{\alpha_s\in\mathbb{Z}} a(k, \alpha_2, \cdots, \alpha_s), \qquad k \in \mathbb{Z}. \tag{3.4.1}$$

If the mask a satisfies the sum rules of order at least $2r-1$, then a_1 is a univariate interpolatory refinement mask satisfying the sum rules of order $2r-1$. Moreover, if the mask a satisfies the sum rules of order $2r$, then the mask a_1 must be the mask b_r, the unique interpolatory refinement mask supported on $[1-2r, 2r-1]$ and satisfying the sum rules of order $2r$.

Proof: By the definition of sum rules given in (3.1.6), it is easily seen that the sequence a_1 satisfies the same order of sum rules as the sequence a does. Hence, to complete the proof, it suffices to prove that a_1 is a univariate interpolatory refinement mask. Namely, we have to prove that $a_1(2k) = 0$ for all $k \in \mathbb{Z}\backslash\{0\}$. To this end, it suffices to prove that for any ε in Ω such that $\varepsilon = (0, \varepsilon_2, \cdots, \varepsilon_s)$,

$$\sum_{\alpha_2\in\mathbb{Z}} \cdots \sum_{\alpha_s\in\mathbb{Z}} a(2k, 2\alpha_2+\varepsilon_2, \cdots, 2\alpha_s+\varepsilon_s) = 0 \qquad \forall\, k \in \mathbb{Z}\backslash\{0\}. \tag{3.4.2}$$

Let b be a sequence on $\mathbb{Z}$ given by

$$b(k) := \sum_{\alpha_2\in\mathbb{Z}} \cdots \sum_{\alpha_s\in\mathbb{Z}} a(2k, 2\alpha_2+\varepsilon_2, \cdots, 2\alpha_s+\varepsilon_s), \qquad k \in \mathbb{Z}.$$

It is evident that b is supported on $[1-r, r-1]$ since a is supported on $[1-2r, 2r-1]^s$. Note that the mask a is an interpolatory refinement mask which satisfies the sum rules of order $2r-1$. By the definition of sum rules given in (3.1.6), for any integer j such that $0 \le j < 2r-1$, we deduce that

$$\sum_{k\in\mathbb{Z}} b(k)(2k)^j = \sum_{k\in\mathbb{Z}} \sum_{\alpha_2\in\mathbb{Z}} \cdots \sum_{\alpha_s\in\mathbb{Z}} a(2k, 2\alpha_2+\varepsilon_2, \cdots, 2\alpha_s+\varepsilon_s)(2k)^j = \delta(j).$$

This gives us

$$\sum_{k=1-r}^{r-1} b(k)k^j = \delta(j), \qquad 0 \le j < 2r-1. \tag{3.4.3}$$

This linear system has $2r-1$ unknowns $b(1-r), \ldots, b(r-1)$ and $2r-1$ equations, and its coefficient matrix is a Vandermonde matrix. Hence, it has a unique solution. It is easily seen that $b(j) = \delta(j)$, $j = 1-r, \ldots, r-1$ is a solution to the above linear system. This verifies (3.4.2), thereby completing the proof. ∎

Lemma 3.11 *Let a function ϕ in $L_p(\mathbb{R}^s)$ $(1 \le p \le \infty)$ be the normalized solution of the refinement equation* (3.1.1) *with a finitely supported refinement mask a on $\mathbb{Z}^s$. Let the sequence a_1 be given by Equation* (3.4.1) *and ϕ_{a_1} be the normalized solution of the refinement equation* (3.1.1) *with the refinement mask a_1. If the subdivision scheme associated with a converges in the L_p norm, then the subdivision scheme associated with the mask a_1 also converges in the L_p norm. In addition, if the shifts of ϕ are stable, then $\nu_p(\phi) \le \nu_p(\phi_{a_1})$.*

Proof: In the following, we shall prove that for any nonnegative integer k, there exists a positive constant C such that

$$\|\nabla_1^k S_{a_1}^n \delta\|_p \le C 2^{n(1-s)/p} \|\nabla_1^k S_a^n \delta\|_p \qquad \forall\, n \in \mathbb{N}. \tag{3.4.4}$$

From the definition of the subdivision operator given in (3.1.4), we observe that $\widetilde{S_a^n\delta}(z) = \prod_{j=0}^{n-1} \widetilde{a}(z^{2^j})$ for any z in $\mathbb{T}^s$. Therefore, $\widetilde{S_{a_1}^n\delta}(z) = 2^{(1-s)n} \widetilde{S_a^n\delta}(z, 1, \cdots, 1)$ for any z in $\mathbb{T}$ since $\widetilde{a_1}(z) = 2^{1-s}\widetilde{a}(z, 1, \cdots, 1)$ and $\widetilde{S_{a_1}^n\delta}(z) = \prod_{j=0}^{n-1} \widetilde{a_1}(z^{2^j})$ for any z in $\mathbb{T}$. That is,

$$S_{a_1}^n \delta(j) = 2^{(1-s)n} \sum_{\beta_2 \in \mathbb{Z}} \cdots \sum_{\beta_s \in \mathbb{Z}} S_a^n \delta(j, \beta_2, \cdots, \beta_s) \qquad \forall\, j \in \mathbb{Z},\, n \in \mathbb{N}. \tag{3.4.5}$$

Since $\nabla_1 \lambda(\beta) = \lambda(\beta) - \lambda(\beta - e_1)$, $\lambda \in \ell_0(\mathbb{Z}^s)$ where e_1 is the first coordinate unit vector, we have

$$\nabla_1^k S_{a_1}^n \delta(j) = 2^{(1-s)n} \sum_{\beta_2 \in \mathbb{Z}} \cdots \sum_{\beta_s \in \mathbb{Z}} \nabla_1^k S_a^n \delta(j, \beta_2, \cdots, \beta_s).$$

Since the mask a is finitely supported, there exists a positive integer r such that $\operatorname{supp} a \subseteq [-r, r]^s$. It is easily seen that $\operatorname{supp} S_a^n \delta \subseteq [-2^n r, 2^n r]$. Therefore, the above equality can be rewritten as

$$\nabla_1^k S_{a_1}^n \delta(j) = 2^{(1-s)n} \sum_{\beta_2 = -2^n r}^{2^n r} \cdots \sum_{\beta_s = -2^n r}^{2^n r} \nabla_1^k S_a^n \delta(j, \beta_2, \cdots, \beta_s), \qquad j \in \mathbb{Z}.$$

Applying the Hölder inequality to the above sum, we obtain

$$
\begin{aligned}
|\nabla_1^k S_{a_1}^n \delta(j)|^p &\le 2^{n(1-s)p}(2^{n+1}r+1)^{(s-1)p/q} \sum_{\beta_2\in\mathbb{Z}} \cdots \sum_{\beta_s\in\mathbb{Z}} |\nabla_1^k S_a^n \delta(j,\beta_2,\cdots,\beta_s)|^p \\
&\le C_1 2^{n(1-s)} \sum_{\beta_2\in\mathbb{Z}} \cdots \sum_{\beta_s\in\mathbb{Z}} |\nabla_1^k S_a^n \delta(j,\beta_2,\cdots,\beta_s)|^p,
\end{aligned}
$$

where $1/p+1/q=1$ and $C_1=(2r+1)^{(s-1)p/q}$. It follows from the above inequality that

$$
\|\nabla_1^k S_{a_1}^n \delta\|_p \le C_1^{1/p} 2^{n(1-s)/p} \|\nabla_1^k S_a^n \delta\|_p \qquad \forall\, n\in\mathbb{N}.
$$

Therefore, the inequality (3.4.4) holds true. Since the subdivision scheme associated with a converges in the L_p norm, by Theorem 1.7 in Chapter 1, we have

$$
\lim_{n\to\infty} \max\{\ \|\nabla_i S_a^n \delta\|_p^{1/n}\ :\ i=1,\cdots,s\ \} < 2^{s/p}.
$$

Taking $k=1$ in (3.4.4), we get

$$
\lim_{n\to\infty} \|\nabla_1 S_{a_1}^n \delta\|_p^{1/n} \le 2^{(1-s)/p} \lim_{n\to\infty} \max\{\ \|\nabla_i S_a^n \delta\|_p^{1/n}\ :\ i=1,\cdots,s\ \} < 2^{1/p}.
$$

Hence, the subdivision scheme associated with the mask a_1 converges in the L_p norm. In particular, we have $\phi_{a_1}\in L_p(\mathbb{R})$.

Note that $\sigma_{k,p}(a_1) := \lim_{n\to\infty} \|\nabla_1^k S_{a_1}^n \delta\|_p^{1/n}$ and

$$
\sigma_{k,p}(a) := \lim_{n\to\infty} \max\{\ \|\nabla_i^k S_a^n \delta\|_p^{1/n}\ :\ i=1,\cdots,s\ \} \ge \lim_{n\to\infty} \|\nabla_1^k S_a^n \delta\|_p^{1/n}.
$$

Hence, the inequality (3.4.4) gives rise to

$$
\sigma_{k,p}(a_1) \le 2^{(1-s)/p}\sigma_{k,p}(a) \qquad \forall\, k\in\mathbb{N}\cup\{0\}.
$$

Take k to be a positive integer greater than $\nu_p(\phi)$. Since the shifts of ϕ are stable, it follows from Theorem 3.5 that

$$
\nu_p(\phi_{a_1}) \ge 1/p - \log_2 \sigma_{k,p}(a_1) \ge s/p - \log_2 \sigma_{k,p}(a) = \nu_p(\phi),
$$

as desired. ∎

Combining the above results and Theorem 3.9, we have the following result:

Corollary 3.12 *Suppose a function ϕ is a fundamental real-valued function and satisfies the refinement equation* (3.1.1) *with an interpolatory refinement mask a supported on $[-3,3]^s$. Then $\nu_\infty(\phi) \le 2$ and therefore, ϕ does not belong to $C^2(\mathbb{R}^s)$.*

Proof: Let the sequence a_1 on $\mathbb{Z}$ be given in (3.4.1). Suppose $\nu_\infty(\phi) > 2$. Then the mask a must satisfy the sum rules of order at least 3. Therefore, it follows from Lemma 3.10 that a_1 is an interpolatory mask. Let ϕ_{a_1} be the normalized solution of (3.1.1) with the mask a_1. Then by Lemma 3.11, the subdivision scheme associated with a_1 converges in the L_∞ norm which implies that the function ϕ_{a_1} is a fundamental function. From Lemma 3.11, we also have $\nu_\infty(\phi) \le \nu_\infty(\phi_{a_1})$. It follows from Theorem 3.9 that $\nu_\infty(\phi) \le \nu_\infty(\phi_{a_1}) \le 2$. This is a contradiction to our assumption $\nu_\infty(\phi) > 2$. Therefore, the inequality $\nu_\infty(\phi) \le 2$ holds true. ∎

Corollary 3.12 says that there is no C^2 fundamental refinable function supported on $[-3,3]^s$. This result also implies that if a function ϕ is an orthogonal scaling function supported on $[0,3]^s$, then $\nu_2(\phi) \le 1$ and therefore, $\phi \notin C^1(\mathbb{R}^s)$.

Theorem 3.13 *Let ϕ be a fundamental refinable function with a finitely supported interpolatory mask a. Suppose a is supported on $[1-2r, 2r-1]^s$ for some positive integer r and the mask a satisfies the sum rules of order $2r-1$. Let a sequence a_1 on $\mathbb{Z}$ be given by Equation* (3.4.1) *and let ϕ_{a_1} be the normalized solution of the refinement equation* (3.1.1) *with the mask a_1. Then the function ϕ_{a_1} is a fundamental function and*

$$\nu_p(\phi) \le \nu_p(\phi_{a_1}) \qquad \forall\, 1 \le p \le \infty.$$

Moreover, if the mask a satisfies the sum rules of order $2r$, then

$$\nu_p(\phi) \le \nu_p(\phi_{b_r}) \qquad \forall\, 1 \le p \le \infty.$$

In other words, the inequality $\nu_p(\phi) \le \nu_p(\phi_{t_r})$ holds true where t_r is the tensor product interpolatory mask given in (3.1.7).

Proof: By Lemma 3.10, we see that the mask a_1 is an interpolatory refinement mask. Since the function ϕ is fundamental, the shifts of ϕ are stable. By Lemma 3.11, the subdivision scheme associated with the mask a_1 converges in the L_p norm for any p such that $1 \le p \le \infty$. Hence ϕ_{a_1}, the normalized solution of the refinement equation (3.1.1) with the interpolatory refinement mask a_1, is continuous and therefore fundamental. It follows from Lemma 3.11 that $\nu_p(\phi) \le \nu_p(\phi_{a_1})$ for any $1 \le p \le \infty$.

If the mask a satisfies the sum rules of order $2r$, by Lemma 3.10, then the sequence a_1 must be the mask b_r. Hence, by Lemma 3.11, $\nu_p(\phi) \le \nu_p(\phi_{b_r})$ for any p such that $1 \le p \le \infty$. ∎

3.5 Construction of Bivariate Optimal Interpolatory Masks

Our construction of optimal interpolatory masks relies on solvability of certain linear systems of equations. To facilitate our discussion, we establish two auxiliary lemmas first. In what follows, the set of nonnegative integers is denoted by $\mathbb{Z}_+$, and the cardinality of a set E is denoted by $\#E$.

Lemma 3.14 *Let $l_1, \ldots, l_r$ be distinct parallel lines in $\mathbb{R}^2$, and let T be a subset of $l_1 \cup \cdots \cup l_r$ such that $\#(T \cap l_j) = j$ for each $j = 1, \ldots, r$. Suppose p is a polynomial in two variables of (total) degree at most $r-1$. If p vanishes on T, then p vanishes everywhere. Consequently, the square matrix $(t_1^{\nu_1} t_2^{\nu_2})_{(t_1,t_2)\in T, 0\le \nu_1+\nu_2 \le r-1}$ is nonsingular.*

Proof: The proof proceeds by induction on r. The statements are obviously true for $r = 1$. Let $r > 1$ and assume that the lemma has been verified for $r - 1$.

After a suitable coordinate transform, we may assume without loss of generality that the equations of the lines $l_1, \dots, l_r$ are given by

$$x_1 - \lambda_j = 0, \qquad j = 1, \dots, r,$$

where $\lambda_1, \dots, \lambda_r$ are pairwise distinct real numbers. We observe that $p(\lambda_r, x_2)$ is a polynomial in x_2 of degree at most $r-1$. But it has at least r zeros. Hence $p(\lambda_r, x_2) = 0$ for all $x_2 \in \mathbb{R}$. This shows that $p(x_1, x_2)$ is divisible by $x_1 - \lambda_r$. Suppose

$$p(x_1, x_2) = (x_1 - \lambda_r) q(x_1, x_2).$$

Then q is a polynomial of degree at most $r-2$. Let $T' := T \setminus l_r$. Then q vanishes on T'. By the induction hypothesis we obtain $q = 0$. It follows that $p = 0$, as desired.

In order to prove that the matrix $(t_1^{\nu_1} t_2^{\nu_2})_{(t_1,t_2)\in T, 0 \le \nu_1+\nu_2 \le r-1}$ is nonsingular, it suffices to show that the linear system of homogeneous equations

$$\sum_{0 \le \nu_1+\nu_2 \le r-1} c_{\nu_1,\nu_2} t_1^{\nu_1} t_2^{\nu_2} = 0, \qquad (t_1, t_2) \in T,$$

only has the trivial solution for c_{ν_1,ν_2} $(0 \le \nu_1 + \nu_2 \le r-1)$. For this purpose, let

$$p(x_1, x_2) := \sum_{0 \le \nu_1+\nu_2 \le r-1} c_{\nu_1,\nu_2} x_1^{\nu_1} x_2^{\nu_2}.$$

Then $p(x_1, x_2)$ is a polynomial of total degree at most $r-1$ and it vanishes on T. By what has been proved, $p = 0$. This completes the proof. ∎

Lemma 3.15 *For a positive integer r, let*

$$\Gamma_r := \{(\mu_1, \mu_2) \in \mathbb{Z}_+^2 : \mu_1 + \mu_2 \le 2r - 1\} \setminus \{(0, 2j-1) : j = 1, \dots, r\},$$

and let p be a linear combination of the monomials $x_1^{\mu_1} x_2^{\mu_2}$, $(\mu_1, \mu_2) \in \Gamma_r$. Let $l_1, \dots, l_{2r}$ be the lines $x_1 - \lambda_j = 0$, $j = 1, \dots, 2r$, where $\lambda_1, \dots, \lambda_{2r}$ are mutually distinct nonzero real numbers. Suppose T is a subset of the union of these lines such that $\#(T \cap l_{2j-1}) = \#(T \cap l_{2j}) = 2j-1$ for each $j = 1, \dots, r$. If p vanishes on T, then p vanishes everywhere. Consequently, the square matrix $(t_1^{\mu_1} t_2^{\mu_2})_{(t_1,t_2)\in T, (\mu_1,\mu_2)\in\Gamma_r}$ is nonsingular.

Proof: The proof proceeds by induction on r. The statements are obviously true for $r = 1$. Let $r > 1$ and assume that the lemma has been verified for $r - 1$.

Since p does not contain a term associated to x_2^{2r-1}, the degree of the univariate polynomial $p(\lambda_{2r}, x_2)$ is at most $2r-2$. But it has at least $2r-1$ zeros. So $p(\lambda_{2r}, x_2) = 0$ for all $x_2 \in \mathbb{R}$. This shows that $p(x_1, x_2)$ is divisible by $x_1 - \lambda_{2r}$. Suppose

$$p(x_1, x_2) = (x_1 - \lambda_{2r})u(x_1, x_2).$$

Then u is a polynomial of (total) degree at most $2r - 2$. But $p(\lambda_{2r-1}, x_2)$ has at least $2r - 1$ zeros; hence so does $u(\lambda_{2r-1}, x_2)$. This shows that $u(x_1, x_2)$ is divisible by $x_1 - \lambda_{2r-1}$. Suppose $u(x_1, x_2) = (x_1 - \lambda_{2r-1})q(x_1, x_2)$. It follows that

$$p(x_1, x_2) = (x_1 - \lambda_{2r})(x_1 - \lambda_{2r-1})q(x_1, x_2).$$

Since $\lambda_{2r}\lambda_{2r-1} \neq 0$, we see that q is a linear combination of the monomials $x_1^{\mu_1}x_2^{\mu_2}$, $(\mu_1, \mu_2) \in \Gamma_{r-1}$. Moreover, q vanishes on $T' := T \cap (l_1 \cup \cdots \cup l_{2r-2})$. By the induction hypothesis we obtain $q = 0$. It follows that $p = 0$, as desired. The proof for the last statement is analogous to that for Lemma 3.14. ∎

We are in a position to describe a general method for the construction of bidimensional optimal interpolatory masks.

Theorem 3.16 *For each positive integer r, there exists a unique interpolatory mask g_r with the following properties:*

(a) *g_r is supported on the square $\{(\alpha_1, \alpha_2) \in \mathbb{Z}^2 : |\alpha_1| + |\alpha_2| \leq 2r\}$;*

(b) *g_r is symmetric about the two coordinate axes;*

(c) *g_r satisfies the sum rules of order $2r$.*

Proof: Let Ω denote the set $\{(0,0),(1,0),(0,1),(1,1)\}$. An interpolatory mask a satisfies the sum rules of order $2r$ if and only if

$$\sum_{\beta_1,\beta_2\in\mathbb{Z}} a(\varepsilon_1+2\beta_1,\varepsilon_2+2\beta_2)(\varepsilon_1+2\beta_1)^{\mu_1}(\varepsilon_2+2\beta_2)^{\mu_2}=\delta_{\mu 0} \tag{3.5.1}$$

for all $(\varepsilon_1,\varepsilon_2)\in\Omega$ and all $(\mu_1,\mu_2)\in\mathbb{Z}_+^2$ with $\mu_1+\mu_2\le 2r-1$. If a is symmetric about the two coordinate axes, then (3.5.1) is valid whenever one of μ_1 and μ_2 is an odd number. Thus, in such a case, we only have to verify (3.5.1) when both μ_1 and μ_2 are even.

Let us construct the desired mask. Set $g_r(0):=1$ and $g_r(2\beta):=0$ for $\beta\in\mathbb{Z}^2\backslash\{0\}$. Then g_r is an interpolatory mask and satisfies (3.5.1) for $(\varepsilon_1,\varepsilon_2)=(0,0)$ and all $\mu\in\mathbb{Z}_+^2$. Set $g_r(\alpha_1,\alpha_2):=0$ for $|\alpha_1|+|\alpha_2|>2r$. Then g_r satisfies condition (a). Furthermore, set

$$g_r(2\beta_1+1,2\beta_2)=\begin{cases} b_r(2\beta_1+1), & \beta_2=0,\\ 0, & \beta_2\in\mathbb{Z}\setminus\{0\},\end{cases}$$

where b_r is the unique interpolatory mask supported on $[1-2r,2r-1]$ and satisfying the sum rules of order $2r$. Since b_r satisfies the sum rules of order $2r$, we deduce that g_r satisfies (3.5.1) for $(\varepsilon_1,\varepsilon_2)=(1,0)$ and $\mu_1+\mu_2\le 2r-1$. Similarly, set

$$g_r(2\beta_1,2\beta_2+1)=\begin{cases} b_r(2\beta_2+1), & \beta_1=0,\\ 0, & \beta_1\in\mathbb{Z}\setminus\{0\}.\end{cases}$$

Then g_r satisfies (3.5.1) for $(\varepsilon_1,\varepsilon_2)=(0,1)$ and $\mu_1+\mu_2\le 2r-1$.

We assume that g_r is symmetric about the two coordinate axes. Thus, it remains to determine $g_r(1+2\beta_1,1+2\beta_2)$ for $0\le\beta_1+\beta_2\le r-1$. Suppose $\mu_1=2\nu_1$ and $\mu_2=2\nu_2$, where ν_1,ν_2 are nonnegative integers. For $(\varepsilon_1,\varepsilon_2)=(1,1)$, (3.5.1) reduces to the following system of equations for $a(1+2\beta_1,1+2\beta_2)$ $(0\le\beta_1+\beta_2\le r-1)$:

$$\sum_{0\le\beta_1+\beta_2\le r-1} a(1+2\beta_1,1+2\beta_2)(1+2\beta_1)^{2\nu_1}(1+2\beta_2)^{2\nu_2}=1/4\delta_{\nu 0}, \tag{3.5.2}$$

where $\nu = (\nu_1, \nu_2)$. In this case, (3.5.1) is valid for $0 \le \mu_1 + \mu_2 \le 2r - 1$ if and only if (3.5.2) is true for $0 \le \nu_1 + \nu_2 \le r - 1$. Let T be the set

$$\{((1+2\beta_1)^2, (1+2\beta_2)^2) \,:\, (\beta_1, \beta_2) \in \mathbb{Z}_+^2,\, \beta_1 + \beta_2 \le r - 1\}.$$

Then T intersects the line $x_1 - (2j-1)^2 = 0$ at exactly $r + 1 - j$ points for each $j = 1, \ldots, r$. Thus, the conditions of Lemma 3.14 are satisfied. By Lemma 3.14 the square matrix

$$\left((1+2\beta_1)^{2\nu_1}(1+2\beta_2)^{2\nu_2}\right)_{0 \le \beta_1+\beta_2 \le r-1, 0 \le \nu_1+\nu_2 \le r-1}$$

is nonsingular. Therefore, the linear system of equations in (3.5.2) with $(\nu_1, \nu_2) \in \mathbb{Z}_+^2$ and $\nu_1 + \nu_2 \le r - 1$ is uniquely solvable. Let

$$\left(g_r(1+2\beta_1, 1+2\beta_2)\right)_{0 \le \beta_1+\beta_2 \le r-1}$$

be the solution. This completes our construction of g_r. Obviously, g_r satisfies conditions (a), (b), and (c).

Finally, let us show the uniqueness of such a mask. Let a be an interpolatory mask satisfying conditions (a), (b), and (c). We wish to verify that for $(\varepsilon_1, \varepsilon_2) \in \Omega$

$$a(\varepsilon_1 + 2\beta_1, \varepsilon_2 + 2\beta_2) = g_r(\varepsilon_1 + 2\beta_1, \varepsilon_2 + 2\beta_2) \qquad \forall\, (\beta_1, \beta_2) \in \mathbb{Z}^2. \tag{3.5.3}$$

From the preceding analysis, this is certainly true for $(\varepsilon_1, \varepsilon_2) = (0, 0)$ or $(1, 1)$. Consider the case $(\varepsilon_1, \varepsilon_2) = (1, 0)$. By symmetry, (3.5.1) reduces to the following system of equations for c_{β_1,β_2} $(0 \le \beta_1 + \beta_2 \le r - 1)$:

$$\sum_{0 \le \beta_1+\beta_2 \le r-1} c_{\beta_1,\beta_2}(1+2\beta_1)^{2\nu_1}(2\beta_2)^{2\nu_2} = \delta_{\nu 0}, \qquad \nu_1 + \nu_2 \le r - 1, \tag{3.5.4}$$

where

$$c_{\beta_1,\beta_2} = \begin{cases} 2a(2\beta_1+1, 0), & \beta_2 = 0, \\ 4a(2\beta_1+1, 2\beta_2), & \beta_2 > 0. \end{cases}$$

Since the matrix

$$((1+2\beta_1)^{2\nu_1}(2\beta_2)^{2\nu_2})_{0\le\beta_1+\beta_2\le r-1, 0\le\nu_1+\nu_2\le r-1}$$

is nonsingular, the linear system of equations in (3.5.4) is uniquely solvable. On the other hand, if we choose

$$c_{\beta_1,\beta_2} = \begin{cases} 2b_r(2\beta_1+1,0), & \beta_2 = 0, \\ 0, & \beta_2 > 0, \end{cases}$$

then $(c_{\beta_1,\beta_2})_{0\le\beta_1+\beta_2\le r-1}$ is a solution of (3.5.4). By the uniqueness of the solution, this shows that (3.5.3) is true for $(\varepsilon_1,\varepsilon_2) = (1,0)$. For the case $(\varepsilon_1,\varepsilon_2) = (0,1)$, the proof is similar. ∎

From the above proof we see that g_r is minimally supported among all the optimal interpolatory masks having the indicated symmetry and supported on the square $[1-2r, 2r-1]^2$. However, if we relax the requirement on symmetry, then there exists a family of optimal interpolatory masks with smaller support.

Theorem 3.17 *For each positive integer r, there exists a unique interpolatory mask h_r with the following properties:*

(a) *h_r is supported on the rectangle $\{(\alpha_1,\alpha_2) : |\alpha_1+\alpha_2| \le 2r,\ |\alpha_1-\alpha_2| \le 2r-1\}$;*

(b) *h_r satisfies the sum rules of order $2r$.*

Proof: Suppose a is an interpolatory mask satisfying conditions (a) and (b). In particular, $a(0) = 1$ and $a(2\alpha) = 0$ for $\alpha \in \mathbb{Z}\setminus\{0\}$. Let $\Omega := \{(0,0),(1,0),(0,1),(1,1)\}$. For $(\varepsilon_1,\varepsilon_2) \in \Omega$, let

$$T_{\varepsilon_1,\varepsilon_2} := ((\varepsilon_1,\varepsilon_2) + 2\mathbb{Z}^2) \cap \{(\alpha_1,\alpha_2) : |\alpha_1+\alpha_2| \le 2r,\ |\alpha_1-\alpha_2| \le 2r-1\}.$$

Let

$$\Gamma := \{(\mu_1,\mu_2) \in \mathbb{Z}_+^2 : \mu_1+\mu_2 \le 2r-1\} \setminus \{(0,2j-1) : j = 1,\ldots,r\}.$$

Since a satisfies the sum rules of order $2r$, the following relations are valid for $(\varepsilon_1, \varepsilon_2) \in \Omega$:

$$\sum_{(\alpha_1,\alpha_2)\in T_{\varepsilon_1,\varepsilon_2}} a(\alpha_1, \alpha_2)\alpha_1^{\mu_1}\alpha_2^{\mu_2} = \delta_{\mu 0} \qquad \forall\, (\mu_1, \mu_2) \in \Gamma. \tag{3.5.5}$$

We observe that $T_{1,0}$ intersects the lines $x_1 \pm (2j-1) = 0$ at exactly $2r-(2j-1)$ points for each $j = 1, \ldots, r$. Thus, the conditions of Lemma 3.15 are satisfied. Hence the square matrix $(\alpha_1^{\mu_1}\alpha_2^{\mu_2})_{(\alpha_1,\alpha_2)\in T_{1,0},(\mu_1,\mu_2)\in\Gamma}$ is nonsingular. Therefore, the values of $a(\alpha)$ for $\alpha \in T_{1,0}$ are uniquely determined by (3.5.5). On the other hand, a solution for the values of a on $T_{1,0}$ can be easily found as follows: $a(\alpha_1, 0) = b_r(\alpha_1)$ for $\alpha_1 \in 1+2\mathbb{Z}$ and $a(\alpha_1, \alpha_2) = 0$ for $\alpha_1 \in 1 + 2\mathbb{Z}$ and $\alpha_2 \in 2\mathbb{Z} \setminus \{0\}$. In the same way we can show that $a(0, \alpha_2) = b_r(\alpha_2)$ for $\alpha_2 \in 1+2\mathbb{Z}$ and $a(\alpha_1, \alpha_2) = 0$ for $\alpha_1 \in 2\mathbb{Z} \setminus \{0\}$ and $\alpha_2 \in 1+2\mathbb{Z}$. It remains to determine the values of a on $T_{1,1}$. We observe that $T_{1,1}$ also intersects the lines $x_1 \pm (2j-1) = 0$ at exactly $2r-(2j-1)$ points for each $j = 1, \ldots, r$. Thus, Lemma 3.15 is applicable and $a(\alpha)$ $(\alpha \in T_{1,1})$ are uniquely determined by (3.5.5) with $(\varepsilon_1, \varepsilon_2) = (1, 1)$. This completes the proof for uniqueness.

Let h_r denote the unique solution determined in the preceding process. We claim that h_r is symmetric about the origin, *i.e.*, $h_r(\alpha) = h_r(-\alpha)$ for all $\alpha \in \mathbb{Z}$. This is certainly true for $\alpha \in T_{0,0} \cup T_{1,0} \cup T_{0,1}$. To verify $h_r(\alpha) = h_r(-\alpha)$ for $\alpha \in T_{1,1}$, we set $a(\alpha) := h_r(-\alpha)$ for $\alpha \in T_{1,1}$. Then $(a(\alpha))_{\alpha\in T_{1,1}}$ satisfies the linear system of equations in (3.5.5) with $(\varepsilon_1, \varepsilon_2) = (1, 1)$. By the uniqueness of the solution, we obtain $a(\alpha) = h_r(\alpha)$ for all $\alpha \in T_{1,1}$, as desired. (A similar reasoning shows that h_r is symmetric about the line $x_1 - x_2 = 0$.)

It remains to verify that h_r satisfies the sum rules of order $2r$. Obviously, h_r satisfies (3.5.1) for all $(\varepsilon_1, \varepsilon_2) \in \Omega \setminus \{(1, 1)\}$ and $(\mu_1, \mu_2) \in \mathbb{Z}_+^2$ with $\mu_1 + \mu_2 \le 2r - 1$. Let us show that this is also true for $(\varepsilon_1, \varepsilon_2) = (1, 1)$. Note that Π_{2r-1} is the linear span of the set

$$\left\{ x_1^{\mu_1}x_2^{\mu_2} \,:\, (\mu_1, \mu_2) \in \Gamma \cup \{(0, 2j-1) \,:\, j = 1, \ldots, r\} \right\}.$$

Thus, it suffices to show that

$$\sum_{(\alpha_1,\alpha_2)\in T_{1,1}} h_r(\alpha_1,\alpha_2)\alpha_1^{\mu_1}\alpha_2^{\mu_2} = 0$$

whenever $(\mu_1, \mu_2) = (0, 2j-1)$ for $j = 1, \dots, r$. This is indeed true, since h_r is symmetric about the origin. The proof of the theorem is complete. ∎

The above proof tells us that h_r is minimally supported among all the masks which are supported on $[1-2r, 2r-1]^2$ and satisfy the sum rules of the optimal order $2r$. If we set $a(\alpha_1, \alpha_2) := (h_r(\alpha_1,\alpha_2) + h_r(-\alpha_1,\alpha_2))/2$ for $(\alpha_1, \alpha_2) \in \mathbb{Z}^2$, then a satisfies all the conditions in Theorem 3.16. By the uniqueness of the solution, we obtain $a = g_r$. This shows that $g_r(\alpha_1,\alpha_2) = (h_r(\alpha_1,\alpha_2) + h_r(-\alpha_1,\alpha_2))/2$ for all $(\alpha_1,\alpha_2) \in \mathbb{Z}^2$. A similar argument shows that g_r is symmetric about the two lines $x_1 - x_2 = 0$ and $x_1 + x_2 = 0$.

From our construction, we see that the numbers of nonzero coefficients of g_r and h_r are $2r^2 + 6r + 1$ and $2r^2 + 4r + 1$, respectively.

3.6 Examples, Figures and Applications

The masks g_1 and h_1 are well known. The mask g_1 induces the bivariate hat function which is defined in (1.1.3) with $s = 2$ and the the mask h_2 induces the well known three direction box spline function, see [5].

In this section we provide details for the interpolatory masks h_2, g_2, h_3, and g_3. In what follows, the refinable function corresponding to a given mask a is denoted by ϕ_a. We shall use the results in Section 3.3 to compute the L_2 smoothness order of ϕ_{g_r} and ϕ_{h_r}. It turns out that ϕ_{g_r} $(r = 1, 2, \dots, 12)$ attain the optimal L_2 smoothness order. Note that $\nu_2(\phi) > k+1$ implies $\phi \in C^k(\mathbb{R}^2)$. In passing, we mention that given any multivariate interpolatory mask a, for any positive integer r, it is easy to obtain a new interpolatory mask b such that $\widetilde{b}(z) = (\widetilde{a}(z))^r \widetilde{c}_r(z), z \in \mathbb{T}^s$ where $\widetilde{c}_r(z)$

can be explicitly expressed by using $\widetilde{a}(z)$. See Proposition 3.7 in Han [48] and [50] for detailed discussion. Such method of constructing interpolatory masks from known ones was further discussed by Ji, Riemenschneider and Shen in [57].

Example 3.18 The refinement mask h_2 is given by

$$\begin{bmatrix} 0 & 0 & 0 & -\frac{1}{16} & -\frac{1}{16} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{8} & \frac{9}{16} & \frac{1}{2} & 0 & -\frac{1}{16} \\ -\frac{1}{16} & 0 & \frac{9}{16} & 1 & \frac{9}{16} & 0 & -\frac{1}{16} \\ -\frac{1}{16} & 0 & \frac{1}{2} & \frac{9}{16} & \frac{1}{8} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{16} & -\frac{1}{16} & 0 & 0 & 0 \end{bmatrix}.$$

Then h_2 satisfies the sum rules of order 4 and $\nu_2(\phi_{h_2}) \approx 2.44077$. Therefore, h_2 induces a C^1 interpolatory subdivision scheme.

Example 3.19 The refinement mask g_2 is given by

$$\begin{bmatrix} 0 & 0 & -\frac{1}{32} & -\frac{1}{16} & -\frac{1}{32} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{32} & 0 & \frac{5}{16} & \frac{9}{16} & \frac{5}{16} & 0 & -\frac{1}{32} \\ -\frac{1}{16} & 0 & \frac{9}{16} & 1 & \frac{9}{16} & 0 & -\frac{1}{16} \\ -\frac{1}{32} & 0 & \frac{5}{16} & \frac{9}{16} & \frac{5}{16} & 0 & -\frac{1}{32} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{32} & -\frac{1}{16} & -\frac{1}{32} & 0 & 0 \end{bmatrix}.$$

Then g_2 satisfies the sum rules of order 4 and $\nu_2(\phi_{g_2}) \approx 2.44077$.

Example 3.20 The refinement mask h_3 is given by

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \frac{3}{256} & \frac{3}{256} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{3}{256} & -\frac{25}{256} & -\frac{3}{32} & 0 & \frac{1}{128} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{3}{256} & 0 & \frac{3}{16} & \frac{75}{128} & \frac{63}{128} & 0 & -\frac{3}{32} & 0 & \frac{3}{256} \\ \frac{3}{256} & 0 & -\frac{25}{256} & 0 & \frac{75}{128} & 1 & \frac{75}{128} & 0 & -\frac{25}{256} & 0 & \frac{3}{256} \\ \frac{3}{256} & 0 & -\frac{3}{32} & 0 & \frac{63}{128} & \frac{75}{128} & \frac{3}{16} & 0 & -\frac{3}{256} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{128} & 0 & -\frac{3}{32} & -\frac{25}{256} & -\frac{3}{256} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{3}{256} & \frac{3}{256} & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then h_3 satisfies the sum rules of order 6, $\nu_2(\phi_{h_3}) \approx 3.04845$ and $\phi_{h_3} \in C^2$.

Example 3.21 The refinement mask g_3 is given by

$$\begin{bmatrix} 0 & 0 & 0 & 0 & \frac{3}{512} & \frac{3}{256} & \frac{3}{512} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{256} & 0 & -\frac{27}{512} & -\frac{25}{256} & -\frac{27}{512} & 0 & \frac{1}{256} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{3}{512} & 0 & -\frac{27}{512} & 0 & \frac{87}{256} & \frac{75}{128} & \frac{87}{256} & 0 & -\frac{27}{512} & 0 & \frac{3}{512} \\ \frac{3}{256} & 0 & -\frac{25}{256} & 0 & \frac{75}{128} & 1 & \frac{75}{128} & 0 & -\frac{25}{256} & 0 & \frac{3}{256} \\ \frac{3}{512} & 0 & -\frac{27}{512} & 0 & \frac{87}{256} & \frac{75}{128} & \frac{87}{256} & 0 & -\frac{27}{512} & 0 & \frac{3}{512} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{256} & 0 & -\frac{27}{512} & -\frac{25}{256} & -\frac{27}{512} & 0 & \frac{1}{256} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{3}{512} & \frac{3}{256} & \frac{3}{512} & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then g_3 satisfies the sum rules of order 6, $\nu_2(\phi_{g_3}) \approx 3.17513$ and $\phi_{g_3} \in C^2$.

Comparison Results								
r	$\nu_2(t_r)$	$\nu_2(h_r)$	$\nu_2(g_r)$	$\nu_2(RS_r)$	$N(t_r)$	$N(h_r)$	$N(g_r)$	$N(RS_r)$
1	1.50000	1.50000	1.50000	1.50000	9	7	9	7
2	2.44077	2.44077	2.44077	2.44077	25	17	21	31
3	3.17513	3.04845	3.17513	3.17513	49	31	37	49
4	3.79313	3.43242	3.79313	3.79313	81	49	57	133
5	4.34408	3.79464	4.34408	4.34408	121	71	81	175
6	4.86202	4.15622	4.86202	4.86202	169	97	109	307
7	5.36283	4.52217	5.36283	5.36283	225	127	141	373
8	5.85293	4.89133	5.85293	5.85293	289	161	177	553
9	6.33524	5.26021	6.33524	5.89529	361	199	217	643
10	6.81144	5.62339	6.81144	6.42641	441	241	261	871
11	7.28260	5.97253	7.28260	6.17848	529	287	309	985
12	7.74953	6.29948	7.74953	6.68093	625	337	361	1261

Table 3.1: Comparison results among interpolatory subdivision schemes by tensor product, Riemenschneider and Shen [89] and Han and Jia [52].

Recall that b_r denotes the interpolatory mask supported on $[1-2r, 2r-1]$ as constructed by Deslauriers and Dubuc in [30]. Let t_r be the tensor product of b_r, *i.e.*, $t_r(\alpha_1, \alpha_2) = b_r(\alpha_1)b_r(\alpha_2)$ for $(\alpha_1, \alpha_2) \in \mathbb{Z}^2$. Let RS_r denote the interpolatory mask supported on $[1-2r, 2r-1]^2$ as constructed by Riemenschneider and Shen in [89]. The following table gives a comparison of our masks h_r and g_r with t_r and RS_r. All these masks are supported on the square $[1-2r, 2r-1]^2$. For convenience, we use $\nu_2(a)$ to denote $\nu_2(\phi_a)$. Also we use $N(a)$ to denote the number of nonzero coefficients in the refinement mask a. The values of $\nu_2(t_r) = \nu_2(b_r)$ are taken from [42] and the values of $\nu_2(RS_r)$ for $r = 2, \cdots, 8$ are taken from [89]. The values of $\nu_2(RS_r)$ for $r = 9, \cdots, 12$ are taken from [74].

From the above table we conclude that ϕ_{g_r} $(r = 1, \ldots, 12)$ attain the optimal L_2 smoothness order. In passing, we mention that without solving any system of linear equations an explicit formula of the interpolatory masks g_r can be easily found by using a similar idea as in [88].

In the following, we shall apply the above various interpolatory subdivision schemes to generate bidimensional surfaces.

By RS_2 we denote the interpolatory mask supported on $[-3,3]^2$ and given by Riemenschneider and Shen [89]. By $Butterfly$ we denote the butterfly interpolatory mask given by Dyn, Gregory and Levin in [39].

Given any interpolatory refinement mask a, for any given sequence $b \in \ell(\mathbb{Z}^s)$, we construct a function

$$f_b(x) = \sum_{\beta \in \mathbb{Z}^s} b(\beta)\phi_a(x-\beta) = \sum_{\beta \in Z_s} S_a^n\delta(\beta)\phi_a(2^n x - \beta), \qquad x \in \mathbb{R}^s,$$

which interpolates the sequence b with $f_b(\beta) = b(\beta)$ for all $\beta \in \mathbb{Z}^s$. Then we calculate the value of f_b at any dyadic rational number by the following iterative subdivision scheme formula:

$$f_b(2^{-n}\beta) = S_a^n b(\beta) \qquad \forall\, n \in \mathbb{N}, \beta \in \mathbb{Z}^s,$$

In our experiment, the initial data is given by

$$b(i,j) = 25 - i^2 + j^2 \qquad \text{for} \quad i^2 + j^2 \le 25,$$

and otherwise $b(i,j) = 0$.

The comparison results are given in Tables 3.2 and 3.3.

In Table 3.2, the first row of numbers refers to the nonzero elements in each mask. The first column of numbers refers to the iteration step in the subdivision schemes and other numbers refer to the actual CPU time in second when applying the subdivision scheme with a mask a to the iteration step i.

step	RS_2	$Butterfly$	t_2	g_2	h_2
Elm	31	25	25	21	17
3	11.67	10.00	10.00	8.33	6.67
4	45.00	38.33	38.33	33.33	28.33
5	188.33	156.67	156.67	135.00	113.33
6	765.00	635.00	631.67	548.33	458.33

Table 3.2: Comparison results on CPU time in second when using the subdivision schemes with the masks RS_2, $Butterfly$, t_2, g_2 and h_2.

step	RS_2	$Butterfly$	t_2	g_2	h_2
Ratio	1.8235	1.4705	1.4705	1.235	1
3	1.7500	1.5000	1.5000	1.2500	1
4	1.5882	1.3529	1.3529	1.1765	1
5	1.6618	1.3824	1.3824	1.1912	1
6	1.6691	1.3855	1.3782	1.1964	1

Table 3.3: Ratio comparison between the masks RS_2, $Butterfly$, t_2, g_2 and h_2.

In Table 3.3, the first row of numbers refers to $N(a)/N(h_2)$. The first column refers to the iteration step. For example, if the step is 3, we calculate the value of f_b at all the points $2^{-3}(\beta_1, \beta_2)$ with $\max\{|\beta_1|, |\beta_2|\} \leq 5 \cdot 2^3$. The number determined by the mask a and the iteration number i means the ratio between the CPU time when applying the subdivision scheme with the mask a to the iteration step i and the CPU time when applying the subdivision scheme with the mask h_2 to the iteration step i. It is convincing that this ratio is very close to the ratio $N(a)/N(h_2)$. Thus, with less nonzero numbers in the mask, we use less CPU time to generate a surface. See Figure 3.2 for the surfaces generated by using different masks.

Finally the fundamental functions in our examples are given in Figures 3.3 – 3.6.

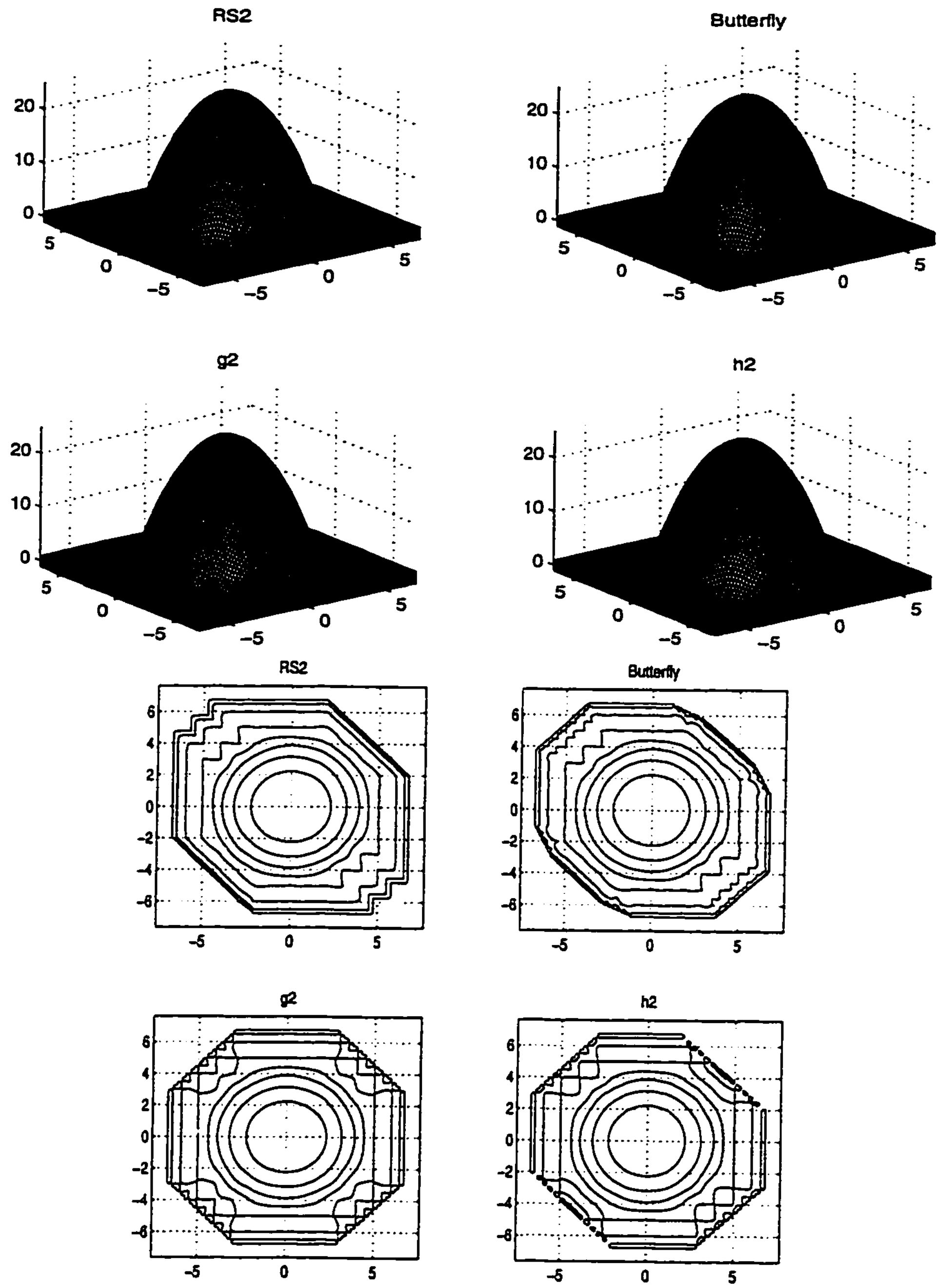

Figure 3.2: The graphs and contours of the generated surfaces by using different interpolatory refinement masks.

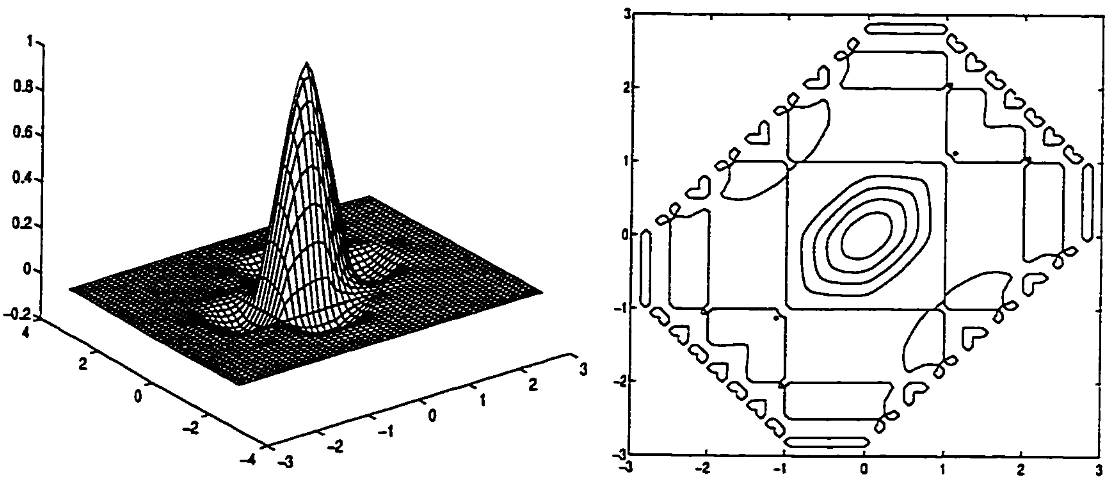

Figure 3.3: The graph and contour of the fundamental function ϕ_{h_2}. It is a C^1 function with $\nu_2(\phi_{h_2}) \approx 2.44077$.

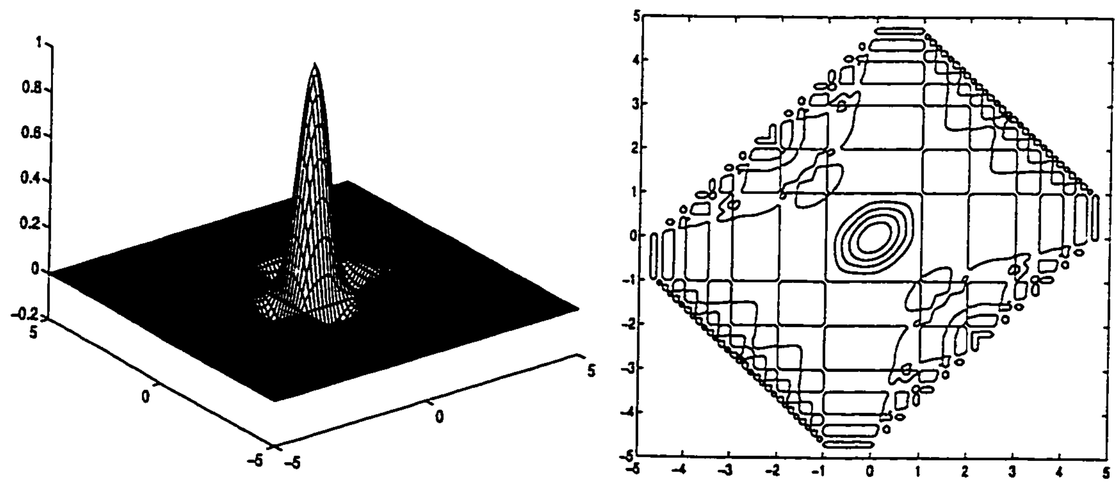

Figure 3.4: The graph and contour of the fundamental function ϕ_{h_3}. It is a C^2 function with $\nu_2(\phi_{h_3}) \approx 3.04845$.

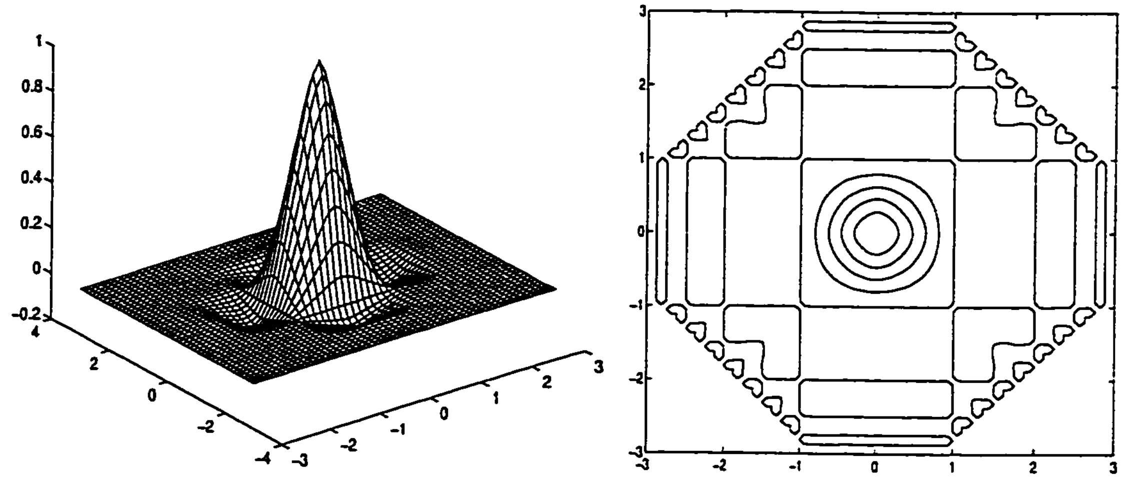

Figure 3.5: The graph and contour of the fundamental function ϕ_{g_2}. It is a C^1 function with $\nu_2(\phi_{g_2}) \approx 2.44077$.

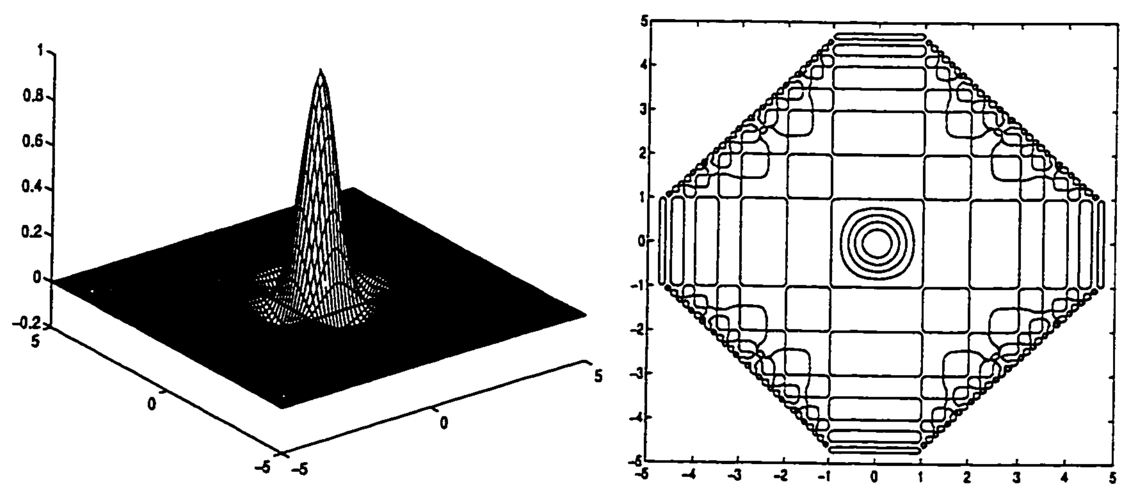

Figure 3.6: The graph and contour of the fundamental function ϕ_{g_3}. It is a C^2 function with $\nu_2(\phi_{g_3}) \approx 3.17513$.

Chapter 4

Multivariate Biorthogonal Wavelets

4.1 Introduction

Based on the results in Chapters 1 and 3, in this chapter we deal with the analysis and construction of multivariate biorthogonal wavelets with some desired properties. It is well known that in various applications high smoothness, small support and high vanishing moments are the three most important properties of a (bi)orthogonal wavelet. On the other hand, there is no C^∞ (bi)orthogonal wavelet with compact support. In this chapter, we shall investigate the mutual relations among these three properties.

Compactly supported (bi)orthogonal wavelets on the real line have been found to be very useful in applications such as signal processing and image compression, for example, see [1, 27, 76, 77]. In [18], Cohen, Daubechies and Feauveau proposed a general way of constructing univariate biorthogonal wavelets. Though the tensor product (bi)orthogonal wavelets provide a family of multivariate (bi)orthogonal wavelets to deal with problems in high dimensions in applications, it has its own advantages and disadvantages. Therefore, as noted in many papers [10, 16, 21, 57, 77, 90] and references cited there, it is of interest in its own right to construct non-tensor product (bi)orthogonal wavelets in high dimensions. In the current literature, there are many papers on constructing multivariate biorthogonal wavelets, especially bivariate biorthogonal wavelets. To only mention a few, see [16, 21, 57, 77, 90] and references therein. Bivariate compactly supported quincunx biorthogonal wavelets were constructed by Cohen and Daubechies in [16]. In [90], a family of bivariate biorthogonal wavelets with the scaling function being a box spline was given by Riemenschneider and Shen.

Usually, a biorthogonal wavelet is derived from a multiresolution analysis generated by a pair of a scaling function and its dual scaling function. The construction of wavelets in multivariate setting is more challenging than its univariate counterpart, see [4, 18, 27, 57, 67, 72, 79, 90] and references therein on construction of (bi)orthogonal wavelets from a multiresolution analysis. To obtain a biorthogonal wavelet, we have to find two refinable functions with some desired properties. Recall that a function ϕ is said to be **refinable** if it satisfies the following refinement equation

$$\phi = \sum_{\beta \in \mathbb{Z}^s} a(\beta)\phi(2\cdot - \beta), \tag{4.1.1}$$

where a is a finitely supported sequence on $\mathbb{Z}^s$, called the **refinement mask**.

As before, we assume that a satisfies $\sum_{\beta \in \mathbb{Z}^s} a(\beta) = 2^s$ and we shall use ϕ_a to denote the normalized solution of the refinement equation (4.1.1) with the mask a.

The concepts of linear independence and approximation order of a function play an important role in the study of biorthogonal wavelets. The shifts of a compactly supported function $\phi : \mathbb{R}^s \to \mathbb{C}$ are said to be **linearly independent** if for any $z \in \mathbb{C}^s$, there exists a multi-integer β in $\mathbb{Z}^s$ such that $\widehat{\phi}(z + 2\pi\beta) \neq 0$. If for any $\xi \in \mathbb{R}^s$, there exists a multi-integer β in $\mathbb{Z}^s$ such that $\widehat{\phi}(\xi + 2\pi\beta) \neq 0$, then the shifts of

ϕ are said to be **stable**. See [68] for discussion on linear independence and stability.

A function is called a **scaling function** if it is refinable and has linearly independent shifts. The general procedure of constructing a biorthogonal wavelet is the following. First, find a scaling function ϕ in $L_2(\mathbb{R}^s)$ which satisfies the refinement equation (4.1.1) with a finitely supported refinement mask a. The next step is to find a refinable function ϕ^d in $L_2(\mathbb{R}^s)$ such that ϕ^d satisfies

$$\phi^d = \sum_{\beta \in \mathbb{Z}^s} a^d(\beta)\phi^d(2\cdot - \beta), \tag{4.1.2}$$

where a^d is a finitely supported sequence on $\mathbb{Z}^s$, and ϕ^d satisfies the following biorthogonal relation

$$\int_{\mathbb{R}^s} \overline{\phi(t-\alpha)}\, \phi^d(t)\, dt = \delta(\alpha) \qquad \forall\, \alpha \in \mathbb{Z}^s, \tag{4.1.3}$$

where $\delta(0) = 1$ and $\delta(\alpha) = 0$ for all $\alpha \in \mathbb{Z}^s \backslash \{0\}$. This function ϕ^d is called a **dual scaling function** of ϕ. If ϕ is the dual scaling function of itself, ϕ is called an **orthogonal scaling function**. Finally, a biorthogonal wavelet is derived from the above ϕ, ϕ^d, a and a^d. The reader is referred to [10, 12, 13, 18, 27, 57, 67, 72, 79, 90] for detail on the construction of a biorthogonal wavelet from a pair of a scaling function and its dual scaling function. It is well known that the smoothness of the scaling function and its dual scaling function will determine the smoothness of their derived wavelets, and the approximation orders of the scaling function and its dual scaling function will determine the vanishing moments of their derived wavelets. For more detail on (bi)orthogonal wavelets, the reader is referred to [4, 10, 12, 13, 16, 18, 21, 22, 25, 27, 57, 67, 72, 77, 90, 99] and references cited there.

By Ω we denote the set of the vertices of the unit cube $[0,1]^s$. For a positive integer k, recall that we say that a sequence a on $\mathbb{Z}^s$ satisfies the **sum rules** of order k if

$$\sum_{\beta \in \mathbb{Z}^s} a(2\beta+\varepsilon)p(2\beta+\varepsilon) = \sum_{\beta \in \mathbb{Z}^s} a(2\beta)p(2\beta) \qquad \forall\, \varepsilon \in \Omega,\ p \in \Pi_{k-1}, \tag{4.1.4}$$

where Π_{k-1} is the set of polynomials with total degree less than k. Let a function ϕ be a refinable function with a mask a. It was proved by Jia in [61, 62] that if the shifts of ϕ are stable, then $S(\phi)$ provides approximation order k if and only if the mask a satisfies the sum rules of order k. Therefore, it is evident that $S(\phi)$ (or $S(\phi^d)$) provides approximation order k if and only if the mask a (or a^d) satisfies the sum rules of order k.

Now it is natural to ask the following question: given a scaling function with compact support, does a dual scaling function with compact support exist? As noted by Lemarié [82] and Jia [63], the answer is yes at least in the univariate case. More precisely, given a scaling function with compact support, a dual scaling function always exists with compact support and arbitrarily high smoothness. Therefore, it is interesting to ask the following question: given any scaling function ϕ, if we fix the size of the support of a dual scaling function of ϕ, then what is the highest approximation order and the highest smoothness of such dual scaling function of ϕ that we can expect? Based on our previous results on interpolatory subdivision schemes in [51, 52], we shall answer the above question in this chapter.

Here is an outline of this chapter. In Section 4.2, given a scaling function, we shall study the relation between the approximation order of its dual scaling function and the support of its dual scaling function. In Section 4.3, we shall prove that for any orthogonal scaling function with its mask supported on $[0, 2r-1]^s$ ($r \in \mathbb{N}$) and satisfying the sum rules of optimal order r, then its L_p smoothness does not exceed that of the univariate Daubechies orthogonal scaling function with its mask supported on $[0, 2r-1]$. An example will be provided to illustrate our result. In Section 4.4, for any given scaling function, we shall study the optimal smoothness of a dual scaling function if its support is fixed and it attains the optimal approximation order. Finally, in Section 4.5, a general CBC (construction by cosets) algorithm is presented to generate all the dual masks of a given interpolatory refinement mask. This algorithm can be easily implemented. In particular, as an application of this general construction, for any bivariate interpolatory mask which is symmetric about the two coordinate axes, we can construct a family of dual masks with arbitrary order of sum rules and symmetry about the two coordinate axes. Finally in Section 4.6, a family of optimal bivariate biorthogonal wavelets is presented with the scaling function being a spline function. In Section 4.7, several examples are provided and comparison results are given.

4.2 Sum Rules

In this section, we shall first introduce some notations. For a given scaling function, we shall study the relation between the approximation order of a dual scaling function and the support of a dual scaling function.

It is well known that there is a close relation between biorthogonal wavelets and fundamental refinable functions. Recall that a function ϕ is said to be **fundamental** if ϕ is continuous, $\phi(0) = 1$, and $\phi(\alpha) = 0$ for all $\alpha \in \mathbb{Z}^s \backslash \{0\}$. If ϕ is a fundamental refinable function with a mask a, then it is necessary that

$$a(0) = 1 \quad \text{and} \quad a(2\beta) = 0 \qquad \forall\, \beta \in \mathbb{Z}^s \backslash \{0\}.$$

Recall that a mask that satisfies the above condition is called an **interpolatory refinement mask.**

The following fact is well known (see [15, 27, 84]) and reveals the relation between a biorthogonal wavelet and a fundamental refinable function.

Lemma 4.1 *Let a function ϕ be a scaling function with a mask a, and let ϕ^d be a dual scaling function of ϕ with a mask a^d. Define*

$$\Phi(x) := \int_{\mathbb{R}^s} \overline{\phi(t-x)}\, \phi^d(t)\, dt, \qquad x \in \mathbb{R}^s \tag{4.2.1}$$

and

$$b(\alpha) := 2^{-s} \sum_{\beta\in\mathbb{Z}^s} \overline{a(\beta-\alpha)}\, a^d(\beta), \qquad \alpha \in \mathbb{Z}^s. \tag{4.2.2}$$

Then the function Φ *is a fundamental refinable function satisfying the refinement equation* (4.1.1) *with the interpolatory mask* b. *In other words, the mask* a *and* a^d *satisfy the following well-known discrete biorthogonal relation:*

$$\sum_{\beta\in\mathbb{Z}^s} \overline{a(\beta-2\alpha)}\, a^d(\beta) = 2^s\delta(\alpha) \qquad \forall\, \alpha \in \mathbb{Z}^s. \tag{4.2.3}$$

Conversely, if the masks a *and* a^d *satisfy the above discrete biorthogonal relation* (4.2.3) *and the subdivision schemes associated with* a *and* a^d *converge in the* L_2 *norm respectively, then the functions* ϕ *and* ϕ^d *lie in* $L_2(\mathbb{R}^s)$ *and satisfy the biorthogonal relation* (4.1.3) *where the functions* ϕ *and* ϕ^d *are the normalized solutions of the refinement equations* (4.1.1) *with the masks* a *and* a^d, *respectively. Therefore, the function* ϕ *is a scaling function and* ϕ^d *is a dual scaling function of* ϕ.

If two sequences a and a^d on $\mathbb{Z}^s$ satisfy the discrete biorthogonal relation (4.2.3), then the mask a^d is called a **dual mask** of the mask a. Throughout, we shall use the following notation:

$$\mathbb{T}^s := \{\, (z_1, \cdots, z_s) \in \mathbb{C}^s \ :\ |z_1| = \cdots = |z_s| = 1 \,\}.$$

Recall that for any sequence λ in $\ell_0(\mathbb{Z}^s)$, its **symbol** $\widetilde{\lambda}$ is given by

$$\widetilde{\lambda}(z) := \sum_{\beta\in\mathbb{Z}^s} a(\beta) z^\beta, \qquad z \in \mathbb{T}^s.$$

By Lemma 4.1, we have $\widehat{\Phi}(\xi) = \overline{\widehat{\phi}(\xi)}\widehat{\phi^d}(\xi), \xi \in \mathbb{R}^s$ and $\widetilde{b}(z) = 2^{-s}\overline{\widetilde{a}(z)}\widetilde{a^d}(z), z \in \mathbb{T}^s$.

By Theorem 3.1 in Chapter 3, in the univariate case ($s = 1$), there is a unique interpolatory mask, which is denoted by b_r, supported on $[1-2r, 2r-1]$ and satisfying the sum rules of order $2r$.

In the multivariate case ($s > 1$), such interpolatory masks are not unique. Let t_r be the sequence on $\mathbb{Z}^s$ given by

$$t_r(\alpha_1, \dots, \alpha_s) := b_r(\alpha_1) \cdots b_r(\alpha_s), \qquad (\alpha_1, \dots, \alpha_s) \in \mathbb{Z}^s. \tag{4.2.4}$$

Then t_r is a tensor product interpolatory refinement mask supported on $[1-2r, 2r-1]^s$ and it satisfies the sum rules of the optimal order $2r$.

Based on the above results, we have the following theorem:

Theorem 4.2 *Let ϕ be a scaling function with its refinement mask a supported on $\Pi_{j=1}^s[-l_j, h_j]$ for some nonnegative integers l_j and h_j, and ϕ^d be its dual scaling function with a mask a^d supported on $\Pi_{j=1}^s[-L_j, H_j]$ for some nonnegative integers L_j and H_j. Suppose a satisfies the sum rules of order k, then a^d can satisfy the sum rules of order at most*

$$\min_{1\le j\le s}\left(\left\lfloor\frac{h_j+L_j+1}{2}\right\rfloor + \left\lfloor\frac{l_j+H_j+1}{2}\right\rfloor\right) - k,$$

where $\lfloor\cdot\rfloor$ is the floor function.

Proof: Let b be the sequence defined in (4.2.2). Then by Lemma 4.1, b is an interpolatory mask and b is supported on $\Pi_{j=1}^s[-h_j - L_j, l_j + H_j]$. From Theorem 3.2 in Chapter 3, we see that b can satisfy the sum rules of order at most $\min_{1\le j\le s}\left(\lfloor\frac{h_j+L_j+1}{2}\rfloor + \lfloor\frac{l_j+H_j+1}{2}\rfloor\right)$. To complete the proof, it suffices to prove that if the mask a^d satisfies the sum rules of order $\widetilde{k}$, then b will satisfy the sum rules of order at least $k + \widetilde{k}$. Denote

$$\mathbb{Z}_+^s := \{(\alpha_1, \cdots, \alpha_s) \in \mathbb{Z}^s \ : \ \alpha_j \ge 0 \quad \forall j = 1, \cdots, s\},$$

and $|\mu| := \mu_1 + \cdots + \mu_s$ for $\mu = (\mu_1, \cdots, \mu_s) \in \mathbb{Z}_+^s$, and $D^\mu := D_1^{\mu_1} \cdots D_s^{\mu_s}$ where D_j denotes the partial derivative with respect to the jth coordinate.

By Lemma 3.3 in [62], the mask a satisfies the sum rules of order k if and only if

$$D^\mu \widetilde{a}(e^{-i\xi})|_{\xi=\pi\omega} = 0 \qquad \forall \omega \in \Omega\backslash\{0\}, |\mu| < k, \mu \in \mathbb{Z}_+^s.$$

If a^d satisfies the sum rules of order $\widetilde{k}$, then

$$D^\mu \widetilde{a}^d(e^{-i\xi})|_{\xi=\pi\omega} = 0 \qquad \forall \omega \in \Omega\backslash\{0\}, |\mu| < \widetilde{k}, \mu \in \mathbb{Z}_+^s.$$

Therefore, by using the Leibniz formula for differentiation, we obtain

$$D^\mu \widetilde{b}(e^{-i\xi})|_{\xi=\pi\omega} = 2^{-s} D^\mu \left[\, \overline{\widetilde{a}(e^{-i\xi})} \widetilde{a}^d(e^{-i\xi}) \,\right]|_{\xi=\pi\omega} = 0 \qquad \forall \omega \in \Omega\backslash\{0\}, |\mu| < k + \widetilde{k}.$$

By Lemma 3.3 in [62], b satisfies the sum rules of order $k + \widetilde{k}$. ■

From the proof of Theorem 4.2, we have the following result:

Corollary 4.3 *If ϕ in $L_2(\mathbb{R}^s)$ is an orthogonal scaling function with its mask a supported on $[0, r]^s$ for some positive integer r, then the mask a can satisfy the sum rules of order at most $\lfloor \frac{r+1}{2} \rfloor$. Therefore, $S(\phi)$ can provide approximation order at most $\lfloor \frac{r+1}{2} \rfloor$.*

4.3 Optimal Orthogonal Wavelets

In [26], Daubechies first constructed a family of compactly supported orthogonal scaling functions on the real line, namely, ϕ_{D_r} $(r \in \mathbb{N})$ where ϕ_{D_r} satisfies the refinement equation (4.1.1) with $s = 1$ and the mask D_r supported on $[0, 2r - 1]$. It is observed (see [84]) that D_r satisfies the sum rules of order r and $\overline{\widetilde{D_r}(z)}\widetilde{D_r}(z) = 2\widetilde{b}_r(z)$ for any z in $\mathbb{T}$ where b_r is the unique univariate interpolatory mask supported on $[1-2r, 2r-1]$ and satisfies the sum rules of order $2r$. Therefore, by Corollary 4.3, the mask D_r attains the sum rules of optimal order r. In the multivariate setting, due to the lack of the Riesz Factorization Theorem, it is much more difficult to construct multivariate orthogonal scaling functions than to construct univariate ones. In the current literature, there are few examples of non-tensor product multivariate orthogonal scaling functions.

From Chapter 3, we know that there is no C^2 fundamental refinable function supported on $[-3,3]^s$. This result also implies that if a function ϕ is an orthogonal scaling function supported on $[0,3]^s$, the $\nu_2(\phi) \le 1$ and therefore, $\phi \notin C^1(\mathbb{R}^s)$.

Let ϕ be an multivariate orthogonal scaling function with its mask supported on $[0, 2r-1]^s$ for some positive integer r. From Corollary 4.3, we see that $S(\phi)$ can provide approximation order at most r. For this case, we shall study the upper bound of the critical exponent $\nu_p(\phi)$ for any p such that $1 \le p \le \infty$. Based on Lemmas 3.10, 3.11 and Theorem 3.5, we have the following result on orthogonal scaling functions.

Theorem 4.4 *Suppose a function ϕ in $L_2(\mathbb{R}^s)$ is an orthogonal scaling function with its refinement mask a supported on $[0,2r-1]^s \cap \mathbb{Z}^s$ for some positive integer r. Define a new sequence a_1 on $\mathbb{Z}$ as follows:*

$$a_1(k) := 2^{1-s} \sum_{\beta_2 \in \mathbb{Z}} \cdots \sum_{\beta_s \in \mathbb{Z}} a(k, \beta_2, \cdots, \beta_s), \qquad k \in \mathbb{Z}.$$

Let ϕ_{a_1} be the normalized solution of the refinement equation (4.1.1) with the mask a_1. If the mask a satisfies the sum rules of order r, then the function ϕ_{a_1} is an orthogonal scaling function with the mask a_1 satisfying

$$\overline{\widetilde{a_1}(z)}\,\widetilde{a_1}(z) = 2\widetilde{b_r}(z), \qquad z \in \mathbb{T}.$$

If in addition, the function ϕ belongs to $L_p(\mathbb{R}^s)$ for some p such that $1 \le p \le \infty$, then

$$\nu_p(\phi) \le \nu_p(\phi_{a_1}).$$

In particular,

$$\nu_2(\phi) \le \nu_2(\phi_{D_r}) \quad \text{and} \quad \nu_2(\phi_{a_1}) = \nu_2(\phi_{D_r}) = \nu_\infty(\phi_{b_r})/2,$$

where ϕ_{D_r} is the Daubechies orthogonal scaling function with its mask D_r supported on $[0, 2r-1]$, and ϕ_{b_r} is the Deslauriers and Dubuc fundamental refinable function with its mask b_r supported on $[1-2r, 2r-1]$.

Proof: Let a sequence b on $\mathbb{Z}^s$ be given by its symbol

$$\widetilde{b}(z) := 2^{-s}\overline{\widetilde{a}(z)}\,\widetilde{a}(z), \qquad z \in \mathbb{T}^s.$$

By Lemma 4.1, the sequence b is an interpolatory refinement mask since ϕ is an orthogonal scaling function. Since the mask a satisfies the sum rules of order r, by the proof of Theorem 4.2, we see that the sequence b must satisfy the sum rules of order at least $2r$. Define a new sequence c on $\mathbb{Z}$ as in Equation (3.4.1) by

$$c(k) = 2^{1-s} \sum_{\beta_2 \in \mathbb{Z}} \cdots \sum_{\beta_s \in \mathbb{Z}} b(k, \beta_2, \cdots, \beta_s), \qquad k \in \mathbb{Z}.$$

By Lemma 3.10, the sequence c must be the mask b_r since the sequence b is supported on $[1-2r, 2r-1]^s$ and satisfies the sum rules of order $2r$. It is easily seen that $\widetilde{c}(z) = 2^{1-s}\widetilde{b}(z, 1, \cdots, 1)$ and $\widetilde{a_1}(z) = 2^{1-s}\widetilde{a}(z, 1, \cdots, 1)$ for any $z \in \mathbb{T}$. Therefore,

$$\overline{\widetilde{a_1}(z)}\,\widetilde{a_1}(z) = 2^{2-s}\widetilde{b}(z, 1, \ldots, 1) = 2\widetilde{c}(z) = 2\widetilde{b}_r(z) \qquad \forall\, z \in \mathbb{T}.$$

Thus, the mask a_1 is the dual mask of itself for $s = 1$. Since the function ϕ is a scaling function, by Lemma 3.11, the subdivision scheme associated with the mask a_1 converges in the L_2 norm. Hence, the function ϕ_{a_1} is an orthogonal scaling function by Lemma 4.1. If ϕ lies in $L_p(\mathbb{R}^s)$ for some p such that $1 \le p \le \infty$, then by Lemma 3.11, we have $\nu_p(\phi) \le \nu_p(\phi_{a_1})$. Note that $\overline{\widetilde{a_1}(z)}\widetilde{a_1}(z) = 2\widetilde{b}_r(z)$ implies that $\nu_2(\phi_{a_1}) = \nu_\infty(\phi_{b_r})/2$. Since $\overline{\widetilde{D_r}(z)}\widetilde{D_r}(z) = 2b_r(z)$ for any z in $\mathbb{T}$,

$$\nu_2(\phi) \le \nu_2(\phi_{a_1}) = \nu_\infty(\phi_{b_r})/2 = \nu_2(\phi_{D_r})$$

which completes the proof. ∎

Note that by using dilation matrix $2I_s$, there is no orthogonal scaling function which can be symmetric about the origin. In the univariate case, by using dilation factor 4, several symmetric C^1 orthogonal scaling functions were reported in Han [46].

In the following, we give two examples to demonstrate that when $s > 1$, such optimal orthogonal scaling functions are not unique.

Example 4.5 The mask a is supported on $[0,3]^2$ and is given by

$$\begin{bmatrix} -\frac{3}{8}+\frac{\sqrt{-10+6\sqrt{3}}}{8}+\frac{\sqrt{3}}{8} & \frac{1}{4}-\frac{\sqrt{3}}{4} & \frac{5}{8}-\frac{3\sqrt{3}}{8}-\frac{\sqrt{-10+6\sqrt{3}}}{8} & 0 \\ -\frac{1}{8}+\frac{\sqrt{3}}{8}-\frac{\sqrt{-10+6\sqrt{3}}}{8} & \frac{1}{2} & \frac{7}{8}-\frac{3\sqrt{3}}{8}+\frac{\sqrt{-10+6\sqrt{3}}}{8} & \frac{1}{4}-\frac{\sqrt{3}}{4} \\ \frac{5}{8}+\frac{\sqrt{3}}{8}-\frac{\sqrt{-10+6\sqrt{3}}}{8} & \frac{1}{2}+\frac{\sqrt{3}}{2} & \frac{1}{8}+\frac{\sqrt{-10+6\sqrt{3}}}{8}+\frac{\sqrt{3}}{8} & \frac{1}{4}-\frac{\sqrt{3}}{4} \\ \frac{3}{8}+\frac{\sqrt{3}}{8}+\frac{\sqrt{-10+6\sqrt{3}}}{8} & \frac{1}{4}+\frac{\sqrt{3}}{4} & -\frac{1}{8}+\frac{\sqrt{3}}{8}-\frac{\sqrt{-10+6\sqrt{3}}}{8} & 0 \end{bmatrix}.$$

Then the function ϕ_a is an orthogonal scaling function and the mask a satisfies the sum rules of order 2. Moreover, by calculation, we have $\nu_2(\phi_a) = 1$. Combining Theorem 4.4 and Corollary 3.12, we see that for any orthogonal scaling ϕ with its mask supported on $[0,3]^s$, the inequality $\nu_2(\phi) \le 1$ holds true. Therefore, the function ϕ_a is an optimal orthogonal scaling function in the L_2 norm sense.

A similar example is the following:

Example 4.6 The mask a is supported on $[0,3]^2$ and is given by

$$\begin{bmatrix} -\frac{3}{8}-\frac{\sqrt{-10+6\sqrt{3}}}{8}+\frac{\sqrt{3}}{8} & \frac{1}{4}-\frac{\sqrt{3}}{4} & 5/8-\frac{3\sqrt{3}}{8}+\frac{\sqrt{-10+6\sqrt{3}}}{8} & 0 \\ -\frac{1}{8}+\frac{\sqrt{3}}{8}+\frac{\sqrt{-10+6\sqrt{3}}}{8} & \frac{1}{2} & \frac{7}{8}-\frac{3\sqrt{3}}{8}-\frac{\sqrt{-10+6\sqrt{3}}}{8} & \frac{1}{4}-\frac{\sqrt{3}}{4} \\ \frac{5}{8}+\frac{\sqrt{3}}{8}+\frac{\sqrt{-10+6\sqrt{3}}}{8} & \frac{1}{2}+\frac{\sqrt{3}}{2} & \frac{1}{8}-\frac{\sqrt{-10+6\sqrt{3}}}{8}+\frac{\sqrt{3}}{8} & \frac{1}{4}-\frac{\sqrt{3}}{4} \\ \frac{3}{8}+\frac{\sqrt{3}}{8}-\frac{\sqrt{-10+6\sqrt{3}}}{8} & \frac{1}{4}+\frac{\sqrt{3}}{4} & -\frac{1}{8}+\frac{\sqrt{3}}{8}+\frac{\sqrt{-10+6\sqrt{3}}}{8} & 0 \end{bmatrix}.$$

Then the function ϕ_a is an orthogonal scaling function and the mask a satisfies the sum rules of order 2. Moreover, by calculation, we have $\nu_2(\phi_a) = 1$.

The graphs and contours of the above examples are given in Figures 4.1 and 4.2. By the results in Chapter 1, it is not difficult to prove that the orthogonal scaling functions in the above two examples are continuous.

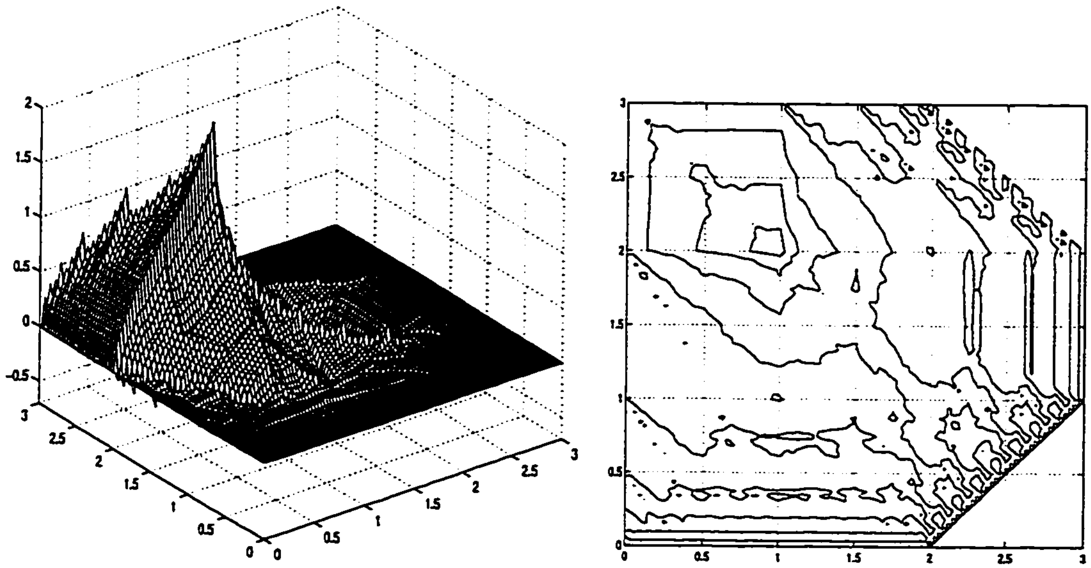

Figure 4.1: The graph and contour of the orthogonal scaling function ϕ_a in Example 4.5.

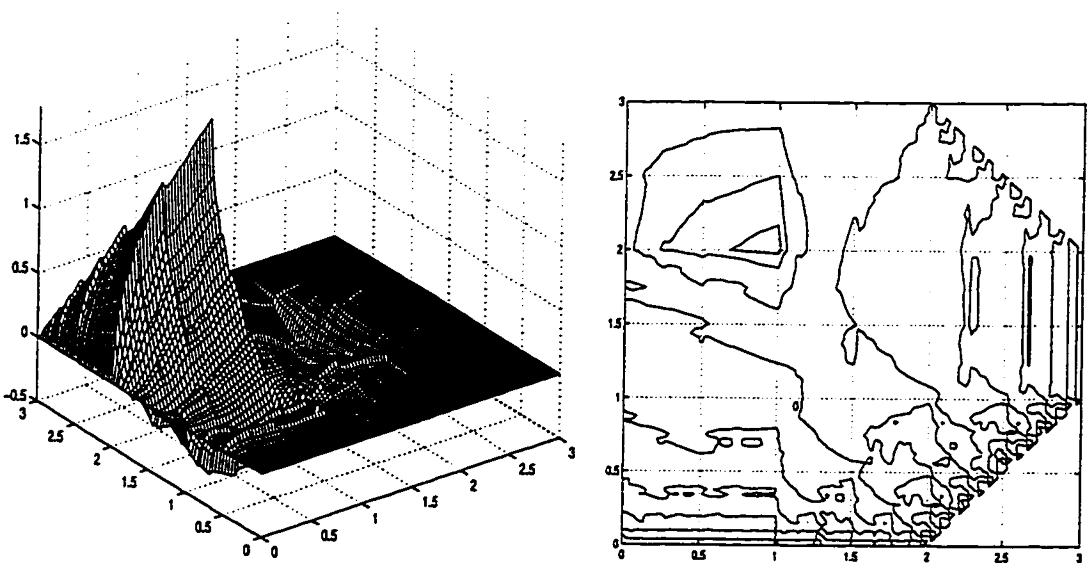

Figure 4.2: The graph and contour of the orthogonal scaling function ϕ_a in Example 4.6.

4.4 Optimal Biorthogonal Wavelets

In this section, we will demonstrate a result similar to Theorem 4.4 for biorthogonal wavelets. A similar result to Theorem 4.4 for a biorthogonal wavelet is the following:

Theorem 4.7 *Let a function ϕ in $L_2(\mathbb{R}^s)$ be a scaling function with a refinement mask a, and a function ϕ^d in $L_2(\mathbb{R}^s)$ be a dual scaling function of ϕ with a refinement mask a^d. Define two new sequences a_1 and a_1^d on $\mathbb{Z}$ as follows:*

$$a_1(k) = 2^{1-s} \sum_{\beta_2 \in \mathbb{Z}} \cdots \sum_{\beta_s \in \mathbb{Z}} a(k, \beta_2, \cdots, \beta_s), \qquad k \in \mathbb{Z}$$

and

$$a_1^d(k) = 2^{1-s} \sum_{\beta_2 \in \mathbb{Z}} \cdots \sum_{\beta_s \in \mathbb{Z}} a^d(k, \beta_2, \cdots, \beta_s), \qquad k \in \mathbb{Z}.$$

By ϕ_{a_1} and $\phi_{a_1^d}$ we denote the normalized solutions of the refinement equation (4.1.1) with $s = 1$ and the masks a_1 and a_1^d respectively. Let a sequence b on $\mathbb{Z}^s$ be given as in (4.2.2) by

$$b(\alpha) := 2^{-s} \sum_{\beta \in \mathbb{Z}^s} \overline{a(\beta - \alpha)}\, a^d(\beta), \qquad \alpha \in \mathbb{Z}^s. \tag{4.4.1}$$

Suppose the sequence b is supported on $[1-2k, 2k-1]^s \cap \mathbb{Z}^s$ for some positive integer k and b satisfies the sum rules of order $2k-1$. Then the function ϕ_{a_1} is a univariate scaling function with $\phi_{a_1^d}$ being a dual scaling function of ϕ_{a_1}. If ϕ belongs to $L_p(\mathbb{R}^s)$ and ϕ^d belongs to $L_q(\mathbb{R}^s)$ for some p, q such that $1 \le p, q \le \infty$, then $\phi_{a_1} \in L_p(\mathbb{R})$, $\phi_{a_1^d} \in L_q(\mathbb{R})$ and

$$\nu_p(\phi) \le \nu_p(\phi_{a_1}) \quad \text{and} \quad \nu_q(\phi^d) \le \nu_q(\phi_{a_1^d}). \tag{4.4.2}$$

In particular, if the sequence b satisfies the sum rules of order $2k$, then

$$\overline{\widetilde{a_1}(z)}\, \widetilde{a_1^d}(z) = 2\widetilde{b_k}(z), \qquad z \in \mathbb{T} \quad \text{and} \quad \nu_q(\phi^d) \le \nu_r(\phi_{b_k}) - \nu_p(\phi),$$

where $1/r = 1/p + 1/q - 1$ and b_k is the unique interpolatory mask which is supported on $[1-2k, 2k-1]$ and satisfies the sum rules of order $2k$.

Proof: By Lemma 4.1, it is easily seen that the sequence b is an interpolatory mask. Let c be a sequence on $\mathbb{Z}$ given by

$$c(k) = 2^{1-s} \sum_{\beta_2 \in \mathbb{Z}} \cdots \sum_{\beta_s \in \mathbb{Z}} b(k, \beta_2, \cdots, \beta_s), \qquad k \in \mathbb{Z}.$$

It follows from Lemma 3.10 that the sequence c is an interpolatory mask since the sequence b is supported on $[1-2k, 2k-1]^s$ and satisfies the sum rules of order $2k-1$. We observe that $\widetilde{c}(z) = 2^{1-s}\widetilde{b}(z, 1\cdots, 1)$, $\widetilde{a_1}(z) = 2^{1-s}\widetilde{a}(z, 1, \cdots, 1)$ and $\widetilde{a_1^d}(z) = 2^{1-s}\widetilde{a^d}(z, 1, \cdots, 1)$ for any z in $\mathbb{T}$. It is easy to see that

$$\overline{\widetilde{a_1}(z)}\, \widetilde{a_1^d}(z) = 2^{2-2s}\overline{\widetilde{a}(z, 1\cdots, 1)}\widetilde{a^d}(z, 1, \cdots, 1) = 2\widetilde{c}(z), \qquad z \in \mathbb{T}. \tag{4.4.3}$$

Therefore, the masks a_1 and a_1^d must satisfy the discrete biorthogonal relation (4.2.3) with $s = 1$ since the sequence c is an interpolatory mask. Since both ϕ and ϕ^d belong to $L_2(\mathbb{R}^s)$ and their shifts are stable, by Lemma 3.11, the subdivision schemes associated with the masks a_1 and a_1^d converge in the L_2 norm respectively. Thus, by Lemma 4.1, the function ϕ_{a_1} is a scaling function with $\phi_{a_1^d}$ being a dual scaling function of ϕ_{a_1}. The inequality (4.4.2) follows directly from Lemma 3.11.

If the sequence b satisfies the sum rules of order $2k$, by Lemma 3.10, the mask c in (4.4.3) must be the mask b_k. Note that $\widetilde{\nabla^{k_1} S_{a_1}^n \delta}(z) = (1-z)^{k_1} \prod_{j=0}^{n-1} \widetilde{a_1}(z^{2^j})$. Therefore, it follows from (4.4.3) that for any positive integers k_1 and k_2, it is easy to verify that

$$2^n \widetilde{\nabla^{k_1+k_2} S_{b_k}^n \delta}(z) = \overline{\widetilde{\nabla^{k_1} S_{a_1}^n \delta}(z)}\, \widetilde{\nabla^{k_2} S_{a_1^d}^n \delta}(z), \qquad z \in \mathbb{T}.$$

Therefore, by applying Young's inequality to the above equation, we have

$$2^n \|\nabla^{k_1+k_2} S_{b_k}^n \delta\|_r \le \|\nabla^{k_1} S_{a_1}^n \delta\|_p \|\nabla^{k_2} S_{a_1^d}^n \delta\|_q \qquad \forall\, n \in \mathbb{N},$$

where $1/r = 1/p + 1/q - 1$. This yields

$$2\sigma_{k_1+k_2,r}(b_k) \le \sigma_{k_1,p}(a_1)\sigma_{k_2,q}(a_1^d) \qquad \forall\, k_1, k_2 \in \mathbb{N}.$$

By Theorem 3.5 in Chapter 3, we have $\nu_r(\phi_{b_k}) \ge \nu_p(\phi_{a_1}) + \nu_q(\phi_{a_1^d})$. Therefore, by $\nu_p(\phi) \le \nu_p(\phi_{a_1})$ and $\nu_q(\phi^d) \le \nu_q(\phi_{a_1^d})$, we have $\nu_r(\phi_{b_k}) \ge \nu_p(\phi_a) + \nu_q(\phi_{a^d})$. ■

Corollary 4.8 *Let ϕ be a scaling function with a refinement mask a supported on $[-l, l]^s$ for some positive integer l, and ϕ^d be a dual scaling function of ϕ with a mask a^d supported on $[1+l-2k, 2k-l-1]^s$ for some positive integer k. Let the sequence b be given in (4.4.1). Suppose the mask a satisfies the sum rules of order m. Then the mask a^d can satisfy the sum rules of order at most $2k-m$. Moreover, if the mask a^d satisfies the sum rules of order $2k-m-1$ (or $2k-m$), then the sequence b can satisfy the sum rules of order at least $2k-1$ (or $2k$) and the corresponding results in Theorem 4.7 hold true.*

Proof: This is a direct consequence of Theorem 4.2 and Theorem 4.6. ∎

Let us consider an example. Let ϕ be a refinable box spline function with its mask a given by its symbol

$$\widetilde{a}(z) = 2^{-s}\Pi_{j=1}^{s}(z_j^{-1} + 2 + z_j), \qquad z \in \mathbb{T}^s$$

or

$$\widetilde{a}(z) = 2^{-1}(1 + z_1^{-1} \cdots z_s^{-1})\Pi_{j=1}^{s}(1 + z_j), \qquad z \in \mathbb{T}^s.$$

It is easy to verify that ϕ is a fundamental function with $\nu_1(\phi) = 2$, its mask a is supported on $[-1, 1]^s$ and a satisfies the sum rules of order 2. Thus, the function ϕ is a scaling function. Then Corollary 3.12 and Corollary 4.7 imply that if a function ϕ^d is a dual scaling function of the scaling function ϕ with its mask supported on $[-2, 2]^s$, then the function ϕ^d can not be continuous. For any dual scaling function ϕ^d of the scaling function ϕ with its mask a^d supported on $[2-2r, 2r-2]^s$ for some positive integer r, by Theorem 4.2, the mask a^d can satisfy the sum rules of order at most $2r-2$. If a^d satisfies the sum rules of order $2r-2$, by Corollary 4.7, then we have

$$\nu_2(\phi^d) \le \nu_2(\phi_{b_r}) - \nu_1(\phi) = \nu_2(\phi_{b_r}) - 2.$$

When $s = 2$, in Section 4.6, we shall construct a family of dual scaling functions $\phi_{\mathcal{H}_r}$ ($r \in \mathbb{N}$) of the bivariate hat function ϕ such that the dual mask $\mathcal{H}_r$ is supported on

$[2-2r, 2r-2]^2$ and satisfies the sum rules of order $2r-2$. In addition, the equality $\nu_2(\phi_{\mathcal{H}_r}) = \nu_2(\phi_{b_r}) - 2$ holds true at least for $r = 3, \cdots, 12$ and each mask $\mathcal{H}_r$ is symmetric about the two coordinate axes, and the lines $x_1 = x_2$ and $x_1 = -x_2$.

4.5 Construction of Biorthogonal Wavelets

In this section, we shall present a general method to construct multivariate biorthogonal wavelets. More precisely, for any scaling function ϕ with an interpolatory refinement mask a, a general CBC (Construction By Cosets) algorithm is given to produce all the dual masks of the mask a. As an application of this general theory, for any bivariate fundamental mask a which is symmetric about the two coordinate axes, we construct a family of dual masks of a which satisfy any desired order of sum rules and are also symmetric about the two coordinate axes. Based on this construction, a family of optimal bivariate biorthogonal wavelets is presented in the next section. Such biorthogonal wavelets have full symmetry (i.e., they are symmetric about the x_1-axis, x_2-axis, and the lines $x_1 = x_2$ and $x_1 = -x_2$), have the optimal order of sum rules, the optimal L_2 smoothness order and relatively small support of the dual masks.

Before proceeding further, we introduce some notation. Recall that

$$\mathbb{Z}_+^s := \{ (\alpha_1, \cdots, \alpha_s) \in \mathbb{Z}^s : \alpha_i \geq 0 \quad \forall\, i = 1, \cdots, s \}.$$

For any $\mu = (\mu_1, \cdots, \mu_s) \in \mathbb{Z}^s$, we denote $|\mu| := |\mu_1| + \cdots + |\mu_s|$ and $\mu! := \mu_1! \cdots \mu_s!$ if $\mu \in \mathbb{Z}_+^s$. For any $\mu = (\mu_1, \cdots, \mu_s), \nu = (\nu_1, \cdots, \nu_s) \in \mathbb{Z}^s$, by $\nu \leq \mu$ we mean $\nu_i \leq \mu_i$ for all $i = 1, \cdots, s$, and by $\nu < \mu$ we mean $\nu \leq \mu$ and $\nu \neq \mu$.

Throughout this section, for any $\nu \in \mathbb{Z}_+^s$, by p_ν we denote the monomial $(\cdot)^\nu$ and

$$\langle \lambda, p_\nu \rangle := \sum_{\alpha \in \mathbb{Z}^s} \lambda(\alpha) p_\nu(\alpha) = \sum_{\alpha \in \mathbb{Z}^s} \lambda(\alpha) \alpha^\nu, \qquad \lambda \in \ell_0(\mathbb{Z}^s).$$

Theorem 4.9 *Let a sequence a on $\mathbb{Z}^s$ satisfy $\sum_{\beta\in\mathbb{Z}^s} a(\beta) = 2^s$. Suppose a sequence a^d on $\mathbb{Z}^s$ is a dual mask of a that satisfies the following relation*

$$\sum_{\beta\in\mathbb{Z}^s} \overline{a(\beta-2\alpha)}a^d(\beta) = 2^s\delta(\alpha) \qquad \forall\, \alpha\in\mathbb{Z}^s. \tag{4.5.1}$$

If the sequence a^d satisfies the sum rules of order k for some positive integer k, then for any $\mu\in\mathbb{Z}^s_+$ such that $|\mu|<k$, the value $h_\mu := 2^{-s}\langle a^d, p_\mu\rangle$ is uniquely determined by the sequence a. More precisely, h_μ is given by the following recursive relation:

$$h_\mu = \delta(\mu) - 2^{-s}\sum_{0\le\nu<\mu}(-1)^{|\mu-\nu|}\frac{\mu!}{\nu!(\mu-\nu)!}\overline{\langle a, p_{\mu-\nu}\rangle}\, h_\nu, \qquad |\mu|<k, \mu\in\mathbb{Z}^s_+. \tag{4.5.2}$$

Proof: Recall that Ω is the set of the vertices of the unit cube $[0,1]^s$. By the definition of the sum rules (4.1.4), we observe that the sequence a^d satisfies the sum rules of order k if and only if

$$\sum_{\beta\in\mathbb{Z}^s} a^d(2\beta+\varepsilon)(2\beta+\varepsilon)^\nu = 2^{-s}\langle a^d,\, p_\nu\rangle = h_\nu \qquad \forall\varepsilon\in\Omega,\, |\nu|<k,\, \nu\in\mathbb{Z}^s_+. \tag{4.5.3}$$

From Equation (4.5.1), we get for any $\mu\in\mathbb{Z}^s_+$,

$$\begin{aligned} 2^s\delta(\mu) &= \sum_{\alpha\in\mathbb{Z}^s}\sum_{\beta\in\mathbb{Z}^s}\overline{a(\beta-2\alpha)}a^d(\beta)(2\alpha)^\mu \\ &= \sum_{\varepsilon\in\Omega}\sum_{\alpha\in\mathbb{Z}^s}\sum_{\beta\in\mathbb{Z}^s}\overline{a(2\beta+\varepsilon-2\alpha)}a^d(2\beta+\varepsilon)(2\alpha)^\mu. \end{aligned}$$

On the other hand, we have

$$(2\alpha)^\mu = \big((2\beta+\varepsilon)-(2\beta+\varepsilon-2\alpha)\big)^\mu = \sum_{0\le\nu\le\mu} c_{\nu,\mu}(2\beta+\varepsilon-2\alpha)^{\mu-\nu}(2\beta+\varepsilon)^\nu,$$

where $c_{\nu,\mu} := (-1)^{|\mu-\nu|}\mu!/(\nu!(\mu-\nu)!)$ and recall that by $\nu\le\mu$ we mean $\nu_i\le\mu_i$ for all $i=1,\cdots,s$. Hence, for any $\mu\in\mathbb{Z}^s_+$, we deduce that

$$\begin{aligned} &2^s\delta(\mu) \\ &= \sum_{0\le\nu\le\mu} c_{\nu,\mu}\sum_{\varepsilon\in\Omega}\sum_{\alpha\in\mathbb{Z}^s}\sum_{\beta\in\mathbb{Z}^s}\overline{a(2\beta+\varepsilon-2\alpha)}(2\beta+\varepsilon-2\alpha)^{\mu-\nu}a^d(2\beta+\varepsilon)(2\beta+\varepsilon)^\nu \\ &= \sum_{0\le\nu\le\mu} c_{\nu,\mu}\sum_{\varepsilon\in\Omega}\sum_{\alpha\in\mathbb{Z}^s}\overline{a(2\alpha+\varepsilon)}(2\alpha+\varepsilon)^{\mu-\nu}\sum_{\beta\in\mathbb{Z}^s}a^d(2\beta+\varepsilon)(2\beta+\varepsilon)^\nu. \end{aligned}$$

Since $\sum_{\beta\in\mathbb{Z}^s} a(\beta) = 2^s$, we have $\langle a,\, p_0\rangle = 2^s$. Note that $c_{\mu,\mu} = 1$ for any $\mu \in \mathbb{Z}_+^s$. From Equation (4.5.3), we conclude that

$$\begin{aligned} 2^s\delta(\mu) &= \sum_{0\le\nu\le\mu} c_{\nu,\mu} \sum_{\varepsilon\in\Omega}\sum_{\alpha\in\mathbb{Z}^s} \overline{a(2\alpha+\varepsilon)}(2\alpha+\varepsilon)^{\mu-\nu}\, h_\nu \\ &= \sum_{0\le\nu\le\mu} c_{\nu,\mu}\overline{\langle a,\, p_{\mu-\nu}\rangle}\, h_\nu = 2^s h_\mu + \sum_{0\le\nu<\mu} c_{\nu,\mu}\overline{\langle a,\, p_{\mu-\nu}\rangle}\, h_\nu \end{aligned}$$

from which Equation (4.5.2) can be easily derived. ∎

By definition, the value $\langle a^d,\, p_\mu\rangle$ in Theorem 4.9 is totally determined by the sequence a^d. But Theorem 4.9 says that if the sequence a^d is a dual mask of the mask a and the sequence a^d satisfies the sum rules of order k, then for any $\mu \in \mathbb{Z}_+^s$ such that $|\mu| < k$, the value $\langle a^d,\, p_\mu\rangle$ is uniquely determined by the sequence a instead of the sequence a^d. Therefore, if a sequence a on $\mathbb{Z}^s$ satisfies $\sum_{\beta\in\mathbb{Z}^s} a(\beta) = 2^s$, by Theorem 4.9, then we can define a sequence h^a on $\mathbb{Z}_+^s$ as follows:

$$h^a(\mu) = \delta(\mu) - 2^{-s}\sum_{0\le\nu<\mu}(-1)^{|\mu-\nu|}\frac{\mu!}{\nu!(\mu-\nu)!}\overline{\langle a, p_{\mu-\nu}\rangle}\, h^a(\nu), \qquad \mu\in\mathbb{Z}_+^s. \quad (4.5.4)$$

An important consequence of Theorem 4.9 is that it allows us to propose a general method to construct a dual mask satisfying the sum rules of arbitrary order for a given interpolatory refinement mask. Since in the following method, we obtain the dual masks a^d by constructing each coset $a^d(2\beta+\varepsilon), \beta\in\mathbb{Z}^s$ separately, we call this method CBC algorithm (Construction by Cosets Algorithm).

CBC Algorithm 4.10 (*Construction by Cosets Algorithm*)

1. *Given a sequence a on $\mathbb{Z}^s$ such that a satisfies the following conditions:*

$$\sum_{\beta\in\mathbb{Z}^s} a(\beta) = 2^s, \quad a(0) = 1 \quad \text{and} \quad a(2\beta) = 0 \qquad \forall\beta\in\mathbb{Z}^s\backslash\{0\}; \qquad (4.5.5)$$

2. *Let k be any fixed positive integer;*

3. *Calculate $h^a(\mu)$ as in Equation* (4.5.4) *for $\mu\in\mathbb{Z}_+^s$ such that $|\mu| < k$;*

4. *Let Ω be the set of vertices of $[0,1]^s$. For each $\varepsilon \in \Omega\backslash\{0\}$, choose an appropriate subset E_ε of $\mathbb{Z}^s$ such that the following linear system*

$$\sum_{\beta \in E_\varepsilon} b_{\varepsilon,\beta}(2\beta+\varepsilon)^\mu = h^a(\mu), \qquad \mu \in \mathbb{Z}^s_+,\, |\mu| < k \tag{4.5.6}$$

has at least one solution for $\{b_{\varepsilon,\beta} : \beta \in E_\varepsilon\}$;

5. *Construct the mask a^d coset by coset as follows: for each $\varepsilon \in \Omega\backslash\{0\}$,*

$$a^d(2\beta+\varepsilon) = b_{\varepsilon,\beta},\ \beta \in E_\varepsilon \quad \text{and} \quad a^d(2\beta+\varepsilon) = 0, \quad \beta \in \mathbb{Z}^s\backslash E_\varepsilon$$

and

$$a^d(2\beta) = 2^s\delta(\beta) - \sum_{\varepsilon\in\Omega\backslash\{0\}}\sum_{\alpha\in\mathbb{Z}^s} \overline{a(2\alpha-2\beta+\varepsilon)}\, a^d(2\alpha+\varepsilon),\ \beta \in \mathbb{Z}^s; \tag{4.5.7}$$

6. *Then the mask a^d is a dual mask of the given interpolatory mask a and satisfies the sum rules of order k.*

Proof: It is easy to verify that if the sequence a is an interpolatory mask, then the discrete biorthogonal relation (4.5.1) is equivalent to (4.5.7). Therefore, the mask a^d is a dual mask of the given mask a. On the other hand, (4.5.6) can be rewritten as

$$\sum_{\beta\in\mathbb{Z}^s} a^d(2\beta+\varepsilon)(2\beta+\varepsilon)^\mu = h^a(\mu), \qquad \varepsilon \in \Omega\backslash\{0\},\, |\mu| < k,\, \mu \in \mathbb{Z}^s_+. \tag{4.5.8}$$

By the definition of sum rules, to verify that the sequence a^d satisfies the sum rules of order k, it suffices to demonstrate that

$$\sum_{\beta\in\mathbb{Z}^s} a^d(2\beta)(2\beta)^\mu = h^a(\mu) \qquad \forall\, |\mu| < k,\, \mu \in \mathbb{Z}^s_+. \tag{4.5.9}$$

As in the proof of Theorem 4.9, from (4.5.7), we have

$$\begin{aligned}&\sum_{\beta\in\mathbb{Z}^s} a^d(2\beta)(2\beta)^\mu \\ &= 2^s\delta(\mu) - \sum_{0\le\nu\le\mu} c_{\nu,\mu} \sum_{\varepsilon\in\Omega\backslash\{0\}}\sum_{\alpha\in\mathbb{Z}^s} \overline{a(2\alpha+\varepsilon)}(2\alpha+\varepsilon)^{\mu-\nu} \sum_{\beta\in\mathbb{Z}^s} a^d(2\beta+\varepsilon)(2\beta+\varepsilon)^\nu,\end{aligned}$$

where $c_{\nu,\mu} := (-1)^{|\mu-\nu|}\mu!/(\nu!(\mu-\nu)!)$. Since the sequence a is an interpolatory mask, it is easily seen that

$$\sum_{\varepsilon\in\Omega\backslash\{0\}}\sum_{\alpha\in\mathbb{Z}^s}\overline{a(2\alpha+\varepsilon)}(2\alpha+\varepsilon)^{\mu-\nu} = \overline{\langle a,\ p_{\mu-\nu}\rangle} - \delta(\mu-\nu).$$

Therefore, it follows from Equation (4.5.8) that for any $\mu \in \mathbb{Z}_+^s$ such that $|\mu| < k$,

$$\begin{aligned}\sum_{\beta\in\mathbb{Z}^s} a^d(2\beta)(2\beta)^\mu &= 2^s\delta(\mu) - \sum_{0\le\nu\le\mu} c_{\nu,\mu}\left(\overline{\langle a,\ p_{\mu-\nu}\rangle} - \delta(\mu-\nu)\right) h^a(\nu)\\ &= (1-2^s)h^a(\mu) + 2^s\delta(\mu) - \sum_{0\le\nu<\mu} c_{\nu,\mu}\overline{\langle a,\ p_{\mu-\nu}\rangle}\, h^a(\nu)\\ &= h^a(\mu),\end{aligned}$$

where in the last equality we used Equation (4.5.4) for $h^a(\mu)$. We are done. ∎

It is evident that the above CBC algorithm can produce all the dual masks for any given interpolatory mask. In general, if the set E_ε is large enough, the equation in Step (4) must have at least one solution. We point out that based on Theorem 4.9, the CBC algorithm can be generalized to the general case. We mention that there is a similar CBC algorithm such that for any given scaling function with a mask a, we can construct a dual mask of the mask a which can satisfy the sum rules of arbitrary order. Based on the Lemma 3.14 in Chapter 3, here we present a concrete way to implement the above general CBC algorithm in the bivariate case.

Now for any bivariate interpolatory mask a which is symmetric about the two coordinate axes, the following algorithm provides us a method to construct a dual mask which satisfies the sum rules of arbitrary order.

TCBC Algorithm 4.11 (*Triangle Construction by Cosets Algorithm*)

1. *Let a bivariate mask a satisfy* $\sum_{\beta\in\mathbb{Z}^2} a(\beta) = 4, a(0) = 1$ *and* $a(2\beta) = 0$ *for all* $\beta \in \mathbb{Z}^2\backslash\{0\}$ *with symmetry about the two coordinate axes, i.e.,*

$$a(\beta_1,\beta_2) = a(-\beta_1,\beta_2) = a(\beta_1,-\beta_2) = a(-\beta_1,-\beta_2) \quad \forall\,(\beta_1,\beta_2)\in\mathbb{Z}^2; \quad (4.5.10)$$

2. *Let k be any fixed positive integer;*

3. *Calculate $h^a(2\mu)$ as in Equation* (4.5.4) *for $\mu \in \mathbb{Z}_+^2$ such that $|\mu| < k$;*

4. *Let $E := \{(\beta_1, \beta_2) \in \mathbb{Z}^2 \,:\, \beta_1 \geq 0,\, \beta_2 \geq 0$ and $\beta_1 + \beta_2 < k\}$;*

5. *Let $\Omega' := \{(1,0), (0,1), (1,1)\}$. For each $\varepsilon \in \Omega'$, there is a unique solution for $\{b_{\varepsilon,\beta}, \beta \in E\}$ to the following linear system:*

$$\sum_{\beta \in E} b_{\varepsilon,\beta}(2\beta + \varepsilon)^{2\mu} = h^a(2\mu)/4, \qquad |\mu| < k, \mu \in \mathbb{Z}_+^2;$$

6. *For each $(\varepsilon_1, \varepsilon_2) \in \Omega'$, set $a^d(2\beta_1 + \varepsilon_1, 2\beta_2 + \varepsilon_2) = 0$ for all $(\beta_1, \beta_2) \in \mathbb{Z}_+^2 \backslash E$, and for any $(\beta_1, \beta_2) \in E$,*

$$a^d(2\beta_1 + \varepsilon_1, 2\beta_2 + \varepsilon_2) = \big(1 + \delta(2\beta_1 + \varepsilon_1)\big)\big(1 + \delta(2\beta_2 + \varepsilon_2)\big) b_{(\varepsilon_1,\varepsilon_2),(\beta_1,\beta_2)};$$

7. *For each $\varepsilon \in \Omega'$, complete each coset $a(2\beta + \varepsilon), \beta \in \mathbb{Z}^2$ by symmetry as in* (4.5.10) *and set*

$$a^d(2\beta) := 4\delta(\beta) - \sum_{\varepsilon \in \Omega'} \sum_{\alpha \in \mathbb{Z}^2} \overline{a(2\alpha - 2\beta + \varepsilon)} a^d(2\alpha + \varepsilon), \quad \beta \in \mathbb{Z}^2;$$

8. *Then the mask a^d is a dual mask of the given mask a, satisfies the sum rules of order* $2k$ *and it is symmetric about the two coordinate axes.*

The above algorithm is called TCBC (Triangle Construction by Cosets) algorithm since we choose a special triangle subset E of $\mathbb{Z}^2$ in the above algorithm. The existence and uniqueness of the solutions in Step (5) of the above TCBC algorithm are guaranteed by Lemma 3.14 in Chapter 3. The claim that the mask a^d satisfies the sum rules of order $2k$ follows from the fact that if the sequence a is symmetric about the two coordinate axes, then $\langle a, p_{(\nu_1,\nu_2)} \rangle = 0$ for any $(\nu_1, \nu_2) \in \mathbb{Z}_+^2$ with either ν_1 or ν_2 being an odd integer. We mention that if in the TCBC algorithm, the mask a is also symmetric about the lines $x_1 = x_2$ and $x_1 = -x_2$, then the resulting dual mask also has such properties. For this case, in Step (5) of the TCBC algorithm, we only need to deal with the coset of a^d at $2\beta + \varepsilon, \beta \in \mathbb{Z}^2$ for $\varepsilon \in \{(1,0),(1,1)\}$. The coset of the mask a^d at $2\beta + (0,1), \beta \in \mathbb{Z}^2$ is obtained by symmetry. In passing, we mention that the unique solution to the linear system in Step (5) of the above TCBC algorithm can be explicitly obtained. Thus, the resulting dual mask can be obtained without solving any equation.

Let us illustrate the above general theory by giving an example. Let φ_h be the bivariate hat function with its mask a_h supported on $[-1,1]^2 \cap \mathbb{Z}^2$ and given by

$$\begin{bmatrix} 1/4 & 1/2 & 1/4 \\ 1/2 & 1 & 1/2 \\ 1/4 & 1/2 & 1/4 \end{bmatrix}. \tag{4.5.11}$$

An easy calculation gives us

$$\langle a, p_{(\mu_1,\mu_2)} \rangle = \begin{cases} (1+\delta(\mu_1))(1+\delta(\mu_2)), & (\mu_1,\mu_2) \in 2\mathbb{Z}_+^2; \\ 0, & \text{otherwise.} \end{cases}$$

Let H_r denote the dual mask of the mask a_h derived by the TCBC algorithm such that H_r satisfies the sum rules of order $2r-2$.

From the TCBC algorithm, it is easily seen that

$$\operatorname{supp} H_r \subseteq \{ (\beta_1,\beta_2) \in \mathbb{Z}^2 \,:\, |\beta_1| \le 2r-2,\ |\beta_2| \le 2r-2,\ |\beta_1| + |\beta_2| \le 2r \},$$

and it is symmetric about the x_1-axis, x_2-axis, and the lines $x_1 = x_2$ and $x_1 = -x_2$.

We shall give an example of the dual mask H_r in the last section of this chapter.

4.6 A Family of Optimal Biorthogonal Wavelets

In this section, we shall modify the TCBC algorithm in the previous section to construct a new family of optimal biorthogonal wavelets by shrinking the support of each

H_r. Since the mask a_h has full symmetry, we only need to deal with $\varepsilon \in \{(1,0),(1,1)\}$ in Step (5) of the TCBC algorithm. The only part we need to modify in the TCBC algorithm is Steps (5) and (6). All other steps are the same. Throughout the rest of this section, the mask a in the TCBC algorithm is assumed to be a_h given in (4.5.11).

Let E be the set given in Step (4) of the TCBC algorithm and let $b_\beta, \beta \in E$ be the unique solution of the following linear system:

$$\sum_{\beta \in E} b_\beta \big(2\beta + (1,1)\big)^{2\mu} = h^a(2\mu)/4, \qquad |\mu| < k, \mu \in \mathbb{Z}_+^2.$$

Set $a^d(2\beta + (1,1)) = b_\beta, \beta \in E$ and $a^d(2\beta + (1,1)) = 0, \beta \in \mathbb{Z}_+^2 \backslash E$. Take F to be the following set:

$$F := \{\, (\beta_1, \beta_2) \in \mathbb{Z}_+^2 \ : \ \beta_1 + \beta_2 = k \quad \text{and} \quad \beta_2 > 0 \,\}.$$

Now we set $a^d(2\beta + (1,0)) = 0$ for any $\beta \in \mathbb{Z}_+^2 \backslash (E \cup F)$ and

$$a^d(2\beta_1 + 1, 2\beta_2) = \big(1 + \delta(\beta_2)\big) c_{(\beta_1,\beta_2)}, \qquad (\beta_1, \beta_2) \in E \cup F$$

with yet-to-be-determined parameters $c_\beta, \beta \in E \cup F$.

This extra freedom $c_\beta, \beta \in F$ given by F will be used to reduce the support of the mask a^d at the coset $(0,0)$ constructed in Step (7) of the TCBC algorithm. More precisely, we shall try to adjust the coefficients of the mask H_{k-1} to be zero on the set $\{(\beta_1, \beta_2) \in \mathbb{Z}^2 \ : \ \beta_1 + \beta_2 = 2k - 2\}$. By using symmetry, after a simple calculation, it is easily seen that this restriction is equivalent to the following linear system

$$c_{(\beta_1,\beta_2)} + c_{(\beta_2 - 1, \beta_1 + 1)} + b_{(\beta_1, \beta_2 - 1)}/2 = 0 \qquad \text{for all} \quad (\beta_1, \beta_2) \in F.$$

By simply setting $c_{(\beta_1,\beta_2)} = 0$ for any $(\beta_1, \beta_2) \in F$ such that $k/2 \le \beta_1 < k$, the above linear system has a unique solution $c_\beta, \beta \in F$. Now the following linear system

$$\sum_{\beta \in E} c_\beta \big(2\beta + (1,0)\big)^{2\mu} = h^a(2\mu)/4 - \sum_{\beta \in F} c_\beta \big(2\beta + (1,0)\big)^{2\mu}, \qquad |\mu| < k, \mu \in \mathbb{Z}_+^2$$

has a unique solution for $c_\beta, \beta \in E$ by Lemma 3.14 in Chapter 3.

Set $a^d(2\beta_1, 2\beta_2+1) = a^d(2\beta_2+1, 2\beta_1), (\beta_1, \beta_2) \in \mathbb{Z}_+^2$. By the TCBC algorithm, we have a dual mask a^d of the mask a_h such that a^d satisfies the sum rules of order $2k$.

We shall use $\mathcal{H}_r$ to denote the dual mask of the mask a_h derived from the above modified TCBC algorithm such that $\mathcal{H}_r$ satisfies the sum rules of order $2r-2$. For each positive integer r, by G_r we denote the following set

$$G_r := \{(\alpha_1, \alpha_2) \in \mathbb{Z}^2 : |\alpha_1| + |\alpha_2| = 2r-1 \text{ and either } |\alpha_1| \text{ or } |\alpha_2| \text{ is an even number less than } r-1\}.$$

To sum up and restate the above construction of the dual masks $\mathcal{H}_r$ of the mask a_h, we have the following theorem.

Theorem 4.12 *Let r be a positive integer greater than two. Then there exists a unique refinement mask $\mathcal{H}_r$ satisfying the following conditions:*

1. *$\mathcal{H}_r$ is supported on*

$$\{(\alpha_1, \alpha_2) \in \mathbb{Z}^2 : |\alpha_1| + |\alpha_2| \le 2r-1, \max\{|\alpha_1|, |\alpha_2|\} \le 2r-2\} \backslash G_r;$$

2. *$\mathcal{H}_r$ is symmetric about the two coordinate axes, the lines $x_1 = x_2, x_1 = -x_2$;*

3. *$\mathcal{H}_r$ satisfies the sum rules of order $2r-2$;*

4. *$\mathcal{H}_r$ and a_h (the mask a_h is given in (4.5.11)) satisfy the dual relation (4.5.1).*

The set G_r looks like a little bit strange here. The reason is that in our modified TCBC algorithm, we set $c_{(\beta_1,\beta_2)} = 0$ for any $(\beta_1, \beta_2) \in F$ with $(r-1)/2 \le \alpha_1 < r-1$. Note that both H_r and $\mathcal{H}_r$ are symmetric about the x_1-axis, x_2-axis, and the lines $x_1 = x_2$ and $x_1 = -x_2$. In passing, we mention that an explicit formula for H_r and $\mathcal{H}_r$ can be easily obtained by a similar idea as in [88].

We provide detail in the next section for the masks $\mathcal{H}_3$ and $\mathcal{H}_4$.

4.7 Examples, Figures and Tables

In this section, we shall provide several examples of the masks H_r and $\mathcal{H}_r$. The graphs of the related dual scaling functions and associated wavelets are presented at the end of this section. Let us give some examples of the dual masks H_r and $\mathcal{H}_r$ as follows:

Example 4.13 The mask H_3 is supported on $[-4,4]^2$ and is given by

$$\begin{bmatrix}
0 & 0 & \frac{3}{256} & 0 & \frac{9}{128} & 0 & \frac{3}{256} & 0 & 0 \\
0 & 0 & 0 & -\frac{3}{64} & -\frac{3}{32} & -\frac{3}{64} & 0 & 0 & 0 \\
\frac{3}{256} & 0 & -\frac{1}{32} & -\frac{1}{32} & -\frac{51}{128} & -\frac{1}{32} & -\frac{1}{32} & 0 & \frac{3}{256} \\
0 & -\frac{3}{64} & -\frac{1}{32} & \frac{11}{32} & \frac{21}{32} & \frac{11}{32} & -\frac{1}{32} & -\frac{3}{64} & 0 \\
\frac{9}{128} & -\frac{3}{32} & -\frac{51}{128} & \frac{21}{32} & \frac{75}{32} & \frac{21}{32} & -\frac{51}{128} & -\frac{3}{32} & \frac{9}{128} \\
0 & -\frac{3}{64} & -\frac{1}{32} & \frac{11}{32} & \frac{21}{32} & \frac{11}{32} & -\frac{1}{32} & -\frac{3}{64} & 0 \\
\frac{3}{256} & 0 & -\frac{1}{32} & -\frac{1}{32} & -\frac{51}{128} & -\frac{1}{32} & -\frac{1}{32} & 0 & \frac{3}{256} \\
0 & 0 & 0 & -\frac{3}{64} & -\frac{3}{32} & -\frac{3}{64} & 0 & 0 & 0 \\
0 & 0 & \frac{3}{256} & 0 & \frac{9}{128} & 0 & \frac{3}{256} & 0 & 0
\end{bmatrix}.$$

The mask H_3 satisfies the sum rules of order 4 and $\nu_2(\phi_{H_3}) \approx 0.42927$.

Example 4.14 The mask $\mathcal{H}_3$ is supported on $[-4,4]^2$ and is given by

$$\begin{bmatrix}
0 & 0 & 0 & \frac{3}{128} & \frac{3}{64} & \frac{3}{128} & 0 & 0 & 0 \\
0 & 0 & 0 & -\frac{3}{64} & -\frac{3}{32} & -\frac{3}{64} & 0 & 0 & 0 \\
0 & 0 & \frac{1}{16} & -\frac{1}{8} & -\frac{3}{8} & -\frac{1}{8} & \frac{1}{16} & 0 & 0 \\
\frac{3}{128} & -\frac{3}{64} & -\frac{1}{8} & \frac{11}{32} & \frac{51}{64} & \frac{11}{32} & -\frac{1}{8} & -\frac{3}{64} & \frac{3}{128} \\
\frac{3}{64} & -\frac{3}{32} & -\frac{3}{8} & \frac{51}{64} & \frac{33}{16} & \frac{51}{64} & -\frac{3}{8} & -\frac{3}{32} & \frac{3}{64} \\
\frac{3}{128} & -\frac{3}{64} & -\frac{1}{8} & \frac{11}{32} & \frac{51}{64} & \frac{11}{32} & -\frac{1}{8} & -\frac{3}{64} & \frac{3}{128} \\
0 & 0 & \frac{1}{16} & -\frac{1}{8} & -\frac{3}{8} & -\frac{1}{8} & \frac{1}{16} & 0 & 0 \\
0 & 0 & 0 & -\frac{3}{64} & -\frac{3}{32} & -\frac{3}{64} & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{3}{128} & \frac{3}{64} & \frac{3}{128} & 0 & 0 & 0
\end{bmatrix}.$$

Then $\mathcal{H}_3$ satisfies the sum rules of order 4 with $\nu_2(\phi_{\mathcal{H}_3}) \approx 1.17513$. Thus, by Theorem 4.2 and Corollary 4.7, the function $\phi_{\mathcal{H}_3}$ is an optimal dual scaling function of the function φ_h in the L_2 norm since $\nu_2(\phi_{\mathcal{H}_3}) \approx \nu_2(\phi_{b_3}) - \nu_1(\varphi_h)$.

Example 4.15 The mask $\mathcal{H}_4$ is supported on $[-6,6]^2$ and the part of $\mathcal{H}_4$ in the first quadrant is supported on $[0,6]^2$ and is given by

$$\begin{bmatrix} -\frac{5}{512} & -\frac{5}{1024} & 0 & 0 & 0 & 0 & 0 \\ \frac{5}{256} & \frac{5}{512} & 0 & 0 & 0 & 0 & 0 \\ \frac{83}{1024} & \frac{145}{4096} & -\frac{15}{2048} & -\frac{9}{4096} & 0 & 0 & 0 \\ -\frac{363}{2048} & -\frac{87}{1024} & \frac{15}{1024} & \frac{9}{1024} & -\frac{9}{4096} & 0 & 0 \\ -\frac{359}{1024} & -\frac{69}{512} & \frac{1}{16} & \frac{15}{1024} & -\frac{15}{2048} & 0 & 0 \\ \frac{1723}{2048} & \frac{401}{1024} & -\frac{69}{512} & -\frac{87}{1024} & \frac{145}{4096} & \frac{5}{512} & -\frac{5}{1024} \\ \frac{493}{256} & \frac{1723}{2048} & -\frac{359}{1024} & -\frac{363}{2048} & \frac{83}{1024} & \frac{5}{256} & -\frac{5}{512} \end{bmatrix}$$

with the number $\frac{493}{256}$ at the bottom-left as $\mathcal{H}_4(0,0)$. Since $\mathcal{H}_4$ is symmetric about the coordinate axes, other part of $\mathcal{H}_4$ is obtained by symmetry as in (4.5.10). By calculation, we have $\nu_2(\phi_{\mathcal{H}_4}) \approx 1.79313$ and the mask $\mathcal{H}_4$ satisfies the sum rules of order 6. Thus, the function $\phi_{\mathcal{H}_4}$ is an optimal dual scaling function of φ_h in the L_2 norm sense since $\nu_2(\phi_{\mathcal{H}_4}) \approx \nu_2(\phi_{b_4}) - \nu_1(\varphi_h)$.

The graphs and contours of all the refinable functions and biorthogonal wavelets of the above mentioned examples are given in Figures 4.3–4.8. From the figures, it is easily seen that all the wavelets are symmetric and they have good time localization.

Recall that by b_r we denote the interpolatory refinement mask supported on $[1-2r, 2r-1]$ as constructed by Deslauriers and Dubuc in [30]. By $\phi_{a^d_{t_r}}$ we denote the tensor product dual scaling function of φ_h with its mask $a^d_{t_r}$ satisfying

$$\overline{\widetilde{a_h}(z_1,z_2)}\widetilde{a^d_{t_r}}(z_1,z_2) = \overline{\widetilde{b_r}(z_1)\widetilde{b_r}(z_2)}, \qquad (z_1,z_2) \in \mathbb{T}^2.$$

Comparison Results							
r	$\nu_2(\phi_{b_r})$	$\nu_2(\phi_{a^d_{t_r}})$	$\nu_2(\phi_{H_r})$	$\nu_2(\phi_{\mathcal{H}_r})$	$N(a^d_{t_r})$	$N(H_r)$	$N(\mathcal{H}_r)$
3	3.17513	1.17513	0.42927	1.17513	81	49	49
4	3.79313	1.79313	0.98084	1.79313	169	97	101
5	4.34408	2.34408	1.46708	2.34408	289	161	161
6	4.86202	2.86202	1.90387	2.86202	441	241	245
7	5.36283	3.36283	2.30033	3.36283	625	337	337
8	5.85293	3.85293	2.66264	3.85293	841	449	453
9	6.33524	4.33524	2.99578	4.33524	1089	557	577
10	6.81144	4.81144	3.30381	4.81144	1369	721	725
11	7.28260	5.28260	3.58991	5.28260	1681	881	881
12	7.74953	5.74953	3.85672	5.74953	2025	1057	1061

Table 4.1: Comparison Results between different dual scaling functions of φ_h.

Let the masks H_r and $\mathcal{H}_r$ be the dual masks constructed by the TCBC algorithm and the modified TCBC algorithm respectively such that both H_r and $\mathcal{H}_r$ satisfy the sum rules of order $2r-2$. In the following, we use $N(a)$ to denote the number of nonzero coefficients in the refinement mask a. The values of $\nu_2(\phi_{b_r})$ are taken from [42]. The following table shows that for $r = 3, \cdots, 12$, the function $\phi_{\mathcal{H}_r}$ is an optimal dual scaling function of φ_h in the L_2 norm sense.

In passing, we mention that there is a very close relation between biorthogonal wavelets and dual wavelet frames. A complete characterization of dual wavelet frames associated with any dilation matrix was given by myself in [48]. Such characterization appeared in my master degree thesis [50] in 1994, and in Long's book [83] published in 1995. In addition, given any multivariate mask a with its dual mask a^d, a new pair of biorthogonal masks can be easily derived. See Proposition 3.7 in Han [48] and [50] for detailed discussion.

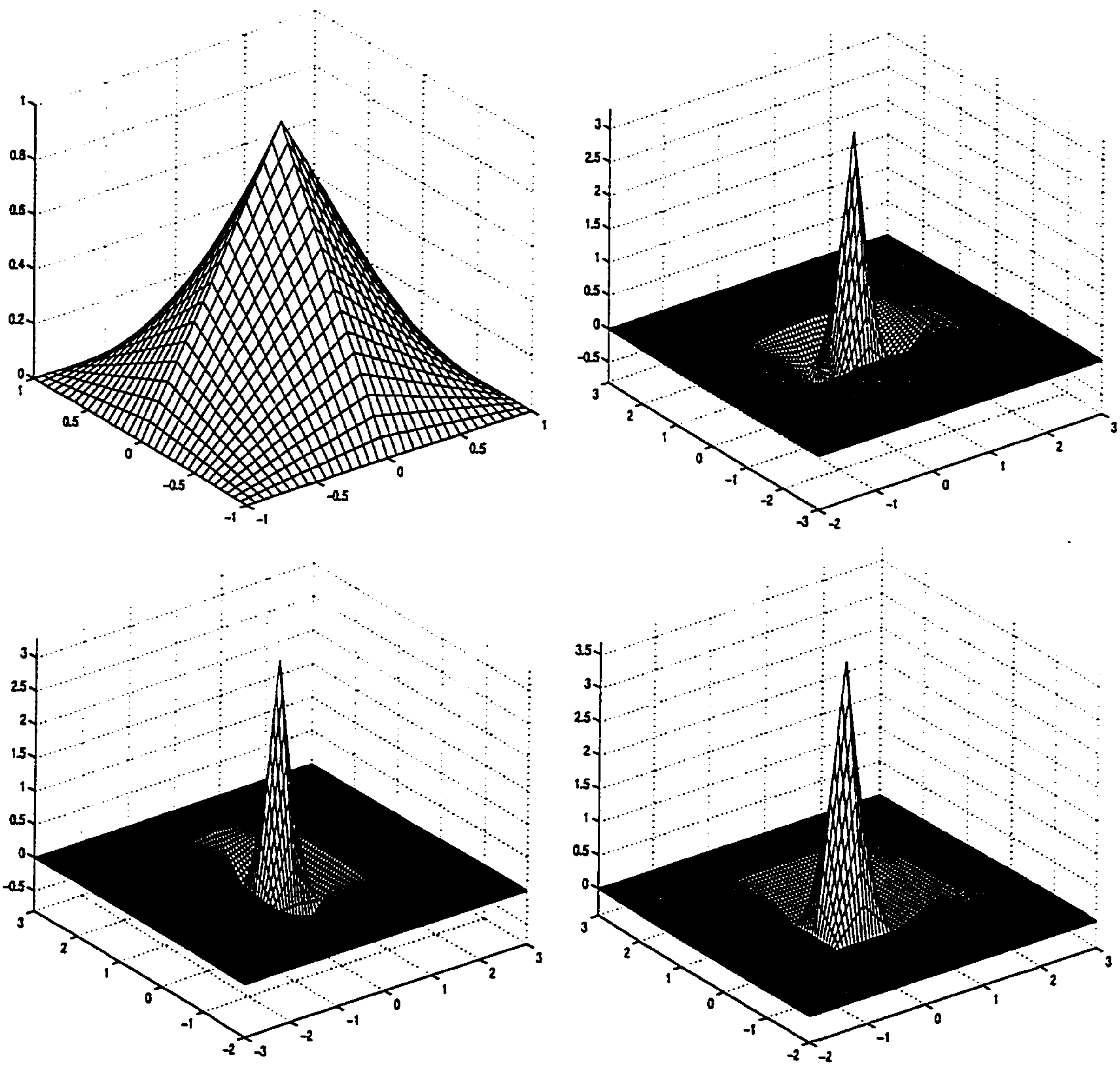

Figure 4.3: The scaling function φ_h and the associated three wavelets ψ_1, ψ_2 and ψ_3 in Example 4.14.

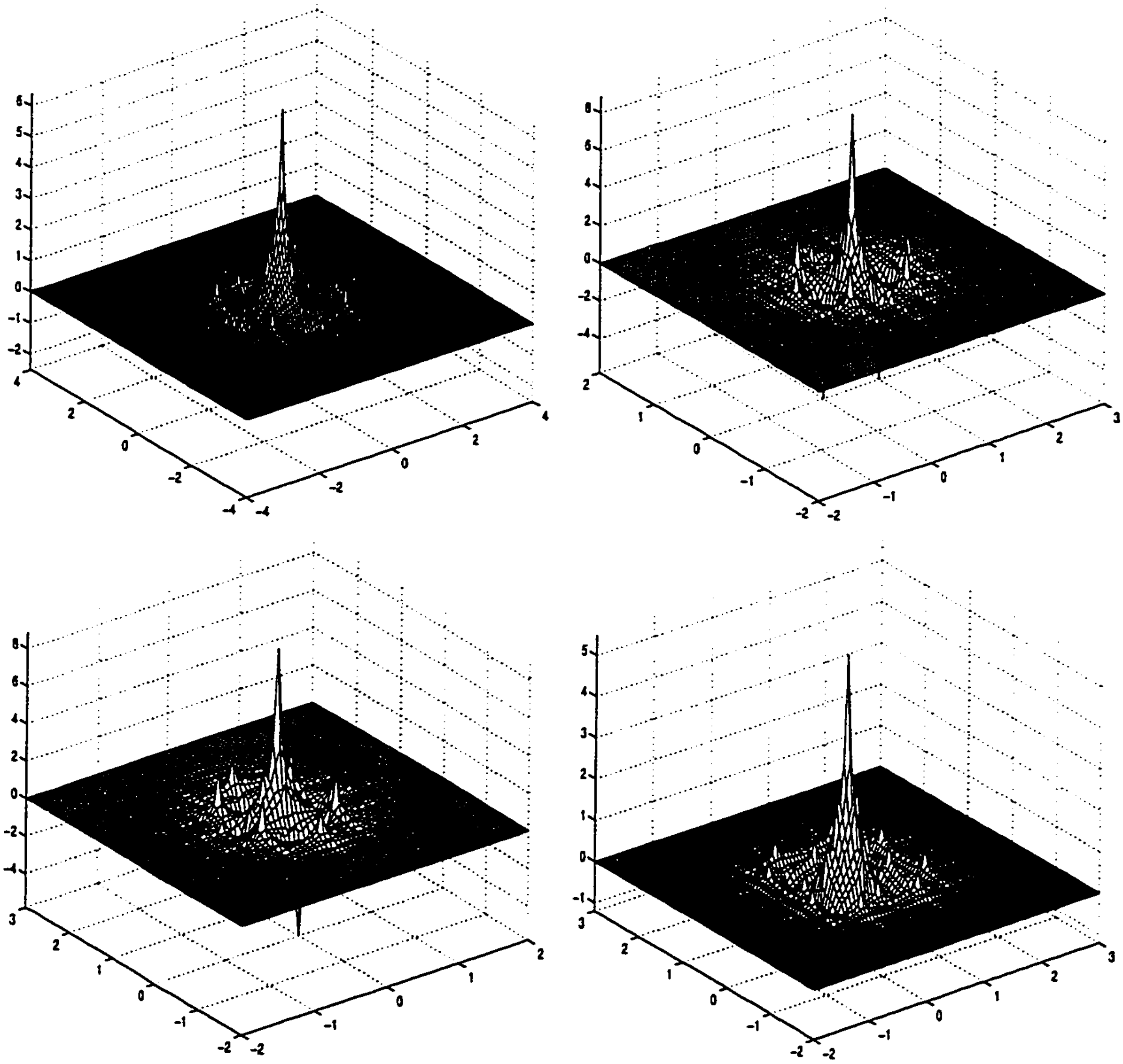

Figure 4.4: The dual scaling function $\phi_{\mathcal{H}_3}$ and the associated three dual wavelets ψ_1, ψ_2 and ψ_3 in Example 4.14.

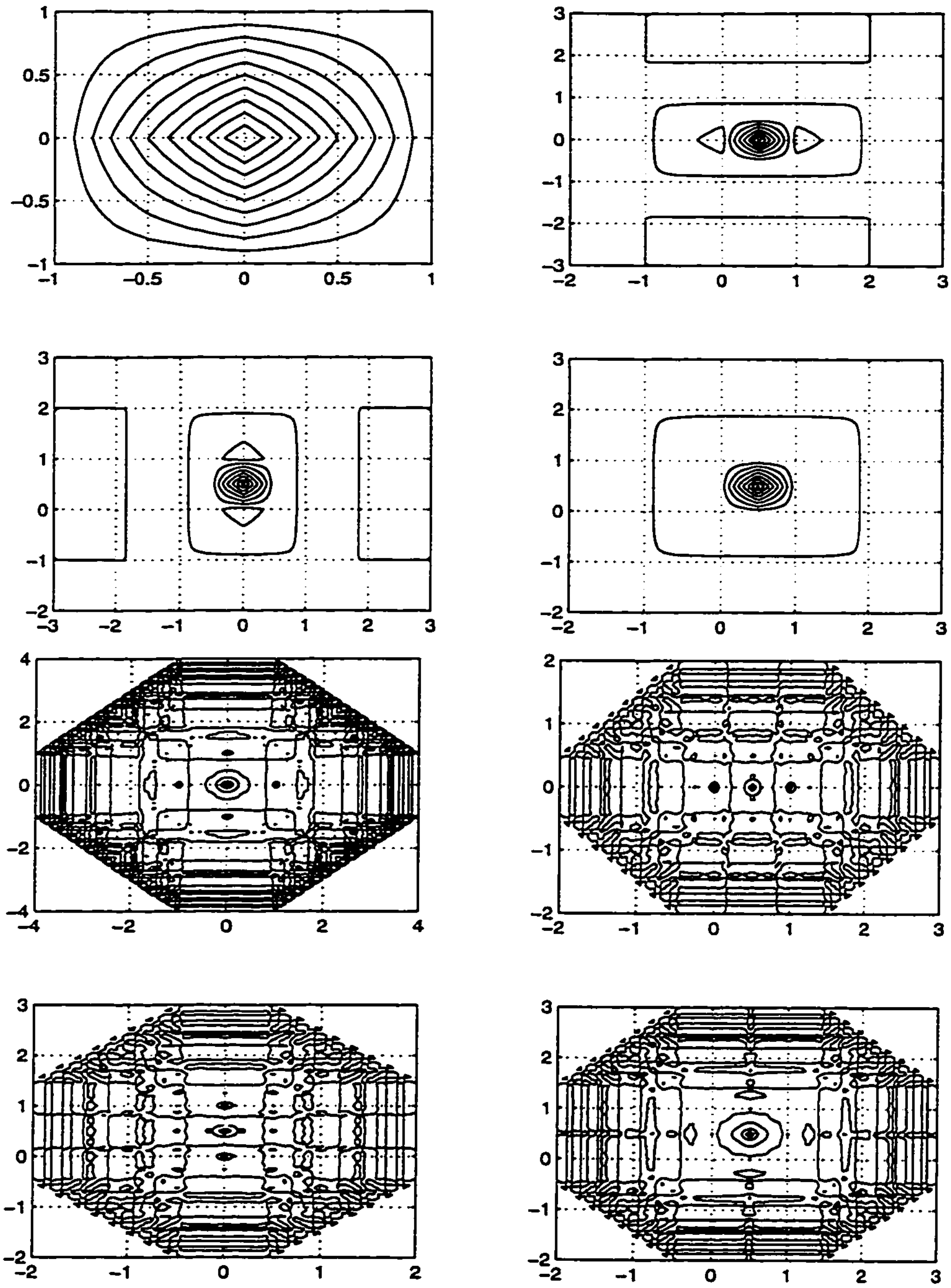

Figure 4.5: The contours of the scaling function φ_h,its dual scaling functions $\phi_{\mathcal{H}_3}$ and the associated wavelets and dual wavelets in Example 4.14.

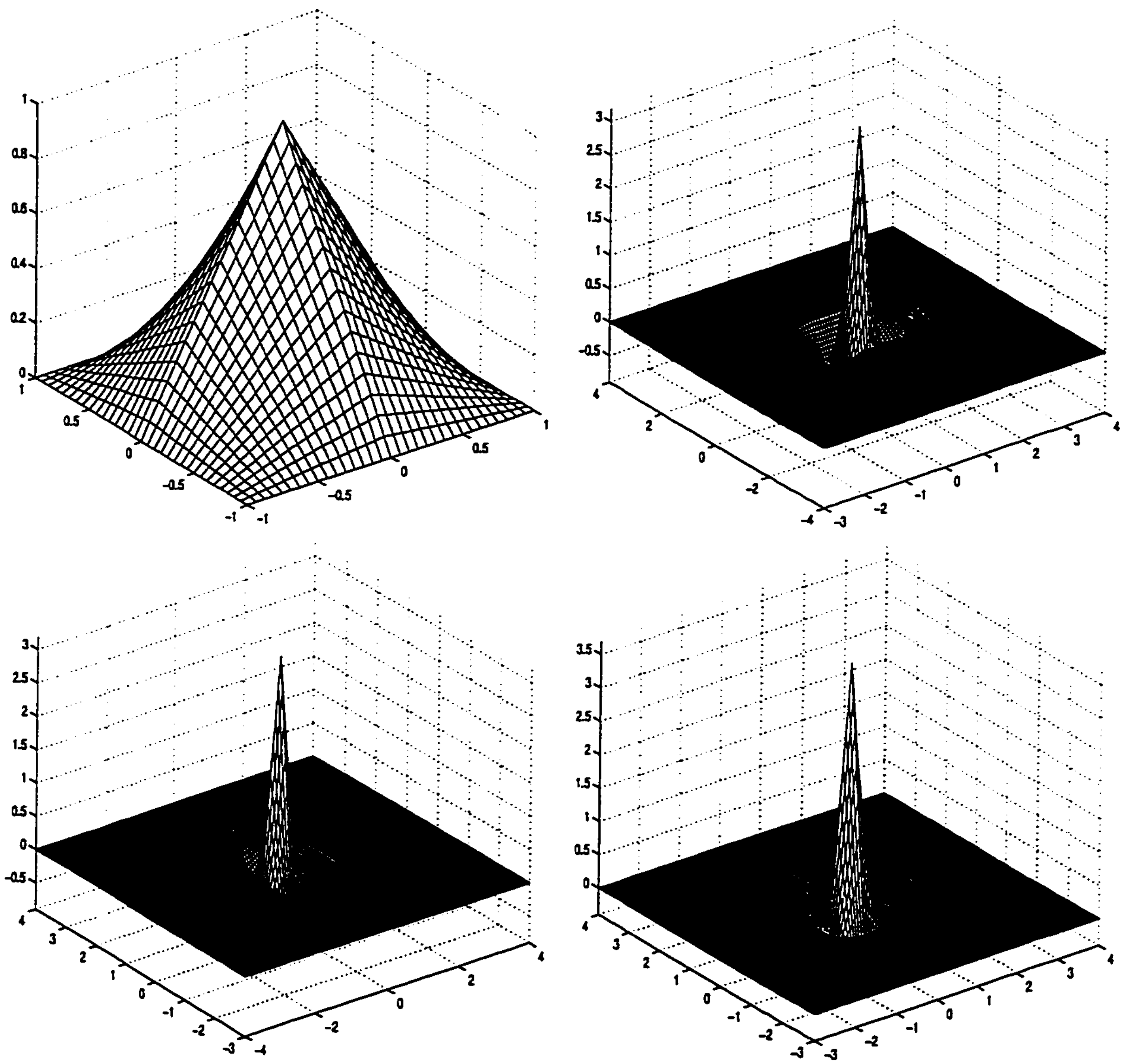

Figure 4.6: The scaling function φ_h and the associated three wavelets ψ_1, ψ_2 and ψ_3 in Example 4.15.

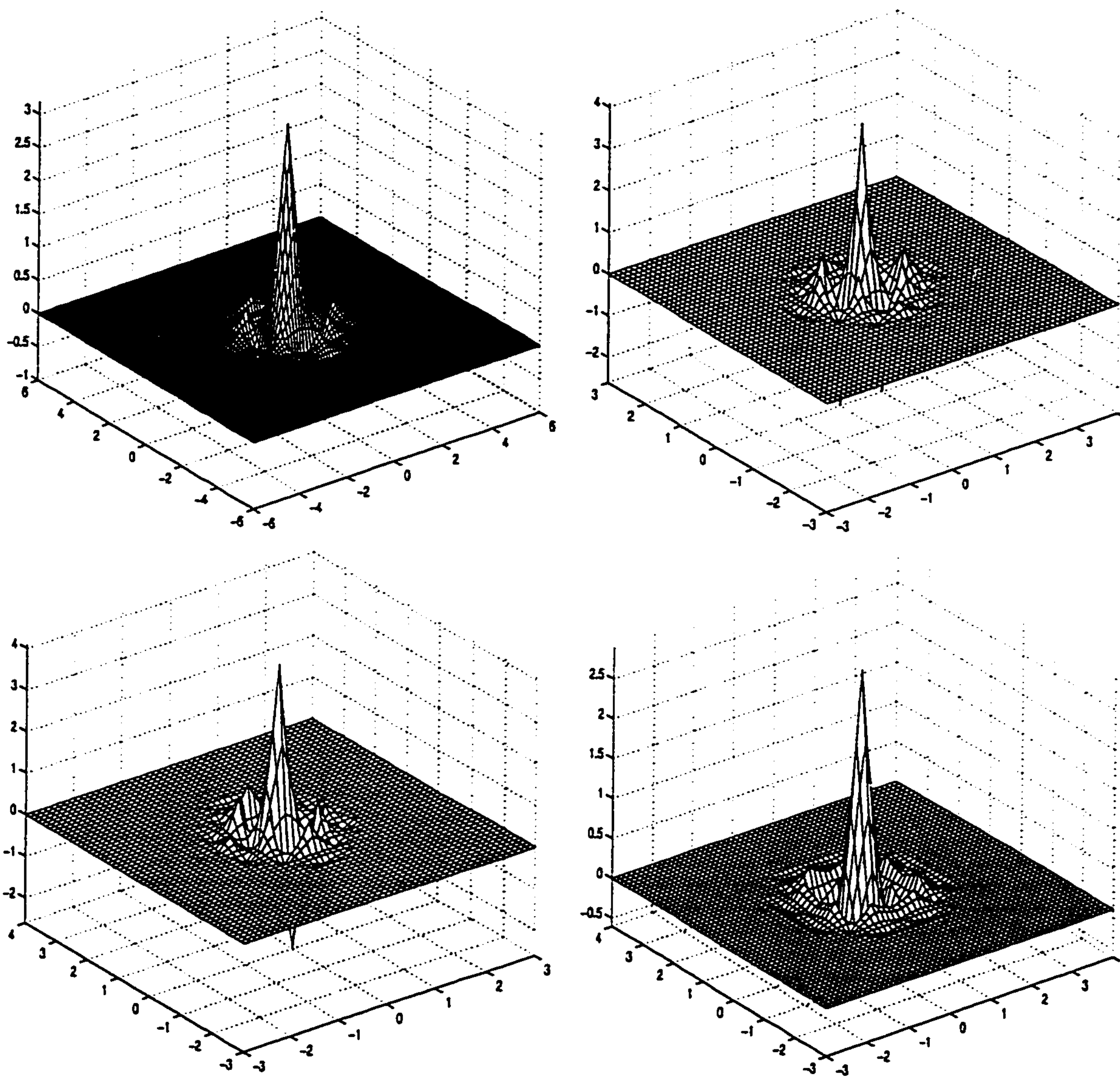

Figure 4.7: The dual scaling function $\phi_{\mathcal{H}_4}$ and the associated three dual wavelets ψ_1, ψ_2 and ψ_3 in Example 4.15.

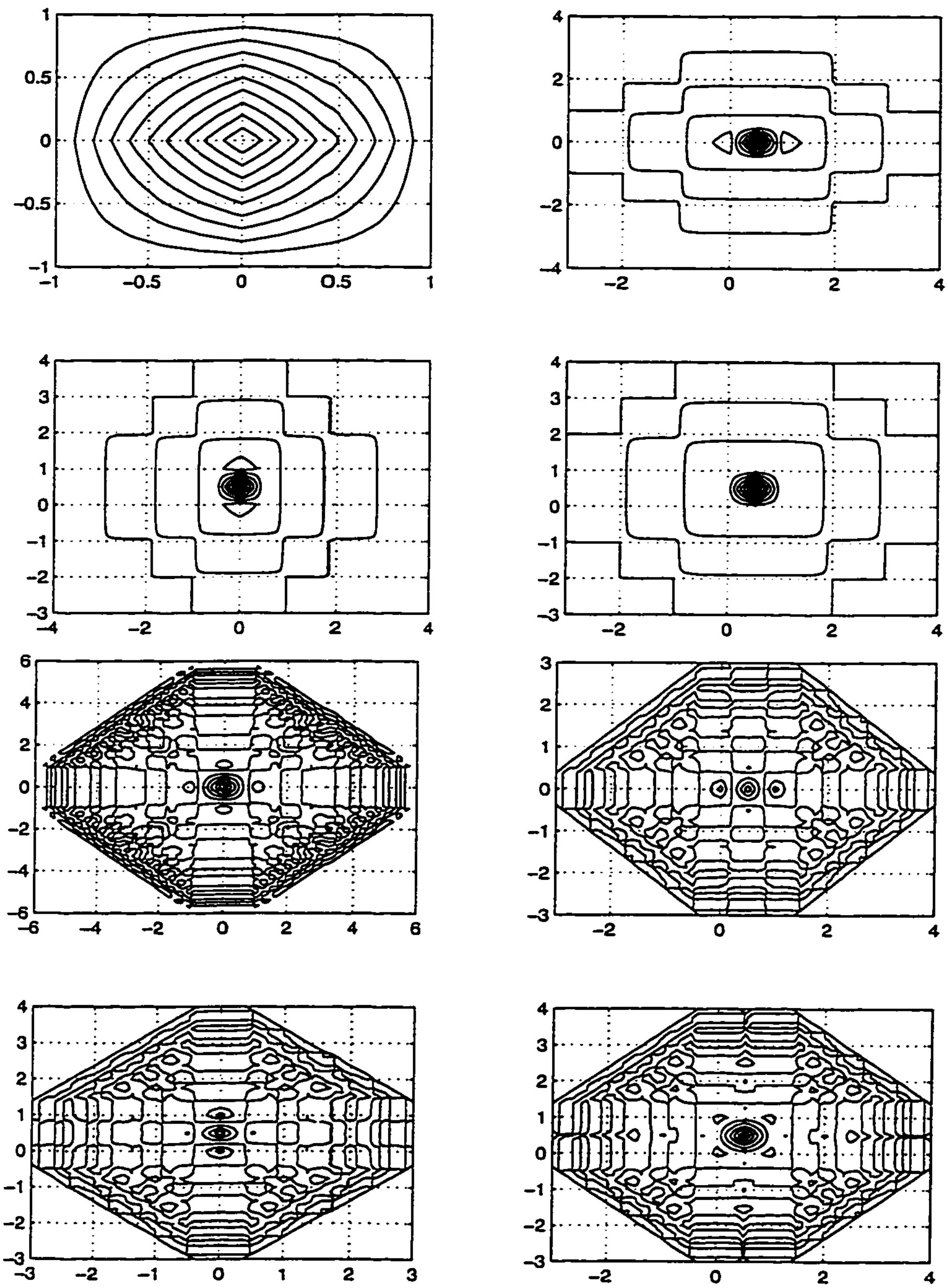

Figure 4.8: The contours of the scaling function φ_h,its dual scaling functions $\phi_{\mathcal{H}_4}$ and the associated wavelets and dual wavelets in Example 4.15.

Chapter 5

Image Compression By Using 2-D Wavelet Filters

5.1 Introduction

In this chapter, we shall implement a wavelet transform algorithm by directly using bivariate (2-D) wavelet filters. Then we shall use this algorithm to test some examples of 2-D wavelet filters given in Chapter 4 on lossy image compression.

A fundamental goal of lossy image compression is to reduce the bit rate (i.e., the number of bits used to store each pixel) to save time in transmission and save storage of digital images while maintaining an acceptable fidelity or image quality. One of the methods to achieve this goal is to employ transform such as the well known discrete cosine transform (DCT) and more recently the wavelet transform. Transform-based image compression can be achieved by transforming the data, projecting it onto a basis of functions, and encoding this transform. Because of the nature of human vision, the transform used must be well localized in both time and frequency domains. A wavelet transform is well localized in both space and frequency domains and is very similar to the mechanisms of human vision system. Thus the wavelet transform matches well with human visual system characteristics. From an image-coding point of view, this tends to contribute to good image quality. Results reported in the literature have already demonstrated that wavelet-based image compression techniques have many advantages and match or outperform many other well known lossy image compression methods, see [94, 95].

Currently, many excellent wavelet based image compression algorithms are proposed in the literature. For example, the EZW algorithm proposed by Shapiro in [95] and the SPIHT algorithm provided by Said and Pearlman in [94] are two well-known examples. Other good quantization schemes are the CB algorithm proposed in [8] and the algorithm given in [103].

In almost all the wavelet-based image compression algorithms proposed in the literature, the tensor product 2-D wavelet transform is employed. That is, performing the 1-D wavelet transform on columns and rows of an image separately. But as noted by many experts, the tensor product wavelet transform gives preference to the horizontal and vertical directions (see [77]). Based on our study of analysis and construction of multivariate biorthogonal wavelets in the previous chapters, we shall try to use 2-D wavelet filters directly to do wavelet transform. Since an image actually consists of 2-D data and has redundancy in a 2-D neighborhood, it is natural to use 2-D wavelet transform on images by using 2-D wavelet filters directly. We acknowledge that our 2-D wavelet transform algorithm is based on the C++ source code from SPIHT in[94] where the tensor product wavelet transform is used.

Here is the outline of this chapter. In Section 5.2, we shall describe the 2-D wavelet transform. In Section 5.3, we illustrate our experimental results on image compression.

5.2 2-D Wavelet Transform

An obvious way of building wavelets in higher dimensions is through tensor products of 1-D wavelets. This gives us separable 2-D wavelet filters. However, this approach gives preferential treatment to the coordinate axes and only allows for rectangular division of the frequency spectrum. Often, symmetric axes and certain nonrectangular division of the frequency spectrum correspond better to the human visual system (see [77]). Thus, it is interesting in its own right to construct 2-D wavelet filters and apply them in image compression.

To perform a 2-D wavelet transform, we need a pair of wavelet filters, one is the primal/synthesis filter h and the other is the dual/analysis filter h^d. In other words, both h and h^d are sequences on $\mathbb{Z}^2$ and satisfy the following conditions:

$$\sum_{j\in\mathbb{Z}^2} h(j) = 2 \quad \text{and} \quad \sum_{j\in\mathbb{Z}^2} h^d(j) = 2 \tag{5.2.1}$$

and

$$\sum_{j\in\mathbb{Z}^2} h(j-2i)\overline{h^d(j)} = \begin{cases} 1, & \text{if } i=(0,0), \\ 0, & \text{if } i \in \mathbb{Z}^2\backslash\{(0,0)\}. \end{cases} \tag{5.2.2}$$

From these two wavelet filters, six more high pass filters including three high pass primal wavelet filters g_1, g_2, g_3 and three high pass dual wavelet filters g_1^d, g_2^d, g_3^d are derived. For example, see [10, 12, 13, 18, 27, 57, 67, 72, 79, 90] on how to derive these six high pass wavelet filters from the primal filter h and the dual filter h^d.

Now for any given 2-D data c, we can decompose it into four subbands as follows: one low frequency subband

$$c^1(i) := \sum_{j\in\mathbb{Z}^2} h^d(2i-j)c(j), \qquad i \in \mathbb{Z}^2 \tag{5.2.3}$$

and three high frequency subbands

$$d_k^1(i) = \sum_{j\in\mathbb{Z}^2} g_k^d(2i-j+\varepsilon_k)c(j), \qquad k=1,2,3, i\in\mathbb{Z}^2, \tag{5.2.4}$$

where $\varepsilon_1 = (0,1), \varepsilon_2 = (1,0), \varepsilon_3 = (1,1)$.

Now from these data c^1 and d_1^1, d_2^1, d_3^1, we can recover the original data c as follows:

$$\sum_{j\in\mathbb{Z}^2} h(2j-i)c^1(j) + \sum_{k=1}^{3}\sum_{j\in\mathbb{Z}^2} g_k(2j-i+\varepsilon_k)d_k^1(j) = c(i), \qquad i \in \mathbb{Z}^2.$$

To the best of our knowledge, there is no program available in the current literature to do 2-D wavelet transform by directly using 2-D wavelet filters. Usually it is harder to design 2-D filters h, h^d, g's and g^d's than to design 1-D wavelet filters and the code to perform 2-D wavelet transform will be more complicated than that in the corresponding 1-D case.

5.3 Preliminary Tests

As we demonstrated in Chapter 4, in the 2-D case, we can construct a dual filter h^d provided that a primal filter h is given. Though we know how to derive the six associated high pass wavelet filters from h and h^d, it is still not clear in the literature that how we should design the high pass primal wavelet filters g_1, g_2, g_3 and the three high pass dual wavelet filters g_1^d, g_2^d, g_3^d such that they are desirable in image compression. In the 1-D case, given a primal filter h and a dual filter h^d, the associated high pass wavelet filters are in some sense uniquely determined. In the two dimensional case, such uniqueness is not known and there is no existing algorithm to derive best possible wavelet filters g's and g^d's from h and h^d such that they have some desirable properties. In this chapter, we shall use the method described in [90] to derive the associated six high pass wavelet filters. It is still an open problem about how to design three high pass primal wavelet filters g_1, g_2, g_3 and the three high pass dual wavelet filters g_1^d, g_2^d, g_3^d from the primal filter h and the dual filter h^d such that they are efficient in image compression.

In the following, we shall try the following trick on tensor product wavelet filters to illustrate why it is very important to design the associated high pass wavelet filters with some desired properties.

Let H, H^d, G, G^d denote the standard D9-7 1-D wavelet filters given in [1] where H is the primal filter, H^d is the dual wavelet filter, G is the associated 1-D high pass primal wavelet filter and G^d is the associated 1-D high pass dual wavelet filter. When we apply these 1-D wavelet filters on images, we actually use the tensor product method to apply the following 2-D wavelet filters:

$$\begin{aligned}
h(i,j) &= H(i)H(j) &\quad \text{and} \quad && h^d(i,j) &= H^d(i)H^d(j)\\
g_1(i,j) &= H(i)G(j) &\quad \text{and} \quad && g_1^d(i,j) &= H^d(i)G^d(j)\\
g_2(i,j) &= G(i)H(j) &\quad \text{and} \quad && g_2^d(i,j) &= G^d(i)H^d(j)\\
g_3(i,j) &= G(i)G(j) &\quad \text{and} \quad && g_3^d(i,j) &= G^d(i)G^d(j).
\end{aligned}$$

Thus h is a 2-D primal filter and h^d is a dual filter of h. The filters $g_1, g_2, g_3, g_1^d, g_2^d, g_3^d$ are the associated high pass wavelet filters.

Now we play a trivial trick here. Let us first choose a parameter c. Then it is easily seen that $cg_1, cg_2, cg_3, c^{-1}g_1^d, c^{-1}g_2^d, c^{-1}g_3^d$ are still the associated high pass wavelet filters (see [27]). We apply the set of wavelet filters $h, h^d, cg_1, cg_2, cg_3, c^{-1}g_1^d, c^{-1}g_2^d, c^{-1}g_3^d$ to image compression and use our modified 2-D SPIHT algorithm. It is evident that when $c = 1$, this is exactly the D9-7 tensor product wavelet filters. Experiment shows the results in Table 5.1 where *bpp* means bit per pixel (the number of bits used to store each pixel) and all the comparison results are in PSNR. For a definition of PSNR, see [94].

Given a primal filter h, we can design a dual filter h^d easily as we demonstrated in Chapter 4. The following tests also demonstrate that the design of the associated high pass wavelet filters are important in image compression. Let h be our primal filter given in (4.5.11) in Chapter 4. Then both $\mathcal{H}_3$ and $a_{t_3}^d$ defined in Chapter 4 are dual filters of h and both of them are supported on $[-4,4]^2$. Since h is an interpolatory filter, we employ the method described by Riemenschneider and Shen in [90] to derive

256 × 256 lena image						
bpp	1	0.5	0.25	0.125	0.0625	0.03125
c=1	36.49	31.99	28.54	25.96	23.99	22.12
c=0.75	36.54	32.06	28.63	26.09	23.93	22.22
256 × 256 monalisa image						
bpp	1	0.5	0.25	0.125	0.0625	0.03125
c=1	34.37	31.86	30.33	28.96	27.69	26.13
c=0.75	34.56	31.91	30.26	29.05	27.71	26.14

Table 5.1: Experimental results on choosing different wavelet filters on lena and monalisa images of size 256 by 256.

their associated high pass wavelet and dual wavelet filters. For simplicity, we still use $\mathcal{H}_3$ and $a^d_{t_3}$ to denote the derived wavelet filter sets. By computation, up to a shift, the three wavelet filters derived from both $\mathcal{H}_3$ and $a^d_{t_3}$ are supported on $[-5,5] \times [-4,4]$, $[-4,4] \times [-5,5]$ and $[-4,4] \times [-4,4]$ respectively, and their three dual wavelet filters are supported on $0 \times [-1,1]$, $[-1,1] \times 0$ and $[-1,1] \times [-1,1]$.

In fact, the three high pass dual wavelet filters derived from both $\mathcal{H}_3$ and $a^d_{t_3}$ are the same if we derive them by using the method in [90]. We shall use $Tensor_3$ to denote the tensor product wavelet filter set such that all the primal filter, the dual filter and the six associated high pass wavelet filters are derived by tensor product as we did before for D9-7. The only difference between this set of wavelet filters $Tensor_3$ and the set of wavelet filters $a^d_{t_3}$ is that their derived high pass wavelet filters are different. In fact, the three associated high pass primal wavelet filters with $Tensor_3$ are supported on $[-4,4] \times [-1,1]$, $[-1,1] \times [-4,4]$ and $[-4,4] \times [-4,4]$ respectively, and the three high pass dual wavelet filters associated with $Tensor_3$ are supported on $[-1,1] \times [-4,4]$, $[-4,4] \times [-1,1]$ and $[-1,1] \times [-1,1]$. In the following D9-7 means the tensor product D9-7 wavelet filter set.

256 × 256 lena image						
bpp	1	0.5	0.25	0.125	0.0625	0.03125
$\mathcal{H}_3$	35.07	30.59	27.48	25.17	23.24	21.39
$a^d_{t_3}$	35.05	30.60	27.46	25.15	23.23	21.38
$Tensor_3$	35.76	31.40	28.18	25.71	23.71	22.02
$D97$	36.49	31.99	28.54	25.96	23.99	22.12
256 × 256 monalisa image						
bpp	1	0.5	0.25	0.125	0.0625	0.03125
$\mathcal{H}_3$	33.73	31.35	29.82	28.55	27.29	25.71
$a^d_{t_3}$	33.72	31.34	29.82	28.55	27.31	25.73
$Tensor_3$	34.04	31.74	30.20	28.91	27.60	25.98
$D97$	34.37	31.86	30.33	28.96	27.69	26.13

Table 5.2: Experimental results by using different wavelet filter sets on lena and monalisa images of size 256 by 256.

The test results are presented in Table 5.2 where we use the dual filter and its associated three high pass dual wavelet filters to do wavelet decomposition.

From Table 5.2, we see that the performance of $\mathcal{H}_3$ and $a^d_{t_3}$ is very similar. The reason is that their associated high pass dual wavelet filters are the same. The difference between $a^d_{t_3}$ and $Tensor_3$ demonstrates that even we have the same set of a primal filter and a dual filter, how to design their associated high pass wavelet filters is very important in image compression. Due to the complicity of the design of good wavelet filters from given pair of scaling and dual scaling filters, we just test the above four sets of biorthogonal wavelet filters in our wavelet library. To get some 2-D wavelet filter sets which have better performance than the well known D9-7 wavelet filter set, more effort and analysis are needed to build a reasonable large 2-D wavelet library.

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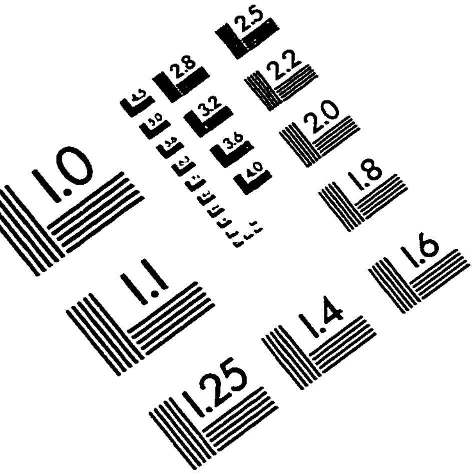
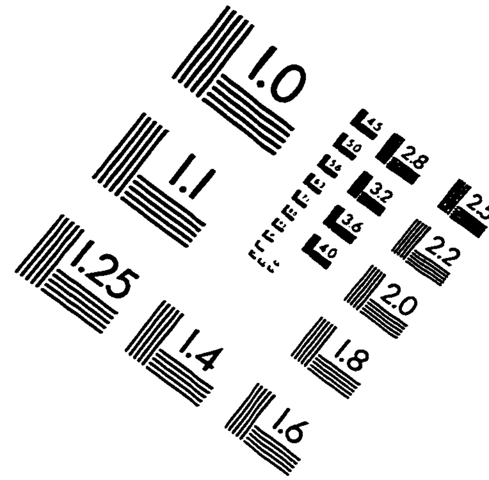
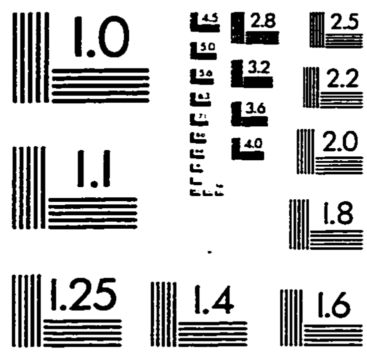
IMAGE EVALUATION TEST TARGET (QA-3)







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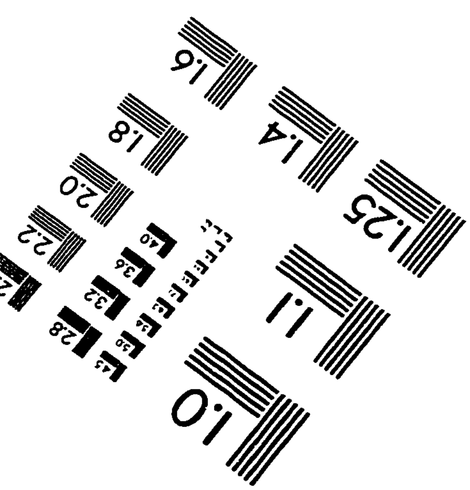

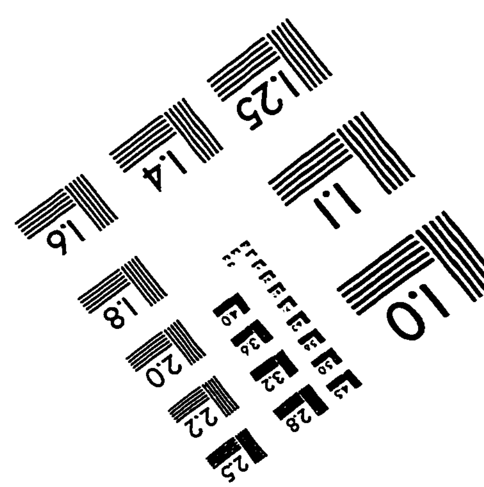

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