SUBGROUPS OF *HNN* GROUPS

Dedicated to the memory of Hanna Neumann

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The purpose of this paper is to give a more precise form of Theorem 1 of [2], which gives a structure theorem for subgroups of HNN groups; we prove the following.

Let H be a subgroup of the HNN group $\langle A, x_i; x_i U_{-i} x_i^{-1} = U_i \rangle$. Then H is an HNN group whose base is a tree product of groups $H \cap wAw^{-1}$ where w runs over a set of double coset representatives of (H, A); the amalgamated and associated subgroups are all of the form $H \cap vU_iv^{-1}$ for some v. We can be more precise about which subgroups occur and about the tree product. We will also obtain stronger forms of other results in [1] and [2].

The main technique is Serre's theory of groups acting on trees. This theory is an important new development in combinatorial group theory; in fact the theorem above follows immediately from Serre's work. As these results have not yet been published (they will appear as Springer Lecture Notes) the main results are stated in section 1. The subgroup theorems are derived in section 2, and section 3 contains some results on finitely generated subgroups of HNN groups and amalgamated free products.

1. Bass and Serre's theory

The usual meaning is given to the word 'graph', except that a graph may have several edges joining a pair of vertices and may have loops, i.e. edges whose vertices coincide. All graphs considered will be connected.

A *graph of groups*, $(\mathscr{G}, Y)$ consists of:

(i) a graph Y,

(ii) for each vertex P and edge e of Y groups G_P and G_e,

(iii) if P and Q are the vertices of the edge e, monomorphisms $G_e \to G_P$ and $G_e \to G_Q$ (if $P = Q$ we require two monomorphisms $G_e \to G_P$).

An isomorphism from the graph of groups $(\mathscr{G}, Y)$ to the graph of groups $(\mathscr{H}, X)$ consists of:

(i) an isomorphism from the graph Y to the graph X,
(ii) isomorphisms $G_P \to H_{P'}$ and $G_e \to H_{e'}$, where P is a vertex and e and edge of Y and P' and e' are their images in X; these isomorphisms must be such that if P is a vertex of e the diagram
$$\begin{array}{ccc} G_e & \to & G_P \\ \downarrow & & \downarrow \\ H_{e'} & \to & H_{P'} \end{array}$$
commutes.

Let $(\mathscr{G}, Y)$ be a graph of groups and T a maximal tree in Y. As we are only considering connected graphs, T contains all the vertices of Y. Let G_T be the tree product of the vertex groups G_P amalgamating for each edge e of T the two images of G_e in the corresponding vertex groups (for general information about tree products and *HNN* groups see [1], [2]). Then G_T contains the groups G_P (up to isomorphism).

The *fundamental group of* $(\mathscr{G}, Y)$ *relative to* T, written $\pi(\mathscr{G}, Y, T)$, is defined to be the *HNN* group with base G_T, with free part having basis $\{t_e\}$ where e runs over the edges of Y not in T, and with the subgroups associated to t_e being the two images of G_e. Plainly if $(\mathscr{G}, Y)$ is isomorphic to $(\mathscr{H}, Y')$ with T' the tree in Y' corresponding to T in Y, then $\pi(\mathscr{G}, Y, T)$ is isomorphic to $\pi(\mathscr{H}, Y', T')$. Also by replacing $(\mathscr{G}, Y)$ by an isomorphic graph of groups over Y, we may assume that the groups G_P are subgroups of $\pi(\mathscr{G}, Y, T)$, that for e in T the maps $G_e \to G_P$ and $G_e \to G_Q$ are inclusions, while for e not in T one map $G_e \to G_P$ is an inclusion (the other map need not be an inclusion).

It can be shown that up to isomorphism $\pi(\mathscr{G}, Y, T)$ is independent of T. The proof is similar to the proof that the fundamental group of a graph Y may be obtained using any maximal tree. This latter is a special case of the general result, obtained by taking all the groups to be trivial.

Let a group G act on a graph X. We say G acts *without inversions* if
(i) X has no loops
(ii) if the edge e has vertices P and Q and $ge = e$ then $gP = P$ and $gQ = Q$ (the alternative possibility $gP = Q$ and $gQ = P$ would be called an inversion).

Let G act without inversions on X. Let Y be the quotient graph, $p: X \to Y$ the projection and T a maximal tree in Y. It is not difficult to find a morphism of graphs $j: T \to Y$ such that pj is the identity on T. If we have a tree $X_1 \subseteq X$ containing one vertex from each G-orbit of vertices then $p: X_1 \to Y$ is one-one and pX_1 will be a maximal tree of Y. We could then take pX_1 for T with j the inverse of p.

We shall define a graph of groups $(\mathscr{G}, Y)$ which we refer to as *associated to the action of* G *on* X. This graph of groups will not be unique but is obviously unique up to isomorphism once T is chosen.

We begin by defining j on the edges of Y not in T (however the resulting map $j: Y \to X$ will not be a morphism of graphs). Let e be an edge of Y not in T with vertices P and Q. Take any edge $\tilde{e}$ in X with vertices $\tilde{P}$ and $\tilde{Q}$ such that $p\tilde{e} = e$,

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$p\tilde{P} = P$, $p\tilde{Q} = Q$. As $p\tilde{P} = pjP$ we can find $g \in G$ such that $g\tilde{P} = jP$. Define je to be $g\tilde{e}$. Then $pje = e$ and jP is one vertex of je. The other vertex will usually not be jQ.

For each vertex P and edge e of Y, let G_P be the stabiliser in G of the vertex jP of X and G_e the stabiliser of the edge je of X. For e in T with vertices P and Q the maps $G_e \to G_P$ and $G_e \to G_Q$ are the inclusions. For e not in T with vertices P and Q such that jP is a vertex of je the map $G_e \to G_P$ is the inclusion. Let $\tilde{Q}$ be the other vertex of je. Then we have $p\tilde{Q} = pjQ$, so that there is an element g_e of G with $\tilde{Q} = g_e(jQ)$. Thus stab $jQ = g_e^{-1}(\text{stab } \tilde{Q})g_e$ and the map $G_e \to G_Q$ is the composite of the inclusion $G_e \to \text{stab}\,\tilde{Q}$ and conjugation. We have now defined $(\mathcal{G}, Y)$.

The group $\pi(\mathcal{G}, Y, T)$ is generated by the groups G_P and symbols t_e for each edge of Y not in T with relations $t_e H_e t_e^{-1} = G_e$, where H_e is the image of G_e in G_Q. Hence we have a homomorphism $\pi(\mathcal{G}, Y, T) \to G$ mapping G_P by inclusion and sending t_e to g_e.

THEOREM 1. (Serre [4]) *The above homomorphism is an isomorphism if and only if X is a tree.*

THEOREM 2. (Serre [4]) *Let $(\mathcal{G}, Y)$ be a graph of groups with fundamental group π (relative to some tree T). Then there is a tree $\tilde{Y}$ on which π acts without inversions such that the associated graph of groups is isomorphic to $(\mathcal{G}, Y)$.*

It is not difficult to see what $\tilde{Y}$ must be. We may assume that each G_P is a subgroup of π, that the maps $G_e \to G_P$ and $G_e \to G_Q$ are inclusions for e in T, and that $G_e \to G_P$ is an inclusion for e not in T. Since there is a π-orbit of vertices above each vertex P of Y and one vertex in this orbit has stabiliser G_P we can take as the vertices of $\tilde{Y}$ the cosets gG_P of G_P in π, where P ranges over the vertices of Y (if $G_P = G_Q$ for $P \neq Q$ then gG_P and gG_Q are to be different vertices of $\tilde{Y}$). Similarly the edges of $\tilde{Y}$ may be taken as the indexed cosets gG_e where e ranges over all edges of Y. For e in T with vertices P and Q the vertices of gG_e are gG_P and gG_Q. For e not in T with vertices P and Q where $G_e \to G_P$ is inclusion the vertices of gG_e are gG_P and gt_eG_Q. It is straightforward to see that this definition of the vertices of an edge is unambiguous so that we have a (possibly not connected) graph $\tilde{Y}$ on which π acts without inversions, the associated graph of groups being $(\mathcal{G}, Y)$. The problems are to show that $\tilde{Y}$ is connected (which is not difficult) and then to show $\tilde{Y}$ is a tree. The latter is a consequence of the normal form for tree products and HNN groups.

2. Subgroup theorems

Let $G = A *_U B$. Then G is the fundamental group of the graph of groups $A \underset{U}{\text{———}} B$. Hence G acts without inversion on a graph X whose vertices are the cosets gA and gB of A and B in G and whose edges are the cosets gU, where the vertices of gU are gA and gB. It is easy to see directly that X is a tree; this also follows from the general theory. G acts transitively on the edges, while there are two transitivity classes of vertices.

Let H be a subgroup of G. We shall construct a set $\{D_\alpha\}$ of double coset representatives for (H, A) in G, a set $\{D_\beta\}$ of double coset representatives of (H, B) in G, and for every α a set $\{E_u\}$ of double coset representatives of $(D_\alpha^{-1}HD_\alpha \cap A, U)$ in A and for every β a set $\{E_v\}$ of double coset representatives of $(D_\beta^{-1}HD_\beta \cap B, U)$ in B.

Given D_α the set $\{E_u\}$ must contain 1 but otherwise can be any set of double coset representatives of $(D_\alpha^{-1}HD_\alpha \cap A, U)$ in A. Similarly for $\{E_v\}$ given D_β. We define the representatives for the double cosets HwA and HwB by induction on the length of the double coset (i.e. the length of the shortest element in the coset). The only double cosets of length 0 are HA and HB, for each of which we choose the representative 1. Let HwA have length r and suppose representatives have been chosen for all double cosets of length less than r. We may assume w has length r, and can write $w = w'b$ for some $b \in B$ and w' of length $r-1$. Let D_β be the representative of $Hw'B = HwB$. We can find a unique element E_v of the set associated with D_β and an element $u \in U$ such that $w \in HD_\beta E_v u$. Then $HwA = HD_\beta E_v A$ and we choose $D_\beta E_v$ as the representative of HwA. Similarly for (H, B) double cosets.

This collection of double coset representatives $\{D_\alpha\}$, $\{D_\beta\}$ and the associated collections $\{E_u\}$, $\{E_v\}$ will be called a *semi-cress*, since it is a weaker form of the cress defined in [1]. Note that the collection $\{D_\beta E_v\}$ over all β and associated v forms a set of double coset representatives of (H, U) in G.

The set X_1 of all vertices $D_\alpha A$, $D_\beta B$ plainly contains one vertex from each H-orbit of vertices of X. Also X_1 is connected (and hence is a tree as X is a tree) since by construction any vertex $D_\alpha A$ with $D_\alpha \neq 1$ is joined to some vertex $D_\beta B$ with D_β shorter than D_α by an edge $D_\beta E_v U$, whence inductively $D_\alpha A$ will be joined to A by a path in X_1.

The set of edges $D_\beta E_v U$ contains exactly one edge from each H-orbit of edges and $D_\beta E_v U$ has at least one vertex, namely $D_\beta B$, in X_1. Given D_β and a corresponding E_v there exists a unique D_α, corresponding E_u and element P in U such that $D_\beta E_v \in HD_\alpha E_u P$. Let $t_{\beta v}$ denote $D_\beta E_v(D_\alpha E_u P)^{-1} \in H$.

If $E_v = 1$, by construction we have D_α and E_u with $D_\beta = D_\alpha E_u$ so $t_{\beta v} = 1$. Suppose $E_v \neq 1$ and $t_{\beta v} = 1$ so $D_\beta E_v = D_\alpha E_u P$ and $D_\beta E_v \in D_\alpha A$. Since, by construction D_β ends in $A - U$ and D_α in $B - U$ (unless they equal 1) while $E_v \in B - U$, if we have $D_\beta E_v \in D_\alpha A$ we must have $D_\beta E_v \in D_\alpha U$. It is then clear from the construction that $D_\alpha = D_\beta E_v$ since $D_\alpha \in D_\beta B$. We will then have $t_{\beta v} = 1$.

We now have the group H acting without inversions on the graph X, and have obtained a tree $X_1 \subseteq X$ which contains exactly one vertex from each H-orbit and have also obtained a set of edges, one on each H-orbit and each with one vertex in X_1. We can now apply Theorem 1 to see that H is isomorphic to the fundamental group of a graph of groups whose construction is easy. Since the H-stabiliser of a vertex gA is $H \cap gAg^{-1}$ we can read off the following theorem, which is a slight generalisation of theorem 5 of [1] (since our coset system is more general than that in [1]).

THEOREM 3. *Let H be a subgroup of $A \overset{*}{_U} B$. Construct a semi-cress as above, and let $t_{\beta v}$ be the associated elements of H. Then H is generated by all $t_{\beta v}$ together with all the subgroups $H \cap D_\alpha A D_\alpha^{-1}$ and $H \cap D_\beta B D_\beta^{-1}$. Further, (1) those $t_{\beta v} \neq 1$ (which correspond to those β and associated v such that $D_\beta E_v$ is not a coset representative) form a basis of a free subgroup of H;*
(2) the group K generated by all $H \cap D_\alpha A D_\alpha^{-1}$ and $H \cap D_\beta B D_\beta^{-1}$ is the tree product of these groups, two such groups being adjacent if $D_\alpha = D_\beta = 1$ or if $D_\alpha = D_\beta b$ or $D_\beta = D_\alpha a$ for some $a \in A$ or $b \in B$: the subgroup amalgamated between two adjacent groups is $H \cap DUD^{-1}$ where D is the longer of D_α and D_β;
(3) H is the HNN group $\langle K, t_{\beta v}; t_{\beta v}(H \cap D_\alpha E_u U E_u^{-1} D_\alpha^{-1}) t_{\beta v}^{-1} = H \cap D_\beta E_v U E_v^{-1} D_\beta^{-1} \rangle$ where in this expression we take all $t_{\beta v} \neq 1$ and the corresponding D_α, E_u.

We now proceed to give a similar analysis for *HNN* groups.

Let G be the *HNN* group $\langle A, x_i; x_i U_{-i} x_i^{-1} = U_i \rangle$. Then every element of G has a normal form $a_1 x_{i_1}^{\varepsilon_1} \cdots a_n x_{i_n}^{\varepsilon_n} a_{n+1}$ where $\varepsilon_r = \pm 1$, $a_r \in A$ and if $i_{r-1} = i_r$ with $\varepsilon_{r-1} = -\varepsilon_r$ then $a_r \notin U_{\varepsilon_r i_r}$. This normal form is not unique; we can replace $a_1, \cdots, a_{n+1}$ by $b_1, \cdots, b_{n+1}$ where $b_1 = a_1 u_1$, $b_2 = v_1^{-1} a_2 u_2, \cdots$ where $u_r \in U_{\varepsilon_r i_r}$ and $x_{i_r}^{\varepsilon} v_r = u_r x_{i_r}^{\varepsilon}$. In particular the integer n is uniquely determined and will be called the length of the element.

G is the fundamental group of a graph of groups with one vertex only and with one edge (which is a loop) for each i. Thus G acts without inversions on a graph X whose vertices are the cosets gA and whose edges are the cosets gU_i. There is one transitivity class of vertices, for each i the edges gU_i form a transitivity class, and gU_i joins gA and gx_iA. Using the normal form (or general theory) X is easily seen to be a tree. If we call the vertices gA and gx_iA of gU_i the initial and final vertices respectively it is clear that the action of G sends the initial and final vertices of an edge to the initial and final vertices respectively of the image edge.

Let H be a subgroup of G. We shall construct a set $\{D_\alpha\}$ of double coset representatives of (H, A) in G and for every α and i sets $\{E_{iu}\}$ and $\{E_{iv}\}$ of double coset representatives of $(D_\alpha^{-1} H D_\alpha \cap A, U_i)$ and of $(D_\alpha^{-1} H D_\alpha \cap A, U_{-i})$ respectively in A. The systems $\{E_{iu}\}$ and $\{E_{iv}\}$ must each contain 1 but are otherwise arbitrary. We define the representative of the double coset HwA by induction on the length of the double coset. The only double coset of length 0 is HA for which we choose the representative 1. Let HwA have length r and suppose that representatives have been chosen for all double cosets of length less than r. We may assume w has length r, and can assume $w = w' x_i^{\pm 1}$. Let D_β be the representative of $Hw'A$. If $w = w'x$ take the unique element E_{iu} associated with D_β and element $u \in U_i$ such that $w' \in HD_\beta E_u u$. Then $HwA = Hw'x_iA = HD_\beta E_{iu}ux_iA = HD_\beta E_{iu}x_iA$, since $ux_i = x_iv$ for some $v \in U_{-i}$; we take $D_\beta E_u x_i$ as the representative of HwA_i. If $w = w'x_i^{-1}$, we similarly obtain a representative $D_\beta E_v x_i^{-1}$ for HwA.

The collection of double coset representatives $\{D_\alpha\}$ and the associated collections $\{E_{iu}\}$ and $\{E_{iv}\}$ will again be called a *semi-cress*. The collection $\{D_\alpha E_{iu}\}$ over all α and all associated u forms a set of double coset representatives of (H, U_i) in G.

The set X_1 of all vertices $D_\alpha A$ plainly contains one vertex from each H-orbit of vertices of X. Also X_1 is connected since by construction any vertex $D_\alpha A$ with $D_\alpha \neq 1$ is joined to a vertex $D_\beta A$ with D_β shorter than D_α by an edge $D_\beta E_{iu} U_i$ or by an edge $D_\beta E_{iv} x_i^{-1} U$, whence inductively $D_\alpha A$ will be joined to A by a path in X_1.

The set of edges $D_\alpha E_u U_i$ contains exactly one edge from each H-orbit of edges labelled i, and $D_\alpha E_{iu} U_i$ has its initial vertex $D_\alpha A$ in X_1. Given D_α and corresponding E_{iu} there exists a unique D_β, corresponding E_{iv} and element $P \in U_{-i}$ such that $D_\alpha E_{iu} x_i \in HD_\beta E_{iv} P$. Let $t_{\alpha iu}$ denote $D_\alpha E_{iu} x_i (D_\beta E_{iv} P)^{-1} \in H$.

If $t_{\alpha iu} = 1$, we get $D_\alpha E_{iu} x_i \in D_\beta A$. Suppose $D_\alpha E_{iu} x_i \in D_\beta A$. As D_β ends in $x_j^{\pm 1}$ while $D_\alpha E_{iu} x_i$ ends in x_i unless $E_{iu} = 1$ and D_α ends in x_i^{-1} we see that $D_\beta \in D_\alpha E_{iu} x_i U_{-i}$ unless $E_{iu} = 1$ and D_α ends in x_i^{-1}. From the construction it is now clear that $D_\beta = D_\alpha E_{iu} x_i$ so that $t_{\alpha iu} = 1$. If $E_{iu} = 1$ and D_α ends in x_i^{-1}, $D_\alpha E_{iu} x_i \in D_\beta A$ gives, from the construction, $D_\alpha = D_\beta E_{iv} x_i^{-1}$ and so $t_{\alpha iu} = 1$.

We now have the group H acting without inversions on the graph X, and have obtained a tree $X_1 \subseteq X$ which contains exactly one vertex from each H-orbit and have also obtained a set of edges, one in each H-orbit and each with initial vertex in X_1. We can now apply theorem 1 to see that H is isomorphic to the fundamental group of a graph of groups whose construction is easy. Since the H-stabiliser of a vertex gA is $H \cap gAg^{-1}$, we obtain the following theorem, which is a significant generalisation of theorem 1 of [2].

THEOREM 4. *Let H be a subgroup of $\langle A, x_i; x_i U_{-i} x_i^{-1} = U_i \rangle$. Construct a semi-cress as above, and let $t_{\alpha iu}$ be the associated elements of H. Then H is generated by all $t_{\alpha iu}$ together with all the subgroups $H \cap D_\alpha A D_\alpha^{-1}$. Further*
(1) those $t_{\alpha iu} \neq 1$ (which correspond to those α and associated u such that $D_\alpha E_{iu} x_i$ is not a D_β and where $E_{iu} = 1$ is omitted if D_α ends in x_i^{-1}) form a basis of a free subgroup of H;
(2) the group K generated by all $H \cap D_\alpha A D_\alpha^{-1}$ is the tree product of these groups, two such groups corresponding to D_α and D_β, with D_β shorter than D_α, being adjacent if $D_\alpha = D_\beta E_{iu} x_i$ or $D_\alpha = D_\beta E_{iv} x_i^{-1}$, the subgroup amalgamated between these two being $H \cap D_\alpha U_{-i} D_\alpha^{-1}$ or $H \cap D_\alpha U_i D_\alpha^{-1}$ respectively;
(3) H is the HNN group

$$\langle K, t_{\alpha iu}; t_{\alpha iu}(H \cap D_\beta E_{iv} U_{-i} E_{iv}^{-1} D_\beta^{-1}) t_{\alpha iu}^{-1} = H \cap D_\alpha E_{iu} U_i E_{iu}^{-1} D_\alpha^{-1} \rangle$$

where in this expression we take all $t_{\alpha iu} \neq 1$ and the associated D_β, E_{iv}.

3. Finitely generated subgroups

Let $G = A *_U B$ and let $A_1 \subseteq A, B_1 \subseteq B$ with $A_1 \cap U = B_1 \cap U = U_1$. Then $\langle A_1, B_1 \rangle = A_1 *_{U_1} B_1$ and $\langle A_1, B_1 \rangle \cap A = A_1$. It follows that if G and U are finitely generated so are A and B, by taking A_1 to be generated by the elements of A occurring in normal forms of the finitely many generators of G and the generators of U, and similarly for B_1 so that $G = \langle A_1, B_1 \rangle$. Similar results hold for HNN groups, either by direct use of the normal form or by embedding an HNN group in an amalgamated free product.

Let X be a tree, O a vertex of X. Let e be an edge with vertices P and Q. Then exactly one of the paths from O to P and from O to Q will contain e. Orient e by defining P to be the final vertex if the path from O to P contains e. We call this orientation of X the orientation *outwards from* O. Plainly for $P \neq O$ there is exactly one edge whose final vertex is P; we will denote it by e_P. If X_1 is a subtree of X containing O then $P \in X_1$ implies $e_P \in X_1$, since e_P must be in the path in X_1 from O to P. Any path in X is of the form $P_0, \cdots, P_n$ where for some r, $0 \leqq r \leqq n$, the edge joining P_{i-1} and P_i is oriented from P_i to P_{i-1} for $i \leqq r$ and from P_{i-1} to P_i for $i > r$. If $P_0, \cdots, P_n$ is a path in a subtree X_1 containing O with each edge directed from P_i to P_{i-1} and $P_0 \in X_1$ then $P_i \in X_1$ for all i.

For the remainder of this section we let G be a group acting without inversions on a tree X. The symbols Y, T, p, j will have the same meaning as in section 1. We will orient X outwards from a vertex O in jT.

An element $g \in G$ will be called *negative* for an edge e if e is oriented from P to Q but ge oriented from gQ to gP. A G-orbit of edges will be called *reversing* if for some (and then for every) edge e in the orbit $\exists g$ negative for e. (If G is a subgroup of $A *_U B$ and X is the standard tree for $A *_U B$ reversing orbits correspond to double-ended cosets GwU.)

LEMMA 1. *There are finitely many reversing orbits if and only if*
(i) *there are finitely many edges of Y not in T and*
(ii) *there are finitely many vertices P of jT for which* stab $P \neq$ stab e_P.

PROOF. In an orbit above an edge not in T there will be an edge e from P to Q with $P \in jT$ and $Q \notin jT$. There will be $g \in G$ with $gQ \in jT$ and then $gP \notin jT$. Then e is oriented from P to Q and ge from gQ to gP, so the orbit is reversing.

If P is a vertex of jT with stab $P \neq$ stab e_P, then plainly any element of stab P not in stab e_P is negative for e_P. Also for P, Q distinct vertices of jT, as all vertices of e_P and e_Q are in jT and distinct vertices of jT are in different G-orbits, we see that e_P and e_Q are in different G-orbits.

Hence (i) and (ii) hold if there are only finitely many reversing orbits.

Suppose (i) and (ii) hold. Let X_1 be a finite subtree of jT containing O, all

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the vertices P of jT with $\operatorname{stab} P \neq \operatorname{stab} e_P$, and all vertices jQ with Q a vertex of an edge of Y not in T. We show that any reversing orbit either lies above an edge of Y not in T or meets X_1.

Take a reversing orbit lying above an edge in T. Take an edge e in this orbit and in jT, and let g be negative for e. Let the path from e to ge be $P_0, \cdots, P_n$.

Suppose first that every edge is oriented from P_{i-1} to P_i. As $P_n = gP_0$ we have $P_n \notin jT$. Take k with $P_0, \cdots, P_k \in jT$ but $P_{k+1} \notin jT$. If the edge P_kP_{k+1} maps to an edge not in T then $P_k \in X_1$ by our choice of X_1. Our general remarks about orientation then show that $e = e_{P_1} \in X_1$. If the edge P_kP_{k+1} maps to an edge in T, $\exists h \in G$ with $h(P_kP_{k+1}) \in jT$, and we must have $hP_k = P_k$ since jT contains only one vertex in each orbit. If $h(P_kP_{k+1}) = e_{P_k}$ then $P_k \in X_1$ and as before $e \in X_1$. Otherwise the path joining e to hge will be $P_0, \cdots, P_k = hP_k, hP_{k+1}, \cdots, hP_n$ and $hgP_0 = hP_n$. Induction on $n - k$ now gives the result.

Now suppose that for some r with $0 < r \leqq n$ the edge joining P_{i-1} and P_i is oriented from P_i to P_{i-1} for $i \leqq r$ and from P_{i-1} to P_i for $i > r$. We cannot have $r = n$ as that would give $P_n \in jT$ although $P_n = gP_0$.

So we may take $r < n$. Since g is negative for e, we must have $gP_0 = P_{n-1}$, $gP_1 = P_n$. If $P_0 \in X_1$ we have $e \in X_1$. If $P_0 \notin X_1$ the edge from $P_0 = g^{-1}P_{n-1}$ to $g^{-1}P_{n-2}$ must lie above an edge in T. As before $\exists h \in G$ such that h maps this edge to an edge in jT and with $hP_0 = P_0$. Let $P_{-1} = hg^{-1}P_{n-2}$. As $P_0 \notin X_1$ we have $\operatorname{stab} P_0 = \operatorname{stab} e_{P_0}$. Thus the edge joining P_0 to P_{-1} is oriented from P_0 to P_{-1} since $g^{-1}P_{n-2} \neq P_1 = g^{-1}P_n$. Then we have a path $P_{-1}, P_0, \cdots, P_{n-1}$ with $P_{-1} \in jT$, the edge joining P_0 and P_{-1} oriented from P_0 to P_{-1} and with $P_{n-1} = gh^{-1}P_0$ and $P_{n-2} = gh^{-1}P_{-1}$. By induction on $n - r$ we may assume $P_{-1} \in X_1$ and can then deduce $P_0 \in X_1$ and so $e \in X_1$, as required.

Lemma 2. *Let G be finitely generated. Then G has finitely many reversing orbits. If, in addition, each edge has finitely generated stabiliser then each vertex has finitely generated stabiliser. Conversely, if G has finitely many reversing orbits and each vertex has finitely generated stabiliser then G is finitely generated.*

Proof. G has the free group with basis $\{t_e, e \text{ an edge of } Y \text{ not in } T\}$ as homomorphic image. Hence there are only finitely many edges not in T if G is finitely generated. The finitely many generators of G will involve the t_e and elements from the stabilisers of finitely many vertices P. Let X_1 be a finite subtree of jT containing O, these vertices, and jQ for all vertices Q of an edge not in T. It follows that G, which is an HNN group with free part $\langle t_e \rangle$ and base group the tree product over jT of stab P is equal to its subgroup which has for base group the tree product over X_1 only. This requires that the tree products over jT and X_1 are the same, and by induction over the distance of P from 0 we see that $\operatorname{stab} P = \operatorname{stab} e_P$ for $P \notin X_1$. The result now follows from Lemma 1.

If, in addition, every edge has finitely generated stabiliser, then, by induction on the number of edges not in T and the number of vertices in X_1, we see first that the tree product over X_1 is finitely generated, and then that stab P is finitely generated for $P \in X_1$. Then stab P is finitely generated for $P \in jT$ (since stab P = stab e_P for $P \in jT$, $P \notin X_1$) and any vertex has stabiliser conjugate to stab P for some $P \in jT$.

By lemma 1, if G has finitely many reversing orbits, the free part of G is finitely generated and the base group is the tree product over a finite subtree of jT of the goups stab P. So G will be finitely generated if it has finitely many reversing orbits and each vertex has finitely generated stabiliser.

We say the group A has the *finitely generated intersection property* (f.g.i.p.) if the intersection of two finitely generated subgroups of A is finitely generated.

THEOREM 5. *The group $A *_U B$ has f.g.i.p. if A and B have f.g.i.p. and U is finite.*

THEOREM 6. *The HNN group $\langle A, t_i; t_i V_i t_i^{-1} = U_i \rangle$ has f.g.i.p. if A has f.g.i.p. and each U_i is finite.*

Theorem 5, which is proved in [1], and theorem 6, which improves on a theorem in [2], are special cases of the next theorem.

THEOREM 7. *Let G act without inversions on a tree X. If the stabiliser of each vertex has f.g.i.p. and the stabiliser of each edge is finite then G has f.g.i.p.*

PROOF. Let H and K be finitely generated subgroups of G. By Lemma 2, there are finitely many reversing H-orbits and finitely many reversing K-orbits, and for any vertex P both $H \cap$ stab P and $K \cap$ stab P are finitely generated. As stab P has f.g.i.p. it follows that $H \cap K \cap$ stab P is finitely generated for any vertex P. Hence it is enough to prove that there are only finitely many reversing $(H \cap K)$-orbits. Since such an orbit is in the intersection of a reversing H-orbit and a reversing K-orbit we need only prove that the intersection of an H-orbit and a K-orbit contains finitely many $(H \cap K)$-orbits.

Let the stabiliser of the edge e be U. Then $ge \in He \cap Ke$ if and only if $g \in HU \cap KU$. As U is finite $HU \cap KU$ is the intersection of finitely many cosets of H and K, and so consists of finitely many double cosets $(H \cap K)wU$, so that ge lies in the $(H \cap K)$-orbit of one of finitely many edges we.

The next lemma will enable us to prove some results on subnormal subgroups.

LEMMA 3. *Let G act without inversions on a tree X. Let e be an edge such that $\{ge; g$ negative for $e\}$ is infinite. If $h \in G$ stabilises no vertex, then $\exists k \in G$ such that $\{ge$; g negative for e and g a power of $k^{-1}hk\}$ is infinite.*

PROOF. Let e be oriented from P_0 to P_1. Suppose first that the path from e to he begins with P_1, P_0. Choose k so that ke is oriented from kP_1 to kP_0 and so that the paths from e to ke and from he to ke both end with kP_1, kP_0. This is possible since we need only choose k negative for e and such that ke is not on the paths joining e and he to the origin O.

If the path from ke to hke begins with kP_1, kP_0, then the path from he to ke will be shorter than the path from he to hke, and this has the same length as the path from e to ke. If the path from ke to $h^{-1}ke$ begins with kP_1, kP_0, then the path from e to ke will be shorter than the path from e to $h^{-1}ke$, and this has the same length as the path from he to ke. Hence (as $ke \neq hke$, for h stabilises no vertex) either the path from ke to hke or the path from ke to $h^{-1}ke$ begins with kP_0, kP_1. Then either the path from e to $k^{-1}hke$ or the path from e to $k^{-1}h^{-1}ke$ will begin with P_0, P_1.

It follows that (replacing h by $k^{-1}hk$ or by $k^{-1}h^{-1}k$ if necessary) we need only consider the case when the path from e to he begins with P_0, P_1.

Suppose this is so, and suppose that h is positive for e. Let the path from e to he be $P_0, P_1, \cdots, P_{m-1} = hP_0, P_m = hP_1$. We show inductively that, for any positive integer r, h^r is positive for e and that the path from e to h^re begins with P_0, P_1, and has $1 + r(m-1)$ edges.

This is true for $r = 1$. Suppose it is true for r, and let the path e to h^re be $P_0, P_1, \cdots, P_{r(m-1)} = h^rP_0, P_{1+r(m-1)} = h^rP_1$. Then the path from he to $h^{r+1}e$ is $hP_0, hP_1, \cdots, hP_{r(m-1)}, hP_{1+r(m-1)}$. Then the path from e to $h^{r+1}e$ will consist of the path from e to he (which ends with hP_0, hP_1) followed by the path from he to $h^{r+1}e$, as required.

The path from e to $h^{-r}e$ will be of length $1 + r(m-1)$, being

$$h^{-r}P_{1+r(m-1)} = P_1, h^{-r}P_{r(m-1)} = P_0, \cdots, h^{-r}P_1, h^{-r}P_0.$$

For large r this will be longer than the path from e to the origin, and so its last edge must be oriented from $h^{-r}P_1$ to $h^{-r}P_0$. Hence h^{-r} is negative for e if r is large, and the edges $h^{-r}e$ are distinct, being at different distances from e.

We are left with the case when h is negative for e, and the path from e to he begins with P_0, P_1. Let this path be $P_0, P_1, \cdots, P_{m-1} = hP_1, P_m = hP_0$. Then $\exists n, 1 \leqq n \leqq m$ with $hP_i = P_{m-i}$ for $i \leqq n$ but $hP_{n+1} \neq P_{m-n-1}$. We must have $n < m/2$. For if $n \geqq m/2$ we would have either $m = 2k$, $hP_k = P_k$ or $m = 2k+1$, $hP_k = P_{k+1}$, $hP_{k+1} = P_k$. The first is impossible as h stabilises no vertex, the second is impossible as G acts without inversions.

We show inductively that h^r is negative for e for any positive integer r and that the path from e to h^re has form $Q_0 = P_0, Q_1 = P_1, \cdots, Q_{n+1} = P_{n+1}, \cdots, Q_{s-1} = h^rP_1, Q_s = h^rP_0$. This holds for $r = 1$. Suppose it holds for some r. Then the path from he to $h^{r+1}e$ is $hQ_0 = P_m, hQ_1 = P_{m-1}, \cdots, hQ_n = P_{m-n}, hQ_{n+1} \neq P_{m-n-1}, \cdots, hQ_{s-1} = h^{r+1}P_1, hQ_s = h^{r+1}P_0$. Hence the path from e to $h^{r+1}e$ will be $P_0, P_1, \cdots, P_{m-n}, hQ_{n+1}, hQ_{n+2}, \cdots, h^{r+1}P_1, h^{r+1}P_0$. As this path starts with P_0, P_1, all its edges are positively oriented, and in particular h^{r+1} is negative for e.

As $n < m/2$, this completes the induction step. Also, as $n < m/2$, the path from e to $h^{r+1}e$ is longer than the path from e to $h^r e$. Consequently the edges $h^r e$ are all distinct and h^r is negative for e for any positive integer r and the proof of the lemma is complete.

The next theorem generalises Theorem 10 of [1] and Theorem 9 of [2]. it can also be proved using the theory of ends (see [3]).

THEOREM 8. *Let G be either* (i) $A*_U B$ *or* (ii) *an HNN group*

$$\langle A, x_i; x_i V_i x_i^{-1} = U_i \rangle.$$

Let G_r be a subnormal subgroup of G such that in case (i) *G_r is contained in no conjugate of A or B and in case* (ii) *G_r is contained in no conjugate of A. Let H be a finitely generated subgroup of G with $G_r \subseteq H$. Then in case* (i) *the double coset index of (H, U) in G is finite while in case* (ii) *the double coset index of (H, U_i) in G is finite for each i.*

COROLLARY 1. *Let G be either $A*_U B$ or the HNN group $\langle A, x; xVx^{-1} = U\rangle$, with U finite in either case. If the finitely generated subgroup H of G contains an infinite subnormal subgroup of G, then H has finite index in G.*

PROOFS. In either case let X be the tree constructed in section 2. Orient X outwards from the vertex $1.A$. In the first case we see that the edge gU is oriented from gA to gB if the last syllable of g is in A, and is oriented from gB to gA if the last syllable of g is in B. In the second case we find that if $r > 0$ the edge $x_i^r U_i$ is oriented from $x_i^r A$ to $x_i^{r+1} A$ while if $a \in A - V_i$ the edge $x_i^{r+1} a x_i^{-1} U_i$ which joins $x_i^{r+1} a x_i^{-1} A (\neq x_i^r A)$ to $x_i^{r+1} a x_i^{-1} x_i A = x_i^{r+1} A$ must be oriented from $x_i^{r+1} A$ to $x_i^{r+1} a x_i^{-1} A$ (since only one edge ends with $x_i^{r+1} A$). Similarly if $r > 0$ the edge $x_i^{-r} U_i$ is oriented from $x_i^{-(r-1)} A$ to $x_i^{-r} A$ and if $a \in A - U_i$ the edge $x_i^{-r} a U_i (\neq x_i^{-r} U_i)$ which joins $x_i^{-r} a A = x_i^{-r} A$ to $x_i^{-r} a x_i A$ must be oriented from $x_i^{-r} A$ to $x_i^{-r} a x_i A$.

It follows that in case (i) we have $\{ge; g \text{ negative for } e\}$ infinite for any edge e while in case (ii) we have $\{ge; g \text{ negative for } e\}$ infinite for any edge e provided we do not have $e = gU_i$ where $U_i = A = V_i$.

Let $G_r \lhd G_{r-1} \lhd \cdots \lhd G_0 = G$. As G_r is contained in no conjugate of A (or B) it contains an element h not in any conjugate of A (or B), and h stabilises no vertex of X.

Let e be an edge such that $\{ge; g \text{ negative for } e\}$ is infinite. We show inductively that for $0 \leqq s \leqq r$ we have $\{ge; g \in G_s \text{ negative for } e\}$ is infinite. This is true for $s = 0$. Suppose it is true for some s. By Lemma 3 $\exists k \in G_s$ such that $\{ge; g$ negative for e, g a power of $k^{-1}hk\}$ is infinite. Since $h \in G_{s+1} \lhd G_s$, we have $\{ge; g \in G_{s+1} \text{ negative for } e\}$ is infinite.

In particular, the G_r-orbit of e will be reversing, and so the H-orbit of e will be reversing. By Lemma 2, H has only finitely many reversing orbits since it is finitely generated.

Since the number of H-orbits containing an edge gU_i fro some g (and fixed i) is the double coset index of (H, U_i) in G, we have the result for those i such that we do not have $U_i = A = V_i$. (We also see that if there are infinitely many i for which we do not have $U_i = A = V_i$, then no such subgroup H can exist).

Plainly the result follows unless $U_i = A = V_i$ for all i. In this case $A \lhd G$ and G/A is free. Let $\alpha : G \to G/A$ be the natural homomorphism. Then the double coset index of (H, A) in G is the index $|\alpha G : \alpha H|$, and αH is a finitely generated subgroup of αG containing the non-trivial subnormal subgroup αG_r. If the free group αG has rank 1 then $|\alpha G : \alpha H|$ is obviously finite while if αG has rank greater than 1 it is a free product and the result follows as in the proof of the corollary (and is in fact well-known).

To prove the corollary (and the above statement) it is enough to show that if $G = A *_U B$ any subnormal subgroup contained in a conjugate of A or B must be contained in a conjugate of U.

Let A_1 be a subgroup of A. If $\exists\, g \in G$ with $g \notin A$ and $g^{-1}A_1 g \leqq A$ it is easy to see A_1 must be contained in a conjugate of U. Hence if A_1 is contained in no conjugate of U its normaliser is contained in A, and we see, as required that A_1 cannot be subnormal in G.

Added in proof

This also has been obtained by Karrass, Pietrowski and Solitar 'An improved subgroup theorem for HNN groups with some applications', Canadian Journal of Mathematics, 26 (1974), 214–224.

Added in proof

Serre informs me that he had obtained spacial cases of theorems 1 and the general forms being due to Bass. Accordingly, the results should be attributed to Bass and Serre rather than to Serre.

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