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Subharmonic solutions of the prescribed curvature equation*

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Abstract

We study the existence of subharmonic solutions of the prescribed curvature equation

$$-\left(u'/\sqrt{1+u'^2}\right)' = f(t,u).$$

According to the behaviour at zero, or at infinity, of the prescribed curvature f, we prove the existence of arbitrarily small classical subharmonic solutions, or bounded variation subharmonic solutions with arbitrarily large oscillations.

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Keywords and Phrases: quasilinear ordinary differential equation, prescribed curvature equation, periodic solution, subharmonic solution, bounded variation function, minimum period.

1 Introduction

In this paper we are concerned with the existence of periodic, in particular subharmonic, solutions of the quasilinear ordinary differential equation

$$-\left(u'/\sqrt{1+u'^2}\right)' = f(t,u). \tag{1}$$

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This equation, together with its N-dimensional counterpart

$$-\mathrm{div}\Big(\nabla u/\sqrt{1+|\nabla u|^2}\Big) = f(x,u),$$

plays a relevant role in various physical and geometrical questions, such as capillarity-type problems, flux limited diffusion phenomena, prescribed mean curvature problems (see, e.g., [1, 2, 3]). The question of the existence of periodic solutions of (1) has received considerable attention in recent years: the existence of classical solutions has been addressed in [4, 5, 6, 7, 8, 9, 10] by using topological methods, whereas the existence of bounded variation solutions has been discussed in [11, 12, 13] by using non-smooth critical point theory. The advisability of considering bounded variation solutions, besides classical solutions, in order to have a complete picture of the solvability patterns of (1), is already evident for the autonomous equation

$$-\Big(u'/\sqrt{1+u'^2}\Big)' = f(u).$$

Indeed, elementary phase-plane analysis and energy arguments, as in [14, 15, 16, 17], show that any solution u, for which $\int_0^{u(\cdot)} f(\xi)d\xi$ exceeds somewhere the threshold 1, exhibits discontinuities and therefore cannot be a solution of (1) in the classical sense. The coexistence of classical and non-classical solutions of (1), according to the terminology introduced in [18, 19, 15, 17, 20], is determined by the specific structure of the curvature operator $\big(u'/\sqrt{1+u'^2}\big)'$, which behaves like the 2-Laplacian u'' near zero and like the 1-Laplacian $\big(\mathrm{sgn}(u')\big)'$ at infinity. These considerations lead us to introduce the following concept of periodic solution for equation (1) that will be considered throughout this paper.

Definition 1.1. *Let $\tau > 0$ be fixed. We say that a function $u \in BV_{\mathrm{loc}}(\mathbb{R})$ is a τ-periodic solution of (1) if u is τ-periodic, $f(\cdot,u) \in L^1(0,\tau)$ and*

$$\int_0^\tau \frac{(Du)^a\,(D\phi)^a}{\sqrt{1+|(Du)^a|^2}}\,dt + \int_0^\tau \mathrm{sgn}\left(\frac{Du}{|Du|}\right)\frac{D\phi}{|D\phi|}\,|D\phi|^s$$
$$+\mathrm{sgn}\big(u(0^+)-u(\tau^-)\big)\,\big(\phi(0^+)\;-\phi(\tau^-)\big) = \int_0^\tau f(t,u)\phi\,dt \qquad (2)$$

holds for every $\phi \in BV_{\mathrm{loc}}(\mathbb{R})$ such that $|D\phi|^s$ is absolutely continuous with respect to $|Du|^s$.

As usual, for any $v \in BV(a,b)$, $Dv = (Dv)^a dt + (Dv)^s$ is the Lebesgue decomposition of the measure Dv in its absolutely continuous part $(Dv)^a dt$, with density function $(Dv)^a$, and its singular part $(Dv)^s$ with respect to the Lebesgue measure in $\mathbb{R}$, $|Dv|$ denotes the total variation of the measure Dv, $|Dv| = |Dv|^a dt + |Dv|^s$ is the Lebesgue decomposition of $|Dv|$, and $\frac{Dv}{|Dv|}$ is the density function of Dv with respect to its total variation $|Dv|$.

It is immediate to verify that if u is a τ-periodic solution of (1) such that $u \in W^{1,1}_{\mathrm{loc}}(\mathbb{R})$, then it is a weak τ-periodic solution of (1), in the sense that

$$\int_0^\tau \frac{u'\phi'}{\sqrt{1+u'^2}}\,dt = \int_0^\tau f(t,u)\phi\,dt$$

for every $\phi \in W^{1,1}(0,\tau)$ with $\phi(0) = \phi(\tau)$. This implies that $u'/\sqrt{1+u'^2} \in W^{1,1}_{\mathrm{loc}}(\mathbb{R})$, $u'/\sqrt{1+u'^2}$ is τ-periodic and $-\big(u'/\sqrt{1+u'^2}\big)' = f(t,u)$ a.e. in $]0,\tau[$. Note that a weak τ-periodic solution u of (1) is continuous, but may present a derivative blow up. However, we have $u' \in C^0([0,\tau],[-\infty,+\infty])$. Hence u' satisfies the periodicity conditions in an extended sense, i.e., with possibly $u'(0) = u'(\tau) = +\infty$ or $u'(0) = u'(\tau) = -\infty$. It is clear that, if $u \in C^1(\mathbb{R})$, then u is a τ-periodic solution of (1) in the Carathéodory sense; if, in addition, $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous and τ-periodic in t, then u is a classical τ-periodic solution of (1).

There exists a huge literature concerning the existence of periodic, specifically subharmonic, solutions of the semilinear equation

$$-u'' = f(t,u), \tag{3}$$

where $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is, say, Lipschitz continuous and T-periodic in t, for some $T > 0$. We refer, e.g., to [21] for a rather exhaustive and updated bibliography on this subject. Although various definitions exist, a subharmonic solution of (3) is usually intended to be a periodic solution of the equation having minimum period mT for some integer $m \geq 2$. If this last piece of information is missing, a subharmonic solution is at least required to be mT-periodic, but not T-periodic. In case the solution is mT-periodic, but not jT-periodic, for any $j = 1, 2, \ldots, m-1$, then it is referred to as a subharmonic solution of order m. As a general rule in this context, one tries to get as much information as possible about the minimality of the period. In particular, in [22] the existence of subharmonic solutions of (3) has been proved assuming that either f is superlinear at 0, i.e.,

$$\lim_{s\to 0^\pm} \frac{f(t,s)}{s} = 0, \tag{4}$$

uniformly in t, or sublinear at infinity

$$\lim_{s\to \pm\infty} \frac{f(t,s)}{s} = 0, \tag{5}$$

uniformly in t. More precisely, it is shown in [22] that condition (4), even assumed only at 0^+, or at 0^-, implies the existence of two sequences of arbitrarily small subharmonic solutions having a prescribed number of zeroes and condition (5), even assumed only at $+\infty$, or at $-\infty$, implies the existence of two sequences of arbitrarily large subharmonic solutions having a prescribed number of zeroes. The proof is performed by a phase-plane analysis and relies on the Poincaré-Birkhoff fixed point theorem; the nodal properties of the solutions are obtained by using the rotation number which counts the number of turns of the solutions around the origin in the phase-plane.

Our aim here is to investigate the existence of subharmonic solutions for (1), taking inspiration from these results, but keeping in mind the behaviour of the curvature operator at 0 and at infinity. The following notions of subharmonic solution of (1) are used in this paper.

Definition 1.2. *We say that u is a subharmonic solution of* (1) *if it is a periodic solution of* (1) *having minimum period $\tau = rT$ for some $r \in \mathbb{Q}$ with $r > 1$.*

Definition 1.3. *We say that u is a subharmonic solution of order m of* (1) *if it is a mT-periodic solution of* (1) *for some $m \in \mathbb{N}$ with $m \geq 2$, but it is not jT-periodic, for any $j = 1, 2, \ldots, m-1$.*

It is easily seen that a subharmonic solution of (1) having minimum period $\tau = \frac{p}{q}T$, for some $p, q \in \mathbb{N}_0$, with p, q coprime and $p > q$, is a subharmonic solution of order p of (1).

Assuming that f is superlinear at 0, we prove in Theorem 3.4 the existence of small classical subharmonic solutions having suitable nodal properties; in this case the proof, which borrows some arguments from [22], is based on the use of the rotation number and on a version of the Poincaré-Birkhoff theorem given in [23, Theorem 8.2] and not requiring uniqueness of solutions for the Cauchy problems associated with (1). In particular, the following result holds.

Theorem 1.1. *Assume that*

(h_0) $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *is* T*-periodic with respect to the first variable, for some* $T > 0$*, and continuous,*

(h_1) $\displaystyle\lim_{s \to 0} \frac{f(t, s)}{s} = 0$*, uniformly in* $t \in [0, T]$*,*

(h_2) *there exists* $\delta > 0$ *such that* $f(t, s)s > 0$*, for all* $t \in [0, T]$ *and for all* $s \in [-\delta, \delta] \setminus \{0\}$*.*

Then there exists a sequence $(u_k)_k$ *of classical subharmonic solutions of* (1) *such that*

$$\lim_{k \to +\infty} \|u_k\|_{C^1} = 0$$

and whose minimum periods diverge.

A parallel result concerning the existence of subharmonic solutions of (1) having large oscillations is obtained supposing that the potential F of f is sublinear and coercive at infinity; in this case bounded variation non-classical solutions are expected. The proof makes use of some tools of non-smooth critical point theory, namely a version of the mountain pass lemma in the space of bounded variation functions given in [12, Lemma 2.13], combined with suitable critical value estimates as introduced for the semilinear problem (3) in [24].

Theorem 1.2. *Assume that*

(k_0) $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *is* T*-periodic with respect to the first variable, for some* $T > 0$*, and satisfies the* L^1*-Carathéodory conditions in* $[0, T] \times \mathbb{R}$*,*

(k_1) $\displaystyle\lim_{|s| \to +\infty} f(t, s) = 0$ *uniformly a.e. in* $t \in [0, T]$*,*

(k_2) $\displaystyle\lim_{|s| \to +\infty} \int_0^T F(t, s)\, dt = +\infty$*, where* $\displaystyle F(t, s) = \int_0^s f(t, \xi)\, d\xi$*,*

(k_3) *there exists* $R > 0$ *such that* $f(t, s)s > 0$ *for a.e.* $t \in [0, T]$ *and every* s *with* $|s| \geq R$*.*

Then there exists a sequence $(u_k)_k$ *of subharmonic solutions of* (1) *such that*

$$\lim_{k \to +\infty} \left(\operatorname*{ess\,sup}_{\mathbb{R}} u_k - \operatorname*{ess\,inf}_{\mathbb{R}} u_k \right) = +\infty$$

and whose minimum periods diverge.

Notations. For any given $a, b \in \mathbb{R}$, with $a < b$, and each $v \in BV(a,b)$ we set, as usual,

$$\int_a^b |Dv| = \sup\Big\{ \int_a^b vw'\,dt : w \in C_0^1(]a,b[),\ \|w\|_{L^\infty(a,b)} \le 1 \Big\}$$

and

$$\int_a^b \sqrt{1+|Dv|^2} = \sup\Big\{ \int_a^b (vw_1' + w_2)\,dt : w_1, w_2 \in C_0^1(]a,b[) \quad \text{and } \|w_1^2 + w_2^2\|_{L^\infty(a,b)} \le 1 \Big\}.$$

Clearly, we have

$$\int_a^b |Dv| \le \int_a^b \sqrt{1+|Dv|^2}.$$

The norm in $BV(a,b)$ is defined by

$$\|v\|_{BV(a,b)} = \int_a^b |v|\,dt + \int_a^b |Dv|.$$

We also denote by $v(t_0^+)$ the right trace of v at $t_0 \in [a,b[$ and by $v(t_0^-)$ the left trace of v at $t_0 \in]a,b]$. Finally, we write $\mathbb{N}_0 = \{n \in \mathbb{N} : n \ge 1\}$, $\mathbb{R}_0^+ = \{x \in \mathbb{R} : x > 0\}$ and $\mathbb{R}_0^- = \{x \in \mathbb{R} : x < 0\}$.

2 The autonomous equation

In this section we discuss the existence of periodic solutions of the autonomous equation

$$-\Big(u'/\sqrt{1+u'^2}\Big)' = f(u), \tag{6}$$

by performing an elementary analysis in the phase-plane. We assume that $f : \mathbb{R} \to \mathbb{R}$ is odd and continuous. We also suppose that $f(s) > 0$ for all $s > 0$ and $\lim_{s\to+\infty} F(s) = +\infty$, with $F(s) = \int_0^s f(\xi)\,d\xi$. Let us define the functions

$$\begin{aligned} &\varphi : \mathbb{R} \to]-1,1[, \quad \varphi(s) = \frac{s}{\sqrt{1+s^2}}, \\ &\psi :]-1,1[\to \mathbb{R}, \quad \psi(s) = \frac{s}{\sqrt{1-s^2}} \end{aligned} \tag{7}$$

and set $\Psi(s) = \int_0^s \psi(\xi)\,d\xi = 1 - \sqrt{1-s^2}$ for every $s \in]-1,1[$. Then equation (6) is equivalent to the planar system

$$\begin{cases} u' = -\psi(v) \\ v' = f(u). \end{cases} \tag{8}$$

The energy function associated with (8) is given by

$$\mathcal{E}(u,v) = \Psi(v) + F(u).$$

Clearly, the solutions (u,v) of (8) parametrize the level curves of $\mathcal{E}$. Let us fix $r>0$. We know by [25] that there is a unique non-extendible solution u of the Cauchy problem

$$\begin{cases} -\big(\varphi(u')\big)' = f(u) \\ u(0) = r \\ u'(0) = 0, \end{cases} \tag{9}$$

which satisfies

$$\Psi\big(\varphi(u'(t))\big) + F\big(u(t)\big) = F(r)$$

for all t belonging to its domain. Let us define

$$C_r = \{(x,y) \in \mathbb{R}^2 : \Psi\big(\varphi(y)\big) + F(x) = F(r)\}.$$

Note that $\Psi(\varphi(y)) = 1 - \frac{1}{\sqrt{1+y^2}}$. The curve C_r is symmetric with respect to the origin and its topology depends on the value $F(r)$. Indeed, since C_r can be represented in the form

$$C_r = \left\{(x,y) \in \mathbb{R}^2 : y = \pm \frac{\sqrt{F(r)-F(x)}}{\chi(F(r)-F(x))}\right\},$$

with

$$\chi : [0,1] \to \mathbb{R}, \quad \chi(s) = \frac{1-s}{\sqrt{2-s}},$$

we see that C_r is connected if and only if $F(r) < 1$; indeed, under this assumption, C_r is homeomorphic to the circle S^1. Otherwise C_r is disconnected and unbounded in the y-component; namely, setting $r_\infty = F^{-1}(F(r)-1) \in [0,r[$, we have that, if $(x,y) \in C_r$ and $x \to \pm r_\infty^\pm$, then $|y| \to +\infty$ (cf. Figure 1).

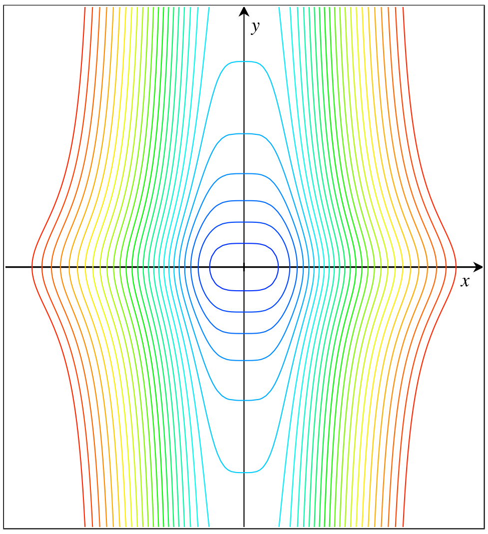

Figure 1: The level sets C_r of the energy $\mathcal{E}$, with $F(x) = \log(1+x^4)$.

If C_r is connected and u is a non-extendible solution of (9) such that the trajectory (u,u') parametrizes C_r, then $u \in C^2(\mathbb{R})$ and is periodic with minimum period $4T(r)$, where $T(r) \in \mathbb{R}_0^+$ is the first positive zero of u, i.e., u is a classical $4T(r)$-periodic solution of (6).

If C_r is disconnected and u is a non-extendible solution of (9) such that the trajectory (u,u') parametrizes $C_r \cap \big(\mathbb{R}_0^+ \times \mathbb{R}\big)$, then there exists $S(r) \in \mathbb{R}_0^+$ such that

$$\lim_{t\to S(r)^-} u(t) = r_\infty \quad \text{and} \quad \lim_{t\to S(r)^-} u'(t) = -\infty$$

and, by symmetry,

$$\lim_{t\to -S(r)^+} u(t) = r_\infty \quad \text{and} \quad \lim_{t\to -S(r)^+} u'(t) = +\infty.$$

Clearly, $u \in W^{1,1}(-S(r),S(r))$. Since, by symmetry, $(-u,-u')$ parametrizes $C_r \cap \big(\mathbb{R}_0^- \times \mathbb{R}\big)$, we can extend u to the interval $]-S(r),3S(r)[$, by setting $u(t) = -u(t-2S(r))$ for all $t \in]S(r),3S(r)[$, and then by $4S(r)$-periodicity all over $\mathbb{R}$. It is clear that $u \in BV_{\rm loc}(\mathbb{R})$ and is periodic, with minimum period $4S(r)$. Let us show that u is a $4S(r)$-periodic solution of (6) according to (2). Without restriction we can also replace u with $u(\cdot + S(r))$.

Let $\phi \in BV_{\rm loc}(\mathbb{R})$ be such that $|D\phi|^s$ is absolutely continuous with respect to $|Du|^s$. Denote by ϕ_1 and ϕ_2 the restrictions of ϕ to $]0,2S(r)[$ and to $]2S(r),4S(r)[$, respectively. By the regularity of u in $]0,2S(r)[$ and in $]2S(r),4S(r)[$, we have $|D\phi_1|^s = |D\phi_2|^s = 0$: this implies that $\phi_1 \in W^{1,1}(0,2S(r))$ and $\phi_2 \in W^{1,1}(2S(r),4S(r))$. Hence, multiplying (6) by ϕ_j, $j=1,2$, and integrating by parts in $]0,2S(r)[$ and in $]2S(r),4S(r)[$, respectively, we obtain

$$\begin{aligned}
-\int_0^{2S(r)} \big(\varphi(u')\big)'\phi_1\,dt &= -\left[\varphi(u')\phi_1\right]_{0^+}^{(2S(r))^-} + \int_0^{2S(r)} \varphi(u')\phi_1'\,dt = \int_0^{2S(r)} f(u)\phi_1\,dt,\\
-\int_{2S(r)}^{4S(r)} \big(\varphi(u')\big)'\phi_2\,dt &= -\left[\varphi(u')\phi_2\right]_{(2S(r))^+}^{(4S(r))^-} + \int_{2S(r)}^{4S(r)} \varphi(u')\phi_2'\,dt = \int_{2S(r)}^{4S(r)} f(u)\phi_2\,dt.
\end{aligned}$$

By the properties of u, we have

$$\begin{aligned}
\left[\varphi(u')\phi_1\right]_{0^+}^{(2S(r))^-} &= -\phi_1\big(0^+\big) - \phi_1\big((2S(r))^-\big),\\
\left[\varphi(u')\phi_2\right]_{(2S(r))^+}^{(4S(r))^-} &= \phi_2\big((2S(r))^+\big) + \phi_2\big((4S(r))^-\big).
\end{aligned}$$

Therefore summing up we get

$$\begin{aligned}
\int_0^{2S(r)} \varphi(u')\phi'\,dt + \int_{2S(r)}^{4S(r)} \varphi(u')\phi'\,dt&\\
+ \big[\phi(0^+) - \phi((4S(r))^-)\big] + \big[\phi\big((2S(r))^-\big) - \phi\big((2S(r))^+\big)\big]&\\
= \int_0^{4S(r)} f(u)\phi\,dt.& \qquad (10)
\end{aligned}$$

Notice that

$$\int_0^{2S(r)} \varphi(u')\phi'\,dt + \int_{2S(r)}^{4S(r)} \varphi(u')\phi'\,dt = \int_0^{4S(r)} \varphi\big((Du)^a\big)(D\phi)^a\,dt$$
$$= \int_0^{4S(r)} \frac{(Du)^a(D\phi)^a}{\sqrt{1+|(Du)^a|^2}}\,dt,$$
$$\phi(0^+) - \phi((4S(r))^-) = \operatorname{sgn}\big(u(0^+) - u((4S(r))^-)\big)\big(\phi(0^+) - \phi((4S(r))^-)\big),$$
$$\phi\big((2S(r))^-\big) - \phi\big((2S(r))^+\big) = -\int_0^{4S(r)} (D\phi)^s = -\int_0^{4S(r)} \frac{(D\phi)^s}{|D\phi|^s}|D\phi|^s$$
$$= -\int_0^{4S(r)} \frac{D\phi}{|D\phi|}|D\phi|^s = \int_0^{4S(r)} \operatorname{sgn}\Big(\frac{Du}{|Du|}\Big)\frac{D\phi}{|D\phi|}|D\phi|^s.$$

Substituting in (10), we obtain (2) with $\tau = S(r)$, that is, u is a $4S(r)$-periodic solution of (6).

We conclude this section by discussing the existence of periodic solutions of the autonomous equation (6) with reference to two model examples for $f = F'$, at 0 or at $\pm\infty$, respectively. Namely we suppose that

(a) $F(s) = |s|^{p+1}$, for some $p > 1$, in a neighbourhood of 0,

or

(b) $F(s) = |s|^{q+1}$, for some $q \in\,]-1, 0[$, in neighbourhoods of $\pm\infty$.

Assume that (a) holds: the expression of the classical time-map $T :]0, F^{-1}(1)[\to]0, +\infty[$, with $F^{-1}(1) = 1$, is

$$T(r) = \int_0^r \frac{\chi\big(F(r) - F(s)\big)}{\sqrt{F(r) - F(s)}}\,ds = r^{\frac{1-p}{2}} \int_0^1 \frac{\chi\big(r^{p+1}(1 - s^{p+1})\big)}{\sqrt{1 - s^{p+1}}}\,ds.$$

The concavity of the function χ implies that

$$\frac{1}{\sqrt{2}}(1 - s) \le \chi(s) \le \frac{1}{\sqrt{2}} \quad \text{in } [0, 1]. \tag{11}$$

Hence we obtain

$$\frac{r^{\frac{1-p}{2}}}{\sqrt{2}} \int_0^1 \frac{1}{\sqrt{1 - s^{p+1}}}\,ds \ge T(r) \ge \frac{r^{\frac{1-p}{2}}}{\sqrt{2}} \int_0^1 \frac{1}{\sqrt{1 - s^{p+1}}}\,ds - \frac{r^{\frac{3+p}{2}}}{\sqrt{2}} \int_0^1 \sqrt{1 - s^{p+1}}\,ds$$

for all $r \in\,]0, 1[$. Then we conclude

$$\lim_{r\to 0^+} T(r) = +\infty.$$

This implies that in case (a) there exists a family of classical periodic solutions of (6), approaching 0 in the C^1-norm and having arbitrarily large minimum periods.

Assume that (b) holds: the expression of the non-classical time-map $S : [F^{-1}(1), +\infty[\to]0, +\infty[$, with $F^{-1}(1) = 1$, is

$$S(r) = \int_{r_\infty}^r \frac{\chi\big(F(r) - F(s)\big)}{\sqrt{F(r) - F(s)}}\,ds = r^{\frac{1-q}{2}} \int_{r_\infty/r}^1 \frac{\chi\big(r^{q+1}(1 - s^{q+1})\big)}{\sqrt{1 - s^{q+1}}}\,ds.$$

By using (11), we have

$$\frac{r^{\frac{1-q}{2}}}{\sqrt{2}}\int_{r_\infty/r}^{1}\frac{1}{\sqrt{1-s^{q+1}}}\,ds \geq S(r) \geq \frac{r^{\frac{1-q}{2}}}{\sqrt{2}}\int_{r_\infty/r}^{1}\frac{1-r^{q+1}(1-s^{q+1})}{\sqrt{1-s^{q+1}}}\,ds$$
$$= \frac{r^{\frac{1-q}{2}}}{(q+1)\sqrt{2}}\int_{0}^{1/r^{q+1}}\frac{1-r^{q+1}s}{\sqrt{s}}(1-s)^{-\frac{q}{q+1}}\,ds,$$

for all $r > r_\infty$. Taking $r > r_\infty$ sufficiently large, we have $(1-s)^{-\frac{q}{q+1}} \geq \frac{1}{\sqrt{2}}$ for all $s \in [0, \frac{1}{r^{q+1}}]$ and then

$$S(r) \geq \frac{r^{\frac{1-q}{2}}}{2(q+1)}\int_{0}^{1/r^{q+1}}\frac{1-r^{q+1}s}{\sqrt{s}}\,ds = \frac{2}{3(q+1)r^{q}}.$$

Hence we conclude that

$$\lim_{r\to+\infty} S(r) = +\infty.$$

This implies that in case (b) there exists a family of periodic solutions of (6) according to Definition 1.1, having arbitrarily large oscillations and arbitrarily large minimum periods.

These simple observations are the starting point of our study of the general non-autonomous equation (1). In particular, the estimates we have produced on $T(r)$ and on $S(r)$ in the model cases (a) and (b) motivate the introduction of the assumptions of superlinearity of f at 0 and of sublinearity of F at infinity.

3 Small classical subharmonic solutions

We start this section with an elementary result concerning a property of the solutions of the first order system in $\mathbb{R}^N$

$$z' = \ell(t, z). \tag{12}$$

We recall that a continuous function $\kappa : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is an Osgood function if

$$\int_0^1 \frac{d\xi}{\kappa(\xi)} = +\infty = \int_1^{+\infty}\frac{d\xi}{\kappa(\xi)}.$$

Let us set, for each $s > 0$,

$$\mathcal{H}(s) = \int_1^s \frac{d\xi}{\kappa(\xi)};$$

it is immediate to see that $\mathcal{H} : \mathbb{R}_0^+ \to \mathbb{R}$ is an increasing diffeomorphism. Then the following conclusion is (essentially) a consequence of a classical result of I. Bihari [26].

Lemma 3.1. *Assume that $\ell : I \times \mathbb{R}^N \to \mathbb{R}^N$ is continuous, with $I \subseteq \mathbb{R}$ an interval, and suppose that there exists an Osgood function κ such that*

$$|\ell(t,\zeta)\cdot\zeta| \leq \kappa(|\zeta|^2),$$

for all $t \in I$ and $\zeta \in \mathbb{R}^N$. Then any non-trivial solution z of (12) is globally defined and satisfies

$$\mathcal{H}^{-1}(-2|t-t_0| + \mathcal{H}(|z(t_0)|^2)) \leq |z(t)|^2 \leq \mathcal{H}^{-1}(2|t-t_0| + \mathcal{H}(|z(t_0)|^2)),$$

for all $t, t_0 \in I$. In particular, z never vanishes.

Throughout this section we suppose that $f : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ satisfies assumptions (h_0), (h_1), (h_2). Let us define $\bar{f} : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ by setting, for any t,

$$\bar{f}(t,s) = \begin{cases} f(t,-\delta) & \text{if } s < -\delta, \\ f(t,s) & \text{if } |s| \le \delta, \\ f(t,\delta) & \text{if } s > \delta. \end{cases}$$

The function $\bar{f}$ satisfies the same assumptions as f does, in particular, (h_2) holds, with f replaced by $\bar{f}$, for all $t \in [0,T]$ and $s \in \mathbb{R}\setminus\{0\}$. Let us also define $\bar{\psi} : \mathbb{R} \to \mathbb{R}$ by

$$\bar{\psi}(s) = \begin{cases} -\psi(\delta) + \psi'(\delta)(s+\delta) & \text{if } s < -\delta, \\ \psi(s) & \text{if } |s| \le \delta, \\ \psi(\delta) + \psi'(\delta)(s-\delta) & \text{if } s > \delta, \end{cases}$$

where ψ has been defined in (7). It is easily checked that the vector field $\ell(t,\zeta) = (-\bar{\psi}(y), \bar{f}(t,x))$ satisfies the assumptions of Lemma 3.1, where the Osgood function κ can be taken to be a non-zero linear function, say, $\kappa(s) = \kappa\, s$ and thus $\mathcal{H}(s) = \frac{1}{\kappa}\ln(s)$. Hence Lemma 3.1 guarantees that any Cauchy problem associated with the system

$$\begin{cases} u' = -\bar{\psi}(v) \\ v' = \bar{f}(t,u) \end{cases} \tag{13}$$

has a global solution $z = (u,v) \in C^1(\mathbb{R})$, which, if non-trivial, never vanishes, since it satisfies

$$|z(t_0)|\, \exp(-\kappa|t-t_0|) \le |z(t)| \le |z(t_0)|\, \exp(\kappa|t-t_0|) \tag{14}$$

for all $t, t_0 \in \mathbb{R}$. This allows, in particular, to represent such a solution in polar coordinates as

$$u(t) = \rho(t)\cos\theta(t), \quad v(t) = \rho(t)\sin\theta(t).$$

Note that the couple (ρ,θ) satisfies

$$\begin{aligned} \rho'(t) &= \bar{f}\big(t, \rho(t)\cos\theta(t)\big)\sin\theta(t) - \bar{\psi}\big(\rho(t)\sin\theta(t)\big)\cos\theta(t), \\ \theta'(t) &= \frac{\bar{\psi}\big(\rho(t)\sin\theta(t)\big)\rho(t)\sin\theta(t) + \bar{f}\big(t,\rho(t)\cos\theta(t)\big)\rho(t)\cos\theta(t)}{\rho(t)^2}, \end{aligned}$$

or equivalently

$$\begin{aligned} \rho'(t) &= \frac{\bar{f}\big(t,u(t)\big)v(t) - \bar{\psi}\big(v(t)\big)u(t)}{\sqrt{u(t)^2+v(t)^2}}, \\ \theta'(t) &= \frac{\bar{\psi}\big(v(t)\big)v(t) + \bar{f}\big(t,u(t)\big)u(t)}{u(t)^2+v(t)^2}, \end{aligned}$$

for all $t \in \mathbb{R}$.

For any fixed $t_0, t_1 \in \mathbb{R}$, with $t_0 < t_1$, and assuming that $z(t_0) = z_0$ for some $z_0 \in \mathbb{R}^2\setminus\{0\}$, we define the rotation number of z in $[t_0,t_1]$ by

$$\mathrm{Rot}(z;[t_0,t_1]) = \frac{\theta(t_1)-\theta(t_0)}{2\pi} = \frac{1}{2\pi}\int_{t_0}^{t_1} \frac{\bar{\psi}\big(v(t)\big)v(t) + \bar{f}\big(t,u(t)\big)u(t)}{u(t)^2+v(t)^2}\, dt.$$

The rotation number counts the counterclockwise turns of the function z around the origin in the time interval $[t_0, t_1]$.

We notice that the sign conditions satisfied by $\bar{f}$ and $\bar{\psi}$, namely $\bar{f}(t,s)s > 0$ and $\bar{\psi}(s)s > 0$, for all $t \in \mathbb{R}$ and $s \in \mathbb{R}\setminus\{0\}$, imply that the function θ is strictly increasing, or equivalently that the rotation number is positive for any non-trivial solution z in any compact time interval.

Lemma 3.2. *Assume (h_0), (h_1), (h_2). For any $k \in \mathbb{N}_0$, there exist $\tau_k^* > 0$ and $r_k^* \in]0, \min\{\frac{1}{k}, \delta\}[$, such that, for any interval $J = [t_0, t_1]$, with $|t_0 - t_1| > \tau_k^*$, any solution z of (13), with $z(t_0) = z_0$ for some $|z_0| = r_k^*$, satisfies*

$$\mathrm{Rot}(z; J) > k. \tag{15}$$

Proof. By condition (h_2) we can find two continuous functions $g, h : \mathbb{R} \to \mathbb{R}$ such that

$$0 < g(s)s \leq h(s)s, \qquad \text{for all } s \in \mathbb{R} \setminus \{0\}, \tag{16}$$

$$g(s)s \leq \bar{f}(t,s)s \leq h(s)s, \qquad \text{for all } t \in [0,T], s \in \mathbb{R}, \tag{17}$$

and

$$\lim_{|s|\to+\infty} G(s) = \lim_{|s|\to+\infty} H(s) = +\infty, \tag{18}$$

where

$$G(s) = \int_0^s g(\xi)\, d\xi, \quad H(s) = \int_0^s h(\xi)\, d\xi.$$

Let us introduce the planar autonomous systems

$$\begin{cases} u' = -\bar{\psi}(v) \\ v' = g(u) \end{cases} \tag{19}$$

and

$$\begin{cases} u' = -\bar{\psi}(v) \\ v' = h(u). \end{cases} \tag{20}$$

The energy functions associated with (19) and (20) are, respectively,

$$\mathcal{E}_G(x,y) = \bar{\Psi}(y) + G(x), \qquad \mathcal{E}_H(x,y) = \bar{\Psi}(y) + H(x),$$

with

$$\bar{\Psi}(s) = \int_0^s \bar{\psi}(\xi)\, d\xi,$$

for any $s \in \mathbb{R}$. By definition of $\bar{\psi}$ and by conditions (16) and (18), the only equilibrium point of (19) and (20) is $(0,0)$ and all level curves of $\mathcal{E}_G$ and $\mathcal{E}_H$ are closed curves around $(0,0)$. Hence global existence and uniqueness of solution hold for every Cauchy problem associated with (19) and (20).

Let us introduce two auxiliary functions $M_\mp : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}$, defined by

$$M_-(x,y) = \begin{cases} \frac{g(x)y - \bar{\psi}(y)x}{g(x)x + \bar{\psi}(y)y}, & \text{if } xy \geq 0, \\ \frac{h(x)y - \bar{\psi}(y)x}{h(x)x + \bar{\psi}(y)y}, & \text{if } xy \leq 0 \end{cases}$$

and

$$M_+(x,y) = \begin{cases} \frac{g(x)y-\bar{\psi}(y)x}{g(x)x+\bar{\psi}(y)y}, & \text{if } xy \le 0, \\ \frac{h(x)y-\bar{\psi}(y)x}{h(x)x+\bar{\psi}(y)y}, & \text{if } xy \ge 0 \end{cases}$$

and consider the equations

$$\frac{dr}{d\theta} = rM_-(r\cos\theta, r\sin\theta) \tag{21}$$

and

$$\frac{dr}{d\theta} = rM_+(r\cos\theta, r\sin\theta). \tag{22}$$

Trajectories associated with non-trivial solutions of (21) and (22) parametrize spirals, surrounding the origin, obtained by alternating the level curves of $\mathcal{E}_G$ and $\mathcal{E}_H$. We see that, for all (θ_0, ρ_0) with $\theta_0 \in \mathbb{R}$ and $\rho_0 > 0$, uniqueness and global continuability of solutions hold for any Cauchy problem associated with (21) and with (22). Let $r_-(\cdot\,;\theta_0,\rho_0), r_+(\cdot\,;\theta_0,\rho_0)$ (in the sequel, sometimes denoted by $r_\mp$ for simplicity) be the solutions of (21) and (22), respectively, satisfying $r_\mp(\theta_0) = \rho_0$. Note, in particular, that any non-trivial solution $r_\mp$ of (21) and (22) satisfies $r_\mp(\theta) > 0$ for all $\theta \in \mathbb{R}$.

Let us fix $k \in \mathbb{N}_0$. For any $\rho_0 > 0$, set

$$m_k^*(\rho_0) = \inf_{\substack{\theta_0 \in [0,2\pi] \\ \theta \in [\theta_0, \theta_0 + 2k\pi]}} r_-(\theta;\theta_0,\rho_0) > 0$$

and

$$M_k^*(\rho_0) = \sup_{\substack{\theta_0 \in [0,2\pi] \\ \theta \in [\theta_0, \theta_0 + 2k\pi]}} r_+(\theta;\theta_0,\rho_0).$$

As $\lim\limits_{\rho_0 \to 0^+} M_k^*(\rho_0) = 0$, there exists $r_k^* > 0$ such that

$$0 < m_k^*(r_k^*) \le r_k^* \le M_k^*(r_k^*) < \min\{\tfrac{1}{k}, \delta\}.$$

Pick $\underline{r_k}, \overline{r_k} > 0$ such that

$$0 < \underline{r_k} < m_k^*(r_k^*) \le r_k^* \le M_k^*(r_k^*) < \overline{r_k} < \min\{\tfrac{1}{k}, \delta\}.$$

Define a continuous function $\mathcal{G} : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}$ by

$$\mathcal{G}(x,y) = \frac{g(x)x + \bar{\psi}(y)y}{x^2+y^2}.$$

Let $\mathcal{A} = \{(x,y) \in \mathbb{R}^2 : \underline{r_k} \le \sqrt{x^2+y^2} \le \overline{r_k}\}$ and $\delta_k^* = \min\limits_{\mathcal{A}} \mathcal{G} > 0$. Set $\tau_k^* = \frac{2k\pi}{\delta_k^*}$ and take any interval $J = [t_0, t_1]$, with $\tau = t_1 - t_0 > \tau_k^*$ and a solution z of (13) with $z(t_0) = z_0$, for some $z_0 \in \mathbb{R}^2$ such that $|z_0| = r_k^*$. We want to prove that (15) holds, i.e.,

$$\operatorname{Rot}(z;J) > k.$$

Without loss of generality, we can assume that $\theta(t_0) = \theta_0 \in [0, 2\pi[$. Therefore the thesis amounts to proving that

$$\theta(t_1) - \theta(t_0) > 2k\pi. \tag{23}$$

Set
$$\sigma = \sup\{s \in [t_0, t_1] : \underline{r_k} \le \rho(t) \le \overline{r_k} \text{ in } [t_0, s]\}.$$
For all $t \in [t_0, \sigma]$, we have
$$|u(t)| \le \rho(t) < \delta. \tag{24}$$
Two cases may occur: either $\sigma = t_1$, or $\sigma < t_1$.

If $\sigma = t_1$, by (24) and (17), we have
$$\begin{aligned}\mathrm{Rot}(z; J) &= \frac{1}{2\pi}\int_{t_0}^{\sigma} \frac{\bar\psi\big(v(t)\big)v(t) + \bar f\big(t, u(t)\big)u(t)}{u(t)^2 + v(t)^2}\, dt \\ &\ge \frac{1}{2\pi}\int_{t_0}^{t_1} \mathcal{G}\big(u(t), v(t)\big)\, dt \ge \frac{\tau}{2\pi}\min_{\mathcal{A}} \mathcal{G} > \frac{\delta_k^* \tau_k^*}{2\pi} = k\end{aligned}$$
and hence (15) follows.

If $\sigma < t_1$, the maximality of σ implies that $\underline{r_k} \le \rho(t) \le \overline{r_k}$ for all $t \in [t_0, \sigma]$ and $\rho(\sigma) \in \{\underline{r_k}, \overline{r_k}\}$. Assume that $\rho(\sigma) = \overline{r_k}$, the other case being treated similarly. In order to prove (23), we only need to show that
$$\theta(\sigma) - \theta(t_0) > 2k\pi,$$
since, as already observed, the function θ is strictly increasing. By contradiction, assume that $\theta(\sigma) - \theta(t_0) \le 2k\pi$. The monotonicity of θ also implies
$$\theta(t) \in [\theta_0, \theta_0 + 2k\pi], \tag{25}$$
for all $t \in [t_0, \sigma]$.

On the other hand, $\zeta = (\rho, \theta)$ satisfies
$$\begin{aligned}\rho'(t) &= \bar f\big(t, \rho(t)\cos\theta(t)\big)\sin\theta(t) - \bar\psi\big(\rho(t)\sin\theta(t)\big)\cos\theta(t), \\ \theta'(t) &= \frac{\bar\psi\big(\rho(t)\sin\theta(t)\big)\sin\theta(t) + \bar f\big(t, \rho(t)\cos\theta(t)\big)\cos\theta(t)}{\rho(t)},\end{aligned} \tag{26}$$
for all $t \in \mathbb{R}$. In this regard, we introduce some more functions $S, U : J \times \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}$, defined by
$$\begin{aligned}S(t, x, y) &= \frac{\bar f(t, x)y - \bar\psi(y)x}{x^2 + y^2}, \\ U(t, x, y) &= \frac{\bar\psi(y)y + \bar f(t, x)x}{x^2 + y^2},\end{aligned}$$
and $R, \Theta : J \times]0, +\infty[\times \mathbb{R} \to \mathbb{R}$, defined by
$$\begin{aligned}R(t, \rho, \theta) &= \rho S(t, \rho\cos\theta, \rho\sin\theta), \\ \Theta(t, \rho, \theta) &= U(t, \rho\cos\theta, \rho\sin\theta).\end{aligned}$$
By definition there holds
$$\frac{\rho S(t, \rho\cos\theta, \rho\sin\theta)}{U(t, \rho\cos\theta, \rho\sin\theta)} = \frac{R(t, \rho, \theta)}{\Theta(t, \rho, \theta)},$$

for all $(t,\rho,\theta) \in J \times]0,+\infty[\times \mathbb{R}$. Moreover we easily see that

$$\frac{S(t,x,y)}{U(t,x,y)} \leq M_+(x,y) \tag{27}$$

is satisfied for all $t \in [t_0,\sigma]$ and $(x,y) \in \mathbb{R}^2 \setminus \{0\}$.

Let $\gamma : \mathbb{R} \to]0,+\infty[$ be the solution of (22) satisfying $\gamma(\theta_0) = r_k^*$. From the definition of M_k^*, we have

$$\gamma(\theta) \leq M_k^*(r_k^*) < \overline{r_k},$$

for all $\theta \in [\theta_0, \theta_0 + 2k\pi]$. By continuity of γ, there exists $\varepsilon > 0$ such that $\gamma(\theta) < \overline{r_k}$ for all $\theta \in]\theta_0 - \varepsilon, \theta_0 + 2k\pi + \varepsilon[$. From (27) and the positivity of γ,

$$\gamma'(\theta) = \gamma M_+(\gamma\cos\theta, \gamma\sin\theta) \geq \frac{\gamma S(t,\gamma\cos\theta,\gamma\sin\theta)}{U(t,\gamma\cos\theta,\gamma\sin\theta)} = \frac{R(t,\gamma,\theta)}{\Theta(t,\gamma,\theta)}$$

holds for all $t \in [t_0,\sigma]$ and $\theta \in \mathbb{R}$. Consider again the function (ρ,θ): when restricted to the interval $[t_0,\sigma]$ and thanks to condition (25), it takes values in $]0,+\infty[\times]\theta_0 - \varepsilon, \theta_0 + 2k\pi + \varepsilon[$. Moreover it is a solution of

$$\begin{cases} \rho' = R(t,\rho,\theta) \\ \theta' = \Theta(t,\rho,\theta) \end{cases}$$

with $\rho(t_0) = r_k^* = \gamma(\theta_0) = \gamma(\theta(t_0))$. We know that $\theta'(t) > 0$ for all $t \in [t_0,\sigma]$, so that the function $\theta : [t_0,\sigma] \to [\theta(t_0),\theta(\sigma)]$ is a C^1-diffeomorphism, with inverse $s : [\theta(t_0),\theta(\sigma)] \to [t_0,\sigma]$. If we set $\varrho(\theta) = \rho(s(\theta))$, so that $\rho(t) = \varrho(\theta(t))$, we find

$$\frac{d\varrho(\theta)}{d\theta} = \frac{\rho'(t)}{\theta'(t)}\Big|_{t=s(\theta)} = \frac{R(s(\theta),\rho(s(\theta)),\theta)}{\Theta(s(\theta),\rho(s(\theta)),\theta)}.$$

Hence, $\varrho(\theta)$ satisfies, for $\theta \in [\theta(t_0),\theta(\sigma)]$, the differential inequality

$$\varrho' \leq \varrho M_+(\varrho\cos\theta, \varrho\sin\theta)$$

and, moreover, $\varrho(\theta_0) = \rho(t_0) = \gamma(\theta_0)$. As uniqueness of solutions holds for any Cauchy problem associated with (22), by a classical result on differential inequalities (see, e.g., [27, Section I.6, Theorem 6.1]), we conclude that $\varrho(\theta) \leq \gamma(\theta)$, for all $\theta \in [\theta(t_0),\theta(\sigma)]$, and hence $\rho(t) \leq \gamma(\theta(t))$, for all $t \in [t_0,\sigma]$. In particular, we have

$$\overline{r_k} = \rho(\sigma) \leq \gamma(\theta(\sigma)) < \overline{r_k},$$

which leads to a contradiction. □

Lemma 3.3. *Assume* (h_0), (h_1), (h_2). *Let* $J \subset \mathbb{R}$ *be a compact interval. Then there exists* $r_0 = r_0(J) > 0$ *such that*

$$\mathrm{Rot}(z;J) < 1, \tag{28}$$

for any solution z *of* (13), *with* $0 < \min_J |z| \leq r_0$.

Proof. Assumptions (h_1) and (h_2) imply in particular that, for any fixed $\varepsilon > 0$, there exists $\delta_\varepsilon \in]0, \delta[$ such that, for all $t \in \mathbb{R}$ and $s \in [-\delta_\varepsilon, \delta_\varepsilon]$,

$$\bar{f}(t,s)s \le \varepsilon s^2.$$

Let $J = [t_1, t_2]$ and $z = (u, v)$ be a non-trivial solution of (13) such that, for all $t \in J$,

$$|u(t)| \le \delta_\varepsilon \tag{29}$$

Denote as usual by (ρ, θ) the polar coordinates of z and set $\theta_1 = \theta(t_1)$ and $\theta_2 = \theta(t_2)$. We want to prove that

$$\theta_2 - \theta_1 < 2\pi. \tag{30}$$

Assume by contradiction that

$$\theta_2 - \theta_1 \ge 2\pi.$$

As we have, by (29),

$$\bar{f}(t, \rho(t)\cos\theta(t))\rho(t)\cos\theta(t) \le \varepsilon(\rho(t)\cos\theta(t))^2$$

and

$$\bar{\psi}\big(\rho(t)\sin\theta(t)\big)\rho(t)\sin\theta(t) \le \psi'(\delta)\big(\rho(t)\sin\theta(t)\big)^2,$$

we obtain, from (26),

$$\frac{\theta'(t)}{\psi'(\delta)\big(\sin\theta(t)\big)^2 + \varepsilon\big(\cos\theta(t)\big)^2} \le 1,$$

for all $t \in [t_1, t_2]$. Setting $c = \psi'(\delta)$ and integrating over $[t_1, t_2]$ yield

$$\begin{aligned}
|J| = t_2 - t_1 &\ge \int_{\theta_1}^{\theta_2} \frac{1}{c(\sin s)^2 + \varepsilon(\cos s)^2}\, ds \\
&\ge \int_0^{2\pi} \frac{1}{c(\sin s)^2 + \varepsilon(\cos s)^2}\, ds = 4\int_0^{\frac{\pi}{2}} \frac{1}{c(\sin s)^2 + \varepsilon(\cos s)^2}\, ds \\
&= \frac{4}{\varepsilon}\int_0^{\frac{\pi}{2}} \frac{1}{(\cos t)^2}\frac{1}{1 + \big(\sqrt{\frac{c}{\varepsilon}}\tan t\big)^2}\, dt = \frac{4}{\sqrt{c\varepsilon}}\int_0^{+\infty} \frac{1}{1+t^2}\, dt = \frac{2\pi}{\sqrt{c\varepsilon}}.
\end{aligned}$$

A contradiction is achieved taking $\varepsilon \in]0, \frac{4\pi^2}{c|J|^2}[$. Hence (30) follows.

In order to conclude, we use Lemma 3.1: choosing $r_0 > 0$ small enough, any solution z, with $0 < \min_J |z| \le r_0$, by (14) satisfies

$$\max_J |u| \le \max_J |z| \le \delta_\varepsilon.$$

Hence (30) holds, implying the validity of (28). □

Remark 3.1 Lemma 3.3 is still valid replacing assumption (h_1) with

$$\lim_{s\to 0^+} \frac{f(t,s)}{s} = 0, \quad \text{uniformly in } t \in [0,T]$$

or

$$\lim_{s\to 0^-} \frac{f(t,s)}{s} = 0, \quad \text{uniformly in } t \in [0,T].$$

The proof requires just few minor modifications, as in [22, Lemma 3.4].

Theorem 3.4. *Assume* (h_0), (h_1), (h_2). *For every* $k \in \mathbb{N}_0$ *there exists* $m_k^* \in \mathbb{N}_0$ *such that for any integer* $m > m_k^*$, *which is coprime with* k, *equation* (1) *has a classical subharmonic solution* u_k *of order* m *with precisely* $2k$ *zeroes in* $[0, mT[$.

Proof. Fix $k \in \mathbb{N}_0$ and let $m_k^* \in \mathbb{N}_0$, with $m_k^* T > k\,\tau_k^*$, and r_k^* be as in Lemma 3.2. Take $m \in \mathbb{N}_0$ such that $m > m_k^*$ and $r_0 = r_0([0, mT]) < r_k^*$ as in Lemma 3.3. The two results just mentioned guarantee, on the one hand, that

$$\mathrm{Rot}(z; [0, mT]) > k,$$

for any solution z with initial value z_0 such that $|z_0| = r_k^*$, and, on the other hand, that

$$\mathrm{Rot}(z; [0, mT]) < k,$$

for any solution z with initial value z_0 such that $|z_0| = r_0$. Since solutions of the Cauchy problems associated with (13) are globally defined, we can apply the Poincaré-Birkhoff theorem, in the version of [23, Theorem 8.2]: there exists in particular a point $z_k^* \in \mathbb{R}^2$ such that $r_0 < |z_k^*| < r_k^*$ and a corresponding solution $z_k = (u_k, v_k)$ of (13) which is mT-periodic and satisfies

$$\mathrm{Rot}(z_k; [0, mT]) = k.$$

By the previous discussion we know that, denoting by (ρ_k, θ_k) the polar coordinates of z_k, the angular displacement θ_k is strictly increasing, and thus u_k has exactly $2k$ zeroes in $[0, mT[$. Moreover, since m, k are coprime, mT is the minimum period of u_k among $T, 2T, \ldots, (m-1)T, mT$. Finally, Lemma 3.2 and Lemma 3.3 imply that z_k satisfies

$$r_0 < |z_k(t)| < r_k^*$$

for all $t \in \mathbb{R}$. This condition assures that u_k is a classical subharmonic solution of (1) of order m, with precisely $2k$ zeroes in $[0, mT[$. □

Remark 3.2 Taking $k = 1$ in Theorem 3.4 we conclude that, for any $m > m_1^*$ there exists at least one subharmonic solution having minimum period mT.

We are now in position of proving Theorem 1.1.

Proof of Theorem 1.1. We keep the same notations as in the proof of Theorem 3.4. Fix any $k \in \mathbb{N}_0$ and take $m_k \in \mathbb{N}_0$, coprime with k, such that $m_k > \max\{k^2, m_k^*\}$. From the proof of Theorem 3.4 we know that there exists at least one point $z_k^* \in \mathbb{R}^2$ such that $r_0 < |z_k^*| < r_k^*$ and a corresponding solution $z_k = (u_k, v_k)$ of (13) which is $m_k T$-periodic and satisfies

$$\mathrm{Rot}(z_k; [0, m_k T]) = k.$$

The minimum period τ_k of z_k, and hence of u_k, satisfies $\tau_k \geq \frac{m_k}{k} T > kT$. This estimate obviously yields

$$\lim_{k \to +\infty} \tau_k = +\infty.$$

The proof of Theorem 3.4 also guarantees that, for any $k \in \mathbb{N}_0$,

$$\max\{\|u_k\|_\infty, \|(\bar{\psi})^{-1}(u_k')\|_\infty\} \leq \|z_k\|_\infty < r_k^*.$$

As we chose $r_k^* < \frac{1}{k}$ in Lemma 3.2, we get

$$\lim_{k \to +\infty} \|u_k\|_{C^1} = 0.$$

This concludes the proof of Theorem 1.1.

4 Large bounded variation subharmonic solutions

We start by proving some auxiliary results.

Lemma 4.1. *Let $\sigma > 0$ and $u \in BV_{\mathrm{loc}}(\mathbb{R})$ be σ-periodic. Then u is a σ-periodic solution of* (1) *if and only if $f(\cdot, u) \in L^1(0, \sigma)$ and the inequality*

$$\mathcal{J}_\sigma(v) - \mathcal{J}_\sigma(u) \geq \int_0^\sigma f(t, u)(v - u)\, dt \tag{31}$$

holds for all $v \in BV(0, \sigma)$, where

$$\mathcal{J}_\sigma(v) = \int_0^\sigma \sqrt{1 + |Dv|^2} + |v(0^+) - v(\sigma^-)|. \tag{32}$$

Proof. For any $\phi \in BV(0, \sigma)$, let us define $\mathcal{T}_\phi : \mathbb{R} \to \mathbb{R}$ by

$$\mathcal{T}_\phi(s) = \mathcal{J}_\sigma(u + s\phi) - \int_0^\sigma f(t, u)(u + s\phi)\, dt.$$

The function $\mathcal{T}_\phi$ is convex. Let us also set $\mathcal{K}_\sigma : BV(0, \sigma) \to \mathbb{R}$ by

$$\mathcal{K}_\sigma(v) = \int_0^\sigma \sqrt{1 + |Dv|^2}.$$

By [28, Theorem 3.6] the functional $\mathcal{K}_\sigma$ is differentiable in the direction $\phi \in BV(0, \sigma)$ if and only if $|D\phi|^s$ is absolutely continuous with respect to $|Du|^s$ and, under this assumption,

$$\frac{d}{ds}\mathcal{K}_\sigma(u + s\phi)|_{s=0} = \int_0^\sigma \frac{(Du)^a\, (D\phi)^a}{\sqrt{1 + |(Du)^a|^2}}\, dt + \int_0^\sigma \operatorname{sgn}\left(\frac{Du}{|Du|}\right) \frac{D\phi}{|D\phi|}\, |D\phi|^s. \tag{33}$$

Fix now $\phi \in BV(0, \sigma)$ as required in [28, Theorem 3.6]. Then $\mathcal{T}_\phi$ is differentiable at $s = 0$ and the following holds

$$\begin{aligned}
{\mathcal{T}_\phi}'(0) &= \frac{d}{ds}\mathcal{J}_\sigma(u + s\phi)|_{s=0} - \frac{d}{ds}\left(\int_0^\sigma f(t, u)(u + s\phi)\, dt\right)|_{s=0} \\
&= \frac{d}{ds}\mathcal{K}_\sigma(u + s\phi)|_{s=0} + \frac{d}{ds}\left|(u + s\phi)(0^+) - (u + s\phi)(\sigma^-)\right|_{s=0} \\
&\qquad\qquad\qquad\qquad\qquad\qquad - \int_0^\sigma f(t, u)\phi\, dt \\
&= \frac{d}{ds}\mathcal{K}_\sigma(u + s\phi)|_{s=0} + \frac{d}{ds}\left|u(0^+) - u(\sigma^-) + s\big(\phi(0^+) - \phi(\sigma^-)\big)\right|_{s=0} \\
&\qquad\qquad\qquad\qquad\qquad\qquad - \int_0^\sigma f(t, u)\phi\, dt \\
&= \frac{d}{ds}\mathcal{K}_\sigma(u + s\phi)|_{s=0} + \operatorname{sgn}\big(u(0^+) - u(\sigma^-)\big)\big(\phi(0^+) - \phi(\sigma^-)\big) \\
&\qquad\qquad\qquad\qquad\qquad\qquad - \int_0^\sigma f(t, u)\phi\, dt.
\end{aligned} \tag{34}$$

From (33) and (34) we obtain

$$\mathcal{T}_\phi{}'(0) = \int_0^\sigma \frac{(Du)^a\,(D\phi)^a}{\sqrt{1+|(Du)^a|^2}}\,dt + \int_0^\sigma \operatorname{sgn}\left(\frac{Du}{|Du|}\right)\,\frac{D\phi}{|D\phi|}\,|D\phi|^s$$
$$+ \operatorname{sgn}\big(u(0^+)-u(\sigma^-)\big)\big(\phi(0^+)-\phi(\sigma^-)\big) - \int_0^\sigma f(t,u)\phi\,dt. \quad (35)$$

Assume now that $f(\cdot,u) \in L^1(0,\sigma)$ and (31) holds for all $v \in BV(0,\sigma)$. Then, for any given $\phi \in BV(0,\sigma)$, the function $\mathcal{T}_\phi$ has a minimum at $s=0$. Fix $\phi \in BV(0,\sigma)$ as required in [28, Theorem 3.6] and such that $\phi(0^+) = \phi(\sigma^-)$ if $u(0^+) = u(\sigma^-)$. Since $s=0$ is a minimum of $\mathcal{T}_\phi$, we have $\mathcal{T}_\phi'(0) = 0$, or equivalently using (35)

$$\int_0^\sigma \frac{(Du)^a\,(D\phi)^a}{\sqrt{1+|(Du)^a|^2}}\,dt + \int_0^\sigma \operatorname{sgn}\left(\frac{Du}{|Du|}\right)\,\frac{D\phi}{|D\phi|}\,|D\phi|^s$$
$$+ \operatorname{sgn}\big(u(0^+)-u(\sigma^-)\big)\,\big(\phi(0^+)\; - \phi(\sigma^-)\big) = \int_0^\sigma f(t,u)\phi\,dt.$$

This means that u is a σ-periodic solution of (1).

Conversely, let us assume that u is a σ-periodic solution of (1) and fix $v \in BV(0,\sigma)$. According to [12, Corollary 2.2], there exists a sequence $(v_n)_n$ in $W^{1,1}(0,\sigma)$ such that $v_n(0^+) = v_n(\sigma^-)$ for all n,

$$\lim_{n\to+\infty} v_n = v$$

in $L^1(0,\sigma)$ and a.e. in $[0,\sigma]$, and

$$\lim_{n\to+\infty} \mathcal{J}_\sigma(v_n) = \mathcal{J}_\sigma(v).$$

For each n, set $\phi_n = v_n - u$. We have that $\phi_n \in BV(0,\sigma)$, with $|D\phi_n|^s = |Du|^s$ and $\phi_n(0^+) - \phi_n(\sigma^-) = u(0^+) - u(\sigma^-)$. As, by assumption, u satisfies the Euler equation (2), we have $\mathcal{T}_{\phi_n}'(0) = 0$; moreover, by convexity of $\mathcal{T}_{\phi_n}$, there holds

$$\mathcal{T}_{\phi_n}(1) \ge \mathcal{T}_{\phi_n}'(0)\cdot 1 + \mathcal{T}_{\phi_n}(0) = \mathcal{T}_{\phi_n}(0),$$

that is,

$$\mathcal{J}_\sigma(u+\phi_n) - \int_0^\sigma f(t,u)(u+\phi_n)\,dt \ge \mathcal{J}_\sigma(u) - \int_0^\sigma f(t,u)u\,dt,$$

or equivalently

$$\mathcal{J}_\sigma(v_n) - \int_0^\sigma f(t,u)v_n\,dt \ge \mathcal{J}_\sigma(u) - \int_0^\sigma f(t,u)u\,dt. \quad (36)$$

As by Lebesgue convergence theorem we have

$$\lim_{n\to+\infty} \int_0^\sigma f(t,u)(v_n)\,dt = \int_0^\sigma f(t,u)v\,dt,$$

we can pass to the limit in (36), as n goes to $+\infty$, obtaining

$$\mathcal{J}_\sigma(v) - \mathcal{J}_\sigma(u) \ge \int_0^\sigma f(t,u)(v-u)\,dt,$$

that is (31). □

Lemma 4.2. *Let $u \in BV_{\mathrm{loc}}(\mathbb{R})$ be a non-constant σ-periodic function, for some $\sigma > 0$. Then u has a minimum period $\tau > 0$ and $\frac{\sigma}{\tau} \in \mathbb{N}_0$.*

Proof. We first prove that u cannot have arbitrarily small periods. Assume by contradiction that there exists a sequence $(\sigma_n)_n$, with $0 < \sigma_n < \frac{\sigma}{n}$, of periods of u. As u is σ_n-periodic, we have by [12, Proposition 2.9]

$$0 < 2(\operatorname*{ess\,sup}_{\mathbb{R}} u - \operatorname*{ess\,inf}_{\mathbb{R}} u) = 2(\operatorname*{ess\,sup}_{[0,\sigma_n]} u - \operatorname*{ess\,inf}_{[0,\sigma_n]} u)$$
$$\leq \int_0^{\sigma_n} |Du| + |u(\sigma_n^-) - u(0^+)| \leq \frac{1}{n}\Big(\int_0^{\sigma} |Du| + |u(\sigma^-) - u(0^+)|\Big),$$

which yields a contradiction by letting $n \to +\infty$.

Let us denote by $\mathcal{T}$ the set of all (positive) periods of u and set $\tau = \inf \mathcal{T}$. We know from the previous step that $\tau > 0$. Let us show that τ is the minimum period. Let $(\sigma_n)_n$ be a sequence in $\mathcal{T}$ converging to τ, with $\sigma_n > \tau$ for all n . Let u_r denote the right continuous representative of the bounded variation function u. As there exists a set $E \subseteq \mathbb{R}$, with zero Lebesgue measure, for which $u_r(t + \sigma_n) = u_r(t)$ for every n and each $t \in \mathbb{R} \setminus E$, we infer that $u_r(t + \tau) = u_r(t)$ for each $t \in \mathbb{R} \setminus E$, that is, $\tau \in \mathcal{T}$.

It is finally clear from the previous steps that, $\sigma > 0$ being a period of u, there exists $N \in N_0$ such that $\sigma = N\tau$. □

Lemma 4.3. *Let $u \in BV_{\mathrm{loc}}(\mathbb{R})$ be a non-constant σ-periodic solution of* (1) *and let $\tau > 0$ be the minimum period of u. Then u is a τ-periodic solution of* (1).

Proof. Suppose that $\sigma > \tau$. By definition of σ-periodic solution of(1), u satisfies

$$\int_0^{\sigma} \frac{(Du)^a \phi'}{\sqrt{1 + |(Du)^a|^2}}\, dt = \int_0^{\sigma} f(t, u)\phi\, dt \tag{37}$$

for every $\phi \in C_0^\infty(]0, \sigma[)$.

Let us prove that the function $f(\cdot, u) \in L^1_{\mathrm{loc}}(\mathbb{R})$ is τ-periodic. As τ is the minimum period of u, there exists $N \in \mathbb{N}$, with $N \geq 2$, such that $\sigma = N\tau$. Assume by contradiction that, e.g., $f(t, u(t)) \neq f(t + \tau, u(t + \tau))$ for all t in a subset of $[0, \tau]$ of positive measure. Take $\phi_1 \in C_0^\infty(]0, \tau[)$ and $\phi_2 \in C_0^\infty(]\tau, 2\tau[)$, with $\phi_1(t) = \phi_2(t + \tau)$ in $[0, \tau]$, such that

$$\int_0^{\tau} f(t, u)\phi_1\, dt \neq \int_{\tau}^{2\tau} f(t, u)\phi_2\, dt.$$

As $(Du)^a = (Du(\cdot - \tau))^a$ a.e. in $\mathbb{R}$ and $\phi_2' = \phi_1'(\cdot - \tau)$ in $[\tau, 2\tau]$, we get from (37)

$$\int_0^{\tau} f(t, u)\phi_1\, dt = \int_0^{\tau} \frac{(Du)^a \phi_1'}{\sqrt{1 + |(Du)^a|^2}}\, dt$$
$$= \int_{\tau}^{2\tau} \frac{(Du)^a \phi_2'}{\sqrt{1 + |(Du)^a|^2}}\, dt = \int_{\tau}^{2\tau} f(t, u)\phi_2\, dt,$$

which is a contradiction.

We next prove that u is a τ-periodic solution of (1). Pick any $w \in BV(0, \tau)$ and let $v_i \in BV((i-1)\tau, i\tau)$ be such that $v_i(t + (i-1)\tau) = w(t)$ for a.e. $t \in [0, \tau]$, for $i = 1, \ldots, N$.

Define $v \in BV(0,\sigma)$ by $v(t) = v_i(t)$ for a.e. $t \in [(i-1)\tau, i\tau]$, for $i = 1,\dots,N$. We have

$$\begin{aligned}
&\int_0^\sigma \sqrt{1+|Dv|^2} + |v(\sigma^-) - v(0^+)| - \int_0^\sigma f(t,u)v\,dt \\
&\quad = \sum_{i=1}^N \int_{(i-1)\tau}^{i\tau} \sqrt{1+|Dv_i|^2} + \sum_{i=1}^{N-1} |v\big((i\tau)^+\big) - v\big((i\tau)^-\big)| + |v(\sigma^-) - v(0^+)| \\
&\qquad\qquad - \sum_{i=1}^N \int_{(i-1)\tau}^{i\tau} f(t,u)v_i\,dt \\
&\quad = N\int_0^\tau \sqrt{1+|Dw|^2} + N|w(\tau^-) - w(0^+)| - N\int_0^\tau f(t,u)w\,dt
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^\sigma \sqrt{1+|Du|^2} + |u(\sigma^-) - u(0^+)| - \int_0^\sigma f(t,u)u\,dt \\
&\qquad = N\int_0^\tau \sqrt{1+|Du|^2} + N|u(\tau^-) - u(0^+)| - N\int_0^\tau f(t,u)u\,dt.
\end{aligned}$$

Hence we conclude that

$$\begin{aligned}
&\int_0^\tau \sqrt{1+|Du|^2} + |u(\tau^-) - u(0^+)| - \int_0^\tau f(t,u)u\,dt \\
&\qquad \le \int_0^\tau \sqrt{1+|Dw|^2} + |w(\tau^-) - w(0^+)| - \int_0^\tau f(t,u)w\,dt.
\end{aligned}$$

Therefore u is a τ-periodic solution of (1). □

The following result guarantees the existence of a sequence of arbitrarily large kT-periodic solutions of (1).

Theorem 4.4. *Assume* $(k_0), (k_2)$,

(k_1') $\displaystyle\lim_{s\to\pm\infty} \frac{F(t,s)}{|s|} = 0$ *uniformly a.e. in* $[0,T]$,

(k_3') *there exists* $R > 0$ *such that* $f(t,s)\,s \ge 0$ *for a.e.* $t \in [0,T]$ *and every* s *with* $|s| \ge R$.

Then there exists a sequence $(u_k)_k$ *of* kT*-periodic solutions of* (1)*, satisfying*

$$\lim_{k\to+\infty} \operatorname*{ess\,sup}_{[0,kT]} u_k = +\infty \quad \text{or} \quad \lim_{k\to+\infty} \operatorname*{ess\,inf}_{[0,kT]} u_k = -\infty. \tag{38}$$

Proof. For any fixed $k \in \mathbb{N}_0$, define a functional $\mathcal{I}_{kT} : BV(0,kT) \to \mathbb{R}$ by setting

$$\mathcal{I}_{kT}(v) = \mathcal{J}_{kT}(v) - \int_0^{kT} F(t,v)\,dt,$$

where $\mathcal{J}_{kT}$ is defined in (32) with $\sigma = kT$. We also set

$$\mathcal{W}_{kT} = \Big\{ w \in BV(0,kT) : \int_0^{kT} w\,dt = 0 \Big\}$$

and, for every $v \in BV(0,kT)$,
$$r = \frac{1}{kT}\int_0^{kT} v\,dt,$$
so that $w = v - r \in \mathcal{W}_{kT}$.

Step 1. *$\mathcal{I}_{kT}$ has a mountain-pass geometry.*

Assumptions (k_0) and (k_1') imply that for every $\varepsilon > 0$ there exists a T-periodic function $c_\varepsilon \in L^1_{\mathrm{loc}}(\mathbb{R})$ such that
$$F(t,s) \le \varepsilon|s| + c_\varepsilon(t), \tag{39}$$
for a.e. $t \in \mathbb{R}$ and every $s \in \mathbb{R}$. For every $w \in \mathcal{W}_{kT}$, we have, using (39) and [12, Corollary 2.7],
$$\begin{aligned}
\mathcal{I}_{kT}(w) &= \mathcal{J}_{kT}(w) - \int_0^{kT} F(t,w)\,dt \\
&\ge \int_0^{kT} |Dw| + |w\big((kT)^-\big) - w(0^+)| - \varepsilon\int_0^{kT} |w|\,dt - \int_0^{kT} c_\varepsilon\,dt \\
&\ge \left(1 - \varepsilon\frac{kT}{4}\right)\left(\int_0^{kT} |Dw| + |w\big((kT)^-\big) - w(0^+)|\right) - \int_0^{kT} c_\varepsilon\,dt.
\end{aligned}$$
This implies that
$$\inf_{\mathcal{W}_{kT}} \mathcal{I}_{kT} > -\infty.$$
On the other hand, by assumption (k_2) there exists $A_k \in \mathbb{R}_0^+$ such that
$$kT - \int_0^{kT} F(t,A_k)\,dt < \inf_{\mathcal{W}_{kT}} \mathcal{I}_{kT}.$$
According to [12, Lemma 2.13], there exist sequences $(v_n)_n$ in $BV(0,kT)$ and $(\varepsilon_n)_n$ in $\mathbb{R}$ such that $\lim\limits_{n\to+\infty} \varepsilon_n = 0$,
$$\mathcal{J}_{kT}(v) - \mathcal{J}_{kT}(v_n) \ge \int_0^{kT} f(t,v_n)(v - v_n)\,dt - \varepsilon_n \left\|v - v_n\right\|_{BV(0,kT)}, \tag{40}$$
for every $v \in BV(0,kT)$, and
$$\lim_{n\to+\infty} \mathcal{I}_{kT}(v_n) = c_k = \inf_{\gamma\in\Gamma_k}\max_{\xi\in[-A_k,A_k]} \mathcal{I}_{kT}\big(\gamma(\xi)\big), \tag{41}$$
where
$$\Gamma_k = \left\{\gamma \in C^0\big([-A_k,A_k], BV(0,kT)\big) : \gamma(\pm A_k) = \pm A_k\right\}. \tag{42}$$

Step 2. *The sequence $(v_n)_n$ is bounded in $BV(0,kT)$.*

Let us write, for each $n \in \mathbb{N}$, $v_n = w_n + r_n$, with $w_n \in \mathcal{W}_{kT}$. Assume by contradiction that, possibly passing to a subsequence that we still denote by $(v_n)_n$,
$$\lim_{n\to+\infty} \|v_n\|_{BV(0,kT)} = +\infty. \tag{43}$$

Using (39), (41) and [12, Corollary 2.7] again, we get for all large n

$$\begin{aligned} c_k + 1 \geq \mathcal{I}_{kT}(v_n) &= \mathcal{J}_{kT}(w_n) - \int_0^{kT} F(t, w_n + r_n)\, dt \\ &\geq \int_0^{kT} |Dw_n| + |w_n\big((kT)^-\big) - w_n(0^+)| - \varepsilon \int_0^{kT} |w_n|\, dt - \varepsilon kT|r_n| - \int_0^{kT} c_\varepsilon\, dt \\ &\geq \left(1 - \varepsilon \frac{kT}{4}\right)\left(\int_0^{kT} |Dw_n| + |w_n\big((kT)^-\big) - w_n(0^+)|\right) - \varepsilon kT|r_n| - \int_0^{kT} c_\varepsilon\, dt. \end{aligned}$$

Hence we deduce that for every $\eta > 0$, there exists $c_\eta > 0$ such that for all large n

$$\frac{1}{2}\left(\int_0^{kT} |Dw_n| + |w_n\big((kT)^-\big) - w_n(0^+)|\right) \leq \eta|r_n| + c_\eta$$

and, by [12, Corollary 2.10],

$$\|w_n\|_{L^\infty(0,kT)} \leq \eta|r_n| + c_\eta. \tag{44}$$

By (43) and (44), we infer that

$$\lim_{n\to+\infty} |r_n| = +\infty.$$

Possibly passing to a further subsequence that we still denote by $(v_n)_n$, we can suppose that either

$$\lim_{n\to+\infty} r_n = +\infty \quad \text{or} \quad \lim_{n\to+\infty} r_n = -\infty.$$

Assume that the former case occurs. From (44) we get

$$\lim_{n\to+\infty} \operatorname*{ess\,inf}_{[0,kT]} v_n = +\infty \tag{45}$$

and hence, by (k_3'), for all large n

$$f\big(t, v_n(t)\big) \geq 0,$$

for a.e. $t \in [0, kT]$. Testing (40) against $v = v_n \pm 1$, we obtain

$$0 \geq \pm \int_0^{kT} f(t, v_n)\, dt - \varepsilon_n kT$$

and then, for all large n,

$$\int_0^{kT} |f(t, v_n)|\, dt = \left|\int_0^{kT} f(t, v_n)\, dt\right| \leq \varepsilon_n kT. \tag{46}$$

Now, test (40) against $v = r_n$. We get, using (46) and [12, Corollary 2.7, Corollary 2.10],

$$\begin{aligned} \mathcal{J}_{kT}(w_n) &\leq \mathcal{J}_{kT}(r_n) + \int_0^{kT} f(t, v_n) w_n\, dt + \varepsilon_n \|w_n\|_{BV(0,kT)} \\ &\leq kT + \|f(\cdot, v_n)\|_{L^1(0,kT)} \|w_n\|_{L^\infty(0,kT)} + \varepsilon_n \|w_n\|_{BV(0,kT)} \\ &\leq \varepsilon_n\left(1 + \frac{3}{4}kT\right)\left(\int_0^{kT} |Dw_n| + |w_n\big((kT)^-\big) - w_n(0^+)|\right) + kT. \end{aligned}$$

Hence we can easily conclude that there exists a constant $c > 0$ such that for all large n

$$\int_0^{kT} |Dw_n| + |w_n\big((kT)^-\big) - w_n(0^+)| \leq c$$

and

$$\mathcal{J}_{kT}(w_n) \leq c.$$

Therefore, using (k_3'), we get for all large n

$$\begin{aligned}\mathcal{I}_{kT}(v_n) &= \mathcal{J}_{kT}(w_n) - \int_0^{kT} F(t, v_n)\,dt \\ &= \mathcal{J}_{kT}(w_n) - \int_0^{kT} \Big(\int_0^{\operatorname*{ess\,inf}_{[0,kT]} v_n} f(t,s)\,ds\Big)\,dt - \int_0^{kT} \Big(\int_{\operatorname*{ess\,inf}_{[0,kT]} v_n}^{v_n(t)} f(t,s)\,ds\Big)\,dt \\ &\leq c - \int_0^{kT} F\big(t, \operatorname*{ess\,inf}_{[0,kT]} v_n\big)\,dt.\end{aligned}$$

Then a contradiction follows, using (45) and (k_2), as we have by (41)

$$\inf_n \mathcal{I}_{kT}(v_n) > -\infty.$$

Step 3. *For each $k \in \mathbb{N}_0$, there exists a kT-periodic solution u_k of* (1)*, with $\mathcal{I}_{kT}(u_k) = c_k$.* Let $k \in \mathbb{N}_0$ be fixed. Since by Step 2 the sequence $(v_n)_n$ is bounded in $BV(0,kT)$, there exists a subsequence, that we still denote by $(v_n)_n$, and a function $u_k \in BV(0,kT)$ such that

$$\lim_{n\to+\infty} v_n = u_k,$$

in $L^1(0,kT)$ and a.e. in $[0,kT]$, and

$$\sup_n \|v_n\|_{L^\infty(0,kT)} < +\infty.$$

Hence, using (k_0), [12, Proposition 2.4] and Lebesgue convergence theorem, we get

$$\begin{gathered}\lim_{n\to+\infty} f(\cdot, v_n) = f(\cdot, u_k), \quad \text{in } L^1(0,kT), \\ \lim_{n\to+\infty} \int_0^{kT} F(t,v_n)\,dt = \int_0^{kT} F(t,u_k)\,dt, \\ \liminf_{n\to+\infty} \mathcal{J}_{kT}(v_n) \geq \mathcal{J}_{kT}(u_k), \\ \lim_{n\to+\infty} \int_0^{kT} f(t,v_n)(v - v_n)\,dt = \int_0^{kT} f(t,u_k)(v-u_k)\,dt, \\ \lim_{n\to+\infty} \varepsilon_n \|v - v_n\|_{BV(0,kT)} = 0,\end{gathered}$$

for every $v \in BV(0,kT)$. Accordingly, we obtain from (40)

$$\begin{aligned}\mathcal{J}_{kT}(v) - \int_0^{kT} f(t,u_k)(v-u_k)\,dt &= \mathcal{J}_{kT}(v) - \lim_{n\to+\infty} \int_0^{kT} f(t,v_n)(v-v_n)\,dt \\ &\quad + \lim_{n\to+\infty} \varepsilon_n \|v - v_n\|_{BV(0,kT)} \geq \liminf_{n\to+\infty} \mathcal{J}_{kT}(v_n) \geq \mathcal{J}_{kT}(u_k),\end{aligned}$$

i.e., u_k is a kT-periodic solution of (1). Moreover, testing (40) against $v = u_k$, we get

$$\mathcal{J}_{kT}(u_k) - \int_0^{kT} f(t, u_k)(u_k - v_n)\, dt + \varepsilon_n \|u_k - v_n\|_{BV(0,kT)} \geq \mathcal{J}_{kT}(v_n).$$

Letting $n \to +\infty$, we have

$$\mathcal{J}_{kT}(u_k) \geq \limsup_{n\to+\infty} \mathcal{J}_{kT}(v_n).$$

As

$$\mathcal{J}_{kT}(u_k) \leq \liminf_{n\to+\infty} \mathcal{J}_{kT}(v_n),$$

we conclude that

$$\mathcal{J}_{kT}(u_k) = \lim_{n\to+\infty} \mathcal{J}_{kT}(v_n)$$

and

$$\mathcal{I}_{kT}(u_k) = \lim_{n\to+\infty} \mathcal{I}_{kT}(v_n) = c_k.$$

Step 4. *The following limits hold*

$$\begin{aligned} \lim_{k\to+\infty} \frac{1}{k} \mathcal{I}_{kT}(u_k) &= -\infty, \\ \lim_{k\to+\infty} \frac{1}{k} \|u_k\|_{L^1(0,kT)} &= +\infty, \\ \lim_{k\to+\infty} \|u_k\|_{L^\infty(0,kT)} &= +\infty. \end{aligned} \tag{47}$$

For each $k \in \mathbb{N}_0$, let $\phi_k : [0, kT] \to \mathbb{R}$ be defined by $\phi_k(t) = k \operatorname{sgn}\Big(t - \dfrac{kT}{2}\Big)$. Note that ϕ_k is an eigenfunction associated with the second eigenvalue $\frac{kT}{4}$ of the 1-Laplace operator with periodic boundary condition on $[0, kT]$ (cf. [29]). Define a path $\gamma_k : [-A_k, A_k] \to BV(0, kT)$ by

$$\gamma_k(\xi) = \xi + \bigg(1 - \frac{|\xi|}{A_k}\bigg)\phi_k.$$

Clearly, we have $\gamma_k \in \Gamma_k$, where Γ_k is defined in (42). Let us compute, for each $\xi \in [-A_k, A_k]$,

$$\mathcal{J}_{kT}\big(\gamma_k(\xi)\big) = \mathcal{J}_{kT}\bigg(\bigg(1 - \frac{|\xi|}{A_k}\bigg)\phi_k\bigg) = 4k\bigg(1 - \frac{|\xi|}{A_k}\bigg) + kT \leq (4 + T)k.$$

Hence we obtain, for each $\xi \in [-A_k, A_k]$,

$$\mathcal{I}_{kT}\big(\gamma_k(\xi)\big) \leq (4 + T)k - \int_0^{kT} F\big(t, \gamma_k(\xi)\big)\, dt. \tag{48}$$

Now we want to estimate the last integral for all large k. Note that we can assume, without restriction, that $A_k \geq k \geq \max\{4, R\}$. Hence, we see that, for each $\xi \in [-A_k, A_k]$, there exists an interval $[a_k, b_k] \subseteq [0, kT]$ with $a_k = a_k(\xi), b_k = b_k(\xi)$ and $b_k - a_k \geq \left\lfloor \frac{k}{2} \right\rfloor T$, such that

$$|\gamma_k\big(\xi\big)(t)| \geq k,$$

for a.e. $t \in [a_k, b_k]$. Indeed, the following statements hold:

- if $\xi = 0$, then
$$|\gamma_k(0)(t)| = \left| k \operatorname{sgn}\left(t - \frac{kT}{2}\right)\right| \geq k,$$
a.e. in $[a_k, b_k] = [0, kT]$;
- if $\xi \in]0, A_k]$, then
$$|\gamma_k(\xi)(t)| = \left|\xi + k\left(1 - \frac{\xi}{A_k}\right)\operatorname{sgn}\left(t - \frac{kT}{2}\right)\right| = k + \xi\left(1 - \frac{k}{A_k}\right) \geq k,$$
a.e. in $[a_k, b_k] = \left[\left\lceil \frac{k}{2}\right\rceil T, kT\right]$.
- if $\xi \in [-A_k, 0[$, then
$$|\gamma_k(\xi)(t)| = \left|\xi + k\left(1 + \frac{\xi}{A_k}\right)\operatorname{sgn}\left(t - \frac{kT}{2}\right)\right| = -\xi\left(1 - \frac{k}{A_k}\right) + k \geq k,$$
a.e. in $[a_k, b_k] = \left[0, \lfloor \frac{k}{2}\rfloor T\right]$.

As we assumed $A_k \geq k \geq R$, we have, using (k_3'),
$$F(t, s) \geq F(t, k),$$
for a.e. $t \in \mathbb{R}$ and every $s \in \mathbb{R}$ with $|s| \geq k$. Moreover, by (k_0), there exists a T-periodic function $h \in L^1_{\mathrm{loc}}(\mathbb{R})$ such that
$$F(t, s) \geq -h(t),$$
for a.e. $t \in \mathbb{R}$ and every $s \in \mathbb{R}$. Therefore we obtain, for every $\xi \in [-A_k, A_k]$,
$$\begin{aligned}
\int_0^{kT} F\big(t, \gamma_k(\xi)\big)\, dt &= \int_{a_k}^{b_k} F\big(t, \gamma_k(\xi)\big)\, dt + \int_{[0,kT]\setminus[a_k,b_k]} F\big(t, \gamma_k(\xi)\big)\, dt \\
&\geq \int_{a_k}^{b_k} F(t, k)\, dt - \int_{[0,kT]\setminus[a_k,b_k]} h\, dt \\
&\geq \int_{a_k}^{b_k} F(t, k)\, dt - k\, \|h\|_{L^1(0,T)}\,.
\end{aligned} \tag{49}$$
Note that, for any $\xi \in [-A_k, A_k]$, we have
$$\int_{a_k}^{b_k} F(t, k)\, dt \geq \left\lfloor \frac{k}{2} \right\rfloor \int_0^T F(t, k)\, dt$$
and then, by (k_2)
$$\lim_{k\to+\infty} \frac{1}{k} \int_{a_k}^{b_k} F(t, k)\, dt = +\infty.$$
Therefore we can conclude, from (48) and (49), that
$$\begin{aligned}
&\lim_{k\to+\infty} \frac{1}{k} \max_{\xi\in[-A_k,A_k]} \mathcal{I}_{kT}\big(\gamma_k(\xi)\big) \\
&\qquad \leq \lim_{k\to+\infty} \left(4 + T + \|h\|_{L^1(0,T)} - \frac{1}{k} \int_{a_k}^{b_k} F(t, k)\, dt\right) = -\infty,
\end{aligned}$$

which in turn implies that

$$\lim_{k\to+\infty} \frac{1}{k}\mathcal{I}_{kT}(u_k) = -\infty, \tag{50}$$

as

$$\mathcal{I}_{kT}(u_k) = \inf_{\gamma\in\Gamma_k} \max_{\xi\in[-A_k,A_k]} \mathcal{I}_{kT}\big(\gamma(\xi)\big).$$

We finally observe that, as

$$\frac{1}{k}\mathcal{I}_{kT}(u_k) \geq -\frac{1}{k}\int_0^{kT} F(t,u_k)\,dt,$$

from (50) it follows

$$\lim_{k\to+\infty} \frac{1}{k}\int_0^{kT} F(t,u_k)\,dt = +\infty.$$

Using (39) with $\varepsilon = 1$, we have

$$\begin{aligned}\frac{1}{k}\int_0^{kT} F(t,u_k)\,dt &\leq \frac{1}{k}\int_0^{kT} |u_k|\,dt + \frac{1}{k}\int_0^{kT} |c_1|\,dt\\ &= \frac{1}{k}\|u_k\|_{L^1(0,kT)} + \|c_1\|_{L^1(0,T)} \leq T\,\|u_k\|_{L^\infty(0,kT)} + \|c_1\|_{L^1(0,T)},\end{aligned}$$

and hence both

$$\lim_{k\to+\infty} \frac{1}{k}\|u_k\|_{L^1(0,kT)} = +\infty$$

and

$$\lim_{k\to+\infty} \|u_k\|_{L^\infty(0,kT)} = +\infty$$

hold true. □

Remark 4.1 Under the assumptions of Theorem 4.4 we cannot exclude that all the solutions u_k are constant. This cannot happen we slightly strengthen assumption (k_3') by replacing it with (k_3), in that case the obtained solutions, if classical, would be subharmonic solutions in the sense, e.g., of [30, p. 426]. This is the content of Theorem 4.5.

Theorem 4.5. *Assume (k_0), (k_1'), (k_2) and (k_3). Then there exists a sequence $(u_k)_{k\in\mathbb{N}_0}$ of kT-periodic solutions of* (1)*, satisfying*

$$\lim_{k\to+\infty} \big(\operatorname*{ess\,sup}_{[0,kT]} u_k - \operatorname*{ess\,inf}_{[0,kT]} u_k\big) = +\infty. \tag{51}$$

Moreover, for each $N \in \mathbb{N}_0$, there exists $\bar{k}$ such that, for every $k \geq \bar{k}$, u_{kN} is not a NT-periodic solution of (1)*.*

Proof. Theorem 4.4 guarantees the existence of a sequence $(u_k)_k$ of kT-periodic solutions of (1) for which (38) holds. Let us prove the validity of (51). Indeed, otherwise from (38) we deduce, possibly passing to a subsequence of $(u_k)_k$, still denoted by $(u_k)_k$, that

$$\lim_{k\to+\infty} \operatorname*{ess\,inf}_{[0,kT]} u_k = +\infty \quad \text{or} \quad \lim_{k\to+\infty} \operatorname*{ess\,sup}_{[0,kT]} u_k = -\infty.$$

Assume that the former case occurs. Hence condition (k_3) implies that, for all large k,

$$f(t, u_k(t)) > 0, \tag{52}$$

for a.e. $t \in [0, kT]$. By Lemma 4.1, testing (31), with $\sigma = kT$ and $u = u_k$, against $v = u_k \pm 1$, we infer

$$\int_0^{kT} f(t, u_k)\, dt = 0.$$

A contradiction then follows from (52).

Next, in order to prove the last conclusion, we suppose by contradiction that, for some $N \in \mathbb{N}_0$, there exists a subsequence $(u_{k_j N})_j$ of $(u_{kN})_k$, such that, for every j, $u_{k_j N}$ is NT-periodic. Let us denote this subsequence by $(u_{kN})_k$ for simplicity. From condition (47) and the NT-periodicity of u_{kN}, we have

$$\lim_{k\to+\infty} \frac{1}{N}\mathcal{I}_{NT}(u_{kN}) = \lim_{k\to+\infty} \frac{1}{kN}\mathcal{I}_{kNT}(u_{kN}) = -\infty.$$

This implies that

$$\sup_k \mathcal{I}_{NT}(u_{kN}) = M < +\infty.$$

The same argument employed in Step 2 in the proof of Theorem 4.4 yields the existence of a further subsequence of $(u_{kN})_N$, which we still denote by $(u_{kN})_N$, such that

$$\lim_{k\to+\infty} \operatorname*{ess\,inf}_{[0,NT]} u_{kN} = +\infty \quad \text{or} \quad \lim_{k\to+\infty} \operatorname*{ess\,sup}_{[0,NT]} u_{kN} = -\infty.$$

Now we proceed as above in order to get a contradiction by means of condition (k_3). $\square$

Remark 4.2 By a diagonal argument, we see that there exists a sequence $(k_j)_j$ of positive integers, with $k_j \geq j$ for every $j \in \mathbb{N}_0$, such that the corresponding solutions $(u_{k_j})_j$ are k_jT-periodic, but not hT-periodic for $h = 1, \dots, j$.

Finally, if both assumptions (k_1') and (k_3') are strengthened into (k_1) and (k_3), respectively, then the obtained solutions exhibit large-amplitude oscillations and have arbitrarily large minimum periods, as stated in Theorem 1.2.

Proof of Theorem 1.2. Theorem 4.5 guarantees the existence of a sequence $(u_k)_k$ of kT-periodic solutions of (1) for which (51) holds. Since, for all large k, u_k is a non-constant kT-periodic solution of (1), Lemma 4.2 and Lemma 4.3 imply that u_k has a minimum period $\tau_k > 0$ and it is a τ_k-periodic solution of (1), i.e., for every $v \in BV(0, \tau_k)$,

$$\mathcal{J}_{\tau_k}(v) - \mathcal{J}_{\tau_k}(u_k) \geq \int_0^{\tau_k} f(t, u_k)(v - u_k)\, dt. \tag{53}$$

We want to prove that

$$\lim_{k\to+\infty} \tau_k = +\infty.$$

Assume by contradiction that there exists a subsequence $(\tau_{k_j})_j$ of $(\tau_k)_k$ such that

$$\sup_j \tau_{k_j} = \tau < +\infty.$$

Let us denote $(\tau_{k_j})_j$ simply by $(\tau_k)_k$. Assumptions (k_0) and (k_1) imply that, for every $\varepsilon > 0$, there exists $c_\varepsilon \in L^1_{\mathrm{loc}}(\mathbb{R})$ such that

$$f(t,s)s \le \varepsilon|s| + c_\varepsilon(t), \tag{54}$$

for a.e. $t \in \mathbb{R}$ and every $s \in \mathbb{R}$. Testing (53) against $v = 0$ and using (54), we get

$$\begin{aligned}\mathcal{J}_{\tau_k}(u_k) &\le \int_0^{\tau_k} f(t,u_k)u_k\,dt + \tau_k\\ &\le \varepsilon\int_0^{\tau_k} |u_k|\,dt + \int_0^{\tau_k} c_\varepsilon\,dt + \tau_k\\ &\le \varepsilon\tau\,\|u_k\|_{L^\infty(0,\tau_k)} + \|c_\varepsilon\|_{L^1(0,\tau)} + \tau.\end{aligned}$$

Set $r_k = \frac{1}{\tau_k}\int_0^{\tau_k} u_k\,dt$ and $w_k = u_k - r_k$. By [12, Corollary 2.10], we obtain

$$\begin{aligned}2\,\|w_k\|_{L^\infty(0,\tau_k)} &\le \int_0^{\tau_k} |Du_k| + |u_k(\tau_k^-) - u_k(0^+)|\\ &\le \varepsilon\tau\,\|w_k\|_{L^\infty(0,\tau_k)} + \varepsilon\tau|r_k| + \|c_\varepsilon\|_{L^1(0,\tau)} + \tau.\end{aligned}$$

Hence we conclude that, for every $\eta > 0$, there exists $c_\eta > 0$, which is independent of k, such that

$$\|w_k\|_{L^\infty(0,\tau_k)} \le \eta|r_k| + c_\eta.$$

This relation is the counterpart of (44) in Step 2 in the proof of Theorem 4.4. We can then proceed as there and obtain, possibly passing to a subsequence of $(u_k)_k$, still denoted by $(u_k)_k$, that

$$\lim_{k\to+\infty}\ \operatorname*{ess\,inf}_{[0,\tau_k]} u_k = +\infty \quad \text{or} \quad \lim_{k\to+\infty}\ \operatorname*{ess\,sup}_{[0,\tau_k]} u_k = -\infty.$$

Arguing as in the first part of the proof of Theorem 4.5 we finally get a contradiction by means of condition (k_3). This concludes the proof of Theorem 1.2.

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