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# SUBGROUPS OF QUASI-HNN GROUPS

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We extend the structure theorem for the subgroups of the class of HNN groups to a new class of groups called quasi-HNN groups. The main technique used is the subgroup theorem for groups acting on trees with inversions.

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**1. Introduction.** Quasi-HNN groups have appeared in [3]. In this paper, we use the results of [8] to obtain the structure theorem for the subgroups of quasi-HNN groups and give some applications. We will use the terminology and notation of [3].

Let $G$ be a group, and let $I$, $J$ be two indexed sets. Let $\{A_i : i \in I\}$, $\{B_i : i \in I\}$, and $\{C_j : j \in J\}$ be families of subgroups of $G$.

For each $i \in I$, let $\phi_i : A_i \to B_i$ be an onto isomorphism and for each $j \in J$, let $\alpha_j : C_j \to C_j$ be an automorphism such that $\alpha_j^2$ is an inner automorphism determined by $c_j \in C_j$ and $c_j$ is fixed by $\alpha_j$. That is, $\alpha_j(c_j) = c_j$ and $\alpha_j^2(c) = c_j c c_j^{-1}$ for all $c \in C_j$.

The group $G^*$ with the presentation

$$G^* = \langle G, t_i, t_j \mid \operatorname{rel} G,\ t_i A_i t_i^{-1} = B_i,\ t_j C_j t_j^{-1} = C_j,\ t_j^2 = c_j,\ i \in I,\ j \in J\rangle \tag{1.1}$$

is called *a quasi-HNN group* with base $G$ and associated pairs $(A_i, B_i)$ and $(C_j, C_j)$, $i \in I$, $j \in J$ of subgroups of $G$, where $\langle G \mid \operatorname{rel} G\rangle$ stands for any presentation of $G$, $t_i A_i t_i^{-1} = B_i$ stands for the set of relations $t_i w(a) t_i^{-1} = w(\phi_i(a))$, and $t_j C_j t_j^{-1} = C_j$ stands for the set of relations $t_j w(c) t_j^{-1} = w(\alpha_j(c))$, where $w(a)$, $w(\phi_i(a))$, $w(c)$, and $w(\alpha_j(c))$ are words in the generating symbols of the presentation of $G$ of values $a$, $\phi_i(a)$, $c$, and $\alpha_j(c)$, respectively, where $a$ runs over the generators of $A_i$ and $c$ over the generators of $C_j$.

It is proved in [3] that $G$ is embedded in $G^*$ (the imbedding theorem of quasi-HNN groups), and every element $g$ of $G^*$, $g \neq 1$ can be written as a reduced word of $G^*$. That is, $g = g_o t_{k_1}^{e_1} g_1 t_{k_2}^{e_2} g_2 \cdots t_{k_n}^{e_n} g_n$, where $g_s \in G$, $e_s = \pm 1$, $k_s \in I \cup J$, for $s = 1, \dots, n$, such that $g$ contains no subword of the following forms:

(1) $t_{k_s} a t_{k_s}^{-1}$, $a \in A_i$, or
(2) $t_{k_s}^{-1} b t_{k_s}$, $b \in B_i$, or
(3) $t_{k_s}^{e} c t_{k_s}^{\delta}$, $c \in C_j$, $e, \delta = \pm 1$, or
(4) $t_{k_s}^{\pm 2}$, for some $k_s \in J$.

We note that if $J = \varnothing$, then $G^* = \langle G, t_i \mid \operatorname{rel} G, t_i A_i t_i^{-1} = B_i, i \in I\rangle$ is an HNN group with base $G$ and associated pairs $(A_i, B_i)$, $i \in I$ of subgroups of $G$.

If $I = \emptyset$, then $G^* = \langle G,\ t_j \mid \mathrm{rel}\,G,\ t_jC_jt_j^{-1} = C_j,\ t_j^2 = c_j,\ j \in J\rangle$. In this case we call $G^*$ a *pure quasi-HNN group* with base $G$ and associated pairs of subgroups $(C_j, C_j)$, $j \in J$ of $G$. The notation of a tree product of groups as defined in [2] will be needed.

In this paper, we form the subgroup theorem for the quasi-HNN group $G^*$, defined above, by applying the subgroup theorem for groups acting on trees with inversions obtained by [8]. In particular, if $H$ is a subgroup of the quasi-HNN group $G^*$ defined above, then $H$ is itself a quasi-HNN group (with possibly trivial free part); its base is a tree product $\pi$ with vertices of the form $xGx^{-1} \cap H$ and amalgamated subgroups either trivial or conjugate of the $A_i$ and $C_j$ intersected with $H$; moreover, every conjugate of $G$ intersected with $H$ is either trivial or is conjugate in $H$ to a vertex of $\pi$.

This paper is divided into seven sections. In Section 2, we introduce the concepts of graphs and the actions of groups on graphs. In Section 3, we have a summary of the structure of groups acting on trees with inversions. In Section 4, we have a summary of the structure of the subgroup theorem of the groups acting on trees with inversions. In Section 5, we associate a tree on which a quasi-HNN group acts and then we form the subgroup theorem for quasi-HNN groups. In Section 6, we apply the results of Section 5 to find the structures of the subgroups of HNN group. In Section 7, we apply the results of Section 5 to find the structures of the subgroups of pure quasi-HNN group.

**2. Basic concepts.** We begin by giving preliminary definitions. We denote by a graph $X$ a pair of disjoint sets $V(X)$ and $E(X)$ with $V(X)$ nonempty, together with a mapping $E(X) \to V(X)\times V(X)$, $y \to (o(y), t(y))$, and a mapping $E(X) \to E(X)$, $y \to \overline{y}$, satisfying the conditions that $\overline{\overline{y}} = y$, and $o(\overline{y}) = t(y)$, for all $y \in E(X)$. The case $\overline{y} = y$ is possible for some $y \in E(X)$. For $y \in E(X)$, $o(y)$, and $t(y)$ are called the ends of $y$, and $\overline{y}$ is called the inverse of $y$. If $A$ is a set of edges of $X$, we define $\overline{A}$ to be the set of inverses of the edges of $A$. That is, $\overline{A} = \{\overline{y} : y \in A\}$. For definitions of subgraphs, trees, morphisms of graphs, and $\mathrm{Aut}(X)$, the set of all automorphisms of the graph $X$ which is a group under the composition of morphisms of graphs, see [5] or [9]. We say that a group $G$ acts on a graph $X$, if there is a group homomorphism $\phi : G \to \mathrm{Aut}(X)$. If $x \in X$ (vertex or edge) and $g \in G$, we write $g(x)$ for $(\phi(g))(x)$. If $y \in E(X)$ and $g \in G$, then $g(o(y)) = o(g(y))$, $g(t(y)) = t(g(y))$, and $g(\overline{y}) = \overline{g(y)}$. The case $g(y) = \overline{y}$ for some $g \in G$ and some $y \in E(X)$ may occur. That is, $G$ acts with inversions on $X$.

We have the following notations related to the action of the group $G$ on the graph $X$:

(1) if $x \in X$ (vertex or edge), we define $G(x) = \{g(x) : g \in G\}$, and this set is called the orbit of $X$ containing $x$;

(2) if $x, y \in X$, we define $G(x,y) = \{g \in G : g(x) = y\}$ and $G(x,x) = G_x$, the stabilizer of $x$. Thus $G(x,y) \neq \emptyset$ if and only if $x$ and $y$ are in the same $G$ orbit. It is clear that if $v \in V(X)$, $y \in E(X)$, and $u \in \{o(y), t(y)\}$, then $G(v,y) = \emptyset$, $G_{\overline{y}} = G_y$, and $G_y \leq G_u$. If $H \leq G$ and $x \in X$, then it is clear that $H_x = H \cap G_x$;

(3) let $X^G$ be the set of elements of $X$ fixed by $G$. That is, $X^G = \{x \in X : G = G_x\}$.

**3. The structure of groups acting on trees with inversions.** In this section, we summarize the presentation for groups of groups acting on trees with inversions obtained by [5].

**Definition 3.1.** Let $G$ be a group acting on a graph $X$, and $T$ and $Y$ be two subtrees of $X$ such that $T \subseteq Y$ and $X^G = \varnothing$. Then $T$ is called a tree of representatives for the action of $G$ on $X$ if $T$ contains exactly one vertex from each $G$ vertex orbit, and $Y$ is called a fundamental domain for the action $G$ on $X$, if each edge of $Y$ has at least one end in $T$, and $Y$ contains exactly one edge (say) $y$ from each $G$ edge orbit such that $G(\overline{y}, y) = \varnothing$, and exactly one pair $x$ and $\overline{x}$ from each $G$-edge orbit such that $G(\overline{x}, x) \neq \varnothing$. It is clear that the properties of $T$ and $Y$ imply that if $u$ and $v$ are two vertices of $T$ such that $G(u,v) \neq \varnothing$, and if $x$ and $y$ are two edges of $Y$ such that $G(x,y) \neq \varnothing$, then $u = v$ and $x = y$ or $x = \overline{y}$.

Let $Y$ be as above. Define the following subsets $Y_0$, $Y_1$, $Y_2$ of edges of $Y$ as follows:

(1) $Y_0 = E(T)$, the set of edges of $T$;

(2) $Y_1 = \{y \in E(Y) : o(y) \in V(T),\ t(y) \notin V(T),\ G(\overline{y}, y) = \varnothing\}$;

(3) $Y_2 = \{x \in E(Y) : o(x) \in V(T),\ t(x) \notin V(T),\ G(\overline{x}, x) \neq \varnothing\}$.

It is clear that $\overline{Y}_0 = Y_0$, $E(Y) = Y_0 \cup Y_1 \cup Y_2 \cup \overline{Y}_1 \cup \overline{Y}_2$, and $G$ acts with inversions on $X$ if and only if $Y_2 \neq \varnothing$.

For the rest of this section $G$ will be a group acting on a graph $X$, $T$ be a tree of representatives for the action of the group $G$ on $X$, and $Y$ be a fundamental domain for the action of $G$ on $X$ such that $T \subseteq Y$ and $X^G = \varnothing$.

We have the following definitions.

**Definition 3.2.** For each vertex $v$ of $X$, define $v^*$ to be the unique vertex of $T$ such that $G(v, v^*) \neq \varnothing$. That is, $v$ and $v^*$ are in the same $G$ vertex orbit. It is clear that if $v$ is a vertex of $T$, then $v^* = v$ and in general, for any two vertices $u$ and $v$ of $X$ such that $G(u,v) \neq \varnothing$, we have $u^* = v^*$.

**Definition 3.3.** For each edge $y$ of $Y_0 \cup Y_1 \cup Y_2$ define $[y]$ to be an element of $G(t(y), (t(y))^*)$. That is, $[y]$ satisfies the condition $[y]((t(y))^*) = t(y)$, to be chosen as follows:

$$[y] = 1 \quad \text{if } y \in Y_0, \qquad [y](y) = \overline{y} \quad \text{if } y \in Y_2. \tag{3.1}$$

Define $[\overline{y}]$ to be the element

$$[\overline{y}] = \begin{cases} [y] & \text{if } y \in Y_0 \cup Y_2, \\ [y]^{-1} & \text{if } y \in Y_1, \end{cases} \tag{3.2}$$

$[y]$ is called *the value of the edge* $y$.

**Definition 3.4.** For each edge $y$ of $Y$ define $-y$ to be the edge $-y = [y]^{-1}(y)$ if $y \in Y_1$, otherwise $-y = y$. It is clear that $t(-y) = (t(y))^*$ and $G_{-y} \le G_{(t(y))^*}$.

**Definition 3.5.** By a reduced word $w$ of $G$ we mean an expression of the form $w = g_0 \cdot y_1 \cdot g_1 \cdot y_2 \cdot g_2 \cdot \dots \cdot y_n \cdot g_n$, $n \ge 0$, $y_i \in E(Y)$, for $i = 1, 2, \dots, n$ such that

(1) $g_0 \in G_{(o(y_1))^*}$;

(2) $g_i \in G_{(t(y_i))^*}$, for $i = 1, 2, \dots, n$;

(3) $(t(y_i))^* = (o(y_{i+1}))^*$, for $i = 1, 2, \dots, n-1$;

(4) $w$ contains no expression of the form $y_i \cdot g_i y_i^{-1}$ if $g_i \in G_{-y_i}$ and $G(y_i, \overline{y}_i) = \varnothing$, or $y_i \cdot g_i \cdot y_i$ if $g_i \in G_{y_i}$ and $G(y_i, \overline{y}_i) \neq \varnothing$.

If $w = g_0$, then $w$ is called a trivial word of $G$. If $(o(y_1))^* = (t(y_n))^*$ then $w$ is called a closed word of $G$.

The value of $w$ denoted $[w]$ is defined to be the element

$$[w] = g_0[y_1]g_1[y_2]g_2\cdots[y_n]g_n \quad \text{of } G. \tag{3.3}$$

Before we state the main result of this section, we have the following notations: let $v \in V(T)$, $m \in Y_0$, $y \in Y_1$, and $x \in Y_2$, then

(1) $\langle G_v \mid \operatorname{rel} G_v\rangle$ stands for any presentation of $G_v$;

(2) $G_m = G_m$ stands for the set of relations $w(g) = w'(g)$, where $w(g)$ and $w'(g)$ are words in the generating symbols of $G_{o(m)}$ and $G_{t(m)}$, respectively of value $g$, and $g$ is in the set of the generators of $G_m$;

(3) $y\cdot[y]^{-1}G_y[y]\cdot y^{-1} = G_y$ stands for the set of relations: $yw([y]^{-1}g[y])y^{-1} = w(g)$, where $w([y]^{-1}g[y])$ and $w(g)$ are words in the generating symbols of $G_{(t(y))^*}$ and $G_{(o(y))^*}$ of values $[y]^{-1}g[y]$ and $g$, respectively, where $g$ is in the set of the generators of $G_y$;

(4) $x\cdot G_x\cdot x^{-1} = G_x$ stands for the set of relations: $xw(g)x^{-1} = w'(g)$, where $w(g)$ and $w'(g)$ are words in the generating symbols of $G_{o(x)}$ of values $g$ and $[x]g[x]^{-1}$, respectively, where $g$ is in the set of the generators of $G_x$;

(5) $x^2 = [x]^2$ stands for the of relation $x^2 = w([x]^2)$ where $w([x]^2)$ is a word in the set of the generators of $G_x$.

**Theorem 3.6.** *Let $G$, $X$, $T$, $Y$, $Y_0$, $Y_1$, and $Y_2$ be as above. Then the following are equivalent:*

(i) *$X$ is tree;*

(ii) *$G$ is generated by the generators of $G_v$ and by the elements $[y]$ and $[x]$ and $G$ has the presentation*

$$\begin{aligned} G = \Big\langle & G_v, y, x \mid \operatorname{rel} G_v,\ G_m = G_m,\ y\cdot[y]^{-1}G_y[y]\cdot y^{-1} = G_y, \\ & x\cdot G_x\cdot x^{-1} = G_x,\ x^2 = [x]^2\Big\rangle \end{aligned} \tag{3.4}$$

*via the map $G_v \to G_v$, $y \to [y]$, and $x \to [x]$ where $v \in V(T)$, $m \in Y_0$, $y \in Y_1$, and $x \in Y_2$;*

(iii) *every element of $G$ is the value of a closed and reduced word of $G$. Moreover, if $w$ is a nontrivial closed and reduced word of $G$, then $[w]$ is not the identity element of $G$.*

**Proof.** (i) implies (ii) follows from [5, Theorem 5.1].
(ii) implies (iii) follows from [6, Corollary 1].
(iii) implies (i) follows from [7, Corollary 3.6]. □

**4. Structure theorem for subgroups of groups acting on trees with inversions.** In this section, we summarize the methods for obtaining generators and presentation for subgroups of groups acting on trees with inversions obtained by [8, Theorem 3, page 28]. In [4, Theorem 2.5, page 91] generators and presentation for a subgroup $B$ of a given group $A$ with given generators and presentation is obtained by a method called the Reidemeister-Schreier methods. In view of such methods we have the following remarks related to subgroups of groups acting on trees with inversions: let $G$ be a group acting on a tree $X$, $v$ be a vertex of $X$, $z$ be an edge of $X$ and, $H$ be a subgroup of $G$, then

(1) the generators and presentation for the vertex stabilizer $G_v$ of $v$ under $G$ are arbitrary;
(2) the generators for the edge stabilizer $G_z$ of $z$ under $G$ are arbitrary;
(3) the generators and presentation for $G$ are those of Theorem 3.6(ii);
(4) the generators and presentation for the vertex stabilizer $H_v = H \cap G_v$ of $v$ under $H$ are those obtained by the Reidemeister-Schreier methods from the generators and presentation for $G_v$;
(5) the generators for the edge stabilizer $H_z = H \cap G_z$ of $z$ under $H$ are those obtained by the Reidemeister-Schreier methods from the generators of $G_z$;
(6) the generators and presentation for $H$ are given in Theorem 4.4.

Now we proceed to obtain the generators and presentation for subgroups of groups acting on trees with inversions as follows.

Let $G$, $X$, $T$, $Y$, $Y_0$, $Y_1$, and $Y_2$ be as above such that $X$ is a tree, and $H$ be a subgroup of $G$. We have the following definitions.

**Definition 4.1.** (1) For each $g \in G$ and $y \in Y_0 \cup Y_1 \cup Y_2$, define $D_y^g$ to be any double coset representative system for $G_{o(y)} \bmod (G_{o(y)} \cap g^{-1}Hg, G_y)$ containing 1 but otherwise arbitrary.

(2) For each $v \in V(T)$, let $D_v$ be a double coset representative system for $G \bmod (H, G_v)$ satisfying the condition that if $g \in D_v$ and

$$g = g_0[y_1]g_1[y_2]g_2 \cdots [y_n]g_n, \tag{4.1}$$

where $g_0 \cdot y_1 \cdot g_1 \cdot y_2 \cdot g_2 \cdot \cdots \cdot y_n \cdot g_n$ is a closed and reduced word of $G$, then $g_0[y_1]g_1[y_2]g_2 \cdots [y_i] \in D_{(o(y_{i+1}))^*}$, and $g_i \in D_{y_i}^{f_i}$, where $f_i = g_0[y_1]g_1[y_2]g_2 \cdots [y_i]$. For more details see [8, page 23].

(3) For each $y \in Y_0 \cup Y_1 \cup Y_2$, define $D^y$ to be the set $D^y = \{de : d \in D_{o(y)},\ e \in D_y^d,\ de[y] \notin D_{(t(y))^*}\}$, and define $D_y$ to be the set $D_y = \{de : d \in D_{0(y)},\ e \in D_y^d,\ de[y] \in D_{(t(y))^*}\}$.

**Definition 4.2.** The collection of double coset representative $\{D_v\}$, and the associated collections $\{D^y\}$, and $\{D_y\}$ defined above will be called a cress for $G \bmod H$. For the existence of a cress for $G \bmod H$ we refer the readers to [8, page 23].

**Proposition 4.3.** *For each $y \in Y_0 \cup Y_1 \cup Y_2$ and $g \in D^y$ there exist unique elements $\overline{g[y]} \in D_{(t(y))^*}$, $\overline{\overline{g[y]}} \in D_y^{\overline{g[y]}}$, and $a_y \in [y]^{-1}G_y[y]$ such that $g[y](\overline{g[y]}\,\overline{\overline{g[y]}}a_y)^{-1} \in H$.*

**Proof.** See [8, page 28]. □

Note that $H \cap \overline{g[y]}\,\overline{\overline{g[y]}}G_y\overline{\overline{g[y]}}^{-1}\,\overline{g[y]}^{-1}$ is a subgroup of $H \cap gG_{o(y)}g^{-1}$.

Before we proceed to state the main result of this section, we introduce the following convention on notation on the generators and defining relations of $H$:

(A) For any vertex $v$ of $T$, $\langle H \cap dG_vd^{-1} \mid \mathrm{rel}(H \cap dG_vd^{-1})\rangle$ is the presentation of the subgroup $H \cap dG_vd^{-1}$ of $G_v$ obtained by the Reidemeister-Schreier methods.

(B) For any $y \in Y_0 \cup Y_1 \cup Y_2$, the relations

$$g(y) \cdot \left(H \cap \overline{g[y]}\,\overline{\overline{g[y]}}G_y\overline{\overline{g[y]}}^{-1}\,\overline{g[y]}^{-1}\right) \cdot (g(y))^{-1} = H \cap gG_{\overline{y}}g^{-1}, \tag{4.2}$$

where $g \in D^y$ are similar to the relations of $G$ as in Theorem 3.6(ii).

The main result of this section is the following theorem. For the proof see [8, Theorem 3].

**Theorem 4.4.** *Let $G$ be a group acting on a tree $X$, and let $T$ be a tree of representatives for the action of the group $G$ on $X$ and $Y$ be a fundamental domain for the action of $G$ on $X$ such that $T \subseteq Y$ and $X^G = \varnothing$. Let $H$ be a subgroup of $G$, and $\{D_u\}$, $\{D^z\}$, and $\{D_z\}$ be a cress for $G \bmod H$, for all $u \in V(T)$, and all $z \in Y_0 \cup Y_1 \cup Y_2$. Then we have the following:*

(I) *$H$ is generated by the following:*

(1) *the generators of the subgroups $H \cap dG_vd^{-1}$ of $G_v$, where $v \in V(T)$ and $d \in D_v$;*

(2) *the elements $g(\overline{g}\,\overline{\overline{g}}a_m)^{-1}$, where $m \in Y_0$ and $g \in D^m$;*

(3) *the elements $g[y](\overline{g[y]}\,\overline{\overline{g[y]}}a_y)^{-1}$, where $y \in Y_1$ and $g \in D^y$;*

(4) *the elements $g[x](\overline{g[x]}\,\overline{\overline{g[x]}}a_x)^{-1}$, where $x \in Y_2$ and $g \in D^x$ such that $H \cap [x]G_xg^{-1} = \varnothing$;*

(5) *the elements $g[x]g^{-1}$, where $x \in Y_2$ and $g \in D^x$ such that $H \cap g[x]G_xg^{-1} \neq \varnothing$.*

(II) *$H$ has the presentation $\langle P \mid R\rangle$, where $P$ is the set of generating symbols of the following forms:*

(1) *the generating symbols of the subgroups $H \cap dG_vd^{-1}$ of $G_v$, where $d \in D_v$;*

(2) *the edges $g(m)$, where $m \in Y_0$ and $g \in D^m$;*

(3) *the edges $g(y)$, where $y \in Y_1$ and $g \in D^y$;*

(4) *the edges $g(x)$, where $x \in Y_2$ and $g \in D^x$ such that $H \cap g[x]G_xg^{-1} \neq \varnothing$.*

(III) *$R$ is the set of relations of the following forms:*

(1) *the relations,* $\mathrm{rel}(H \cap dG_vd^{-1})$ *of the subgroups $H \cap dG_vd^{-1}$ of $G_v$, where $d \in D_v$;*

(2) *the relations $H \cap gG_mg^{-1} = H \cap gG_mg^{-1}$, where $m \in Y_0$ and $g \in D_m$;*

(3) *the relations $H \cap gG_yg^{-1} = H \cap gG_yg^{-1}$, where $y \in Y_0$ and $g \in D_y$;*

(4) *the relations $H \cap gG_xg^{-1} = H \cap gG_xg^{-1}$, where $x \in Y_2$ and $g \in D_x$ such that $H \cap g[x]G_xg^{-1} \neq \varnothing$;*

(5) *the relations $g(m) \cdot (H \cap \overline{g}\,\overline{\overline{g}}G_m\overline{\overline{g}}^{-1}\overline{g}^{-1}) \cdot (g(m))^{-1} = H \cap gG_mg^{-1}$, where $m \in Y_0$ and $g \in D^m$;*

(6) *the relations $g(y) \cdot (H \cap \overline{g[y]}\,\overline{\overline{g[y]}}G_y\overline{\overline{g[y]}}^{-1}\,\overline{g[y]}^{-1}) \cdot (de(y))^{-1} = H \cap gG_yg^{-1}$, where $y \in Y_1$ and $g \in D^y$;*

(7) *the relations $g(x) \cdot (H \cap \overline{g[x]}\,\overline{\overline{g[x]}}G_x\overline{\overline{g[x]}}^{-1}\,\overline{g[x]}^{-1}) \cdot (g(x))^{-1} = H \cap gG_xg^{-1}$, where $x \in Y_2$ and $g \in D^x$ such that $H \cap g[x]G_xg^{-1} \neq \varnothing$;*

(8) *the relations $g(x) \cdot (H \cap gG_xg^{-1}) \cdot (g(x))^{-1} = H \cap gG_xg^{-1}$, where $x \in Y_2$ and $g \in D^x$ such that $H \cap g[x]G_xg^{-1} \neq \varnothing$;*

(9) *the relations $(g(x))^2 = g[x]^2g^{-1}$, where $x \in Y_2$ and $g \in D^x$ such that $H \cap g[x]G_xg^{-1} \neq \varnothing$.*

**Definition 4.5.** A group is called *quasi free group* if it is a free product of a free group and a number of cyclic groups of order 2.

We have the following corollaries of Theorem 4.4.

**Corollary 4.6.** *If $H \cap G_v = \{1\}$ for all $v \in V(X)$, then $H$ is a quasi free group.*

**Corollary 4.7.** *If $H \cap G_y = \{1\}$ for all $y \in E(X)$, then $H$ is a free product of a quasi-HNN group and the intersections of $H$ with $G_v$ for all $v \in V(T)$.*

**Corollary 4.8.** *If $H$ is a nontrivial free product, then $H$ is either infinite cyclic group, a finite cyclic group of order 2, or $H$ is contained in $G_v$ for some $v \in V(X)$.*

**Corollary 4.9.** *If $H$ is a normal subgroup of $G$ such that $H \cap G_v$ is quasi free group for all $v \in V(X)$, and $H \cap G_y = \{1\}$ for all $y \in E(X)$, then $H$ is a quasi free group.*

**Corollary 4.10.** *If $H$ has the property that $D^z = \emptyset$, for all $z \in Y_0 \cup Y_1 \cup Y_2$, then $H$ is a tree product of the subgroups $H \cap dG_v d^{-1}$ of $G_v$, where $v \in V(T)$ and $d \in D_v$ with amalgamation subgroups $H \cap gG_m g^{-1}$, $m \in Y_0$, $g \in D_m$, $H \cap gG_y g^{-1}$, $y \in Y_1$, $g \in D_y$, and $H \cap gG_x g^{-1}$, $x \in Y_2$, $g \in D_x$ such that $H \cap g[x]G_x g^{-1} \neq \emptyset$.*

**5. Subgroups of quasi-HNN groups.** In this section, we construct a tree on which a quasi-HNN group acts with inversions, and then we formulate its subgroups.

The following lemma is essential for the proof of the main theorem of the paper.

**Lemma 5.1.** *A group is a quasi-HNN group if and only if there is a tree on which the group acts with inversion and is transitive on the set of vertices. Moreover, the stabilizer of any vertex is conjugate to the base and the stabilizer of any edge is conjugate to an associate subgroup of the base.*

**Proof.** Let $G$ be a group acting with inversions on a tree $X$ such that $G$ is transitive on $V(X)$. Let $T$ be a tree of representatives, and $Y$ be a fundamental domain for the action of $G$ on $X$ such that $T \subseteq Y$. Since $G$ is transitive on $V(X)$, therefore $T$ consists of exactly one vertex $v$ (say) without edges. So $Y_0 = \emptyset$. Since $G$ acts with inversions on $X$, therefore $Y_2 \neq \emptyset$. Then by Theorem 3.6, $G$ has the presentation

$$G = \langle G_v, y, x \mid \operatorname{rel} G_v,\ y \cdot [y]^{-1} G_y [y] \cdot y^{-1} = G_y,\ x \cdot G_x \cdot x^{-1} = G_x,\ x^2 = [x]^2 \rangle \quad (5.1)$$

via the map $G_v \to G_v$, $y \to [y]$, and $x \to [x]$ where $v \in V(T)$, $y \in Y_1$, and $x \in Y_2$. Then $G$ is a quasi-HNN group of base $G_v$, and associated isomorphic pairs $([y]^{-1}G_y[y], G_y)$ of subgroups of $G_v$ via the isomorphism $\phi_y : [y]^{-1}G_y[y] \to G_y$ defined by $\phi_y([y]^{-1}g[y]) = g$, for all $g \in G_y$ and all $y \in Y_1$ such that $o(y) = v$, and associated isomorphic pairs $(G_x, G_x)$ of subgroups of $G_v$ via the mapping $\alpha_x : G_x \to G_x$ defined by $\alpha_x(g) = [x]g[x]^{-1}$, for all $g \in G_x$ and all $x \in Y_2$ such that $o(x) = v$. Let $g_x = [x]^2$. It is clear that $\alpha_x$ is an automorphism of $G_x$, and $\alpha_x^2$ is inner automorphism of $G_x$ determined by $g_x$, and $\alpha_x([x]^2) = [x]^2$.

Conversely, let $G^*$ be the quasi-HNN group

$$G^* = \langle G, t_i, t_j \mid \operatorname{rel} G,\ t_i A_i t_i^{-1} = B_i,\ t_j C_j t_j^2 = C_j,\ t_j^2 = c_j,\ i \in I,\ j \in J \rangle \quad (5.2)$$

of base $G$, and associated pairs $(A_i, B_i)$, and $(C_j, C_j)$, $i \in I$, $j \in J$ of subgroups of $G$.

Now, we construct a tree $X$ on which $G^*$ acts on $X$ with inversions such that $G^*$ is transitive on $V(X)$, and call it the standard tree associated with the quasi-HNN group. In this construction, define

$$V(X) = \{gG : g \in G^*\}, \tag{5.3}$$

and define

$$\begin{aligned} E(X) = \{(gB_i, t_i) : g \in G^*,\ i \in I\} \cup \{(gA_i, t_i^{-1}) : g \in G^*,\ i \in I\} \\ \cup \{(gC_j, t_j) : g \in G^*,\ j \in J\}. \end{aligned} \tag{5.4}$$

The terminals of the edges are defined as follows:

$$\begin{gathered} o(gB_i, t_i) = o(gA_i, t_i^{-1}) = o(gC_j, t_j) = gG, \\ t(gB_i, t_i) = gt_iG, \qquad t(gA_i, t_i^{-1}) = gt_i^{-1}G, \qquad t(gC_j, t_j) = gt_jG. \end{gathered} \tag{5.5}$$

The inverses of the edges are defined as follows:

$$\overline{(gB_i, t_i)} = (gt_iA_i, t_i^{-1}), \qquad \overline{(gA_i, t_i^{-1})} = (gt_i^{-1}B_i, t_i), \qquad \overline{(gC_j, t_j)} = (gt_jC_j, t_j). \tag{5.6}$$

From above $X$ is a graph. $G^*$ acts on $X$ as follows: let $g' \in G^*$. Then for any vertex $gG$ of $X$ we have $g'(gG) = g'gG$, and for any edge $(gB_i, t_i)$, or $(gA_i, t_i^{-1})$, or $(gC_j, t_j)$ of $X$, we have

$$g'(gB_i, t_i) = (g'gB_i, t_i), \qquad g'(gA_i, t_i^{-1}) = (g'gA_i, t_i^{-1}), \qquad g'(gC_j, t_j) = (g'gC_j, t_j). \tag{5.7}$$

The action of $G^*$ on the vertices of $X$ is transitive because for any two vertices $aG$ and $bG$ of $X$ we have $ba^{-1}(aG) = bG$. That is, the element $ba^{-1}$ of $G^*$ maps the vertex $aG$ to the vertex $bG$. Then there is exactly one $G$ vertex orbit. We take $T = \{G\}$ to be the tree of representatives for the action of $G^*$ on $X$. $G^*$ acts on $X$ with inversions because the element $t_j \in G^*$ maps the edge $(C_j, t_j)$ to its inverse $\overline{(C_j, t_j)}$. That is,

$$t_j(C_j, t_j) = (t_jC_j, t_j) = \overline{(C_j, t_j)}. \tag{5.8}$$

Let $\lambda_i$, $\beta_i$, and $\gamma_j$ stand for the edges $(B_i, t_i)$, $(A_i, t_i^{-1})$, and $(C_j, t_j)$, respectively. Then

$$\begin{gathered} o(\lambda_i) = o(\beta_i) = o(\gamma_j) = G, \qquad t(\lambda_i) = t_iG, \qquad t(\beta_i) = t_i^{-1}G, \qquad t(\gamma_j) = t_jG, \\ \overline{\lambda}_i = (t_iA_i, t_i^{-1}), \qquad \overline{\beta}_i = (t_i^{-1}B_i, t_i), \qquad \overline{\gamma}_j = (t_jC_j, t_j). \end{gathered} \tag{5.9}$$

From above, we see that $\lambda_i$ and $\overline{\lambda}_i$, and $\beta_i$ and $\overline{\beta}_i$ are in different $G^*$ edge orbits, while $\gamma_j$ and $\overline{\gamma}_j$ are in the same $G^*$ edge orbit. Moreover, any edge of $X$ is of the form $g(\lambda_i)$, $g(\overline{\lambda}_i)$, or $g(\gamma_j)$ for some $g \in G^*$.

Let $Y$ be the subgraph of $X$ consisting of the edges $\lambda_i$, $\beta_i$, and $\gamma_j$, and their terminals and inverses for all $i \in I$ and all $j \in J$. Thus, $V(Y) = \{G, t_iG, t_i^{-1}G, t_jG : i \in I,\ j \in J\}$ and $E(Y) = \{\lambda_i, \beta_i, \gamma_j, \overline{\lambda}_i, \overline{\beta}_i, \overline{\gamma}_j : i \in I,\ j \in J\}$. It is clear that $T$ is a tree of representatives for the action $G^*$ on $X$, and $Y$ is a fundamental domain for the action of $G^*$ on $X$ and $T \subseteq Y$. Moreover, $Y_0 = \emptyset$, $Y_1 = \{\lambda_i : i \in I\}$, and $Y_2 = \{\gamma_j : i \in J\}$.

From above, it is easy to verify that the stabilizer of the vertex $v$, where $v = G$ is $G_v^* = G$ and the stabilizers of the edges $\lambda_i$, $\beta_i$, and $\gamma_j$ are $G^*_{\lambda_i} = B_i$, $G^*_{\beta_i} = A_i$, and $G^*_{\gamma_j} = C_j$, respectively. This implies that the stabilizer of any vertex is conjugate to the base $G$ and the stabilizer of any edge is conjugate to an associate subgroup $A_i$, $B_i$, or $C_j$ of the base $G$. The values of the edges $\lambda_i$, $\beta_i$, and $\gamma_j$ are $[\lambda_i] = t_i$, $[\beta_i] = t_i^{-1}$, and $[\gamma_j] = t_j$, respectively. By Theorem 3.6, the presentation of $G^*$ and the action of $G^*$ on $X$ implies that $X$ is a tree.

This completes the proof. $\square$

**Remark 5.2.** The tree $X$, constructed above, will be called the standard tree of the quasi-HNN group $G^*$.

In view of Lemma 5.1 and Definition 4.2 the following concepts is clear.

Let $H$ be a subgroup of the quasi-HNN group

$$G^* = \langle G, t_i, t_j \mid \operatorname{rel} G,\ t_iA_it_i^{-1} = B_i,\ t_jC_jt_j^{-1} = C_j,\ t_j^2 = c_j,\ i \in I,\ j \in J\rangle \tag{5.10}$$

of base $G$ and associated pairs $(A_i, B_i)$, and $(C_j, C_j)$, $i \in I$, $j \in J$ of subgroups of $G$.

We have the following concepts:

(1) for each $i \in I$, $j \in J$, and $d \in G$, let $D_i^d$ and $D_j^d$ be double coset representative system for $G \operatorname{mod}(G \cap d^{-1}Hd, B_i)$, and $G \operatorname{mod}(G \cap d^{-1}Hd, C_j)$, respectively containing 1, but otherwise arbitrary;

(2) let $D$ be a double coset representative system for $G^* \operatorname{mod}(H, D)$ satisfying the condition that if $g = g_o t_{k_1}^{e_1} g_1 t_{k_2}^{e_2} g_2 \cdots t_{k_n}^{e_n} g_n \in D$, then $w_s \in D$ and $g_s \in D_{k_s}^{w_s}$, where

$$w_s = g_o t_{k_1}^{e_1} g_1 t_{k_2}^{e_2} g_2 \cdots t_{k_s}^{e_s} \quad \text{for } s = 1, \dots, n; \tag{5.11}$$

(3) for each $s \in I \cup J$, let $D^s = \{de : d \in D,\ e \in D_s^d,\ \mathrm{det}_s \notin D\}$, and $D_s = \{de : d \in D,\ e \in D_s^d,\ \mathrm{det}_s \in D\}$;

(4) for each $s \in I \cup J$, and $g \in D^s$, let $\overline{gt_s} \in D$ be the representative of $gt_s$ in $D$, and $\overline{\overline{gt_s}} \in D_s^{\overline{gt_s}}$ be the representative of $gt_s$ in $D_s^{\overline{gt_s}}$;

(5) for each $i \in I$, and $g \in D^i$, let $k_i$ be an element of $A_i$ such that $gt_i(\overline{gt_i}\,\overline{\overline{gt_i}}k_i)^{-1} \in H$;

(6) for each $j \in J$, and $g \in D^j$, let $k_j$ be an element of $C_j$ such that $gt_j(\overline{gt_j}\,\overline{\overline{gt_j}}k_j)^{-1} \in H$.

**Definition 5.3.** The collection of double coset representative $\{D\}$, and the associated collections $\{D_s^d\}$, $\{D^s\}$, and $\{D_s\}$ defined above will be called a cress for $G^* \operatorname{mod} H$.

The main result of this section is the following theorem.

**Theorem 5.4.** *Let $H$ be a subgroup of the quasi-HNN group*

$$G^* = \langle G, t_i, t_j \mid \operatorname{rel} G,\ t_i A_i t_i^{-1} = B_i,\ t_j C_j t_j^{-1} = C_j,\ t_j^2 = c_j,\ i \in I,\ j \in J \rangle \tag{5.12}$$

*with base $G$ and associated pairs $(A_i, B_i)$, and $(C_j, C_j)$, $i \in I$, $j \in J$ of subgroups of $G$.*

*Let $\{D\}$, $\{D_s^d\}$, $\{D^s\}$, and $\{D_s\}$, $s \in I \cup J$, be a cress for $G^* \bmod H$.*

*Then, we have the following:*

(I) *$H$ is generated by the following:*

(1) *the generators of the subgroup $H \cap dGd^{-1}$ of $G$, for all $d \in D$;*

(2) *the elements $gt_i(\overline{gt_i}\,\overline{\overline{gt_i}}k_i)^{-1}$, where $i \in I$ and $g \in D^i$;*

(3) *the elements $gt_j(\overline{gt_j}\,\overline{\overline{gt_j}}k_j)^{-1}$, where $j \in J$, $g \in D^j$, and $H \cap gt_jC_jg^{-1} = \varnothing$;*

(4) *the elements $gt_jg^{-1}$, where $j \in J$, $g \in D^j$, and $H \cap gt_jC_jg^{-1} \neq \varnothing$.*

(II) *$H$ has the presentation $\langle P \mid R \rangle$, where $P$ is the set of generating symbols of the following forms:*

(1) *the generating symbols of the subgroup $H \cap dGd^{-1}$ of $G$, for all $d \in D$;*

(2) *the symbols $g(\lambda_i)$, where $i \in I$ and $g \in D^i$;*

(3) *the symbols $g(\lambda_j)$, where $j \in J$ and $g \in D^j$.*

(III) *$R$ is the set of relations of the following forms:*

(1) *the relations $\operatorname{rel}(H \cap dGd^{-1})$ of the subgroup $H \cap dGd^{-1}$ of $G$, for all $d \in D$;*

(2) *the relations $H \cap gA_ig^{-1} = H \cap gB_ig^{-1}$, where $i \in I$ and $g \in D_i$;*

(3) *the relations $H \cap gC_jg^{-1} = H \cap gC_jg^{-1}$, where $j \in J$ and $g \in D_j$;*

(4) *the relations $g(\lambda_i) \cdot (H \cap \overline{gt_i}\,\overline{\overline{gt_i}}A_i\overline{\overline{gt_i}}^{-1}\overline{gt_i}^{-1}) \cdot (g(\lambda_i))^{-1} = H \cap gB_ig^{-1}$, where $i \in I$ and $g \in D^i$;*

(5) *the relations $g(\gamma_j) \cdot (H \cap \overline{gt_j}\,\overline{\overline{gt_j}}C_j\overline{\overline{gt_j}}^{-1}\overline{gt_j}^{-1}) \cdot (g(\gamma_j))^{-1} = H \cap g(C_j)g^{-1}$, where $j \in J$ and $g \in D^j$ such that $H \cap gt_jC_jg^{-1} = \varnothing$;*

(6) *the relations $g(\gamma_j) \cdot (H \cap gC_jg^{-1}) \cdot (g(\gamma_j))^{-1} = H \cap g(C_j)g^{-1}$, where $j \in J$ and $g \in D^j$ such that $H \cap gt_jC_jg^{-1} \neq \varnothing$;*

(7) *the relations $(g(\gamma_j))^2 = gc_jg^{-1}$, where $j \in J$ and $g \in D^j$ such that $H \cap gt_jC_jg^{-1} \neq \varnothing$.*

**Proof.** Let $X$ be the standard tree constructed in Lemma 5.1 on which $G^*$ acts. Then $T = \{G\}$, $Y_0 = \varnothing$, $Y_1 = \{\lambda_i : i \in I\}$, and $Y_2 = \{\gamma_j : i \in J\}$. Let $v = G$. Then $D = D_v$, $D^i = D^{\lambda_i}$, $D^j = D^{\gamma_j}$, $D_i = D_{\lambda_i}$, and $D_j = D_{\gamma_j}$. Since $Y_0 = \varnothing$, therefore $X$ does not contain the edges $m$ of Theorem 4.4. Therefore $H$ is generated by the generators of the forms (I)(1), (3), (4), and (5) of Theorem 4.4. Similarly $H$ has the presentation of generating symbols (II)(1), (3), and (4), and relations (II)(1), (6), (7), (8), and (9) of Theorem 4.4.

This completes the proof. □

We have the following corollaries of Theorem 5.4.

**Corollary 5.5.** *Any subgroup $H$ of $G^*$ having trivial intersection with each conjugate of the base $G$ is a quasi free.*

**Corollary 5.6.** *Any subgroup $H$ of $G^*$ having trivial intersection with the conjugates of $B_i$ and $C_j$ is the free product of a quasi free group and the intersection of $H$ with certain conjugates of $G$.*

**Corollary 5.7.** *If $H$ is a normal subgroup of $G^*$, $H \cap gGg^{-1}$ is a quasi free group, $H \cap gA_ig^{-1} = \{1\}$, and $H \cap gC_jg^{-1} = \{1\}$ for all $i \in I$, $j \in J$, and $g \in G^*$, then $H$ is quasi free.*

**Corollary 5.8.** *If $H$ has the property that $D^s = \varnothing$, for all $s \in I \cup J$, then $H$ is a tree product of the subgroups $H \cap dGd^{-1}$ of $G$, for all $d \in D$ with amalgamation subgroups $H \cap gA_ig^{-1}$, $H \cap gB_ig^{-1}$, where $i \in I$, $g \in D_i$, and $H \cap gC_jg^{-1}$, where $j \in J$ and $g \in D_j$.*

**6. Subgroups of HNN groups.** This section is an application of Theorem 5.4. First we start by finding the structures of subgroups of HNN groups. For different methods of finding the structures of subgroups of HNN groups we refer the readers to [1].

By taking $J = \varnothing$ of Theorem 5.4, we have the following theorem and corollaries.

**Theorem 6.1.** *Let $H$ be a subgroup of the HNN group,*

$$G^* = \langle G, t_i \mid \operatorname{rel} G,\ t_iA_it_i^{-1} = B_i,\ i \in I \rangle, \tag{6.1}$$

*with base $G$ and associated pairs $(A_i, B_i)$, $i \in I$ subgroups of $G$.*

*Let $\{D\}$, $\{D_i^d\}$, $\{D^i\}$, and $\{D_i\}$, $i \in I$, be a cress for $G^* \bmod H$.*

*Then, we have the following:*

(I) *$H$ is generated by the following:*

(1) *the generators of the subgroups $H \cap dGd^{-1}$ of $G$, for all $d \in D$;*

(2) *the elements $gt_i(\overline{gt_i}\,\overline{\overline{gt_i}}k_i)^{-1}$, where $i \in I$ and $g \in D^i$.*

(II) *$H$ has the presentation $\langle P \mid R \rangle$, where $P$ is the set of generating symbols of the following forms:*

(1) *the generating symbols of the subgroups $H \cap dGd^{-1}$ of $G$, for all $d \in D$;*

(2) *the symbols $g(\lambda_i)$, where $i \in I$ and $g \in D^i$.*

(III) *$R$ is the set of relations of the following forms:*

(1) *the relations $\operatorname{rel}(H \cap dGd^{-1})$ of the subgroups $H \cap dGd^{-1}$ of $G$, for all $d \in D$;*

(2) *the relations $H \cap gA_ig^{-1} = H \cap gB_ig^{-1}$, where $i \in I$ and $g \in D_i$;*

(3) *the relations $g(\lambda_i) \cdot (H \cap \overline{gt_i}\,\overline{\overline{gt_i}}A_i\overline{\overline{gt_i}}^{-1}\,\overline{gt_i}^{-1}) \cdot (g(\lambda_i))^{-1} = H \cap gB_ig^{-1}$, where $i \in I$ and $g \in D^i$.*

**Corollary 6.2.** *Any subgroup of $G^*$ having trivial intersection with each conjugate of the base is free.*

**Corollary 6.3.** *Any subgroup $H$ of $G^*$ having trivial intersection with the conjugates of $B_i$ is the free product of a free group and the intersection of $H$ with certain conjugates of $G$.*

**Corollary 6.4.** *If $H$ is a normal subgroup of $G^*$, $H \cap gGg^{-1}$ is a free group, and $H \cap gA_ig^{-1} = \{1\}$, for all $i \in I$ and $g \in G^*$, then $H$ is free.*

**Corollary 6.5.** *If $H$ has the property that $D^i = \varnothing$, for all $i \in I$, then $H$ is a tree product of the subgroups $H \cap dGd^{-1}$ of $G$, for all $d \in D$ with amalgamation subgroups $H \cap gA_ig^{-1}$ and $H \cap gB_ig^{-1}$, where $i \in I$ and $g \in D_i$.*

**7. Subgroups of pure quasi-HNN groups.** This section is an application of Theorem 5.4. First, we start by finding the structures of subgroups of the pure quasi-HNN groups.

By taking $I = \varnothing$ in Theorem 5.4, we have the following theorem and corollaries.

**Theorem 7.1.** *Let $H$ be a subgroup of the pure quasi-HNN group,*

$$G^* = \langle G, t_j \mid \operatorname{rel} G,\ t_j C_j t_j^{-1} = C_j,\ t_j^2 = c_j,\ j \in J\rangle, \tag{7.1}$$

*with base $G$ and associated pairs $(C_j, C_j)$, $j \in J$ of subgroups of $G$.*

*Let $\{D\}$, $\{D_j^d\}$, $\{D^j\}$, and $\{D_j\}$, $j \in J$, be a cress for $G^* \bmod H$.*

*Then, we have the following:*

(I) *$H$ is generated by the following:*

(1) *the generators of the subgroups $H \cap dGd^{-1}$ of $G$, for all $d \in D$;*

(2) *the elements $gt_j(\overline{gt_j}\,\overline{\overline{gt_j}}k_j)^{-1}$ where $j \in J$, $g \in D^j$, and $H \cap gt_jC_jg^{-1} = \varnothing$;*

(3) *the elements $gt_jg^{-1}$, where $j \in J$, $g \in D^j$, and $H \cap gt_jC_jg^{-1} \neq \varnothing$.*

(II) *$H$ has the presentation $\langle P \mid R\rangle$, where $P$ is the set of generating symbols of the following forms:*

(1) *the generating symbols of the subgroups $H \cap dGd^{-1}$ of $G$, for all $d \in D$;*

(2) *the symbols $g(\lambda_j)$, where $j \in J$ and $g \in D^j$.*

(III) *$R$ is the set of relations of the following forms:*

(1) *the relations $\operatorname{rel}(H \cap dGd^{-1})$ of the subgroups $H \cap dGd^{-1}$ of $G$, for all $d \in D$;*

(2) *the relations $H \cap gC_jg^{-1} = H \cap gC_jg^{-1}$, where $j \in J$ and $g \in D_j$;*

(3) *the relations $g(\gamma_j) \cdot (H \cap \overline{gt_j}\,\overline{\overline{gt_j}}C_j\overline{\overline{gt_j}}^{-1}\overline{gt_j}^{-1}) \cdot (g(\gamma_j))^{-1} = H \cap gC_jg^{-1}$, where $j \in J$ and $g \in D^j$ such that $H \cap gt_jC_jg^{-1} = \varnothing$;*

(4) *the relations $g(\gamma_j) \cdot (H \cap gC_jg^{-1}) \cdot (g(\gamma_j))^{-1} = H \cap gC_jg^{-1}$, where $j \in J$ and $g \in D^j$ such that $H \cap gt_jC_jg^{-1} \neq \varnothing$;*

(5) *the relations $(g(\gamma_j))^2 = gc_jg^{-1}$, where $j \in J$ and $g \in D^j$ such that $H \cap gt_jC_jg^{-1} \neq \varnothing$.*

We have the following corollaries of Theorem 7.1.

**Corollary 7.2.** *Any subgroup of $G^*$ having trivial intersection with each conjugate of the base $G$ is quasi free.*

**Corollary 7.3.** *Any subgroup $H$ of $G^*$ having trivial intersection with the conjugates of $C_j$ is the free product of a quasi free group and the intersection of $H$ with certain conjugates of $G$.*

**Corollary 7.4.** *If $H$ is a normal subgroup of $G^*$, $H \cap gGg^{-1}$ is a quasi free group, and $H \cap gC_jg^{-1} = \{1\}$ for all $j \in J$ and all $g \in G^*$, then $H$ is quasi free.*

**Corollary 7.5.** *If $H$ has the property that $D^j = \varnothing$, for all $j \in J$, then $H$ is a tree product of the subgroups $H \cap dGd^{-1}$ of $G$, for all $d \in D$ with amalgamation subgroups $H \cap gC_jg^{-1}$ where $j \in J$ and $g \in D_j$.*

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# Special Issue on Modeling Experimental Nonlinear Dynamics and Chaotic Scenarios

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Thinking about nonlinearity in engineering areas, up to the 70s, was focused on intentionally built nonlinear parts in order to improve the operational characteristics of a device or system. Keying, saturation, hysteretic phenomena, and dead zones were added to existing devices increasing their behavior diversity and precision. In this context, an intrinsic nonlinearity was treated just as a linear approximation, around equilibrium points.

Inspired on the rediscovering of the richness of nonlinear and chaotic phenomena, engineers started using analytical tools from "Qualitative Theory of Differential Equations," allowing more precise analysis and synthesis, in order to produce new vital products and services. Bifurcation theory, dynamical systems and chaos started to be part of the mandatory set of tools for design engineers.

This proposed special edition of the *Mathematical Problems in Engineering* aims to provide a picture of the importance of the bifurcation theory, relating it with nonlinear and chaotic dynamics for natural and engineered systems. Ideas of how this dynamics can be captured through precisely tailored real and numerical experiments and understanding by the combination of specific tools that associate dynamical system theory and geometric tools in a very clever, sophisticated, and at the same time simple and unique analytical environment are the subject of this issue, allowing new methods to design high-precision devices and equipment.

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