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Subideality and Ascendancy in Generalized Solvable Lie Algebras

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Introduction

Wielandt [8] has given some criteria for a subgroup to be subnormal in a finite group. Peng [6, 7] and Hartley and Peng [3] have given similar criteria for not necessarily finite groups. Furthermore Chao and Stitzinger [2] have given conditions for a subalgebra to be a subideal in a finite-dimensional solvable Lie algebra.

In this paper we shall investigate some criteria for subideality and ascendancy in not necessarily finite-dimensional Lie algebras.

Let L be a Lie algebra over a field $\mathfrak{k}$ and let H be a subalgebra of L. When $L/\mathrm{Core}_L(H)$ is solvable, H is a subideal of L if either (a) there exists some integer $n \geq 0$ such that $[L, {}_nH] \subseteq H$, or (b) there exists some integer $n \geq 0$ such that $[L, {}_nx] \subseteq H$ for any $x \in H$ and the characteristic of $\mathfrak{k}$ is 0 or $p > n$ (Theorem 4 and Theorem 7). When $L/\mathrm{Core}_L(H)$ is hyperabelian, H is an ascendant subalgebra of L if one of the following conditions is satisfied: (c) For any $a \in L$ there exists an integer $n = n(a)$ such that $[a, {}_nH] \subseteq H$; (d) $\mathfrak{k}$ is of characteristic 0, H is solvable, and for any $a \in L$ there exists $n = n(a)$ such that $[a, {}_nx] \in H$ for any $x \in H$ (Theorem 12 and Theorem 14). Finally when $L/\mathrm{Core}_L(H)$ has an ascending abelian series, H is an ascendant subalgebra of L if $\langle a^H \rangle$ is finitely generated for any $a \in L$ and one of the above conditions (c) and (d) is satisfied (Theorem 17 and Theorem 18).

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1. Preliminaries

Throughout the paper Lie algebras are not necessarily finite-dimensional over a field $\mathfrak{k}$ of arbitrary characteristic unless otherwise specified. We mostly follow [1] for the use of notations and terminology.

Let L be a Lie algebra over $\mathfrak{k}$. L belongs to the class $\acute{\mathrm{E}}\mathfrak{A}$ if L has an ascending abelian series $(L_\alpha)_{\alpha \leq \lambda}$. If each L_α $(\alpha \leq \lambda)$ is furthermore an ideal of L, then L belongs to the class $\acute{\mathrm{E}}(\lhd)\mathfrak{A}$, that is, L is hyperabelian. For an integer $n \geq 0$ and an ordinal λ, $H \leq L$, $H \lhd L$, H si L, $H \lhd^n L$, H asc L and $H \lhd^\lambda L$ mean that H is respectively a subalgebra, an ideal, a subideal, an n-step subideal, an ascendant subalgebra and a λ-step ascendant subalgebra of L. If $H \triangleleft^n L$ then n is called the subideal index of H. For any subsets A, B of L we denote by $\langle A^B \rangle$ the smallest B-invariant subalgebra of L containing A. The core $\mathrm{Core}_L(H)$ of a subalgebra H in L is the largest ideal of L contained in H. For any $x, y \in L$ and for any subsets A, B of L we define inductively $[x, {}_0 y] = x$ and $[x, {}_{i+1} y] = [[x, {}_i y], y]$ $(i \in \mathbf{N})$; $[A, {}_0 B] = A$, $[A, {}_{i+1} B] = [[A, {}_i B], B]$ $(i \in \mathbf{N})$.

Let (H, K) be an ordered pair of subalgebras of L. We say (H, K) to be an *N_k-pair* $(k \in \mathbf{N})$ if $[K, {}_k H] \subseteq H$, and to be an *N_∞-pair* if for each $a \in K$ there exists $k = k(a) \in \mathbf{N}$ such that $[a, {}_k H] \subseteq H$. The fact that (H, L) is an N_k-pair means that H is a k-step weak ideal of L in the sense of Maruo [5]. We define

$$N_k(H) = \{a \in L \mid [a, {}_k H] \subseteq H\} \qquad (k \in \mathbf{N}),$$

$$N_\infty(H) = \cup_{k \in \mathbf{N}} N_k(H).$$

It is then clear that (H, K) is an N_k-pair (resp. N_∞-pair) if and only if $K \subseteq N_k(H)$ (resp. $K \subseteq N_\infty(H)$). We say (H, K) to be an *E_k-pair* $(k \in \mathbf{N})$ if $[K, {}_k x] \subseteq H$ for any $x \in H$, and to be an *E_∞-pair* if for each $a \in K$ there exists $k = k(a) \in \mathbf{N}$ such that $[a, {}_k x] \in H$ for any $x \in H$. We define

$$E_k(H) = \{a \in L \mid [a, {}_k x] \in H \quad \text{for any} \quad x \in H\} \qquad (k \in \mathbf{N}),$$

$$E_\infty(H) = \cup_{k \in \mathbf{N}} E_k(H).$$

It is then clear that (H, K) is an E_k-pair (resp. E_∞-pair) if and only if $K \subseteq E_k(H)$ (resp. $K \subseteq E_\infty(H)$). Let $n_1, \dots, n_r$ be integers ≥ 0. We say (H, K) to be an *$E_{n_1, \dots, n_r}$-pair* if

$$[K, {}_{n_1} x_1, \dots, {}_{n_r} x_r] \subseteq H$$

for any $x_1, \dots, x_r \in H$, and we define

$$E_{n_1, \dots, n_r}(H) = \{a \in L \mid [a, {}_{n_1} x_1, \dots, {}_{n_r} x_r] \in H \quad \text{for any} \quad x_1, \dots, x_r \in H\}.$$

We first show the following

LEMMA 1. *Let L be a Lie algebra over a field $\mathfrak{k}$ and H be a subalgebra of L. Then*

(a) *$N_\infty(H)$ is a subalgebra of L and $(H, N_\infty(H))$ is an N_∞-pair.*

(b) *H^ω is an ideal of $N_\infty(H)$.*

PROOF. (a) If $x, y \in N_\infty(H)$, then $x, y \in N_n(H)$ for some $n > 0$. Put $m = 2n - 1$. Then

$$[[x, y], {}_m H] \subseteq \textstyle\sum_{i+j=m} [[x, {}_i H], [y, {}_j H]],$$

and either $i \geq n$ or $j \geq n$. If $i \geq n$, then

$$[x, {}_iH] = [x, {}_nH, {}_{i-n}H] \subseteq H^{i-n+1},$$

and therefore

$$\begin{aligned}[[x, {}_iH], [y, {}_jH]] &\subseteq [[y, {}_jH], H^{i-n+1}] \\ &\subseteq [y, {}_jH, {}_{i-n+1}H] \\ &= [y, {}_nH] \subseteq H.\end{aligned}$$

If $j \geq n$, then we similarly have

$$[[x, {}_iH], [y, {}_jH]] \subseteq H.$$

Therefore

$$[[x, y], {}_mH] \subseteq H,$$

whence $[x, y] \in N_m(H)$. Thus $N_\infty(H)$ is a subalgebra of L.

(b) If $x \in N_\infty(H)$, then $x \in N_n(H)$ for some $n > 0$. Hence for any $m > 0$

$$\begin{aligned}[x, H^\omega] &\subseteq [x, H^{n+m-1}] \\ &\subseteq [x, {}_nH, {}_{m-1}H] \\ &\subseteq [H, {}_{m-1}H] = H^m.\end{aligned}$$

It follows that

$$[x, H^\omega] \subseteq \cap_{m>0} H^m = H^\omega.$$

Therefore H^ω is an ideal of $N_\infty(H)$.

By the same way as in Lemma 1 of [4] we have

LEMMA 2. *Let L be a Lie algebra over a field $\mathfrak{k}$ and H be a subalgebra of L.*

(a) *If the characteristic of $\mathfrak{k}$ is either 0 or $p > \max_{1 \leq i \leq r} n_i$, then $E_{n_1,\dots,n_r}(H)$ is an H-invariant subspace of L.*

(b) *If the characteristic of $\mathfrak{k}$ is 0, then $E_\infty(H)$ is an H-invariant subspace of L.*

PROOF. (a) Put $x^* = \mathrm{ad}_L x$ for any $x \in H$. Let $a \in E_{n_1,\dots,n_r}(H)$. Then

$$a x_1^{*n_1} \cdots x_r^{*n_r} \in H$$

for any $x_1, \dots, x_r \in H$. Replace each x_i by $x_i + t y_i$ where $t \in \mathfrak{k}$ and $y_1, \dots, y_r \in H$, and take the coefficient of t. Then by the argument similar to the linearization in [4] we have

$$a\sum_{i=1}^{r} f_i(x_1^*,\dots, x_r^*;\ y_i^*)\in H \tag{$*$}$$

for any $y_1,\dots, y_r\in H$, where

$$\begin{aligned} &f_i(x_1^*,\dots, x_r^*;\ y_i^*)\\ &\quad = x_1^{*n_1}\cdots x_{i-1}^{*n_{i-1}}\Big(\sum_{j=0}^{n_i-1} x_i^{*n_i-j-1} y_i^* x_i^{*j}\Big)x_{i+1}^{*n_{i+1}}\cdots x_r^{*n_r}. \end{aligned}$$

Let $z\in H$ and substitute $[x_i, z]$ for $y_i\ (i=1,\dots, r)$. Then

$$\begin{aligned} &f_i(x_1^*,\dots, x_r^*;\ [x_i, z]^*)\\ &\quad = x_1^{*n_1}\cdots x_{i-1}^{*n_{i-1}}\Big(\sum_{j=0}^{n_i-1} x_i^{*n_i-j-1}(x_i^* z^* - z^* x_i^*)x_i^{*j}\Big)x_{i+1}^{*n_{i+1}}\cdots x_r^{*n_r}\\ &\quad = x_1^{*n_1}\cdots x_{i-1}^{*n_{i-1}}(x_i^{*n_i} z^* - z^* x_i^{*n_i})x_{i+1}^{*n_{i+1}}\cdots x_r^{*n_r}, \end{aligned}$$

and therefore

$$\begin{aligned} &a\sum_{i=1}^{r} f_i(x_1^*,\dots, x_r^*;\ [x_i, z]^*)\\ &\quad = a(x_1^{*n_1}\cdots x_r^{*n_r} z^* - z^* x_1^{*n_1}\cdots x_r^{*n_r}). \end{aligned}$$

By ($*$) we have

$$\begin{aligned} &[a, z]x_1^{*n_1}\cdots x_r^{*n_r}\\ &\quad = a x_1^{*n_1}\cdots x_r^{*n_r} z^* - a\sum_{i=1}^{r} f_i(x_1^*,\dots, x_r^*;\ [x_i, z]^*)\in H. \end{aligned}$$

It follows that for any $z\in H$

$$[a, z]\in E_{n_1,\dots,n_r}(H).$$

Thus $E_{n_1,\dots,n_r}(H)$ is H-invariant.

(b) is immediately obtained from (a), since $E_\infty(H)=\bigcup_{n\in\mathbf{N}} E_n(H)$.

2. Criteria for subideality

In this section we investigate some conditions for a subalgebra to be a subideal. We need the following simple and useful

LEMMA 3. *Let L be a Lie algebra over a field $\mathfrak{k}$. Let H be a subalgebra of L and A be an abelian ideal of L.*

(a) *If (H, A) is an N_n-pair for some $n\in\mathbf{N}$, then $H\lhd^n A+H$.*

(b) *If (H, A) is an N_∞-pair, then $H\lhd^\omega A+H$.*

PROOF. Put $A_i = A \cap N_i(H)$ for any $i \in \mathbf{N}$. By the definition of $N_i(H)$ it is clear that $[A_i, H] \subseteq A_{i-1}$ and so A_i is H-invariant. Therefore

$$A_i + H \lhd A_{i+1} + H \qquad (i \in \mathbf{N}).$$

If (H, A) is an N_n-pair, then $A \subseteq N_n(H)$. Hence $A_n = A$. It follows that

$$H = A_0 + H \lhd^n A + H.$$

If (H, A) is an N_∞-pair, then $A \subseteq N_\infty(H)$. Hence

$$A = A \cap (\cup_{i \in \mathbf{N}} N_i(H)) = \cup_{i \in \mathbf{N}} A_i.$$

It follows that

$$\cup_{i \in \mathbf{N}} (A_i + H) = \cup_{i \in \mathbf{N}} A_i + H = A + H.$$

Thus we have

$$H = A_0 + H \lhd^\omega A + H.$$

THEOREM 4. *Let L be a Lie algebra over a field $\mathfrak{k}$ and H be a subalgebra of L. If $L/\mathrm{Core}_L(H) \in \mathfrak{A}^m$ and $(H, L^{(1)})$ is an N_n-pair for some $m, n > 0$, then $H \lhd^{n(m-1)+1} L$. In particular, if L is solvable and (H, L) is an N_n-pair for some $n \in \mathbf{N}$, then H is a subideal of L.*

PROOF. For $i > 1$, put $\bar{L} = L/L^{(i)}$ and $\bar{H} = (H + L^{(i)})/L^{(i)}$. Then $\bar{L}^{(i-1)}$ is an abelian ideal of $\bar{L}$ and $(\bar{H}, \bar{L}^{(i-1)})$ is an N_n-pair. By Lemma 3

$$\bar{H} \lhd^n \bar{L}^{(i-1)} + \bar{H},$$

whence

$$L^{(i)} + H \lhd^n L^{(i-1)} + H.$$

Therefore

$$H = L^{(m)} + H \lhd^{n(m-1)} L^{(1)} + H \lhd L.$$

If $m = 2$ in Theorem 4, then the subideal index of H becomes $n+1$. It will be shown by Example 1 in Section 4 that this bound is best possible.

To consider E_n-pair we modify Theorem 1 in [4] and obtain the following

LEMMA 5. *Let L be a Lie algebra over a field $\mathfrak{k}$. Let H be a solvable subalgebra of L of derived length $\leq m$ and A be an ideal of L. Let $n > 0$ be an integer such that $[A, {}_n x] = 0$ for any $x \in H$. If the characteristic of $\mathfrak{k}$ is either* 0 *or $p > n$, then $[A, {}_k H] = 0$ with $k = n^m$.*

PROOF. We use induction on m. The case $m=0$ being trivial, let $m>0$ and assume that the result holds for $m-1$. Then we have

$$[A, {}_rH^{(1)}] = 0$$

with $r=n^{m-1}$. Put $A_i=[A, {}_iH^{(1)}]$ for $i=0,\dots, r$. Then $A_0=A$ and $A_r=0$. It suffices to show that

$$[A_i, {}_nH] \subseteq A_{i+1} \tag{*}$$

for $i=0,\dots, r-1$. In fact, we then have

$$[A_0, {}_{nr}H] \subseteq A_r,$$

that is,

$$[A, {}_kH] = 0 \quad \text{for} \quad k = n^m,$$

as required.

To show (*) we claim that A_i is H-invariant for any i. It is obvious for $i=0$. Assume inductively that A_i is H-invariant. Then

$$\begin{aligned}[A_{i+1}, H] &= [A_i, H^{(1)}, H] \\ &\subseteq [A_i, H, H^{(1)}] + [A_i, [H^{(1)}, H]] \\ &\subseteq [A_i, H^{(1)}] = A_{i+1},\end{aligned}$$

whence A_{i+1} is H-invariant. Now we show (*). Put $x^*=\mathrm{ad}_A x$ for any $x\in H$. Then by the hypothesis of the lemma

$$x^{*n} = 0.$$

By repeated use of the linearization we have

$$\sum_{\pi\in S_n} x^*_{\pi(1)}\cdots x^*_{\pi(n)} = 0$$

for any $x_1,\dots, x_n\in H$. Now $x_i^*x_j^*=x_j^*x_i^*+[x_i, x_j]^*$. Therefore

$$n!x_1^*\cdots x_n^* + f = 0, \tag{**}$$

where f is a linear combination of the element of form

$$x^*_{\pi(1)}\cdots x^*_{\pi(i-1)}[x_{\pi(i)}, x_{\pi(i+1)}]^*x^*_{\pi(i+2)}\cdots x^*_{\pi(n)}.$$

Since A_i is H-invariant,

$$A_ix^*_{\pi(1)}\cdots x^*_{\pi(i-1)}[x_{\pi(i)}, x_{\pi(i+1)}]^*x^*_{\pi(i+2)}\cdots x^*_{\pi(n)}$$

$$\subseteq A_i[x_{\pi(i)}, x_{\pi(i+1)}]^* x^*_{\pi(i+2)} \cdots x^*_{\pi(n)}$$
$$\subseteq A_{i+1} x^*_{\pi(i+2)} \cdots x^*_{\pi(n)}$$
$$\subseteq A_{i+1}.$$

By (**) it follows that

$$A_i x_1^* \cdots x_n^* \subseteq A_i f \subseteq A_{i+1},$$

which is to be shown.

REMARK. In the above lemma the assumption that A is an ideal of L can be replaced by the assumption that A is an H-invariant subspace of L.

We can now state a relation between E_n-pairs and N_k-pairs in the following

LEMMA 6. *Let L be a Lie algebra over a field $\mathfrak{k}$. Let H be a solvable subalgebra of L of derived length $\leq m$ and A be an abelian ideal of L. Let (H, A) be an E_n-pair for some $n>0$. If the characteristic of k is either 0 or $p>n$, then (H, A) is an N_k-pair with $k=n^m$.*

PROOF. Since $A \cap H \lhd A+H$, we take $\overline{A+H}=(A+H)/A \cap H$. Then for any $\bar{x} \in \bar{H}$

$$[\bar{A}, {}_n\bar{x}] \subseteq \bar{A} \cap \bar{H} = \bar{0}.$$

By Lemma 5 we have

$$[\bar{A}, {}_k\bar{H}] = \bar{0}$$

with $k=n^m$. Therefore (H, A) is an N_k-pair.

By making use of Lemma 6 we can prove the following

THEOREM 7. *Let L be a Lie algebra over a field $\mathfrak{k}$. Let H be a subalgebra of L such that $L/\mathrm{Core}_L(H) \in \mathfrak{A}^m$ and $(H, L^{(1)})$ is an E_n-pair for some $m, n>0$. If the characteristic of $\mathfrak{k}$ is either 0 or $p>n$, then H is an h-step subideal of L, where $h=\sum_{i=0}^{m-1} n^i$. In particular, if L is solvable and (H, L) is an E_n-pair for some $n \in \mathbf{N}$, then H is a subideal of L.*

PROOF. We use induction on m. The assertion is clear for $m=1$. Let $m>1$ and assume that the assertion holds for $m-1$. Put $\bar{L}=L/L^{(m-1)}$. Then $\bar{L}^{(m-1)} \subseteq \bar{H}$. Clearly $(\bar{H}, \bar{L}^{(1)})$ is an E_n-pair. By the inductive hypothesis

$$\bar{H} \lhd^r \bar{L}$$

with $r=\sum_{i=0}^{m-2} n^i$, and so

$$L^{(m-1)}+H \lhd^{r} L.$$

We may assume that $L^{(m)}=0$ by considering $L/\mathrm{Core}_L(H)$. Hence $A=L^{(m-1)}$ is an abelian ideal of L. It is clear that $A\cap H \lhd A+H$. Put $\overline{A+H}=(A+H)/A\cap H$. Then $(\bar{H}, \bar{A})$ is an E_n-pair and $\bar{H}^{(m-1)}=0$ since $H^{(m-1)}\subseteq A\cap H$. By Lemma 6 $(\bar{H}, \bar{A})$ is an N_k-pair for $k=n^{m-1}$. Therefore by Lemma 3

$$\bar{H} \lhd^{k} \overline{A+H},$$

whence

$$H \lhd^{k} A+H.$$

Thus we obtain

$$H \lhd^{h} L,$$

where $h=k+r=\sum_{i=0}^{m-1} n^i$.

If $m=2$ in the above theorem, then the subideal index $\sum_{i=0}^{m-1} n^i$ of H becomes $n+1$. It will be shown by Example 1 in Section 4 that this bound is best possible.

In the case that $n=2$, the subideal index can be improved in the following

THEOREM 8. *Let L be a Lie algebra over a field $\mathfrak{k}$ of characteristic $\neq 2$ and H be a subalgebra of L. If $(H, L^{(1)})$ is an E_2-pair and $L/\mathrm{Core}_L(H)\in \mathfrak{A}^m$ for some $m>1$, then H is a $3(m-1)$-step subideal of L.*

PROOF. We use induction on m. If $m=2$ then $H \lhd^{3} L$ by Theorem 7. Let $m>2$ and assume that the assertion holds for $m-1$. Put $\bar{L}=L/L^{(m-1)}$. Then by the inductive hypothesis

$$\bar{H} \lhd^{3(m-2)} \bar{L},$$

and so

$$L^{(m-1)}+H \lhd^{3(m-2)} L.$$

Now $(H, L^{(m-1)})$ is an E_2-pair. By the argument similar to the proof of Theorem 7.3.2 in [1] it is easily seen that $(H, L^{(m-1)})$ is an N_3-pair. By using Lemma 3 we obtain

$$H \lhd^{3} L^{(m-1)}+H.$$

Therefore H is a $3(m-1)$-step subideal of L.

We generalize Theorem 7 by using the following

LEMMA 9. *Let L be a Lie algebra over a field $\mathfrak{k}$. Let H be a solvable* *subalgebra of L of derived length $\leq m$ and A be an abelian ideal of L. Let (H, A) be an $E_{n_1,\dots,n_r}$-pair. If the characteristic of $\mathfrak{k}$ is either 0 or $p > \max_{1\leq i\leq r} n_i$, then (H, A) is an N_k-pair with $k=\sum_{i=1}^r n_i^m$.*

PROOF. We use induction on r. For $r=1$ the assertion holds by Lemma 6. Let $r>1$ and assume the assertion holds for $r-1$. By definition

$$[A, {}_{n_1}x_1, {}_{n_2}x_2, \dots, {}_{n_r}x_r] \subseteq H$$

for any $x_1, x_2, \dots, x_r \in H$. Put $B = A \cap E_{n_2,\dots,n_r}(H)$. Then B is H-invariant by Lemma 2. Thus B is an ideal of $A+H$. Clearly (H, B) is an $E_{n_2,\dots,n_r}$-pair. By the inductive hypothesis we obtain that (H, B) is an N_h-pair for $h=\sum_{i=2}^r n_i^m$. Hence

$$[B, {}_hH] \subseteq H.$$

In $\overline{A+H}=(A+H)/B$ we have

$$[\bar{A}, {}_{n_1}\bar{x}] = \bar{0}$$

for any $x \in H$. By Lemma 5

$$[\bar{A}, {}_{n_1^m}\bar{H}] = \bar{0}$$

and so

$$[A, {}_{n_1^m}H] \subseteq B.$$

It follows that

$$[A, {}_{n_1^m+h}H] \subseteq [B, {}_hH] \subseteq H.$$

Therefore (H, A) is an N_k-pair, where

$$k = n_1^m + h = \sum_{i=1}^r n_i^m.$$

THEOREM 10. *Let L be a Lie algebra over a field $\mathfrak{k}$ and H be a subalgebra of L. Let $L/\mathrm{Core}_L(H) \in \mathfrak{A}^m$ and $(H, L^{(1)})$ be an $E_{n_1,\dots,n_r}$-pair for some $m, n_1, \dots, n_r > 0$. If the characteristic of $\mathfrak{k}$ is either 0 or $p > \max_{1\leq i\leq r} n_i$, then H is an h-step subideal of L where $h=\sum_{j=0}^{m-1}\sum_{i=1}^r n_i^j$. In particular, if L is solvable and (H, L) is an $E_{n_1,\dots,n_r}$-pair, then H is a subideal of L.*

This theorem will be proved in the same way as in Theorem 7, by using Lemma 9 instead of Lemma 6. Hence we omit the proof.

We combine some of the above results in the following theorem, generalizing a result [2, Theorem 1] for finite-dimensional case.

THEOREM 11. *Let L be a Lie algebra over a field $\mathfrak{k}$. Let H be a subalgebra of L such that $L/\mathrm{Core}_L(H)$ is solvable. Then the following conditions are equivalent*:

(a) *H is a subideal of L.*

(b) *There exists $n \in \mathbf{N}$ such that $H \lhd^n \langle H, x \rangle$ for any $x \in L$.*

(c) *(H, L) is an N_n-pair for some $n \in \mathbf{N}$.*

If the field $\mathfrak{k}$ is of characteristic 0, then the above conditions are equivalent to each of the following:

(d) *There exist r, $n = n(r) > 0$ such that $[L, {}_nK] \subseteq H$ for any r-generated subalgebra K of H.*

(e) *(H, L) is an E_n-pair for some $n \in \mathbf{N}$.*

(f) *(H, L) is an $E_{n_1,\dots,n_r}$-pair for some $n_1,\dots, n_r \in \mathbf{N}$.*

PROOF. It is clear that (a)⇒(b)⇒(c) and (a)⇒(d)⇒(e)⇒(f). (c)⇒(a) and (f)⇒(a) follow from Theorem 4 and Theorem 10 respectively.

3. Criteria for ascendancy

In this section we investigate some conditions for a subalgebra to be an ascendant subalgebra for a Lie algebra in the classes $\acute{\mathrm{E}}(\lhd)\mathfrak{A}$ and $\acute{\mathrm{E}}\mathfrak{A}$ of generalized solvable Lie algebras.

THEOREM 12. *Let L be a Lie algebra over a field $\mathfrak{k}$. Let H be a subalgebra of L such that $L/\mathrm{Core}_L(H) \in \acute{\mathrm{E}}(\lhd)\mathfrak{A}$. If (H, L) is an N_∞-pair, then H is an ascendant subalgebra of L.*

PROOF. We may assume that $L \in \acute{\mathrm{E}}(\lhd)\mathfrak{A}$. Let $(L_\alpha)_{\alpha \leq \lambda}$ be an ascending abelian series of ideals of L. For any $\alpha < \lambda$ put $\bar{L} = L/L_\alpha$. Then $\bar{L}_{\alpha+1}$ is an abelian ideal of $\bar{L}$ and $(\bar{H}, \bar{L}_{\alpha+1})$ is an N_∞-pair. By Lemma 3

$$\bar{H} \lhd^\omega \bar{L}_{\alpha+1} + \bar{H}$$

so that

$$L_\alpha + H \lhd^\omega L_{\alpha+1} + H.$$

If $\mu \leq \lambda$ is a limit ordinal, then

$$\cup_{\beta<\mu}(L_\beta + H) = \cup_{\beta<\mu} L_\beta + H = L_\mu + H.$$

Thus we have

$$H = L_0 + H \text{ asc } L_\lambda + H = L.$$

REMARK. Let L be a Lie algebra over $\mathfrak{k}$ and H be a subalgebra of L. If

$L/\mathrm{Core}_L(H) \in \acute{\mathrm{E}}(\lhd)\mathfrak{A}$, then H asc $N_\infty(H)$. In fact, by Lemma 1 $N_\infty(H)$ is a subalgebra of L and $(H, N_\infty(H))$ is an N_∞-pair. Hence the assertion follows from Theorem 12.

We here state a relation between E_∞-pairs and N_∞-pairs in the following

LEMMA 13. *Let L be a Lie algebra over a field $\mathfrak{k}$ of characteristic 0. Let H be a solvable subalgebra of L and A be an abelian ideal of L. If (H, A) is an E_∞-pair, then (H, A) is an N_∞-pair.*

PROOF. For any $n \in \mathbf{N}$ put

$$A_n = A \cap E_n(H).$$

By Lemma 2 A_n is an H-invariant subalgebra of L. Clearly (H, A_n) is an E_n-pair. By Lemma 6 it follows that (H, A_n) is an N_k-pair for some k, so that $A_n \subseteq N_k(H)$. Therefore

$$A = \cup_{n \in \mathbf{N}} A_n \subseteq \cup_{k \in \mathbf{N}} N_k(H) = N_\infty(H).$$

Thus (H, A) is an N_∞-pair.

THEOREM 14. *Let L be a Lie algebra over a field $\mathfrak{k}$ of characteristic 0. Let H be a solvable subalgebra of L such that $L/\mathrm{Core}_L(H) \in \acute{\mathrm{E}}(\lhd)\mathfrak{A}$. If (H, L) is an E_∞-pair, then H is an ascendant subalgebra of L.*

PROOF. We may assume that $L \in \acute{\mathrm{E}}(\lhd)\mathfrak{A}$. Let $(L_\alpha)_{\alpha \leq \lambda}$ be an ascending abelian series of ideals of L. For any $\alpha < \lambda$ put $\bar{L} = L/L_\alpha$. Then $\bar{L}_{\alpha+1}$ is an abelian ideal of $\bar{L}$ and $(\bar{H}, \bar{L}_{\alpha+1})$ is an E_∞-pair. By Lemma 13 $(\bar{H}, \bar{L}_{\alpha+1})$ is an N_∞-pair, and by Lemma 3

$$\bar{H} \lhd^\omega \bar{L}_{\alpha+1} + \bar{H}.$$

Therefore

$$L_\alpha + H \lhd^\omega L_{\alpha+1} + H.$$

If $\mu \leq \lambda$ is a limit ordinal, then

$$\cup_{\beta < \mu} (L_\beta + H) = L_\mu + H.$$

Thus we obtain

$$H = L_0 + H \quad \text{asc} \quad L.$$

Finally we give some criteria for ascendancy in the case that $L/\mathrm{Core}_L(H) \in \acute{\mathrm{E}}\mathfrak{A}$. To this end we show the following

LEMMA 15. *Let L be a Lie algebra over a field $\mathfrak{k}$. Let H, K be subalgebras of L. Then there exists the largest H-invariant subalgebra of K.*

PROOF. Let M be the sum of H-invariant subspaces of K. Then it is clear that M is the largest H-invariant subspace of K. M^2 is also H-invariant since

$$[M^2, H] \subseteq [[M, H], M] \subseteq M^2.$$

By the definition of M

$$M^2 \subseteq M.$$

Hence M is a subalgebra of K and therefore the largest H-invariant subalgebra of K.

LEMMA 16. *Let L be an $\acute{\mathrm{E}}\mathfrak{A}$-algebra over a field $\mathfrak{k}$. Let H be a subalgebra of L such that $\langle a^H \rangle$ is finitely generated for any $a \in L$. Then there exists an ascending abelian series of H-invariant subalgebras of L.*

PROOF. Let $(L_\alpha)_{\alpha \leq \lambda}$ be an ascending abelian series of L. By Lemma 15 there exists the largest H-invariant subalgebra K_α of L_α for any $\alpha \leq \lambda$. Clearly $K_0 = 0$ and $K_\lambda = L$. For any $\alpha < \lambda$

$$K_{\alpha+1}^2 \leq L_{\alpha+1}^2 \leq L_\alpha,$$

and $K_{\alpha+1}^2$ is an H-invariant subalgebra of L_α. Hence by the definition of K_α

$$K_{\alpha+1}^2 \leq K_\alpha.$$

Therefore we have

$$K_\alpha \lhd K_{\alpha+1}, \quad K_{\alpha+1}/K_\alpha \in \mathfrak{A}.$$

Let $\mu \leq \lambda$ be a limit ordinal. Then clearly $K_\mu \geq \cup_{\beta<\mu} K_\beta$. For any $a \in K_\mu$ $\langle a^H \rangle$ is a finitely generated subalgebra of L_μ, and hence there exists an ordinal $\beta < \mu$ such that

$$\langle a^H \rangle \leq L_\beta.$$

Since $\langle a^H \rangle$ is H-invariant,

$$\langle a^H \rangle \leq K_\beta,$$

and hence

$$K_\mu = \cup_{\beta<\mu} K_\beta.$$

Therefore $(K_\alpha)_{\alpha \leq \lambda}$ is an ascending abelian series of H-invariant subalgebras of L.

THEOREM 17. *Let L be a Lie algebra over a field $\mathfrak{k}$. Let H be a subalgebra of L such that $L/\mathrm{Core}_L(H)\in \acute{\mathrm{E}}\mathfrak{A}$ and that $\langle a^H\rangle$ is finitely generated for any $a\in L$. If (H, L) is an N_∞-pair, then H is an ascendant subalgebra of L.*

PROOF. We may assume that $L\in \acute{\mathrm{E}}\mathfrak{A}$. Then by Lemma 16 there exists an ascending abelian series $(L_\alpha)_{\alpha\le\lambda}$ of H-invariant subalgebras of L. We claim that for any $\alpha<\lambda$

$$L_\alpha + H \lhd^\omega L_{\alpha+1} + H. \qquad (*)$$

Clearly $L_{\alpha+1}+H$ is a subalgebra of L, and L_α is an ideal of $L_{\alpha+1}+H$. Furthermore $\bar{L}_{\alpha+1}$ is an abelian ideal of $\overline{L_{\alpha+1}+H}=(L_{\alpha+1}+H)/L_\alpha$ and $(\bar{H}, \bar{L}_{\alpha+1})$ is an N_∞-pair. Hence by Lemma 3

$$\bar{H}\lhd^\omega \bar{L}_{\alpha+1}+\bar{H},$$

and we have $(*)$. It is now easy to see that

$$H = L_0 + H \ \mathrm{asc}\ L_\lambda + H = L.$$

By the same argument as in the proof of Theorem 17 and by using Lemma 13 we can show the following

THEOREM 18. *Let L be a Lie algebra over a field $\mathfrak{k}$ of characteristic 0. Let H be a solvable subalgebra of L such that $L/\mathrm{Core}_L(H)\in \acute{\mathrm{E}}\mathfrak{A}$ and that $\langle a^H\rangle$ is finitely generated for any $a\in L$. If (H, L) is an E_∞-pair, then H is an ascendant subalgebra of L.*

REMARK. Let L be a Lie algebra over a field $\mathfrak{k}$ (resp. a field $\mathfrak{k}$ of characteristic 0). Let H be a subalgebra of L such that (H, L) is an N_∞-pair (resp. E_∞-pair). If $H\in\mathfrak{F}_\omega$ (resp. $H\in\mathfrak{F}_\omega\cap \mathrm{E}\mathfrak{A}$), then $(\langle a^H\rangle+H^\omega)/H^\omega$ is finitely generated for any $a\in L$.

We shall give the proof only for an N_∞-pair and omit the proof of the other case. By Lemma 1 $H^\omega\lhd N_\infty(H)=L$, so that we can consider the quotient algebra L/H^ω. Since $H/H^\omega\in\mathfrak{F}$, we may assume that H is finite-dimensional and nilpotent. Let X be a basis of H. Then for any $a\in L$ there exists $n=n(a)$ such that

$$[a, {}_nX]\subseteq H.$$

Since H is nilpotent,

$$[a, {}_{n+m}X]\subseteq H^{m+1}=0$$

for a sufficiently large m. Therefore

$$\langle a^H\rangle = \langle [a, {}_iX] \mid 0\le i<n+m\rangle$$

is finitely generated.

4. Examples

In Theorems 4 and 7 we observe the case where $m=2$. In this case the assertions become $H \lhd^{n+1} L$. The subideal index $n+1$ of H is best possible for $n>1$.

In Theorem 7 (resp. Theorem 8) we assumed that the characteristic of the basic field $\mathfrak{k}$ is either 0 or $p>n$ (resp. is not 2). These restrictions cannot be removed.

We shall show these facts in the following examples.

EXAMPLE 1. Let $\mathfrak{k}$ be any field and n be an integer >1. Define V to be the vector space over $\mathfrak{k}$ with basis $\{e_i \mid i=1, 2,\dots, 3n\}$, and define endomorphisms f and g of V by

$$e_i f = \begin{cases} e_{i+1} & \text{if } i \neq n, 2n, 3n, \\ 0 & \text{if } i = n, 2n, 3n; \end{cases}$$

$$e_i g = \begin{cases} e_{n+i} & \text{if } i = 1, 2,\dots, 2n, \\ 0 & \text{if } i = 2n+1, 2n+2,\dots, 3n. \end{cases}$$

Clearly f and g are commutative. Consider V as an abelian Lie algebra so that f and g are derivations of V. Define

$$L = V \dotplus (f, g),$$

and put

$$H = (e_1, e_2,\dots, e_n, e_{2n}) + (f).$$

It is clear that H is a subalgebra of L. It is easy to see that for $1 \leq i \leq n-1$

$$[L, {}_iH] = (e_{i+1}, e_{i+2},\dots, e_n, e_{n+i}, e_{n+i+1},\dots, e_{2n}, e_{2n+i+1}, e_{2n+i+2},\dots, e_{3n})$$

and

$$[L, {}_nH] = (e_{2n}) \subseteq H.$$

Let H_i be an i-th ideal closure of H in L for $i=1, 2, 3,\dots$. Then we easily see that

$$H_1 = \langle H^L \rangle = V + (f),$$

$$H_i = \langle H^{H_{i-1}} \rangle$$

$$= (e_1, e_2, \dots, e_n, e_{n+i}, e_{n+i+1}, \dots, e_{2n}, e_{2n+i}, e_{2n+i+1}, \dots, e_{3n}) + (f) \quad \text{for} \quad 2 \le i \le n,$$

and

$$H_{n+1} = \langle H^{H_n} \rangle = (e_1, e_2, \dots, e_n, e_{2n}) + (f) = H.$$

Therefore (H, L) is both an N_n-pair and E_n-pair. But $H \lhd^{n+1} L$ and H is not an n-step subideal of L.

EXAMPLE 2. Let $\mathfrak{k}$ be a field of characteristic $p>0$, and let $\mathbf{Z}[t]$ be a polynomial ring. Define V to be the vector space over $\mathfrak{k}$ with basis $\{e_a \mid a \in S\}$, where

$$S = \{\textstyle\sum_i a_i t^i \in \mathbf{Z}[t] \mid 0 \le a_i < p \text{ for any } i \in \mathbf{N}\}.$$

For each $n \in \mathbf{N}$ define an endomorphism f_n of V as follows: For any $a = \sum_i a_i t^i \in S$

$$e_a f_n = \begin{cases} e_{a+t^n} & \text{if} \quad a_n \ne p-1, \\ 0 & \text{if} \quad a_n = p-1. \end{cases}$$

Then for any $n, m \in \mathbf{N}$

$$f_n^p = 0, \quad f_n f_m = f_m f_n,$$

and for any $\alpha_r \in \mathfrak{k}$ $(r \in \mathbf{N})$

$$(\textstyle\sum_r \alpha_r f_r)^p = \sum_r \alpha_r^p f_r^p = 0. \tag{$*$}$$

Put $H = (f_n \mid n \in \mathbf{N})$. Then H is an abelian Lie subalgebra of $\mathrm{End}_k(V)$. Consider V as an abelian Lie algebra so that each of the elements in H is a derivation of V. Define

$$L = V \dotplus H.$$

Then L is a solvable Lie algebra of derived length 2, and H is a subalgebra of L. By $(*)$

$$[L, {}_p x] = [V, {}_p x] = V x^p = 0$$

for any $x \in H$. Therefore (H, L) is an E_n-pair for any $n \ge p$. However, since

$$I_L(H) = H,$$

H is neither a subideal nor an ascendant subalgebra of L.

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