Subgroups of p^5-bounded groups

Fred Richman
Florida Atlantic University
Boca Raton, FL 33431

Elbert A. Walker
New Mexico State University
Las Cruces, NM 88003

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Abstract

Each v-module B with $B(5) = 0$ is a direct sum of simply presented v-modules and copies of two v-modules which come from (finite) hung trees. There are infinite-rank indecomposable v-modules B with $B(6) = 0$.

1 Valuated modules

By a **module** we will mean a module over a fixed discrete valuation domain with prime p. The reader is assumed to be familiar with the notion of a **valuated module**, or **v-module**, which is a module B together with a filtration $B = B(0) \supset B(1) \supset B(2) \supset \cdots$ such that $pB(n) \subset B(n+1)$. We need not consider arbitrary ordinal values of n because we are interested in the case $B(5) = 0$. If $x \in B(n)$ and $x \notin B(n+1)$ we write $vx = n$ and say the **value** of x is n.

If B is a subgroup of a p^5-bounded group G, then B is naturally a module over the ring of integers localized at p, and the module B is filtered by setting $B(n) = B \cap p^n G$. Classifying such subgroups, up to isomorphism of G, is equivalent to classifying the associated v-modules, because bounded modules (with the filtration $B(n) = p^n B$) are injective in the category of v-modules [5, Theorem 9].

We classify v-modules by writing them as direct sums. These direct sums must **respect values**, that is, they must respect the filtration: $(A \oplus B)(n) = A(n) \oplus B(n)$. The following lemma aids in verifying that a sum respects values.

Lemma 1 (respect value) *If A and H are submodules of a torsion v-module, and $A \cap H = 0$, then $A \oplus H$ respects values provided*

$$(A \oplus H)(n+1) = A(n+1) \oplus H(n+1)$$

whenever n is an Ulm invariant of A.

Proof. Contrapositively, we will show that if the equation fails for some n, then it fails for some n that is an Ulm invariant of A. Suppose

$$v(a+h) > va = n.$$

By induction on the order of $a+h$, we may assume that $v(pa+ph) \leq vpa$. So $vpa > n+1$, whence n is an Ulm invariant of A. ∎

Note that this lemma and its proof are valid for n any ordinal. We will also be worried about respecting the height filtration. The next lemma is a modification of [3, Theorem 1.3].

Lemma 2 (respect height) *Let A be a reduced torsion submodule of a module B, and K a submodule of B. If $\mathrm{ht}_B(a+k) \leq \mathrm{ht}_A a$ whenever $a \in A[p]$ and $k \in K$, then*

$$(A \oplus K) \cap p^n B = p^n A \oplus (K \cap p^n B).$$

If, in addition, A is bounded, then $B = A \oplus H$ for some $H \supset K$.

Proof. Clearly the hypothesis implies that $A \cap K = 0$. Suppose $a + k \in (A \oplus K) \cap p^n B$. We want to show that $a \in p^n A$. By induction on the order of a, we have $pa \in p^{n+1} A$. Let $pa = p^{n+1} a'$, so $a - p^n a' \in A[p]$. Then

$$a - p^n a' + k \in (A[p] \oplus K) \cap p^n B$$

so $a - p^n a' \in p^n A$, by hypothesis. Thus $a \in p^n A$.

If A is bounded, then we can extend K to a complementary summand H of A because algebraically compact modules, such as A, are injective in the category of v-modules [5, Theorem 9]. ∎

We combine these two lemmas in one, which will be used frequently.

Lemma 3 *Let A be a bounded submodule of a v-module B, and K a submodule of B. Suppose*

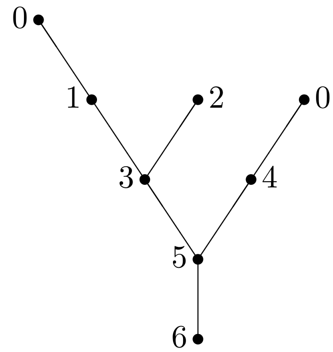
- *$B(m+1) = A(m+1) \oplus K(m+1)$ whenever m is an Ulm invariant of the v-module A, and*
- *if $a \in A[p]$ and $k \in K$, then $\mathrm{ht}_B(a+k) \leq \mathrm{ht}_A a$.*

Then $B = A \oplus H$ for some $H \supset K$.

Proof. Lemma 2 says that we can find $H \supset K$ such that $A \cap H = 0$ and $A + H = B$. Lemma 1 says that $A \oplus H$ respects values. ∎

2 Finite valuated trees

By a **tree** we will mean a valuated tree with values in ω. To fix notation and terminology, consider the tree



We denote this tree by 65(3(10)(2))(40). It is obtained by adjoining a node of value 6 as parent of the root of the tree 5(3(10)(2))(40). This tree, in turn is obtained by adjoining a node of value 5 as the parent of the roots of the trees 3(10)(2) and 40. And so on. A **forest** is a family of trees. A **pole** is a tree with no branching.

If x is a node in a tree, then px denotes the parent of x, if any. If $p^n x$ is the root of the tree, then we say that the **level** of x is n, and write $\ell(x) = n$. A **map** of trees is a function f such that

- $f(px) = pf(x)$ whenever $\ell(x) > 0$,
- $vf(x) \geq vx$.

A map f of trees is **order preserving** if $\ell(f(x)) = \ell(x)$ for all x. The set of trees is pre-ordered by setting $T_1 \leq T_2$ if there is a map (which we can take to be order preserving) from T_1 to T_2. A tree is **irretractible** if the only idempotent map to itself is the identity. The irretractible trees form a partially ordered subset of the trees. The tree 0 is the smallest tree under this partial ordering.

Each tree T gives rise to a simply presented v-module $\langle T \rangle$.

Let F be a forest. The forest F^* is defined to be the set of trees that cannot be mapped into (any tree of) F. The set F^* is an up-set in the set of all trees which is generated by a finite number of minimal elements.

We can compute F^* inductively. If F is empty, then F^* is all trees—the up-set generated by the tree 0. If

$$T = nF$$

is a tree, then T^* is the up-set generated by the tree $n + 1$ together with all trees of the form kT' where $T' \in F^*$ and $k - 1$ is the value of the root of T'. If

$$F = (T_1)(T_2) \cdots (T_k)$$

then the minimal elements of F^* are the minimal elements of the finite set

$$\{t_1 \vee t_2 \vee \cdots \vee t_k : t_i \text{ is a minimal element of } T_i^*\}.$$

Note that the minimal elements of T^* cannot branch at the root.

If T is the pole $n_1 n_2 \cdots n_k$, then the minimal elements of T^* are the poles

$$n_j + j,\ n_j + j - 1, \ldots,\ n_j + 2,\ n_j + 1$$

if $j = 1$ or $n_{j-1} > n_j + 1$, and

$$k, k-1, \ldots, 1, 0$$

if $n_k > 0$. These are gapless poles. There is one for each Ulm invariant of the pole T, and one more if there is no node of value 0 (does this imply an Ulm invariant at -1?).

Some examples:

- If $T = (20, 9, 8, 7, 4, 1, 0)$, then T^* is generated by the poles (21), $(11, 10)$, $(9, 8, 7, 6, 5)$, and $(7, 6, 5, 4, 3, 2)$.
- If $T = 3(2)(01)$, then T^* is generated by 4 and 320.
- If $T = 4(310)(32)$, then T^* is generated by 5 and $43(10)(2)\}$.
- If $T = 6(430)(521)$, then T^* is generated by $\{7, 54(21)(3), 653, 43210\}$.

A v-module B with $\bigcap B(n) = 0$ is an honorary tree. The root of B is 0, has value ∞, and is its own parent. Technically, it's not a tree, but it's clear what a tree-map $T \to B$ is. If T is a tree, then $B(T)$ is defined to be the set of images of the root of T under tree-maps $T \to B$. If T is the gapless pole $(\alpha + n, \ldots, \alpha)$, then $B(T) = p^n(B(\alpha))$. If F is a forest, then $B(F)$ is the submodule of B generated by $\{B(T) : T \in F\}$.

The **v-height** of an element x of finite value in a v-module B is given by the equivalence class of the branch $\{b \in B : p^n b = x \text{ for some } n\}$ above x. There is a unique irretractible tree in this equivalence class, the smallest T such that $x \in B(T)$.

Call an irretractible tree **hangable** if it has distinct nodes t_0 and t_1 such that $vt_0 = vt_1$ and either $pt_0 = pt_1$ or $\min(vpt_0, vpt_1) > vt_0 + 1$. Poles are not hangable, nor is any tree in which distinct nodes have distinct values. The tree 4(310)(32) is the only hangable tree with all values less than 5.

Theorem 4 *Let B be a reduced v-module and T an irretractible subtree of B, with exactly two leaves, that is the v-height of its root. Then*

1. *T is a p-basis for the (unvaluated) submodule A that it generates.*

2. *If T is unhangable, then T is a p-basis for the v-module A.*

Proof. For 1, suppose contrapositively, that $\sum_i u_i t_i = 0$ where the u_i are units and the t_i are distinct nodes of T. We will construct a retraction of T. Consider the nodes t_i of minimum value. There must be at least two of them, lest the sum have value different from ∞, and, because T has only two leaves, there must be exactly two of them, t_0 and t_1. Clearly every other node that appears in the sum is a multiple of either pt_0 or pt_1. So we can write $t_0 = ut_1$ for some unit u. Therefore t_0 and t_1 have the same v-height, and $vp^i t_0 = vp^i t_1$ for each i. So there is a retraction of T that takes t_1 to t_0.

For 2, suppose contrapositively that $v(u_0t_0 + u_1t_1) > vt_0 = vt_1$. We may assume that this is the maximum value of $vt_0 = vt_1$ where this occurs. Either $pt_0 = pt_1$ or

$$vt_0 + 1 < v(u_0pt_0 + u_1pt_1) \leq \min(vpt_0, vpt_1).$$

In either case, T is hangable. ■

3 Szele trees

For C a cyclic valuated p-group, the functor F_C was defined in [3, page 23] as

$$F_C(B) = \sum_{p^n(C(\alpha))=0} p^n(B(\alpha))$$

If C comes from the pole T, then $p^n(C(\alpha)) = 0$ if and only if the gapless pole $(\alpha + n, \ldots, \alpha)$ is in T^*, so $F_C(B) = B(T^*)$. The following is a paraphrase of [3, Theorem 2.5].

Theorem 5 *Let B be a v-module, T a pole, and A a submodule of B which is a direct sum of copies of $\langle T \rangle$. Then*

1. *A is a summand of B if and only if $A \cap B(T^*) = 0$.*

2. *If $A + K = B$ and $A \cap K = 0$ and $K \supset B(T^*)$, then $B = A \oplus K$.*

For T the gapless pole $n \cdots 10$, and B an unvaluated module, Part 1 is the theorem of Szele [1, Prop. 27.1] stating that if A is a direct sum of cyclic groups of order p^n, and $A \cap p^n B = 0$, then A is a summand of B. Much of our work here consists of extending Part 1 to other trees.

A tree T that satisfies Part 1 of Theorem 5 will be called a **Szele tree**. So poles are Szele trees. The tree 3(10)(2) was shown to be a Szele tree in [3, Lemma 4.1]. Note that if A is a summand of B, then $A \cap B(T^*) = 0$ because $B(T^*)$ is an additive functor in B.

Part 2 of Theorem 5 does not carry over to Szele trees. Let $T = 3(10)(2)$ and $B = \langle T\rangle \oplus \langle 32\rangle$. Then $B(T^*) = 0$. Let x be the node of T of value 1, and y the node of 32 of value 2. The submodule K generated by $x - y$ satisfies the hypothesis but not the conclusion of Part 2.

We can rephrase, and slightly strengthen, Theorem 5.

Corollary 6 *Let B be a v-module, T a pole, and A a submodule of B which is a direct sum of copies of $\langle T\rangle$. If K' is a submodule of B such that $K' \cap A = 0$ and $B(T^*) \subset K'$, then $B = A \oplus K$ for some $K \supset K'$.*

Proof. We want a submodule K containing K' such that $A + K = B$ and $A \cap K = 0$. Then Theorem 5 Part 2 finishes the job. As A is bounded, it suffices to show that $\mathrm{ht}_B(a + k) \le \mathrm{ht}_A\, a$ for each $a \in A[p]$ and $k \in K'$ (see [3, Theorem 1.3]). But if $\mathrm{ht}_B(b) > \mathrm{ht}_A\, a$ for $a \in A[p]$, then $b \in B(T^*) \subset K'$. ■

If T is a tree, then the T-th **valuated Ulm invariant** of a valuated module G is defined in [4] to be

$$U_T G = \frac{G(T)[p]}{G(T^*)[p] \cap G(T)[p]}$$

Theorem 7 *Let B be a v-module and T an unhangable Szele tree. Then $B = A \oplus K$ where A is a direct sum of copies of $\langle T\rangle$, and $U_T K = 0$.*

Proof. Consider families of submodules S_i of B with the properties that S_i is isomorphic to $\langle T\rangle$ for each i, the sum $\sum S_i$ is direct, and $B(T^*) \cap \sum S_i = 0$. Zorn's lemma applies, so there is a maximal such family S_i. Let $A = \sum S_i$. As T is a Szele tree,, we can write $B = A \oplus K$. It remains to show that $U_T K = 0$.

Suppose $U_T K \neq 0$. Let c be an element of $K(T)[p]$ that is not in $K(T^*) = B(T^*)$. As T is unhangable, Theorem 4 says that c is contained in a submodule of K isomorphic to $\langle T\rangle$. As T is a Szele tree, this submodule is a summand of K, contradicting the maximality of the family S_i. ■

No doubt any Szele tree is unhangable, but we don't need that.

For the purpose of showing that T is a Szele tree, we may assume that $B(n+1) = 0$, where n is the value of the root of T. Indeed, $B = A \oplus K$ follows easily from

$$\frac{B}{B(n+1)} = A \oplus \frac{K}{B(n+1)}.$$

The smallest tree, $3(10)(2)$, has two Ulm invariants, 1 and 3. The next theorem is effectively a generalization of [3, Lemma 4.1] from the smallest tree to any tree with two Ulm invariants.

Theorem 8 *If T is an irretractible tree with exactly two Ulm invariants, then T is a Szele tree.*

Proof. Let B be a v-module and A a submodule which is a direct sum of copies of $\langle T\rangle$. Suppose $A\cap B(T^*)=0$. We must show that A is a summand of B.

Let the Ulm invariants of T be $k<n$, so

$$T=n\ldots m(m-1\ldots i)(k\ldots j)$$

where $k<m-1$ and $k-j>m-1-i$, the dots indicate no gaps. Note that T^* consists of the poles $n+1$ and $k+n-m+2\ldots k+1, m-1-i\ldots 0$, and, if $j>0$, the pole $n-m+k-j+2\ldots 0$.

Let $T_0=T(k+1)$. Then T_0 is the gapless pole $n\ldots s$ where $s=\max(k+1,i)$, so T_0^* consists of the poles $n+1$ and $n-s+1\ldots 0$. For any v-module K,

$$K(T_0^*)=K(n+1)+p^{n-s+1}K.$$

Thus

$$B(k+1)(T_0^*)=B(n+1)+p^{n-s+1}B(k+1)\subset B(T^*),$$

because $p^{n-s+1}B(k+1)\subset B(n-s+k+2)\subset B(n-m+k+2)$, so

$$B(k+1)(T_0^*)\subset B(T^*)(k+1).$$

As $A\cap B(T^*)=0$, we can write

$$B(k+1)=A(k+1)\oplus H_{k+1}$$

where $H_{k+1}\supset B(T^*)(k+1)$, by Corollary 6.

To write $B=A\oplus H$ with $H\supset H_{k+1}$, it suffices, by Lemma 3, to show that if a in $A[p]$ and $h_{k+1}\in H_{k+1}$, then $\mathrm{ht}_B(a+h_{k+1})\le \mathrm{ht}_A\, a$. Suppose $a+h_{k+1}\in p^tB$. If $va=n$ and $t>\mathrm{ht}_B\, a$, then $t>\mathrm{ht}_A\, a$ so $p^tB\subset B(T^*)$. Thus $a+h_{k+1}\in p^tB\cap B(k+1)\subset H_{k+1}$, and $a\in H_{k+1}$, a contradiction.

So suppose $va=k$ and $a+h_{k+1}\in p^{m-i}B$. Then $a=a_{m-1}-a_k$ where $va_{m-1}=m-1$ and $a_k\in p^{m-i}A$ has value k. Note that $\mathrm{ht}_A\, a\ge m-i-1$. Then $a_{m-1}+h_{k+1}\in p^{m-i}B\cap B(k+1)$ so

$$p^{n-m+1}(a_{m-1}+h_{k+1})\in B(T^*)(k+1)\subset H_{k+1},$$

whence $p^{n-m+1}a_{m-1}\in H_{k+1}$, a contradiction. ■

4 Unhangable trees in $\mathcal{T}_4$ are Szele trees

Let $\mathcal{T}_4$ denote the set of trees with values less than 5 The only hangable tree in $\mathcal{T}_4$ is 4(32)(310). It is not a Szele tree, as we shall see (5.3 Example). In this section we will prove that the other trees in $\mathcal{T}_4$ are Szele trees. Before starting, we note that the restriction to $\mathcal{T}_4$ is essential. Consider the tree $T=5(41)(32)$. Let $A=\langle T\rangle$ and let

x and y be the leaves of T. Adjoin z to A with $pz = x - y$ to get B. Then A is not pure in B, but $B(T^*) = 0$.

Aside from the poles, which we know are Szele trees, the eleven unhangable trees in $\mathcal{T}_4$ are

43(2)(10)	4(210)(32)	4(210)(31)	4(210)(30)	4(210)(3)
4(30)(21)	4(21)(3)	4(20)(3)	4(10)(3)	4(10)(2)

and 3(10)(2). A computer count of $\mathcal{T}_4$ came up with 43 trees [2]. There are $2^5 - 1$ poles in $\mathcal{T}_4$, so it looks like we've listed all the rest here.

Theorem 8 takes care of all but these five

4(210)(31)	4(210)(30)	4(30)(21)
4(20)(3)	4(10)(2)	

4.0.1 The tree 4(30)(21)

The star is generated by 5, 431 and 3210. We may assume $B(5) = 0$, so $B(T^*) \subset (p^2B)(3)$. Note that $A(31) = A(4)$ is a direct sum of copies of $\langle 4 \rangle$, and $(4)^*$ is generated by (5) and (10). So $B(31)((4)^*) \subset B(T^*) \subset B(31)$, whence Corollary 6 says that we can write

$$B(31) = A(31) \oplus H_{31}$$

with $H_{31} \supset B(T^*)$.

As $A(3)$ is a direct sum of copies of $\langle 43 \rangle$, and $p^2(B(3)) = 0$, and $A(3) \cap H_{31} = 0$, Corollary 6 says that we can write

$$B(3) = A(3) \oplus H_3$$

with $H_3 \supset H_{31}$.

We want to write

$$B(1) = A(1) \oplus H_1$$

with $H_1 \supset H_3$. It suffices, by Lemma 3, to show that if $a \in A(1)[p]$, and $h_3 \in H_3$, then $\mathrm{ht}_{B(1)}(a + h_3) \leq \mathrm{ht}_{A(1)}\, a$. Write $a = a_{30} + a_{21}$ where $a_{30} \in A(30)$ and $a_{21} \in A(21)$. If $a_{30} + a_{21} + h_3 \in p(B(1))$, then $a_{30} + h_3 \in p(B(1))$, so

$$a_{30} + h_3 \in B(31) = A(31) \oplus H_{31} \subset A(31) \oplus H_3.$$

Therefore $a_{30} \in A(31)$, so $a \in A(4) \subset p^2(A(1))$.

4.0.2 The tree 4(210)(31)

The star is generated by 5, 432, and 4310, so $B(T^*) \subset B(4)$. First write

$$(p^2B)(4) = A(4) \oplus K_4$$

where $K_4 \supset B(T^*)$. This is just a vector space argument. Note that

$$(p^2B)(3)(10) = (p^2B)(43) = B(4310) \subset B(T^*).$$

As $A(4)$ is a direct sum of copies of $\langle 4\rangle$, and $A(4) \cap K_4 = 0$, Corollary 6 says that we can write

$$(p^2B)(3) = A(4) \oplus K_3$$

where $K_3 \supset K_4$. As $A(3)$ is a direct sum of copies of $\langle 43\rangle$, and $B(3)(210) = B(543) \subset B(T^*) \subset K_3$, Corollary 6 says that we can write

$$B(3) = A(3) \oplus H_3$$

with $H_3 \supset K_3$.

Now we want to show that if $a \in A(2)[p]$, and $h_3 \in H_3$, then $\mathrm{ht}_{B(2)}(a + h_3) \leq \mathrm{ht}_{A(2)}\, a$. In particular, $A(2) \cap H_3 = 0$. Every element of $A(2)[p]$ can be written as $a_3 + a_2$ with $a_3 \in A(3)$ and $a_2 \in p^2A$. Suppose

$$a_3 + a_2 + h_3 \in p(B(2)).$$

Then $a_2 \in B(3)$, so $a_2 \in (p^2B)(3) = A(4) \oplus K_3$, so $a_2 \in A(4)$ because $K_3 \cap A = 0$. So $a_3 + a_2 \in A(4) \subset p(A(2))$. If

$$a_3 + a_2 + h_3 \in p^2(B(2)) \subset B(T^*) \subset H_3$$

then, as before, $a_3 + a_2 \in A(4)$. But $a_3 + a_2 \in H_3$, so $a_3 + a_2 = 0$.

So Lemma 3 says that we can write

$$B(2) = A(2) \oplus H_2$$

where $H_2 \supset H_3$. We want to show that this respects heights in B. If $a_3 + a_2 \in p^2B$, then $a_3 \in p^2B$, so $a_3 \in A(4)$, so $a_3 + a_2 \in A(4) \subset p^2A$.

So Lemma 3 says that we can write

$$B = A \oplus H$$

with $H \supset H_2$.

4.0.3 The tree 4(210)(30)

The star is generated by 5 and 431, so $B(T^*) \subset B(4)$. As $B(31)(40) = B(431) \subset B(T^*)$, and $A(4)$ is a direct sum of copies of $\langle 4\rangle$, Corollary 6 says that we can write

$$B(31) = A(4) \oplus K_3$$

with $K_3 \supset B(431)$.

We want to write

$$B(3) = A(3) \oplus H_3$$

with $H_3 \supset K_3$. It suffices, by Lemma 3, to show that if $a \in A(3)[p]$, and $k_3 \in K_3$, then $\mathrm{ht}_{B(3)}(a + k_3) \leq \mathrm{ht}_{A(3)}\, a$. But if $a \neq 0$, then $\mathrm{ht}_{A(3)}\, a = 1$, while $p^2(B(3)) \subset K_3$, so $\mathrm{ht}_{B(3)}(a + k_3) \leq 1$.

Now we want to write

$$B(21) = A(21) \oplus H_{21}$$

where $H_{21} \supset H_3(31)$. We can do this as before because each nonzero element of $A(21)[p]$ has height 1 in $A(21)$, and $p^2(B(21)) \subset B(431) \subset H_3(31)$.

Now we want to write

$$B(1) = A(1) \oplus H_1$$

with $H_1 \supset H_3 + H_{21}$. We can do this, by Lemma 3, if we can show that, for $a \in A(1)[p]$ and $h_3 \in H_3$ and $h_{21} \in H_{21}$, that $\mathrm{ht}_{B(1)}(a + h_3 + h_{21}) \leq \mathrm{ht}_{A(1)}\, a$.

We can write $a = a_3 - a_2$ where $a_3 \in A(3)$ and $a_2 \in p^2A$. Suppose $a_3 - a_2 + h_3 + h_{21} \in p(B(1))$, as otherwise we're done. Then $a_3 + h_3 \in p(B(1))$, so $a_3 + h_3 \in B(31) = A(4) \oplus K_3$, which says that $a_3 \in A(4) + H_3$, hence $a_3 \in A(4)$ and so $a \in A(4)$. Then $\mathrm{ht}_{A(1)}\, a \geq 2$ while $p^3(B(1)) \subset B(431) \subset H_3$, so if $\mathrm{ht}_{B(1)}(a + h_3 + h_{21}) > 2$, then $a \in H_3 + H_{21}$, so $a \in H_3$, so $a = 0$.

Finally, to apply Lemma 3 one more time, we want to show that, for $a \in A[p]$ and $h_1 \in H_1$, that $\mathrm{ht}_B(a + h_1) \leq \mathrm{ht}_A\, a$. Suppose, in the notation of the preceding paragraph, that $a_3 - a_2 + h_1 \in p^2B$, so $a_3 + h_1 \in p^2B \subset B(21) = A(21) \oplus H_{21}$. So $a_3 + h_1 + h_{21} \in A(21)$. But that implies $h_1 + h_{21} \in A$, hence $h_1 + h_{21} = 0$, so $a_3 \in A(21)$, so $a \in A(4)$. Thus $\mathrm{ht}_A\, a \geq 3$ while $p^4B \subset B(5) \subset H_1$, so if $\mathrm{ht}_B(a + h_1) > 3$, then $a \in H_1$, so $a = 0$.

4.0.4 The trees $4(20)(3)$ and $4(10)(2)$

We can handle all the four-element trees at once.

Theorem 9 *The tree $T = n(mj)(i)$ with $j < m < i < n$ is a Szele tree.*

Proof. Suppose $A \cap B(T^*) = 0$. We may assume $B(n+1) = 0$. Note that T^* is generated by

$$(n+1),\ (i+2, i+1),\ (m+2, m+1, 0),\ (j+3, j+2, j+1),\ 3210$$

where some of these poles may be redundant. Also, $(ni)^*$ is generated by

$$(n+1),\ (i+2, i+1),\ 210$$

so

$$B(T^*)(m+1) \supset B(m+1)((ni)^*).$$

First we want to write

$$B(m+1) = A(m+1) \oplus H_{m+1}$$

where $H_{m+1} \supset B(T^*)(m+1)$. As $A(m+1)$ is a direct sum of copies of $\langle ni \rangle$, Corollary 6 says that we can do that.

Next we want to write

$$pB = pA \oplus K$$

with $K \supset (H_{m+1} + B(T^*)) \cap pB = (H_{m+1} \cap pB) + B(T^*)$. Now pA is a direct sum of copies of $\langle nm \rangle$, and $(nm)^*$ is generated by $(n+1)$ and $(m+2, m+1)$ and (210). So $(pB)((nm)^*) \subset B(T^*)$. Moreover $pA \cap (H_{m+1} + B(T^*)) = 0$ because if $a = h_{m+1} + b^* \in (pA)[p]$, then $b^* \in B(T^*)(m+1) \subset H_{m+1}$, so $a \in H_{m+1}$ whence $a = 0$. So Corollary 6 says that we can write $pB = pA \oplus K$.

Next we want to write

$$B(j+1) = A(j+1) \oplus H_{j+1}$$

with $H_{j+1} \supset H_{m+1} + K(j+1) + B(T^*)(j+1)$. The Ulm invariants of $A(j+1)$ are all at least m so, by Lemma 1 it suffices that $H_{j+1} \cap A(j+1) = 0$ and $H_{j+1} + A(j+1) = B(j+1)$. By Lemma 2 we can do this if,

$$\mathrm{ht}_{B(j+1)}(a + h_{m+1} + k_{j+1} + b^*) \leq \mathrm{ht}_{A(j+1)}\, a$$

for each $a \in A(j+1)[p]$. We can write $a = a_i - a_m$ where $a_i \in A(i)$ and $a_m \in p(A(j))$. Suppose

$$a_i - a_m + h_{m+1} + k_{j+1} + b^* \in p(B(j+1))$$

then

$$a_i + h_{m+1} \in pB$$

so

$$pa_i + ph_{m+1} \in B((m+2, m+1, 0),$$

so $pa_i \in H_{m+1}$, whence $pa_i = 0$, so $a \in A(n) \subset p(A(j+1))$. Suppose, in addition, that

$$a + h_{m+1} + k_{j+1} + b^* \in p^2(B(j+1)) \subset B(T^*) \subset K.$$

Then $a + h_{m+1} \in K(m+1)$. As $K \subset pB$, we have $h_{m+1} \in H_{m+1} \cap pB \subset K$, so $a \in K$, whence $a = 0$.

Finally, $p^t B = p^t A \oplus p^{t-1} K$ for $t > 0$. We will show that

$$(A \oplus H_{j+1}) \cap p^t B \subset p^t A \oplus H_{j+1}.$$

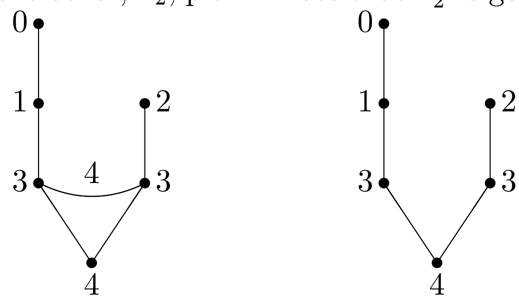
If $a + h_{j+1} = p^t a' + k$, then $pa + ph_{j+1} = p^{t+1} a' + pk$, so $pk \in K(j+1) \subset H_{j+1}$ whence $pa = p^{t+1} a'$. This can only happen if $a \in A(j+1)$. Therefore $k \in K(j+1)$, so $a = p^t a'$. ■

5 The structure of B when $B(5) = 0$

Suppose B is a v-module with $B(5) = 0$. The plan is to remove summands that are direct sums of copies of $\langle T \rangle$, for various trees T, until $B(T) = 0$ for all trees T, hence $B = 0$.

By Theorem 7 we may assume that $U_T B = 0$ for any unhangable Szele tree T. So, for such a tree, if $B(T^*) = 0$, then $B(T) = 0$. Thus $B(T) = 0$ for $T = 43210$, so $p^4 B = 0$. Similarly $B(T) = 0$ for T successively equal to 4321, 4320, and $43(2)(10) = T_0$. Note that T_0^* is generated by 5 and 4320.

The next tree in line, 4(310)(32), comes in two forms: one, T_1, with a hang of 4 across the 3's, and the other, T_2, plain. Note that T_2^* is generated by 5 and 43(2)(10).



Case 1 Case 2

It's clear what we mean by $B(T_1)$, and $\langle T_1 \rangle$, even though T_1 has a hang. We will treat Case 1 first, and in Case 2 we will be able to assume that $B(T_1) = 0$.

Suppose $B(5) = p^4 B = B(T_0) = 0$. Let A be a direct sum of copies of $\langle T \rangle$, with $T = T_1$ or $T = T_2$.

5.1 Case 1, the tree T_1

As $p^2 A$ is pure in $p^2 B$ (because $p^4 B = 0$), we can write

$$p^2 B = p^2 A + H'' \text{ with } p^2 A \cap H'' = 0.$$

Then

$$A \cap H'' \subset A \cap p^2 B = p^2 A$$

so $A \cap H'' = 0$.

We want to write

$$B(2) = A(2) \oplus H'$$

where $H' \supset H''$. Note that 4 is the only Ulm invariant of $A(2)$ (because of the hang), and $B(5) = 0$. So, by Lemma 3, it suffices to show that if $a_2 \in A(2)[p] = A[p]$, and $h'' \in H''$, then $\mathrm{ht}_{B(2)}(a_2 + h'') \leq \mathrm{ht}_{A(2)} a_2$.

There is $a_3 \in p^2 A$ such that $a_3 \in a_2 + p(A(2))$, so if $a_2 + h'' \in p(B(2))$, then $a_3 + h'' \in p(B(2))$. But

$$p(p(B(2)) \cap p^2 B) = B(T_0) = 0,$$

so $pa_3 = 0$. Therefore $a_3 \in A(4310) = p^2(A(2))$, whence $a_2 \in p(A(2))$, so $a_2 \in A(4310) = p^2(A(2))$.

The only Ulm invariants of A are 1 and 4. To finish the proof, it suffices, by Lemma 3, to show that if $z \in A[p]$, and $h' \in H'$, then $\mathrm{ht}_B(z + h') \leq \mathrm{ht}_A z$. If $z + h' \in p^2B$, then $z + h' = p^2a + h''$ so $z = p^2a$.

5.2 Case 2, the tree T_2

We may assume that $B(T_1) = 0$ because $B(T_0) = 0$, so if $B(T_1) \neq 0$, then B contains a copy of $\langle T_1 \rangle$. It follows that

$$(p(B(2)) + B(4)) \cap (p^2B + B(4)) \subset B[p] \qquad (*)$$

because the intersection is contained in $B(3)$, and if $x = pb_2 + b_4 = p^2b + b'_4$ is in it, then $px = p^2b_2 = p^3b$, and $pb_2 - p^2b \in B(4)$, so $px \in B(T_1) = 0$.

As p^2A is an absolute summand of p^2B, we can write

$$p^2B = p^2A + H''' \text{ with } p^2A \cap H''' = 0$$

and

$$(p^2B)(4) = A(4) \oplus H'''(4).$$

Because $B(5) = 0$, we can write

$$B(4) = A(4) \oplus K_4$$

with $K_4 \supset H'''(4)$. Let $H'' = H''' + K_4$. Note that $H''(4) = K_4$. Then

$$A \cap H''' \subset A \cap p^2B = p^2A$$

so $A \cap H''' = 0$. We will show that $A \cap H'' = 0$.

If $a = h''' + k_4$, then $pa \in A \cap H''' = 0$. So we can find $a_1 \in p(A(2))$ and $a_2 \in p^2A$ such that $a = a_1 - a_2$. So

$$a_1 - k_4 = a_2 + h''' \in p^2B.$$

>From $(*)$ it follows that $p(a_1 - k_4) = 0$, so $pa_1 = 0$, whence $a \in A(4)$. Thus $a = 0$ because $A(4) \cap H'' = A(4) \cap K_4 = 0$. We have shown that $A(2) \cap H'' = 0$.

We want to write

$$B(2) = A(2) \oplus H'$$

with $H' \supset H'' = H''' + K_4$. The Ulm invariants of $A(2)$ are 3 and 4, so by Lemma 3 it suffices to show that if $a \in A[p]$, then $\mathrm{ht}_{B(2)}(a + h''' + k_4) \leq \mathrm{ht}_{A(2)}\, a$. Suppose $a + h''' + k_4 \in p(B(2))$. By adding an element in $p(A(2))$ to a, we can get an element a_3 in $(p^2A)(3)$ such that $a_3 + h''' + k_4 \in p(B(2))$. From $(*)$ it follows that $p(a_3 + h''') = 0$, so $pa_3 = 0$. This means $a_3 \in A(4)$, so $a \in A(4) \subset p^2(A(2))$.

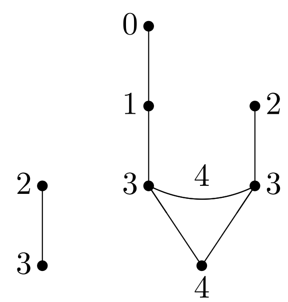
We want to write

$$B = A \oplus H$$

with $H \supset H'$. By Lemma 3 it suffices to show that if $a \in A[p]$, then $\mathrm{ht}_B(a + h') \leq \mathrm{ht}_A a$. If $a + h' \in p^2 B = p^2 A \oplus H'''$, then $a + h' = p^2 a' + h'''$ so $a = p^2 a'$. That completes the proof of case 2.

5.3 Example

Here is an example showing that the hung tree T_1 must be eliminated before eliminating the unhung tree T_2. Alternatively, that T_2 is not a Szele tree unless T_1 is included in its star. Consider $B = (32) \oplus T_1$.



If x generates (32), and y and z are the generators of 4310 and 432 respectively of T_1, set $A = \langle z - x, y \rangle$. Then $B(T_2^*) = 0$, and $A = S(T_2)$ with the v-height of 4 equal to T_2, but A is not a summand.

5.4 The schedule for removal

Here are the trees, in order of removal, together with the relevant generators of their stars. After step T, we have $B(T) = 0$. Why? For unhangable trees we know that $B(T)[p] \subset B(T^*)[p]$. If there's at most one pole left in $B(T^*)$, it will be easy to see that $B(T) \cap B(T^*) = 0$. The hangable tree gets a separate treatment.

43210		pole
4321	43210	pole
4320	4321	pole
43(2)(10)	4320	two Ulm invariants
4(310)(32)		the hangable tree: first hung, then unhung
4310	432	pole
4(210)(32)	4310	two Ulm invariants
432	3210	pole
4(210)(31)	432 4310	
431	432 3210	pole
4(210)(30)	431	
4(30)(21)	431 3210	
430	321	pole

4(210)(3)	430	two Ulm invariants
4210	43	pole
3210	4	pole. At this point $p^3B = 0$.
4(21)(3)	430	four nodes (and two Ulm invariants)
421	43	pole
321	4	pole
4(20)(3)	430 321	four nodes
420	43 321	pole
320	321 4	pole
4(10)(3)	320	four nodes (and two Ulm invariants)
43	210	pole
4(10)(2)	320 43	four nodes
410	32	pole
42	43 210	pole
3(10)(2)	320 4	four nodes (and two Ulm invariants)

At this point all elements of order p^3 have type 310. Elements of type 210 miss $B(3)$, so they split out.

5.4.1 The hung forest (32)(310)

We may now assume that $B(T) = 0$ unless T is a pole, that $(p^2B)[p] = B(310)$, and that $B(42) = B(410) = 0$. Write

$$B(4) = (B(4) \cap (B(310) + B(32))) \oplus K_4$$

Let X be a direct sum of copies of $\langle 310 \rangle$ such that $X[p] = B(310)$. Let Y be a direct sum of copies of $\langle 32 \rangle$ such that $Y[p] = B(32)$. Then $X + Y$ is an absolute direct summand of the unvaluated module B, so we can write

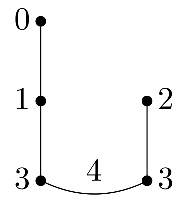
$$B = X + Y + Z$$

an unvaluated direct sum, where $K_4 \subset Z$. Then

$$B = (X + Y) \oplus Z$$

because $X + Y$ has Ulm invariants only at 1, 3 and 4. So we need only check the filtration at 2 and 4. For 2, note that $B(2) = (X + Z)(2) + Y$, and if $x + z \in B(2)$, then $px + pz \in B(32) = Y[p]$, so $px = pz = 0$, so $x \in B(3)$. For 4 we have arranged that $B(4) = (X + Y)(4) \oplus K_4$.

Let $T = 32$. As $Y[p] = B(T)$, and $U_T B = 0$, it follows that $Y[p] \subset B(T^*) = X[p] + B(4)$. Similarly $X[p] \subset Y[p] + B(4)$. Because $B(4) \cap X = B(4) \cap Y = 0$, there is natural isomorphism $f : X[p] \to Y[p]$ such that $x - f(x) \in B(4)$ for all $x \in X[p]$. Let E be a basis for $X[p]$. Then E and $f(E)$ support bases for X and Y, showing that $X + Y$ is a direct sum of copies of $\langle F \rangle$, where F is the hung forest



The complementary summand Z is a p^2-bounded v-module with finitely many values, hence a direct sum of cyclics [3, Theorem 3.2]. Alternatively, we can easily show that Z is a direct sum of cyclics by continuing the process of eliminating poles.

6 Uniqueness

So each v-module B with $B(5) = 0$ is the direct sum of a simply presented v-module with a direct sum of copies of $\langle T_1 \rangle$, and a direct sum of copies of $\langle F \rangle$, where T_1 is the hung tree 4(32)(310), and F is the hung forest (32)(310) (so $T_1 = 4F$). If we extend our notion of Ulm invariant slightly, to cover T_1 and F, then the number of copies of each indecomosable $\langle T \rangle$ is equal to the dimension of $U_T B$, the T-th Ulm invariant, hence is an invariant of B.

We have already used the submodule $B(T_1)$, which has the obvious meaning. For T_2, the unhung tree 4(32)(310), we must extend T_2^* to include T_1, while T_1^* is simply the old T_2^*. Then the definitions of U_{T_1} and U_{T_2} are formally the same as for any other Ulm invariant. Finally, we define

$$B(F) = B(4) \cap (B(32) + B(310))$$

and let $F^* be$ generated by 5, 42, and 3(10)(2). These definitions are all natural—they could be formulated in a general context of certain kinds of hung forests—and do the trick.

7 Indecomposable pairs bounded by p^6

We present a simplification of the categorical equivalence of [3, Cor. 5.3]. Let k be a field, and $\mathcal{C}_1$ the category of modules over $k[X]$ (not a discrete valuation domain). The category $\mathcal{C}_2$ consists of vector spaces V over k, together with a (labeled) family of four distinguished subspaces V_1, V_2, V_3 and V_4 such that

$$V = V_1 \oplus V_2 = V_2 \oplus V_3 = V_1 \oplus V_3 = V_4 \oplus V_2.$$

This implies that $V_1 \cong V_2 \cong V_3 \cong V_4$. Given the object (V, V_1, V_2, V_3, V_4) in $\mathcal{C}_2$, we get a linear transformation $f : V_1 \to V_1$ by setting $fx = \pi_2 \pi_4 x$, where π_4 is the projection on V_4 that kills V_2, and π_2 is the projection on V_2 that kills V_1. Conversely, given $f : V_1 \to V_1$, define

$$\begin{aligned} V_2 &= V_1 \\ V &= V_1 \oplus V_2 \\ V_3 &= \{(x, x) : x \in V_1\} \\ V_4 &= \{(x, fx) : x \in V_1\}. \end{aligned}$$

But $f : V_1 \to V_1$ is simply a $k[X]$-module on V_1, where f gives the action of X.

There are indecomposable modules in $\mathcal{C}_1$ of every finite dimension over k, and we know that there are a ton of infinite-dimensional ones. Let k be the residue class field of our discrete valuation domain. For each indecomposable object (V, V_1, V_2, V_3, V_4) in $\mathcal{C}_2$, we will construct an indecomposable v-group B, with $B(6) = 0$, such that if $C = B/B(5)$, then (V, V_1, V_2, V_3, V_4) is isomorphic to

$$C[p],\ C(32),\ C(310),\ B[p]/B(5),\ B(4)/B(5).$$

The dimension of V is $2m$, where m could be infinite. Let B be the direct sum of m copies of $\langle 5(32)(310) \rangle$, and $C = B/B(5)$. In $C[p]$, let

$$\begin{aligned} W_1 &= C(32) = p(C(2)) \\ W_2 &= C(310) = p^2 C \\ W_3 &= B[p]/B(5). \end{aligned}$$

Choose the subspace W_4 of $C[p] = C(3)$ that makes (V, V_1, V_2, V_3, V_4) isomorphic to $(C[p], W_1, W_2, W_3, W_4)$, and redefine $B(4)$ to be the preimage if W_4. This doesn't affect W_1, W_2 or W_3. Then C is an indecomposable v-module. If $B = B' \oplus B''$, then either B' or B'' is contained in $B(5)$ because C is indecomposable. But $B(5) \subset pB$, so such a summand must be zero.

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