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SUBEXPONENTIAL LOSS RATE ASYMPTOTICS FOR LÉVY PROCESSES

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Abstract

We consider a Lévy process reflected in barriers at 0 and $K > 0$. The loss rate is the mean time spent at the upper barrier K at time 1 when the process is started in stationarity, and is a natural continuous-time analogue of the stationary expected loss rate for a reflected random walk. We derive asymptotics for the loss rate when K tends to infinity, when the mean of the Lévy process is negative and the positive jumps are subexponential. In the course of this derivation, we achieve a formula, which is a generalization of the celebrated Pollaczeck-Khinchine formula.

Keywords finite buffer, heavy tails, Lévy process, local times, loss rate, Pollaczeck-Khinchine formula, subexponential distributions.

1 Introduction

In the papers Jelenković [13] and Pihlsgård [18], the authors examine the loss rate associated with a stochastic process obtained by reflecting a random walk in two barriers at 0 and $K > 0$, and derive asymptotic expressions for the loss rate as K tends to infinity. In particular, Jelenković [13] derives the asymptotics of the loss rate in the case of heavy tails. The continuous-time analogue of the loss rate associated with a reflected random walk, is the loss rate associated with a reflected Lévy process which is examined in Asmussen and Pihlsgård [4], where an explicit expression for the loss rate in terms of the characteristic triplet of the Lévy process is provided. Furthermore, [4] gives the asymptotic behavior of the loss rate as K tends to infinity in the case where the mean of the Lévy process is positive as well the case where the mean is negative and the jumps of the process are light-tailed, and in the Andersen and Asmussen [1] the authors examine loss rate asymptotics for centered Lévy processes. In this paper we derive asymptotics where the mean is negative and the process is heavy-tailed.

Reflected processes may be used to model waiting time processes in queues with finite capacity (Cohen [8], Cooper et al. [9], Bekker and Zwart [5], Daley [10]).

It may be used to model a finite dam or fluid model (Asmussen [3],Moran [17], Stadje [21]). Furthermore, it is used in models of network traffic or telecommunications systems involving a finite buffer (Jelenković [13], Zwart [22], Kim and Shroff [14]) and in this context the loss rate can be interpreted as the bit loss rate in a finite data buffer.

The main contribution of this paper is Theorem 3.1 which provides an asymptotic expression for the loss rate in the heavy-tailed case. In the course of the derivation of this expression, we also obtain a formula, (3.2), which is a generalization of the celebrated Pollaczeck-Khinchine formula.

The outline of the paper is as follows: In Section 2 we provide the essential background on Lévy processes, and give the formal definition of the loss rate. With the definitions and previous results settled we can state the main results in Section 3. The proofs are given in Section 4.

2 Preliminaries

2.1 Lévy Processes and the Loss rate

We consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. A *Lévy process* $\mathbf{S} := \{S_t\}$ is a real-valued stochastic process on $\mathbb{R}$ with stationary independent increments which is continuous in probability and with $S_0 = 0$ $\mathbb{P} - a.s.$ Every Lévy process $\mathbf{S}$ is associated with a unique *characteristic triplet* (θ, σ, ν), where $\theta \in \mathbb{R}$, $\sigma \geq 0$ and ν is a measure (*the Lévy measure*) with $\int_{-\infty}^{\infty}(1 \wedge y^2)\nu(\mathrm{d}y) < \infty$ and $\nu(\{0\}) = 0$. The *Lévy exponent* is given by

$$\kappa(\alpha) = \theta\alpha + \frac{\sigma^2\alpha^2}{2} + \int_{-\infty}^{\infty} \left[e^{\alpha x} - 1 - \alpha I(|x| \leq 1)\right] \nu(\mathrm{d}x)$$

and is defined for α in $\Theta := \{\alpha \in \mathbb{C} \mid \mathbb{E}e^{\Re(\alpha)S_1} < \infty\}$. The Lévy exponent is the unique function satisfying $\mathbb{E}e^{\alpha X_t} = e^{t\kappa(\alpha)}$ and $\kappa(0) = 0$. We assume throughout this paper that $\mathbb{E}|S_1| < \infty$. We use the cadlag version of $\mathbf{S}$, which exists because of stochastic continuity. We note that this implies that $\Delta S_t := S_t - S_{t-}$ is well-defined. Standard references for Lévy processes are Bertoin [6], Kyprianou [15] and Sato [20].

We are given a Lévy process through its characteristic triplet, and reflect it in barriers at 0 and $K > 0$. The reflected process is given as part of the solution to a Skorokhod problem and is denoted $\mathbf{V^K}$. We have a decomposition

$$V_t^K = x + S_t + L_t^0 - L_t^K \tag{2.1}$$

of the reflected process started at $x \in [0, K]$ where $\mathbf{L^0} := \{L_t^0\}$ and $\mathbf{L^K} := \{L_t^K\}$ are the local times at 0, K respectively. Note that the reflected process and the local times are cadlag, so that objects such as $\Delta L_t^0 := L_t^0 - L_{t-}^0$ are well-defined and by way of being increasing, the local times are of bounded variation which allow is to decompose them into a continuous part and a jump part. For more information on Skorokhod problems, see Asmussen [3], Asmussen and Pihlsgård [4] and Andersen and Mandjes [2].

Because of the independent, identically distributed increments of $\mathbf{S}$, $\mathbf{V^K}$ has a regenerative structure which yields a stationary distribution denoted π_K. The stationary distribution satisfies:

$$\overline{\pi}_K(y) = \pi_K[y, K] = \mathbb{P}\big(S_{\tau[y-K,y)} \geq y\big), \quad 0 \leq y \leq K \tag{2.2}$$

where $\tau[u, v) = \inf\big\{t > 0 \mid S_t \notin [u, v)\big\}$. See Asmussen [3, pp. 393-394] for a derivation of this representation. When $K = \infty$, we have one-sided reflection (See Asmussen [3, IX 2a]). In this case $L_t^K \equiv 0$, and $L_t^0 := (-\inf_{0 \leq v \leq t} S_v - y)^+$, and we have a result similar to (2.2) of the one-sided stationary distribution which follows from Cor. 2. IX p. 253 in Asmussen [3]:

$$\overline{\pi}_\infty(y) = \mathbb{P}\big(\sup_{t \geq 0} S_t \geq y\big) = \mathbb{P}(\tau(y) < \infty) \tag{2.3}$$

where $\tau(y) = \inf\{t > 0 : S_t \geq y\}$. Furthermore, for notational convenience we set $L_t^0 := L_t$ when $K = \infty$.

We follow the standard definitions of the classes $\mathcal{S}$ and $\mathcal{S}^*$ of distribution functions. The class $\mathcal{S}$ is defined by the requirement that $\overline{F^{*n}}(x) \sim n\overline{F}(x)$ ($F^{*n} = n$th convolution power), and $\mathcal{S}^*$ by

$$\lim_{x \to \infty} \frac{1}{\mu} \int_0^x \frac{\overline{F}(x-y)}{\overline{F}(x)} \overline{F}(y) \mathrm{d}y = 2$$

where μ is the first moment of F. It is well-known that $\mathcal{S}^* \subseteq \mathcal{S}$ and using (2.3) we may apply Theorem 4.1 from Maulik and Zwart [16] to get

$$\overline{\nu}_I(K) := \int_K^\infty \overline{\nu}(y)\, \mathrm{d}y \sim |\mathbb{E}S_1| \overline{\pi}_\infty(K) \tag{2.4}$$

when $\mathbb{E}S_1 < 0$ and $\overline{\nu}_I(x) \sim \overline{F}(x)$ for some $F \in \mathcal{S}$. The latter condition is ensured by requiring that $\overline{\nu}(x) \sim \overline{F}(x)$ for some $\overline{F}(x) \in \mathcal{S}^*$.

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The loss rate is defined as

$$\ell^K = \mathbb{E}_{\pi_K} L_1^K\,, \tag{2.5}$$

that is, as the mean of L_1^K when the process is started in stationarity.

According to Theorem 3.6 in Asmussen and Pihlsgård [4] we have the following expression of the loss rate, in terms of the characteristic triplet of the Lévy processes:

$$\ell^K = \frac{\mathbb{E}S_1}{K} \int_0^K \overline{\pi}_K(x)\, \mathrm{d}x + \frac{\sigma^2}{2K} + \frac{1}{2K} \int_0^K \pi_K(\mathrm{d}x) \int_{-\infty}^\infty \varphi_K(x, y) \nu(\mathrm{d}y), \tag{2.6}$$

where

$$\varphi_K(x, y) = \begin{cases} -(x^2 + 2xy) & \text{if } y \leq -x \\ y^2 & \text{if } -x < y < K - x \\ 2y(K - x) - (K - x)^2 & \text{if } y \geq K - x\,. \end{cases} \tag{2.7}$$

3 Main results

We start by stating the main results. The first result provides the asymptotics in the case of heavy tails and negative drift.

Theorem 3.1. *Let* $\mathbf{S}$ *be a Lévy process with Lévy measure* ν *such that* $\overline{\nu}_I(x) \sim \overline{B}(x)$ *for some* $B \in \mathcal{S}$, *and with finite negative mean:* $\mathbb{E}S_1 = \mu < 0$. *Define the conditions*

(I) $\mathbb{E}S_1^2 < \infty$ *and* $\int_K^\infty \overline{\nu}_I(y)\,\mathrm{d}y / \overline{\nu}_I(K) \in O(K)$.

(II) $\overline{\nu}(K) \sim L(K)K^{-\alpha}$ *where* L *is a locally bounded slowly varying function and* $1 < \alpha < 2$.

If either (I) or (II) holds, then

$$\ell^K \sim \int_K^\infty \overline{\nu}(y)\mathrm{d}y \tag{3.1}$$

We remark that the requirement $\int_K^\infty \overline{\nu}_I(y)\,\mathrm{d}y/\overline{\nu}_I(K) \in O(K)$ in Theorem 3.1 is very weak. Indeed, suppose $\overline{\nu}_I(x) \sim \overline{B}(x)$ where B is either lognormal, Benktander or heavy-tailed Weibull. Then we recognize $a(x) := \int_x^\infty \overline{B}(y)\,\mathrm{d}y/\overline{B}(x)$ as the mean-excess function and it is known (see Goldie and Klüppelberg [12]), that $a(x) \in o(x)$. Furthermore, it is easily checked that the condition is satisfied when B is a Pareto or Burr distribution, provided that the second moment is finite.

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We also derive the following theorem, giving an expression for the moment generating function of the stationary distribution in the case of one-sided reflection. Recall our decomposition of the one-sided reflected process $V_t(x) = x + S_t - L_t(x)$ and let $\{L_t^c\}$ be the continuous part of the local time.

Theorem 3.2. *Suppose* $-\infty < \mu = \mathbb{E}S_1 < 0$ *then* $V := \lim_t V_t$ *exists in distribution and for* $\alpha \in \Theta$ *with* $\kappa(\alpha) < \infty$ *we have:*

$$\mathbb{E}\big[e^{\alpha V}\big] = -\frac{\alpha\mathbb{E}_{\pi_\infty} L_1^c + \mathbb{E}_{\pi_\infty}\big[\sum_{0\le s\le 1}(1 - e^{-\alpha\Delta L_s})\big]}{\kappa(\alpha)} \tag{3.2}$$

If $\mathbf{S}$ has no negative jumps, the term $\mathbb{E}_{\pi_\infty}\big[\sum_{0\le s\le 1}(1 - e^{-\alpha\Delta L_s})\big]$ disappears, and $\mathbb{E}_{\pi_\infty} L_1^c = \mathbb{E}_{\pi_\infty} L_1 = \mu$, and we see that Theorem 3.2 indeed is a generalization of Corollary 3.4 in Asmussen [3, Chap. IX] which is itself a generalization of the Pollaczeck-Khinchine formula.

4 Loss rate asymptotics in the case of negative drift and heavy tails

In this section we prove Theorem 3.1 and in the pursuit of this, we prove Theorem 3.2. We first prove Proposition 4.1, which is a set of inequalities which allow us to compare the stationary distributions in the cases of one and two-sided reflection. Next, we prove Proposition 4.2 showing that 1 is a lower bound for

$\liminf_K \ell^K/\overline{\nu}_I(K)$, which is essentially half of Theorem 3.1. Lemma 4.1 and Proposition 4.3 provide a martingale, and using optional stopping of this martingale yields Theorem 3.2, which gives the m.g.f. $\mathbb{E}[e^{\alpha V}]$. We differentiate this transform in Corollary 4.1 to give us the mean of V, which is needed in the proof of Theorem 3.1.

Proposition 4.1. *Let* $\mathbf{S}$ *be a Lévy process, and let* $\overline{\pi}_K(y), \overline{\pi}_\infty(y)$ *be the tails of the reflected (one/two-sided) distributions. Then we have the following inequalities for* $x > 0, K > 0$

$$0 \leq \overline{\pi}_\infty(x) - \overline{\pi}_K(x) \leq \overline{\pi}_\infty(K)\,. \tag{4.1}$$

Proof. The inequalities are trivial for $x > K$. Let $0 \leq x \leq K$. The inequality $\overline{\pi}_K(x) \leq \overline{\pi}_\infty(x)$ follows from the representations (2.2) and (2.3). The inequality $\overline{\pi}_\infty(x) - \overline{\pi}_K(x) \leq \overline{\pi}_\infty(K)$, follows by dividing the sample paths of $\mathbf{S}$ which cross above x into those which do so by first passing below $K - x$, and those which stay above $K - x$. To be precise, define $\tau(y) := \inf\{t > 0 : S_t \geq y\}$ and $\sigma(y) := \inf\{t > 0 : S_t < y\}$. Then, since any path which passes below $K - x$ and then above x must pass an interval of length at least K, we have by the strong Markov property:

$$\begin{aligned}\mathbb{P}(\sigma(x-K) < \tau(x) < \infty) &\leq \mathbb{P}\big(\sup_{t>0} S_{\sigma(x-K)+t} - S_{\sigma(x-K)} > K\big)\\ &= \mathbb{P}(\tau(K) < \infty) = \overline{\pi}_\infty(K)\,.\end{aligned}$$

And therefore:

$$\begin{aligned}\overline{\pi}_\infty(x) &= \mathbb{P}(\tau(x) < \infty)\\ &= \mathbb{P}(\sigma(x-K) < \tau(x) < \infty) + \mathbb{P}(\tau(x) < \sigma(x-K) < \infty)\\ &\leq \overline{\pi}_K(x) + \overline{\pi}_\infty(K)\,.\end{aligned}$$

□

In our effort to prove that $\ell^K \sim \overline{\nu}_I(K)$ we need to prove that 1 is a lower bound for $\liminf_K \ell^K/\overline{\nu}_I(K)$ and an upper bound for $\limsup_K \ell^K/\overline{\nu}_I(K)$. The former holds without any regularity conditions.

Proposition 4.2. *For any Lévy process we have*

$$1 \leq \liminf_K \frac{\ell^K}{\overline{\nu}_I(K)}$$

Proof. We have

$$\int_0^K \pi_k(\mathrm{d}x) \int_K^\infty (y - K + x)\nu(\mathrm{d}y) \leq \ell^K$$

since the left hand side is the contribution to the local time by the jumps larger than K. Since

$$\begin{aligned}\overline{\nu}_I(K) &\leq \int_K^\infty (y-K)\nu(\mathrm{d}y) + \int_0^K x\pi_k(\mathrm{d}x)\\ &= \int_0^K \pi_k(\mathrm{d}x) \int_K^\infty (y - K + x)\nu(\mathrm{d}y)\end{aligned}$$

we are done. □

Recall our decomposition $V_t(x) = x + S_t - L_t(x)$ of the one-sided reflection of the Lévy process started at x and reflected in 0 and we let $L_t^c(x)$ and $L_t^j(x)$ denote the continuous and jump parts of the local time respectively. We suppress the x's for ease of notation.

Lemma 4.1. *For $\alpha \in \Theta$ and $t > 0$ we have*

$$\mathbb{E}\Big[\sum_{0\le s\le t} \big|1 - e^{-\alpha\Delta L_s}\big|\Big] < \infty \tag{4.2}$$

Proof. Setting $\underline{\Delta} L_s = \Delta L_s I(L_s \le 1)$ and $\overline{\Delta} L_s = \Delta L_s I(\Delta L_s > 1)$ we can split the sum into the contribution from the jumps of size ≤ 1 and those of size > 1 by writing

$$\mathbb{E}\Big[\sum_{0\le s\le t} \big|1 - e^{-\alpha\Delta L_s}\big|\Big] = \mathbb{E}\Big[\sum_{0\le s\le t} \big|1 - e^{-\alpha\underline{\Delta} L_s}\big|\Big] + \mathbb{E}\Big[\sum_{0\le s\le t} \big|1 - e^{-\alpha\overline{\Delta} L_s}\big|\Big],$$

and we note that first sum on the r.h.s. is bounded, since there exists a constant c such that $|1 - e^{\alpha x}| \le c|\alpha| x$ for $x \in [0, 1]$ and therefore

$$\mathbb{E}\Big[\sum_{0\le s\le t} \big|1 - e^{-\alpha\underline{\Delta} L_s}\big|\Big] \le c|\alpha|\mathbb{E}\Big[\sum_{0\le s\le t} \underline{\Delta} L_s|\Big] \le c|\alpha|\mathbb{E}L_t < \infty,$$

where the last inequality follows from Lemma 3.3 in Asmussen [3, Chap IX].

Since

$$|1 - e^{-\alpha\overline{\Delta} L_s}| = \big|I(\overline{\Delta} L_s > 0) - e^{-\alpha\overline{\Delta} L_s}\big| \le I(\overline{\Delta} L_s > 0) + e^{-\Re(\alpha)\overline{\Delta} L_s} I(\overline{\Delta} L_s > 0)$$

we have

$$\begin{aligned}
&\mathbb{E}\Big[\sum_{0\le s\le t} \big|1 - e^{-\alpha\overline{\Delta} L_s}\big|\Big] \\
&\qquad \le \mathbb{E}\Big[\sum_{0\le s\le t} I(\overline{\Delta} L_s > 0)\Big] + \mathbb{E}\Big[\sum_{0\le s\le t} e^{-\Re(\alpha)\overline{\Delta} L_s} I(\overline{\Delta} L_s > 0)\Big]
\end{aligned}$$

A jump of size of > 1 at time s of the local time can only occur if the process itself makes a negative jump of size > 1, and therefore $I(\overline{\Delta} L_s > 0) \le I(\overline{\Delta} S_s < 0)$, where $\overline{\Delta} S_s := \Delta S_s I(\Delta S_s < -1)$, which implies

$$\mathbb{E}\Big[\sum_{0\le s\le t} I(\overline{\Delta} L_s > 0)\Big] \le \mathbb{E}\Big[\sum_{0\le s\le t} I(\overline{\Delta} S_s < 0)\Big] = t\int_{-\infty}^{-1} \nu(\mathrm{d}y) < \infty$$

where the last number is finite because $\mathbb{E}|S_1| < \infty$. Regarding the remaining sum, we observe that if $\Re(\alpha) \ge 0$ we have

$$\mathbb{E}\Big[\sum_{0\le s\le t} e^{-\Re(\alpha)\overline{\Delta} L_s} I(\overline{\Delta} L_s > 0)\Big] \le \mathbb{E}\Big[\sum_{0\le s\le t} I(\overline{\Delta} L_s > 0)\Big]$$

and the sum is finite by the inequalities above. If $\Re(\alpha) < 0$ we have

$$\begin{aligned}
&\mathbb{E}\Big[\sum_{0\le s\le t} e^{-\Re(\alpha)\overline{\Delta} L_s} I(\overline{\Delta} L_s > 0)\Big] \\
&\qquad \le \mathbb{E}\Big[\sum_{0\le s\le t} e^{\Re(\alpha)\overline{\Delta} S_s} I(\overline{\Delta} S_s < 0)\Big] = t\int_{-\infty}^{-1} e^{\Re(\alpha)y}\nu(\mathrm{d}y) < \infty\,,
\end{aligned}$$

where the last inequality follows from Theorem 25.3 i Sato [20] and the fact that $\alpha \in \Theta$. Putting everything together we have that (4.2) is finite. □

The lemma above is used in the following generalization of Cor. 3.2 in Asmussen [3, Chap. IX].

Proposition 4.3. *Consider a Lévy process* $\mathbf{S}$*, and let* $\mathbf{V}$ *be the process reflected at* 0 *and let* $\mathbf{L^c} := \{L_t^c\}$ *and* $\mathbf{L^j} := L_t^j$ *be the continuous and jump part of the local time* $\mathbf{L}$*. Then for* $\alpha \in \Theta$ *and* $x \geq 0$

$$M_t := \kappa(\alpha)\int_0^t e^{\alpha V_s(x)}\,\mathrm{d}s + e^{\alpha x} - e^{\alpha V_t(x)} + \alpha L_t^c(x) + \sum_{0\leq s\leq t}(1 - e^{-\alpha \Delta L_s(x)}) \tag{4.3}$$

is a martingale.

Proof. For notational convenience, we write $V_s := V_s(x)$ and $L_s^c = L_s^c(x)$. Since the local time is of bounded variation, we may apply Theorem 3.1 in Asmussen [3, Chap. IX], to obtain that

$$\kappa(\alpha)\int_0^t e^{\alpha V_s}\,\mathrm{d}s + e^{\alpha x} - e^{\alpha V_t} + \alpha\int_0^t e^{\alpha V_s}\,\mathrm{d}L_s^c + \sum_{0\leq s\leq t} e^{\alpha V_s}(1 - e^{-\alpha \Delta L_s})$$

is a local martingale. Since L_t^c can only increase when $V_t = 0$ and $\Delta L_t > 0 \Rightarrow V_t = 0$, the expression above is equal to M_t, so that M_t is a local martingale. According to Lemma 3.3 in Protter [19, p. 35] it will be a martingale if we can prove that $\mathbb{E}\sup_{s\leq t}|M_s| < \infty$. But this follows from

$$\begin{aligned}&\mathbb{E}\big[\sup_{0\leq s\leq t}|M_s|\big]\\ &\qquad \leq \kappa(\alpha)t\mathbb{E}\sup_{0\leq s\leq t}|e^{\alpha V_s}| + |e^{\alpha x}| + \mathbb{E}|e^{\alpha V_t}| + |\alpha|\mathbb{E}[L_t^c] + \mathbb{E}\sum_{0\leq s\leq t}|(1-e^{-\alpha\Delta L_s})|\end{aligned}$$

which is finite according to lemma 3.3 in Asmussen [3, Chap. IX] and Lemma 4.1 above. □

We are now ready to prove Theorem 3.2

Proof. The existence of V follows from Cor. 2.6 in Asmussen [3, p. 253]. Let V_0 be a r.v. independent of $\mathbf{S}$ and distributed as V, and set $x = V_0, t = 1$ in (4.3). Then $\mathbf{V}$ is stationary and by taking expectation we get

$$0 = \kappa(\alpha)\mathbb{E}_{\pi_\infty}\Big[\int_0^1 e^{\alpha V_s}\,\mathrm{d}s\Big] + \alpha\mathbb{E}_{\pi_\infty}L_1^c + \mathbb{E}_{\pi_\infty}\Big[\sum_{0\leq s\leq 1}(1-e^{-\alpha\Delta L_s})\Big]$$

$$\Downarrow$$

$$\kappa(\alpha)\int_0^1 \mathbb{E}_{\pi_\infty}\big[e^{\alpha V}\big]\,\mathrm{d}s + \alpha\mathbb{E}_{\pi_\infty}L_1^c + \mathbb{E}_{\pi_\infty}\Big[\sum_{0\leq s\leq 1}(1-e^{-\alpha\Delta L_s})\Big]$$

$$\Downarrow$$

$$\mathbb{E}\big[e^{\alpha V}\big] = -\frac{\alpha\mathbb{E}_{\pi_\infty}[L_1^c] + \mathbb{E}_{\pi_\infty}\big[\sum_{0\leq s\leq 1}(1-e^{-\alpha\Delta L_s})\big]}{\kappa(\alpha)}.$$

□

Next, we use the results above to obtain an expression for the mean of the stationary distribution in the case of one-sided reflection.

Corollary 4.1. *If* $\mathbf{S}$ *is square integrable then* V *is integrable and we have*

$$\mathbb{E}[V] = \frac{\mathbb{E}_{\pi_\infty}[\sum_{0\le s\le 1}\Delta L_s^2] - Var(S_1)}{2\mathbb{E}S_1} \tag{4.4}$$

$$= \frac{\int_{-\infty}^{\infty} y^2\nu(\mathrm{d}y) + \sigma^2 - \int_0^\infty \int_{-\infty}^{-x}(x+y)^2\nu(\mathrm{d}y)\pi_\infty(\mathrm{d}x)}{2|\mathbb{E}S_1|} \tag{4.5}$$

Proof. Since S_1 is non-degenerate, we have by Lemma 4 in Feller [11] that there exists $\epsilon > 0$ such that $\kappa(\mathrm{i}t) \neq 0$ for $t \in (-\epsilon, \epsilon)\setminus\{0\}$, and we may use (3.2) to obtain the characteristic function φ of V and we wish to show that φ is differentiable at 0. Define $g(t) := \mathbb{E}_{\pi_\infty}\big[\sum_{0\le s\le 1}(1-e^{-\mathrm{i}t\Delta L_s})\big]$ and set $\ell_1 := \mathbb{E}_{\pi_\infty} L_1^c$. By Doob's inequality, we have that $\mathbb{E}S_1^2 < \infty$ implies $\mathbb{E}L_1^2 < \infty$ and therefore $\mathbb{E}_{\pi_\infty} L_1^2 < \infty$, and this implies that g is twice differentiable at 0 and we see that $g'(0) = \mathrm{i}\mathbb{E}_{\pi_\infty}\big[\sum_{0\le s\le t}\Delta L_s\big] = \mathrm{i}\mathbb{E}_{\pi_\infty} L_t^j$, $g''(0) = \mathbb{E}_{\pi_\infty}\big[\sum_{0\le s\le t}\Delta L_s^2\big]$ and $\mathrm{i}\ell_1 + g'(0) = \mathrm{i}\mathbb{E}_{\pi_\infty} L_1 = -\mathrm{i}\mathbb{E}S_1$.

Applying Proposition 4.2 and using l'Hospital's rule twice (see Prop. 4.1 the in Appendix), we have:

$$\begin{aligned}
\lim_{t\to 0}\frac{\mathbb{E}e^{\mathrm{i}tV}-1}{t} &= \lim_{t\to 0}\frac{-t\mathrm{i}\ell_1 - g(t) - \kappa(\mathrm{i}t)}{t\kappa(\mathrm{i}t)}\\
&= \lim_{t\to 0}\frac{-\mathrm{i}\ell_1 - g'(t) - \mathrm{i}\kappa'(\mathrm{i}t)}{\kappa(\mathrm{i}t) + t\mathrm{i}\kappa'(t\mathrm{i})}\\
&= \lim_{t\to 0}\frac{-g''(t) + \kappa''(\mathrm{i}t)}{\mathrm{i}\kappa'(\mathrm{i}t) + \mathrm{i}\kappa'(t\mathrm{i}) - t\kappa''(t\mathrm{i})}\\
&= \frac{-g''(0) + \kappa''(0)}{2\mathrm{i}\kappa'(0)}.
\end{aligned}$$

We see that φ is differentiable. In itself, this does not entail integrability of V, but a short argument using the Law of Large Numbers and the fact that V is non-negative, yeilds that V is integrable. The first moment is

$$\mathbb{E}V = \frac{-g''(0) + \kappa''(0)}{2(-1)\kappa'(0)}$$

which is (4.4). We obtain (4.5) by conditioning on the value of the process prior to a jump. □

We are now ready for the proof of Theorem 3.1. The proof has two distinct cases, depending on whether or not the Lévy process is square integrable. If this is the case we require only mild regularity conditions. However, if the Lévy process has infinite variance, we impose stronger regularity conditions.

Proof. Because of Proposition 4.2, we only need to prove

$$\limsup_K \ell^K/\overline{\nu}_I(K) \le 1\,.$$

Define the following:

$$\mathcal{I}_1 := \frac{\mathbb{E}S_1}{K}\int_0^K x\pi_K(\mathrm{d}x)$$
$$\mathcal{I}_2 := \frac{\sigma^2}{2K}$$
$$\mathcal{I}_3 := \frac{1}{2K}\int_0^K \pi_K(\mathrm{d}x)\int_{-\infty}^{\infty}\varphi_K(x,y)\nu(\mathrm{d}y)\,.$$

Then, because of the expression for the loss rate given by (2.6) and the inequality from Proposition 4.1 we have the following inequality:

$$\ell^K \le \frac{\mathbb{E}S_1}{K}\int_0^K \overline{\pi}_\infty(x)\,\mathrm{d}x - \mathbb{E}S_1\overline{\pi}_\infty(K) + \mathcal{I}_2 + \mathcal{I}_3\ . \tag{4.6}$$

First, we assume (I) holds. By (2.4) we have

$$\lim_K \frac{-\mathbb{E}S_1\overline{\pi}_\infty(K)}{\overline{\nu}_I(K)} = 1\,, \tag{4.7}$$

so we will be done, if we can show

$$\limsup_K \frac{1}{\overline{\nu}_I(K)}\left[\frac{\mathbb{E}S_1}{K}\int_0^K \overline{\pi}_\infty(y)\,\mathrm{d}y + \mathcal{I}_2 + \mathcal{I}_3\right] = 0\,. \tag{4.8}$$

We start by rewriting the term in the brackets above. Using Cor. 4.1 and the assumption that $\mathbb{E}S_1^2 < \infty$ we have that $\int_0^\infty \overline{\pi}_\infty(y)\mathrm{d}y < \infty$ and using (4.4)

$$\begin{aligned}
&\frac{\mathbb{E}S_1}{K}\int_0^K \overline{\pi}_\infty(y)\,\mathrm{d}y \\
&\quad= \frac{\mathbb{E}S_1}{K}\int_0^\infty \overline{\pi}_\infty(y)\,\mathrm{d}y - \frac{\mathbb{E}S_1}{K}\int_K^\infty \overline{\pi}_\infty(y)\,\mathrm{d}y \\
&\quad= \frac{\mathbb{E}_{\pi_\infty}[\sum_{0\le s\le 1}\Delta L_s^2] - \mathbb{V}\mathrm{ar}(S_1)}{2K} + \frac{|\mathbb{E}S_1|}{K}\int_K^\infty \overline{\pi}_\infty(y)\,\mathrm{d}y\,.
\end{aligned}$$

Furthermore,

$$
\begin{aligned}
&\mathcal{I}_2+\mathcal{I}_3\\
&\quad=\frac{\sigma^2}{2K}+\frac{1}{2K}\int_0^K\pi_K(\mathrm{d}x)\Big(\int_{-\infty}^{-x}-(x^2+2xy)\nu(\mathrm{d}y)+\frac{1}{2K}\int_{-x}^{K-x}y^2\nu(\mathrm{d}y)\\
&\qquad+\frac{1}{2K}\int_{K-x}^{\infty}\big[2y(K-x)-(K-x)^2\big]\nu(\mathrm{d}y)\Big)\\
&\quad=\frac{\sigma^2}{2K}+\frac{1}{2K}\int_{-\infty}^{\infty}y^2\nu(\mathrm{d}y)+\frac{1}{2K}\int_0^K\pi_K(\mathrm{d}x)\int_{-\infty}^{-x}\big[-(x^2+2xy)-y^2\big]\nu(\mathrm{d}y)\\
&\qquad+\frac{1}{2K}\int_0^K\pi_K(\mathrm{d}x)\int_{K-x}^{\infty}\big[2y(K-x)-(K-x)^2-y^2\big]\nu(\mathrm{d}y)\\
&\quad=\frac{\sigma^2}{2K}+\frac{1}{2K}\int_{-\infty}^{\infty}y^2\nu(\mathrm{d}y)-\frac{1}{2K}\int_0^K\pi_K(\mathrm{d}x)\int_{-\infty}^{-x}(x+y)^2\nu(\mathrm{d}y)\\
&\qquad-\frac{1}{2K}\int_0^K\pi_K(\mathrm{d}x)\int_{K-x}^{\infty}(y-(K-x))^2\nu(\mathrm{d}y)\\
&\quad=\frac{\mathbb{V}\mathrm{ar}(S_1)-\mathbb{E}_{\pi_K}[\sum_{0\le s\le 1}\Delta L_s^2]}{2K}-\frac{1}{2K}\int_0^K\pi_K(\mathrm{d}x)\int_{K-x}^{\infty}(y-(K-x))^2\nu(\mathrm{d}y).
\end{aligned}
$$

We note the fact that

$$
\mathbb{E}_{\pi_\infty}\Big[\sum_{0\le s\le 1}\Delta L_s^2\Big]\le\mathbb{E}_{\pi_K}\Big[\sum_{0\le s\le 1}\Delta L_s^2\Big]
$$

which can be verified using partial integration and (4.1). Using this in the last equation above, we may continue our calculation and obtain:

$$
\begin{aligned}
\mathcal{I}_2+\mathcal{I}_3\le{}&\frac{\mathbb{V}\mathrm{ar}(S_1)-\mathbb{E}_{\pi_\infty}[\sum_{0\le s\le 1}\Delta L_s^2]}{2K}\\
&-\frac{1}{2K}\int_0^K\pi_K(\mathrm{d}x)\int_{K-x}^{\infty}(y-(K-x))^2\nu(\mathrm{d}y)\,.
\end{aligned}
$$

Comparing the expressions above we see that fractions cancel, and the expression in the brackets in (4.8) is less than

$$
\frac{|\mathbb{E}S_1|}{K}\int_K^\infty\overline{\pi}_\infty(y)\,\mathrm{d}y-\frac{1}{2K}\int_0^K\int_{K-x}^\infty(y-(K-x))^2\nu(\mathrm{d}y)\pi_K(\mathrm{d}x)\,.
$$

Applying partial integration

$$
\begin{aligned}
&\frac{|\mathbb{E}S_1|}{K}\int_K^\infty\overline{\pi}_\infty(y)\,\mathrm{d}y-\frac{1}{2K}\int_0^K\int_{K-x}^\infty(y-(K-x))^2\nu(\mathrm{d}y)\pi_K(\mathrm{d}x)\\
&\quad=\frac{|\mathbb{E}S_1|}{K}\int_K^\infty\overline{\pi}_\infty(y)\,\mathrm{d}y-\frac{1}{2K}\int_K^\infty(y-K)^2\nu(\mathrm{d}y)-\frac{1}{K}\int_0^K\overline{\pi}_K(x)\overline{\nu}_I(K-x)\mathrm{d}x\\
&\quad\le\frac{|\mathbb{E}S_1|}{K}\int_K^\infty\overline{\pi}_\infty(y)\,\mathrm{d}y-\frac{1}{2K}\int_K^\infty(y-K)^2\nu(\mathrm{d}y)\\
&\quad=\frac{|\mathbb{E}S_1|}{K}\int_K^\infty\overline{\pi}_\infty(y)\,\mathrm{d}y-\frac{1}{K}\int_K^\infty\overline{\nu}_I(y)\,\mathrm{d}y.
\end{aligned}
$$

Returning to (4.8) and applying the results above we get

$$\begin{aligned}
&\limsup_K \frac{1}{\overline{\nu}_I(K)}\left[\frac{\mathbb{E}S_1}{K}\int_0^K \overline{\pi}_\infty(y)\,\mathrm{d}y + \mathcal{I}_2 + \mathcal{I}_3\right] \\
&\quad \le \limsup_K \frac{1}{\overline{\nu}_I(K)}\left[\frac{|\mathbb{E}S_1|}{K}\int_K^\infty \overline{\pi}_\infty(y)\,\mathrm{d}y - \frac{1}{K}\int_K^\infty \overline{\nu}_I(y)\,\mathrm{d}y\right] \\
&\quad = \limsup_K \frac{\int_K^\infty \overline{\nu}_I(y)\,\mathrm{d}y}{K\overline{\nu}_I(K)}\left[\frac{\int_K^\infty |\mathbb{E}S_1|\overline{\pi}_\infty(y)\,\mathrm{d}y}{\int_K^\infty \overline{\nu}_I(y)\,\mathrm{d}y} - 1\right] = 0,
\end{aligned}$$

where the last equality follows since the term in the brackets tends to 0, and the fraction outside it is bounded by assumption. This proves that (3.1) holds under condition (I).

We now assume condition (II)

We start by noticing the following consequences of the assumptions:

$$\int_K^\infty \overline{\nu}(y)\,\mathrm{d}y \sim \int_K^\infty \frac{L(y)}{y^\alpha}\,\mathrm{d}y \sim \frac{K^{-\alpha+1}L(K)}{\alpha-1} \qquad K\to\infty \tag{4.9}$$

where the last equivalence follows by Proposition 1.5.10 of Bingham et al. [7] and the fact that $\alpha > 1$. Since by Proposition 1.3.6 of Bingham et al. [7], we have $K^{-\alpha+2}L(K)\to\infty$, (4.9) implies $K\overline{\nu}_I(K)\to\infty$.

The inequality (4.6) still holds, as does the limit in (4.7), so we proceed to analysis of $\mathbb{E}S_1\int_0^K \overline{\pi}_\infty(y)\mathrm{d}y/(\overline{\nu}_I(K)K)$

Since $K\overline{\nu}_I(K)\to\infty$ $K\to\infty$ we see that for any A we have

$$\lim_{K\to\infty}\frac{\mathbb{E}S_1}{K\overline{\nu}_I(K)}\int_0^A \overline{\pi}_\infty(y)\,\mathrm{d}y = 0\,. \tag{4.10}$$

Because of the result above we have for any A

$$\lim_{K\to\infty}\frac{\mathbb{E}S_1}{K\overline{\nu}_I(K)}\int_0^K \overline{\pi}_\infty(y)\,\mathrm{d}y = \lim_{K\to\infty}\frac{\mathbb{E}S_1}{K\overline{\nu}_I(K)}\int_A^K \overline{\pi}_\infty(y)\,\mathrm{d}y$$

and using $|\mathbb{E}S_1|\overline{\pi}_\infty(K)\sim\overline{\nu}_I(K)\sim K^{-\alpha+1}L(K)/(\alpha-1)$ we have

$$\begin{aligned}
\lim_{K\to\infty}\frac{\mathbb{E}S_1}{K\overline{\nu}_I(K)}\int_A^K \overline{\pi}_\infty(y)\,\mathrm{d}y &= \lim_{K\to\infty} -\frac{1}{K\overline{\nu}_I(K)}\int_A^K \overline{\nu}_I(y)\,\mathrm{d}y \\
&= -\lim_{K\to\infty}\frac{1}{K\overline{\nu}_I(K)}\int_A^K \frac{y^{-\alpha+1}L(y)}{(\alpha-1)}\,\mathrm{d}y
\end{aligned}$$

in the sense that if either limit exits so does the other and they are equal. Furthermore, since $-\alpha+1>-1$ and L is locally bounded, we may apply Proposition 1.5.8 in Bingham et al. [7] to obtain

$$\begin{aligned}
&-\lim_{K\to\infty}\frac{1}{K\overline{\nu}_I(K)}\int_A^K \frac{y^{-\alpha+1}L(y)}{(\alpha-1)}\,\mathrm{d}y \\
&\qquad = -\lim_{K\to\infty}\frac{1}{K\overline{\nu}_I(K)}\frac{K^{-\alpha+2}L(K)}{(-\alpha+2)(\alpha-1)} = -\frac{1}{-\alpha+2}.
\end{aligned}$$

That is, we obtain

$$\lim_{K\to\infty} \frac{\mathbb{E}S_1}{K\overline{\nu}_I(K)} \int_0^K \overline{\pi}_\infty(y)\,\mathrm{d}y = -\frac{1}{-\alpha+2}. \tag{4.11}$$

Returning to (4.6) we have

$$\begin{aligned}
&\limsup_K \frac{\ell^K}{\overline{\nu}_I(K)} \\
&\quad= \limsup_K \frac{\mathbb{E}S_1}{K\overline{\nu}_I(K)} \int_0^K \overline{\pi}_\infty(y)\,\mathrm{d}y - \frac{\mathbb{E}S_1\overline{\pi}_\infty(K)}{\overline{\nu}_I(K)} + \frac{\mathcal{I}_2}{\overline{\nu}_I(K)} + \frac{\mathcal{I}_3}{\overline{\nu}_I(K)} \\
&\quad= -\frac{1}{-\alpha+2} + 1 + \limsup_K \frac{\mathcal{I}_2}{\overline{\nu}_I(K)} + \frac{\mathcal{I}_3}{\overline{\nu}_I(K)}. \qquad (4.12)
\end{aligned}$$

Since $K\overline{\nu}_I(K) \to \infty$ we have

$$\mathcal{I}_2/\overline{\nu}_I(K) = \frac{\sigma^2}{2K\overline{\nu}_I(K)} = 0$$

and we may continue our calculation from (4.12)

$$-\frac{1}{-\alpha+2} + 1 + \limsup_K \frac{\mathcal{I}_2}{\overline{\nu}_I(K)} + \frac{\mathcal{I}_3}{\overline{\nu}_I(K)} = -\frac{1}{-\alpha+2} + 1 + \limsup_K \frac{\mathcal{I}_3}{\overline{\nu}_I(K)} \tag{4.13}$$

So we turn our attention to $\mathcal{I}_3$. First we divide the integral into two:

$$\begin{aligned}
&2K\mathcal{I}_3 = &(4.14)\\
&\underbrace{\int_0^K \pi_K(\mathrm{d}x) \int_{-\infty}^{-x} -(x^2+2xy)\nu(\mathrm{d}y) + \int_{-x}^0 y^2\nu(\mathrm{d}y)}_{A(K)} &(4.15)\\
&\underbrace{\int_0^K \pi_K(\mathrm{d}x) \int_0^{K-x} y^2\nu(\mathrm{d}y) + \int_{K-x}^\infty 2(K-x)y - (K-x)^2\nu(\mathrm{d}y)}_{B(K)}\,. &(4.16)
\end{aligned}$$

We may assume ν is bounded from below, otherwise we may truncate ν at $-L$ for some $L > 0$ which is chosen large enough to ensure that the mean of S_1 remains negative. This truncation may increase the loss rate, which is not a problem, since we are proving an upper bound. Thus, we may assume that $A(K)$ is bounded:

$$A(K) \le \int_0^K \pi_K(\mathrm{d}x) \int_{-\infty}^0 y^2\nu(\mathrm{d}y) \le \int_{-\infty}^0 y^2\nu(\mathrm{d}y) < \infty$$

And therefore, since $K\overline{\nu}_I(K) \to \infty$, we have

$$\frac{A(K)}{2K\overline{\nu}_I(K)} \to 0\,. \tag{4.17}$$

Turning to $B(K)$, we first perform partial integration

$$\begin{aligned} B(K) &= \int_0^K y^2\nu(\mathrm{d}y) + \int_K^\infty 2Ky - K^2\nu(\mathrm{d}y) \\ &\quad - \int_0^K \overline{\nu}_I(K-x)\overline{\pi}_K(x)\,\mathrm{d}x \\ &\le \int_0^K y^2\nu(\mathrm{d}y) + \int_K^\infty 2Ky - K^2\nu(\mathrm{d}y) \\ &= \int_0^K 2y\overline{\nu}(y)\,\mathrm{d}y - K^2\overline{\nu}(K) + \int_K^\infty 2Ky - K^2\nu(\mathrm{d}y) \\ &= \int_0^K 2y\overline{\nu}(y)\,\mathrm{d}y + 2K\int_K^\infty \overline{\nu}(y)\,\mathrm{d}y\,. \end{aligned}$$

Since $y\overline{\nu}(y) \sim y^{-\alpha+1}L(y)$ way may apply Proposition 1.5.8 from [7]:

$$\int_0^K 2y\overline{\nu}(y)\,\mathrm{d}y \sim 2\frac{L(K)K^{-\alpha+2}}{2-\alpha}$$

and therefore:

$$\lim_K \frac{1}{2K\overline{\nu}_I(K)}\int_0^K 2y\overline{\nu}(y)\,\mathrm{d}y = \frac{\alpha-1}{2-\alpha}\,.$$

Combining this with our inequality for $B(K)$ above, we have:

$$\limsup_K \frac{B(K)}{2K\overline{\nu}_I(K)} \le \frac{\alpha-1}{2-\alpha} + 1 = \frac{1}{2-\alpha}\,.$$

Finally, by combining this with (4.12), (4.17) and (4.13) we have get

$$\limsup_K \frac{\ell^K}{\overline{\nu}_I(K)} \le 1$$

and we are done. □

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