SUBGRADIENT OPTIMIZATION, MATROID PROBLEMS AND HEURISTIC EVALUATION

F. Maffioli
Istituto di Elettrotecnica ed Elettronica
Politecnico di Milano, 20133 Milano (Italy)

Abstract

Many polynomial complete problems can be reduced efficiently to three matroids intersection problems. Subgradient methods are shown to yield very good algorithms for computing tight lower bounds to the solution of these problems. The bounds may be used either to construct heuristically guided (branch-and-bound) methods for solving the problems, or to obtain an upper bound to the difference between exact and approximate solutions by heuristic methods. The existing experience tend to indicate that such bounds would be quite precise.

1. FOREWORD

Consider the following three apparently unrelated problems.

3-dimensional assignment . Given n men for n jobs and n different time slots and a weighting c_{ijk} for assigning the i-th man to the j-th job in the k-th time slot, find an assignment of maximum total weight.

Traveling salesman problem. A salesman has to go from city 1 to n-1 other different cities. Given the distance matrix of the n cities find a path of minimum total length going through each city at most once.

A Sequencing problem. Let there be n jobs to be processed on a single machine and let job i be requiring T_i units of time, having a deadline D_i after which a penalty P_i has to be paid. Find the sequencing of the jobs which minimizes the overall penalty to be paid.

As it will appear in the following these are three instances of a general class of problems, namely those reducible to 3-matroid intersection problems. The purpose of this work is to present a general method to obtain tight bounds to the optimal solution to these problems in order to be able to estimate fairly accurately the error by which any heuristically obtained solution would be affected, In section 3 a few notions from matroid theory will be reviewed. Section 4 will be devoted to matroid problem reduction. Section 5 will develop the main algorithm for calculating the bounds. Section 6 will show how the three above mentioned problems can be approached as matroid problems. Some conclusions and areas for further research will be outlined in section 7.

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2. COMPLEXITY OF ALGORITHMS AND PROBLEMS REDUCTION

A problem is said to be in P if an algorithm for its solution exists whose computing time is a polynomial function in the size of the problem. Karp [1] has shown that many problems which are (probably) not in P can be reduced one to the other so that one of them being in P would imply that all of them are. We say that a problem "reduces" to another if there exists an algorithm in P which would yield the solution to the second once the first is solved. The three problems mentioned above are (probably) not in P. Even the best algorithms known are of exponential complexity and for large problems only heuristic methods can be used successfully.

3. MATROID AXIOMATICS [2]

Let $E = \{e_1, e_2, \ldots, e_n\}$ be a finite set of elements and $\mathcal{J}$ a non-empty family of subsets of E such that:

1) if $I \subset J \in \mathcal{J}$ then $I \in \mathcal{J}$;
2) if $I, J \in \mathcal{J}$ and $|I| = |J|+1$, then there exists an element $e \in I-J$ such that $J+e \in \mathcal{J}$.

Then $M = (E,\mathcal{J})$ is a matroid and the members of $\mathcal{J}$ are called its independent sets. A maximal independent set is called a base. A minimal dependent set is called a circuit. All bases of a matroid have the same cardinality. As an example let E be the set of edges of a linear graph and $\mathcal{J}$ the set of forests of the graph: this is the graphic matroid of the graph. Else let E be the set of columns of a matrix and $\mathcal{J}$ the family of sets of columns which are linearly independent (over any field): this is a matric matroid of the matrix. As a third example let E be any finite set and let π be a partition of E into r disjoint subsets $S_1, S_2, \ldots, S_r$. Let $d = (d_1, d_2, \ldots, d_r)$ be a r-dimensional vector and

$$\mathcal{J} = \{I: I \subseteq E \ \& \ |I \cap S_i| \le d_i \ , \ i = 1,2,\ldots,r\} .$$

Then $M = (E,\mathcal{J})$ is a partition matroid.

Let there be a weighting function $w : E \rightarrow R^{+}$. The problem of finding an independent set $\bar{I}$ of M having maximum total weight (or equivalently a base of minimum total weight) is solved by the "greedy" algorithm: "include in $\bar{I}$ the element of maximum weight among those not yet included, disregarding an element only if it would destroy independence once included into $\bar{I}$" [3].

Let $m = |E|$ and $c(m)$ be the complexity order of the method for testing independence (TI) in M. Then the greedy algorithm has a complexity at worst of order $mc(m)$ and if TI $\in$ P the greedy algorithm also belongs to P.

Let M_1 and M_2 be two given matroids. Then $M = (E_1 \cup E_2, \mathfrak{I})$ where $\mathfrak{I} = \{I : I = I_1 \cup I_2 \ \& \ I_1 \in \mathfrak{I}_1, I_2 \in \mathfrak{I}_2\}$ is a matroid called the sum of M_1 and M_2.

4. MATROID PROBLEMS REDUCTION

Let there be k matroids $M_1, M_2, \ldots, M_k$ over the same set E. A subset I of E which is independent in all of them is called an intersection. Consider also a partition of E into p disjoint subsets $P_1, P_2, \ldots, P_p$ and let h be the maximum cardinality of them. A very general matroid problem is the following. Let $\mathfrak{I}_i$ be the family of independent sets of the i-th matroid. Find the subset I of E of maximum weight such that

$$I \in \bigcap_{i=1}^{k} \mathfrak{I}_i \qquad (3$$

and such that $|I \cap P_j| = \{|P_j| \text{ else } 0\}$ for $j = 1,2,\ldots,p$. (4

Any set obeying (4) is called a h-parity set, where

$$h = \max_j |P_j| .$$

It can be shown that due to the results of Lawler [4] this problem can be reduced to a 3-matroid intersection problem on a set E' containing 2km elements where two of the matroids are partition matroids. The reader is referred to [4] for the corresponding reductions.

5. A SUBGRADIENT ALGORITHM

The method for obtaining bounds will be formulated for the special instance of the 3-matroid intersection problem which is yielded by the reductions mentioned in the previous section.

Let $M_i = (E, \mathfrak{I}_i)$, $i = 1,2,3$ be three matroids defined over the same set of elements E and having respectively ranks n,n and n+r where $2n = |E|$. Let $w : E \to R^+$ be a given weighting function. The first two matroids, M_1 and M_2, are particular partition matroids corresponding to the following partitions of E.

$$E = \{A_1, A_2, \ldots, A_r\} = \{B_1, B_2, \ldots, B_r\}$$

$$|A_i| = |B_i| = 2 \ , \ i = 1,2, \ldots, r.$$

So that

$$\mathfrak{I}_1 = \{I : I \subseteq E \ \& \ |I \cap A_i| \leq 1, \quad i=1,2,\ldots,r\}$$

$$\mathfrak{I}_2 = \{I : I \subseteq E \ \& \ |I \cap B_i| \leq 1, \quad i=1,2,\ldots,r\}$$

Let $\pi = (\pi_1, \pi_2, \ldots, \pi_r)$ be a real r-dimensional vector. Then a new weighting function w' may be defined for each $e \in E$ as

$$w'(e) = w(e) + \sum_{i=1}^{r} \pi_i \left[|\{e\} \cap A_i| + |\{e\} \cap B_i| \right]$$

Then for any $X \subseteq E$,

$$w'(X) = w(X) + \pi \cdot \mu(X)$$

where

$$\mu(X) = (|A_1 \cap X| + |B_1 \cap X|, \ \ldots, \ |A_r \cap X| + |B_r \cap X|)$$

Let now $\bar{I}$ be such that

$$w'(\bar{I}) = \max \{w'(X) : X \in \mathfrak{I}_3 \text{ and } |X| \leq r\}$$

It is always impossible to find $\bar{I}$ by the greedy algorithm. Obviously $|\bar{I}| = r$. Let now I_1 and I_2 be two intersections of maximum cardinality of the three given matroids. Then

$$|I_i \cap A_j| \leq 1 \qquad i = 1,2$$

$$|I_i \cap B_j| \leq 1 \qquad j = 1,2,\ldots,r.$$

On the other hand, since $|I_1| = |I_2| = r$,

$$\mu(I_1) = \mu(I_2) = \varepsilon$$

where ε is a vector whose entries are all equal to 2. Let now I° be a maximum weight intersection. Then by definition

$$w'(I^\circ) \leq w'(\bar{I})$$

so that

$$w(I^\circ) + \pi \cdot \tilde{\mu}(I^\circ) \leq w'(\bar{I})$$

i.e.

$$w(I^\circ) \leq w'(\bar{I}) - \pi \cdot \varepsilon = f(\pi)$$

and $f(\pi)$ is a valid upper bound to the value of the optimum solution for any π. The tightest of such bounds will be obtained by

$$f(\pi^*) = \min_{\pi} f(\pi)$$

Subgradient methods have been extensively studied and used in recent literature [5,6,7,8]. In our case the gradient of $f(\pi)$ is

$$\nabla f(\pi) = \mu(\bar{I}) - \varepsilon$$

and can easily be obtained applying the greedy algorithm to M_3 under the weighting w'. A simple iterative scheme for computing π^* is the following.

Step 1 $\pi = 0$, $i = 0$.

Step 2 $i+1 \rightarrow i$, $w(e) + \sum_{j=1}^{r} \pi_j \left[|A_j \cap \{e\}| + |B_j \cap \{e\}| \right] \rightarrow w(e)$ for all e E.

Step 3 Find a maximum weight independent set $\bar{I}$ of M_3 by the greedy algorithm. Then $f(\pi) = w(\bar{I}) - \pi \cdot \varepsilon$ and $g = \mu(\bar{I}) - \varepsilon$.

Step 4 If $g = 0$ stop: $I^\circ = \bar{I}$. If i = maximum number of iterations stop: $f(\pi)$ is the best obtainable upper bound.

Step 5 $\pi + tg \rightarrow \pi$, t being a suitable scalar. Go to step 2.

A maximum number of iterations has to be chosen since the method could fail to converge to a case for which $g = 0$. The reader is referred to [5,6] for further informations about this behaviour. For choosing t one can adopt a rough criterion as f.i. $t = 1$ as in [5], or more refined criteria as suggested in [5] and used in [7]. Also a modified subgradient direction can be used to speed up the search [9].

6. EXAMPLES

The three dimensional assignment and the traveling salesman problems are already in the form of 3-matroid intersection problems. For the first one has to consider three partition matroids one for each index of the cost coefficients. For the second problem two of the matroids are partition matroids generated by out and in-degrees at every node of the graph, while the third matroid is the graphic matroid.

As far as the sequencing problem is concerned consider the graph of figure 1 together with the following theorem.

Theorem (Edmonds & Fulkerson [10]). Let G = (N,A) be a graph and let $E \subset N$ be any subset of N. Consider the family $\mathfrak{J}$ of all the subsets I of E such that there exists a matching of G covering all the nodes of I. Then M = (E, $\mathfrak{J}$) is a matroid called the matching matroid of G.

Apply this theorem to the graph of figure 1, E being the set of nodes on the left corresponding to the various jobs identified by as many nodes as the required units of time, the penalty of each job being divided in any arbitrary way among its nodes. If each job corresponds at maximum to h nodes, one has a h-parity matroid problem on a matching matroid, since a subset of E of maximum weight which can be covered by a matching corresponds to an optimum scheduling.

7. CONCLUSIONS

An immediate development of this work would be the implementation of a computer code in order to test the algorithm for various problem instances. The present experience although quite promising has been almost entirely confined to the traveling salesman problem: there are no reasons however against the hope of obtaining as good results on other problems. The elementary iteration of section (5) could be improved following [7,9] and [11]. Further research is needed to explore the full class of problems to which this method applys as well as to see which modifications if any would be needed to approach problems apparently not belonging to this class such as the Steiner network problem and the quadratic assignment.

i	D_i	P_i	T_i
1	2	9	2
2	4	7	3
3	2	5	1
4	6	3	2

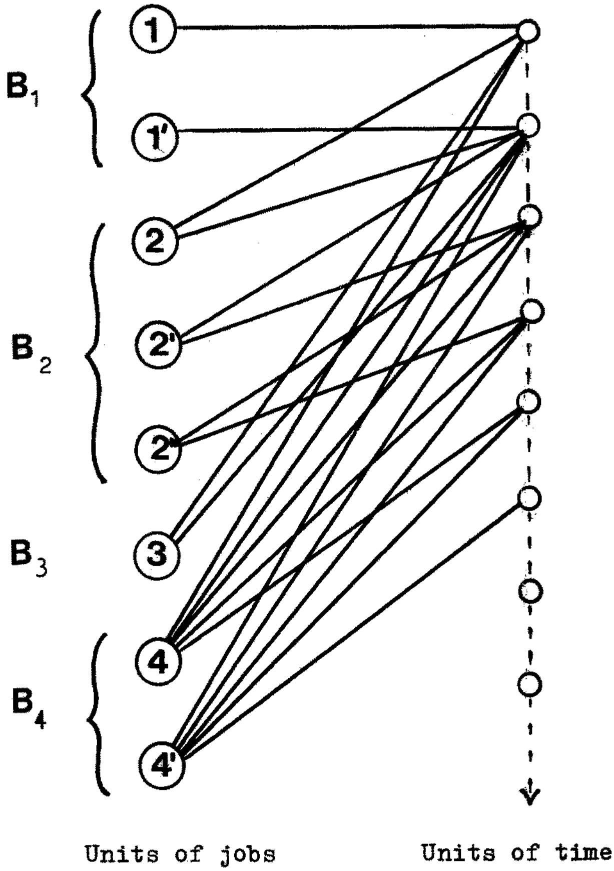

Fig. 1

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