ECONOMETRICA

JOURNAL OF THE ECONOMETRIC SOCIETY

An International Society for the Advancement of Economic Theory in its Relation to Statistics and Mathematics

http://www.econometricsociety.org/

Econometrica, Vol. 81, No. 1 (January, 2013), 255–284

SUBJECTIVE EXPECTED UTILITY WITH INCOMPLETE PREFERENCES

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Econometrica, Vol. 81, No. 1 (January, 2013), 255–284

SUBJECTIVE EXPECTED UTILITY WITH INCOMPLETE PREFERENCES

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This paper extends the subjective expected utility model of decision making under uncertainty to include incomplete beliefs and tastes. The main results are two axiomatizations of the multiprior expected multiutility representations of preference relations under uncertainty. The paper also introduces new axiomatizations of Knightian uncertainty and the expected multiutility model with complete beliefs.

KEYWORDS: Incomplete preferences, Knightian uncertainty, multiprior expected multiutility representations, incomplete beliefs, incomplete tastes.

1. INTRODUCTION

FACING A CHOICE BETWEEN ALTERNATIVES that are not fully understood or not readily comparable, decision makers may find themselves unable to express preferences for one alternative over another or to choose between alternatives in a coherent manner. This problem was recognized by von Neumann and Morgenstern, who stated that "[i]t is conceivable—and may even in a way be more realistic—to allow for cases where the individual is neither able to state which of two alternatives he prefers nor that they are equally desirable" (von Neumann and Morgenstern (1947, p. 19)).[2] Aumann (1962, p. 446) went further when he said "[o]f all the axioms of utility theory, the completeness axiom is perhaps the most questionable. Like others of the axioms, it is inaccurate as a description of real life; but unlike them, we find it hard to accept even from a normative viewpoint." In the same vein, when discussing the axiomatic structure of what became known as the Choquet expected utility theory, Schmeidler (1989, p. 576)[3] said "[o]ut of the seven axioms listed here, the completeness of the preferences seems to me the most restrictive and most imposing assumption of the theory." A natural way to accommodate such situations while maintaining the other aspects of the theory of rational choice is to relax the assumption that the preference relations are complete.

[1]We are grateful to Juan Dubra, Robert Nau, Teddy Seidenfeld, Wolfgang Pesendorfer, and three anonymous referees for their useful comments.

[2]Later von Neumann and Morgenstern (1947, pp. 28–29) added "[w]e have to concede that one may doubt whether a person can always decide which of two alternatives ... he prefers." In a letter to Wold dated October 28, 1946, von Neumann discussed the issue of complete preferences, noting that "[t]he general comparability of utilities, i.e., the completeness of their ordering by (one person's) subjective preferences, is, of course, highly dubious in many important situations" (Redei (2005)).

[3]Schmeidler (1989) went as far as to suggest that the main contributions of all other axioms is to allow the weakening of the completeness assumption. Yet he maintained this assumption in his theory.

 DOI: 10.3982/ECTA9621

Presumably, preferences among uncertain prospects, or acts, reflect the decision maker's beliefs regarding the likelihoods of alternative events and his tastes for their consequences contingent on these events. In this context, the incompleteness of the preference relation may be due to the incompleteness of the decision maker's beliefs, the incompleteness of his tastes, or both.

Our objective of studying the representations of incomplete preferences under uncertainty is to identify preference structures on the set of acts that admit multiprior expected multiutility representation. In such a representation, the set of priors represents the decision maker's incomplete beliefs, and the set of utility functions represents her incomplete tastes. More formally, according to the multiprior expected multiutility representation, an act f is strictly preferred over another act g if and only if there is a nonempty set Φ of pairs (π, U) consisting of a probability measure π on the set of states S and an affine, real-valued function U on the set $\Delta(X)$ of probability measures on the set X of outcomes such that

$$\sum_{s\in S}\pi(s)U\big(f(s)\big) > \sum_{s\in S}\pi(s)U\big(g(s)\big) \quad \text{for all } (\pi, U) \in \Phi. \tag{1}$$[4]

Incomplete beliefs and their representation by a set of probabilities were first explored in the context of statistical decision theory. Koopman (1940) showed that, without completeness, the set of axioms for comparative probabilities entails a representation of beliefs in terms of upper and lower probabilities. Upper and lower probabilities were also studied by Smith (1961), Williams (1976), and Walley (1981, 1982, 1991).[5] These studies are concerned with the structure of binary relations on events, or propositions, interpreted as the intuitive (or subjective) beliefs about likelihoods that these events, or propositions, are true.

A different approach to the definition of subjective probabilities, properly described as choice-based or behavioral, was pioneered by Ramsey (1931) and de Finetti (1937), and culminated in the seminal theories of Savage (1954) and Anscombe and Aumann (1963). According to this approach, beliefs and tastes govern choice behavior and may be inferred from the structure of preferences. Bewley (1986) was the first to study the implications of incomplete beliefs in the context of choice theory. Invoking the Anscombe–Aumann (1963) model and departing from the assumption that the preference relation is complete, Bewley axiomatized the multiprior expected utility representation, which he

[4]This representation may be interpreted as if the decision maker embodies multiple subjective expected-utility-maximizing agents, each of which is characterized by a unique subjective probability and a unique von Neumann–Morgenstern utility function, and one alternative is preferred over another if and only if they all agree.

[5]Bewley (1986) and Nau (2006) discussed these contributions and their relations to the multiprior expected utility representation. The study of multiprior expected utility representations is motivated, in part, by the interest in robust Bayesian statistics (see Seidenfeld, Schervish, and Kadane (1995)).

dubbed Knightian uncertainty. Bewley's model attributes the incompleteness of the preference relations solely to the incompleteness of beliefs. This incompleteness is represented by a closed convex set of probability measures on the set of states. Accordingly, one act is preferred over another (or the status quo) if its associated subjective expected utility exceeds that of the alternative (or the status quo) according to every probability measure in the set. In terms of representation (1), Bewley's work corresponds to the case in which $\Phi = \Pi \times \{U\}$, where Π is a closed convex set of probability measures on the set of states and U is a von Neumann–Morgenstern utility function.[6] Ok, Ortoleva, and Riella (2012) axiomatized a preference structure in which the source of incompleteness is either beliefs or tastes, but not both. In terms of representation (1), Ok, Ortoleva, and Riella (2012) axiomatized the cases in which $\Phi = \Pi \times \{U\}$ or $\Phi = \{\pi\} \times \mathcal{U}$.

Seidenfeld, Schervish, and Kadane (1995) and Nau (2006) studied the representation of incomplete preferences that reflects indeterminacy of both probabilities and utilities (that is, beliefs and tastes). We defer the discussion of their works to Section 4.

This paper provides new axiomatizations of preference relations that exhibit incompleteness in both beliefs and tastes. Invoking the analytical framework of Anscombe and Aumann (1963), we analyze the structure of partial strict preferences on a set of acts whose consequences are lotteries on a finite set X of outcomes. Our main result provides necessary and sufficient conditions characterizing the preference structures that admit multiprior expected multiutility representations (1).[7] The first set of conditions includes the familiar von Neumann–Morgenstern axioms without completeness. To these we add a dominance axiom à la Savage's postulate P7. Specifically, let g and f be any two acts, and denote by f^s the constant act whose payoff is $f(s)$ in every state. The axiom requires that if g is strictly preferred over f^s for every s, then g is strictly preferred over f. These axioms together with the existence of the best and the worst acts yields the representation in (1). Since the sets of probability measures that figure in the representation are "utility dependent," the beliefs and tastes are not entirely separated.

Building upon this result, we axiomatize three special cases. The first case entails a complete separation of beliefs and tastes (that is, Φ is the Cartesian

[6] Aumann (1962) was the first to address this issue in the context of expected utility theory under risk. Baucells and Shapley (2008) proved that a preference relation on a mixture space satisfies the von Neumann–Morgenstern axioms without completeness if and only if it has affine multiutility representation. Dubra, Maccheroni, and Ok (2004) studied the existence and uniqueness properties of the representations of preference relations over lotteries whose domain is a compact metric space.

[7] The set Φ is depicted as $\{(\pi, U) \mid U \in \mathcal{U}, \pi \in \Pi^U\}$ (i.e., each utility in $\mathcal{U}$ is paired with its own set of probability measures). Invoking the metaphor of a decision maker that embodies multiple subjective expected-utility-maximizing agents, this case corresponds to the case in which each agent is characterized by Knightian uncertainty preferences.

product of a set of probability measures $\mathcal{M}$ and a set of utility functions $\mathcal{U}$).[8] This case requires a definition of a set of distributions on S consistent with the preferences and an additional axiom, dubbed belief consistency. Belief consistency asserts that if an act g is strictly preferred over another act f, then every constant act obtained by reduction of g under every compound lottery involving a distribution on S that is consistent with the preference relation is preferred over the corresponding reduction of f. The representation in this case is as in (1), where the set Φ is a product set $\mathcal{M} \times \mathcal{U}$, where $\mathcal{M}$ is a set of probability measures on S and $\mathcal{U}$ is as above.

The second case is Knightian uncertainty. This case requires that the basic model be amended by an axiom requiring that the restriction of the preference relation to constant acts be negatively transitive. The third case is that of expected multiutility representation with complete beliefs. This case requires the formulation of a new behavioral postulate depicting the completeness of beliefs.

The remainder of the paper is organized as follows: In the next section we present our main result. In Section 3 we present the three special cases: the multiprior expected multiutility product representation, a Knightian uncertainty model, and its dual, the subjective expected multiutility model with complete beliefs. Further discussion and concluding remarks appear in Section 4. The proofs appear in Section 5.

2. THE MAIN RESULT

Our results extend the model of Anscombe–Aumann (1963) to include incomplete preferences. As mentioned earlier, the incompleteness in this model may stem from two distinct sources, namely, beliefs and tastes. The main result, Theorem 1 below, is a general model in which these sources of incompleteness are represented by sets of priors and utilities. In this model, beliefs and tastes are not entirely separated, and the representation involves sets of priors that are utility dependent.

2.1. *The Analytical Framework and the Preference Structure*

Let S be a finite set of *states.* Subsets of S are *events*. Let X be a finite set of outcomes, or prizes, and denote by $\Delta(X)$ the set of all probability measures on X. For each $\ell, \ell' \in \Delta(X)$ and $\alpha \in [0, 1]$, define $\alpha\ell + (1-\alpha)\ell' \in \Delta(X)$ by $(\alpha\ell + (1-\alpha)\ell')(x) = \alpha\ell(x) + (1-\alpha)\ell'(x)$ for all $x \in X$.

[8] Invoking the metaphor of the preceding footnote, in this case, there are two sets of agents. One set of agents is responsible for assessing beliefs in terms of probability measures and the second set is responsible for assessing tastes in terms of utility functions. The decision maker's preferences require agreement among all possible pairings of agents from the two sets.

Let $H := \{h \mid h : S \to \Delta(X)\}$ be the set of all functions from S to $\Delta(X)$. Elements of H are referred to as *acts*. For all $h, h' \in H$ and $\alpha \in [0,1]$, define $\alpha h + (1-\alpha)h' \in H$ by $(\alpha h + (1-\alpha)h')(s) = \alpha h(s) + (1-\alpha)h'(s)$ for all $s \in S$, where the convex mixture $\alpha h(s) + (1-\alpha)h'(s)$ is defined as above. Under this definition, H is a convex subset of the linear space $\mathbb{R}^{|X|\cdot|S|}$.

Let $\succ$ be a binary relation on H. The set H is said to be *$\succ$-bounded* if there exist h^M and h^m in H such that $h^M \succ h \succ h^m$ for all $h \in H - \{h^M, h^m\}$.

The following axioms depict the structure of the *preference relation* $\succ$. The first three axioms are well known and require no elaboration.

A.1—Strict Partial Order: *The preference relation $\succ$ is transitive and irreflexive.*

A.2—Archimedean: *For all $f, g, h \in H$, if $f \succ g$ and $g \succ h$, then $\beta f + (1-\beta)h \succ g$ and $g \succ \alpha f + (1-\alpha)h$ for some $\alpha, \beta \in (0,1)$.*

A.3—Independence: *For all $f, g, h \in H$ and $\alpha \in (0,1]$, $f \succ g$ if and only if $\alpha f + (1-\alpha)h \succ \alpha g + (1-\alpha)h$.*

The difference between the preference structure above and that of expected utility theory is that the induced relation $\neg(f \succ g)$ is reflexive but not necessarily transitive (hence, it is not necessarily a preorder).

For every $h \in H$, denote by $B(h) := \{f \in H \mid f \succ h\}$ and $W(h) := \{f \in H \mid h \succ f\}$ the (strict) upper and lower contour sets of h, respectively. The relation $\succ$ is said to be convex if the upper contour set is convex. Note that the $\succ$-boundedness of H implies that for $h \neq h^M, h^m$, $B(h)$ and $W(h)$ have nonempty algebraic interior in the linear space generated by H. It can be shown that if $\succ$ satisfies A.1–A.3, then it is convex and, in addition, the lower contour set is also convex.[9]

LEMMA 1: *Let $\succ$ be a binary relation on H. Then the following conditions are equivalent*:

(i) *H is $\succ$-bounded and $\succ$ satisfies* A.1–A.3.

(ii) *There exists a nonempty closed set $\mathcal{W}$ of real-valued functions w on $X \times S$, such that*

$$\sum_{s\in S}\sum_{x\in X} h^M(x,s)w(x,s) > \sum_{s\in S}\sum_{x\in X} h(x,s)w(x,s) > \sum_{s\in S}\sum_{x\in X} h^m(x,s)w(x,s)$$

[9]The proof is by two applications of A.3.

for all $h \in H - \{h^M, h^m\}$ *and* $w \in \mathcal{W}$, *and for all* $h, h' \in H$,

$$h \succ h' \quad \Leftrightarrow \quad \sum_{s\in S}\sum_{x\in X} h(x,s)w(x,s) > \sum_{s\in S}\sum_{x\in X} h'(x,s)w(x,s) \quad \forall w \in \mathcal{W}. \tag{2}$$

UNIQUENESS: To describe the uniqueness properties of the representation in Lemma 1, we introduce the following notation: Let δ_s be the vector in $\mathbb{R}^{|X|\cdot|S|}$ such that $\delta_s(t,x) = 0$ for all $x \in X$ if $t \neq s$ and $\delta_s(t,x) = 1$ for all $x \in X$ if $t = s$. Let $D = \{\theta\delta_s \mid s \in S, \theta \in \mathbb{R}\}$. Let $\mathcal{U}$ be a set of real-valued functions on $\mathbb{R}^{|X|\cdot|S|}$. Fix $x^0 \in X$ and for each $u \in \mathcal{U}$, define a real-valued function $\hat{u}$ on $\mathbb{R}^{|X|\cdot|S|}$ by $\hat{u}(x,s) = u(x,s) - u(x^0,s)$ for all $x \in X$ and $s \in S$. Let $\widehat{\mathcal{U}}$ be the normalized set of functions corresponding to $\mathcal{U}$ (that is, $\widehat{\mathcal{U}} = \{\hat{u} \mid u \in \mathcal{U}\}$). We denote by $\langle\widehat{\mathcal{U}}\rangle$ the closure of the convex cone in $\mathbb{R}^{|X|\cdot|S|}$ generated by all the functions in $\widehat{\mathcal{U}}$ and D.

LEMMA 2: *If* $\mathcal{W}'$ *is another set of real-valued functions on* $X \times S$, *representing* $\succ$ *in the sense of* (2), *then* $\langle\widehat{\mathcal{W}'}\rangle = \langle\widehat{\mathcal{W}}\rangle$.

REMARK 1: Seidenfeld, Schervish, and Kadane (1995) showed that a strict partial order, defined by strict first-order stochastic dominance, has an expected multi-utility representation, satisfies the independence axiom, and violates the Archimedean axiom.[10] To bypass this problem, Seidenfeld, Schervish, and Kadane (1995) and Nau (2006) invoked alternative continuity axioms that, unlike the Archimedean axiom, require the imposition of a topological structure. We maintain the Archimedean axiom as our continuity postulate at the cost of restricting the upper contour sets associated with the strict preference relation $B(p) := \{q \in C \mid q \succ p\}$ to be algebraically open. (In the example of Seidenfeld, Schervish, and Kadane (1995), these sets are closed.)

Like Nau (2006), we assume that the choice set has best and worst elements.[11] Doing so buys us two important properties. First, it implies that the upper (and lower) contour sets have full dimensionality. Second, the intersection of the upper (and lower) contour sets corresponding to the different acts is nonempty. Both properties are used in the proofs of our results. We recognize that this assumption restricts the degree of incompleteness of the preference relations under consideration.

[10]See Example 2.1 in their paper.

[11]Seidenfeld, Schervish, and Kadane (1995) proved the existence of such elements in their model. For more details, see Section 4.

2.2. *Dominance and the Main Representation Theorem*

For each $f \in H$ and every $s \in S$, let f^s denote the constant act whose payoff is $f(s)$ in every state. Formally, $f^s(s') = f(s)$ for all $s' \in S$. The next axiom is a weak version of Savage's (1954) postulate P7. It asserts that if an act g is strictly preferred over every constant act f^s associated with the consequences of another act f, then g is strictly preferred over f. To grasp the intuition underlying this assertion, note that any possible consequence of f, taken as an act, is an element of the lower contour set of g. Convexity of the lower contour sets implies that any convex combination of the consequences of f is dominated by g. Think of f as representing a set of such combinations whose elements correspond to the implicit set of subjective probabilities of the states that the decision maker may entertain. Since any such combination is dominated by g, so is f.[12] This concept is formally stated as follows.

A.4—Dominance: *For all $f, g \in H$, if $g \succ f^s$ for every $s \in S$, then $g \succ f$.*

The dominance axiom (sometimes referred to as the sure thing principle) is usually described as "technical," to be applied when the set of states is infinite. In our model, the state space is finite, but the dominance axiom has important substantive implications. We show in Section 2.3 that, in conjunction with the other axioms, dominance implies that the preference relation must satisfy state independence and monotonicity. We also show, as part of the proof of Theorem 1 below, that in conjunction with the other axioms, dominance implies that if a decision maker prefers one act over another under all conceivable beliefs about the likelihoods of the states, then he prefers the former act over the latter.

Theorem 1 shows that a preference relation satisfies the axioms A.1–A.4 if and only if there is a nonempty set of utility functions on X and, corresponding to each utility function, a set of probability measures on S such that, when presented with a choice between two acts, the decision maker prefers the act that yields higher expected utility according to every utility function and every probability measure in the corresponding set. Let the set of probability–utility pairs that figure in the representation be $\Phi := \{(\pi, U) \mid U \in \mathcal{U}, \pi \in \Pi^U\}$. Each $(\pi, U) \in \Phi$ defines a hyperplane $w := \pi \cdot U$. We denote by $\mathcal{W}$ the set of all these hyperplanes and define $\langle \widehat{\Phi} \rangle = \langle \widehat{\mathcal{W}} \rangle$.

[12]A slight variation of this axiom, in which the implied preference is $g \succcurlyeq f$ rather than $g \succ f$, appears in Fishburn's (1970) axiomatization of the infinite-state version of the model of Anscombe and Aumann (1963) (see Fishburn (1970, Theorem 13.3)). Fishburn's formulation of Savage's expected utility theorem (Fishburn (1970, Theorem 14.1)), includes axiom P7, which expressed in our notation says $g \succ (\prec) f^s$ given $A \subset S$, for every $s \in A$, implies $g \succcurlyeq (\preccurlyeq) f$ given A. Our version of dominance is weaker in the sense that it is required to hold only for $A = S$. It is stronger in the sense that the implication holds with the strict rather than the weak preference.

THEOREM 1: *Let $\succ$ be a binary relation on H. Then the following conditions are equivalent*:

(i) *H is $\succ$-bounded and $\succ$ satisfies* A.1–A.4.

(ii) *There exists a nonempty closed set $\mathcal{U}$ of real-valued functions on X and nonempty closed sets Π^U, $U \in \mathcal{U}$, of probability measures on S such that*,

$$\sum_{s\in S}\pi(s)\sum_{x\in X}h^M(x,s)U(x) > \sum_{s\in S}\pi(s)\sum_{x\in X}h(x,s)U(x) > \sum_{s\in S}\pi(s)\sum_{x\in X}h^m(x,s)U(x)$$

for all $h \in H$ and $(\pi, U) \in \Phi$, and for all $h, h' \in H$, $h \succ h'$ if and only if

$$\sum_{s\in S}\pi(s)\sum_{x\in X}h(x,s)U(x) > \sum_{s\in S}\pi(s)\sum_{x\in X}h'(x,s)U(x) \quad \forall(\pi,U)\in\Phi, \tag{3}$$

where $\Phi = \{(\pi, U) \mid U \in \mathcal{U}, \pi \in \Pi^U\}$.

Moreover, if $\Phi' = \{(\pi', V) \mid V \in \mathcal{V}, \pi' \in \Pi^V\}$ represents $\succ$ in the sense of (3), *then $\langle\widehat{\Phi'}\rangle = \langle\widehat{\Phi}\rangle$, and for all $U \in \mathcal{U}$ and $\pi \in \Pi^U$, $\pi(s) > 0$ for all s.*

2.3. *State Independence and Monotonicity*

Consider the following additional notation and definitions. For each $h \in H$ and $s \in S$, denote by $h_{-s}p$ the act obtained by replacing the sth coordinate of h, $h(s)$, with p. Define the conditional preference relation $\succ_s$ on $\Delta(X)$ by $p \succ_s q$ if there exists h_{-s} such that $h_{-s}p \succ h_{-s}q$ for all $p, q \in \Delta(X)$. A state s is said to be *nonnull* if $p \succ_s q$ for some $p, q \in \Delta(X)$, and it is *null* otherwise.

A preference relation $\succ$ on H is said to display *state independence* if for any h, h', p, q and for all nonnull $s, s' \in S$, $h_{-s}p \succ h_{-s}q$ if and only if $h'_{-s'}p \succ h'_{-s'}q$. It is said to display *monotonicity* if for all $f, g \in H$, $f^s \succ g^s$ for all $s \in S$ implies $f \succ g$.

If a preference relation is an Archimedean weak order satisfying independence, then state independence and monotonicity are equivalent axioms. However, Ok, Ortoleva, and Riella (2012) demonstrated that if the preference relation is incomplete, they are not. We show below that the dominance axiom A.4 implies both state independence and monotonicity.

LEMMA 3: *Let $\succ$ be a nonempty binary relation on H and suppose that H is $\succ$-bounded. If $\succ$ satisfies* A.1–A.4, *then it displays state-independent preferences. Moreover, all states are nonnull, and $h^M = (\delta_{x_1}, \dots, \delta_{x_1})$ and $h^m = (\delta_{x_2}, \dots, \delta_{x_2})$ for some $x_1, x_2 \in X$.*

The proof is an immediate implication of Theorem 1 and is omitted.

LEMMA 4: *If $\succ$ is a strict partial order on H satisfying independence* A.3 *and dominance* A.4, *then it satisfies monotonicity*.[13]

The proof is given in Section 5.

3. SPECIAL CASES

In this section, we examine three special cases, each of which involves tightening the axiomatic structure by adding a different axiom to the basic preference structure depicted by A.1–A.4. The first is an axiomatic structure that entails a complete separation of beliefs from tastes. The second, Knightian uncertainty, is the case in which tastes are complete but beliefs are incomplete. The third is the case of complete beliefs and incomplete tastes.

3.1. *Belief Consistency and Multiprior Expected Multiutility Product Representation*

One of the features of the Anscombe and Aumann (1963) model is the possibility it affords for transforming uncertain prospects (subjective uncertainty) into risky prospects (objective uncertainty) by comparing acts to their reduction under alternative measures on $\Delta(S)$. In particular, there is a measure $\alpha^* \in \Delta(S)$ such that every act f is indifferent to the constant act f^{α^*} obtained by the reduction of the compound lottery represented by (f, α^*).[14] In fact, the measure α^* is the subjective probability measure on S that governs the decision maker's choice. It is, therefore, natural to think of an act as a tacit compound lottery in which the probabilities that figure in the first stage are, implicitly, the subjective probabilities that govern choice behavior. When, as in this paper, the set of subjective probabilities that govern choice behavior is not a singleton, an act f corresponds to a set of implicit compound lotteries, each of which is induced by a (subjective) probability measure. The set of measures represents the decision maker's indeterminate beliefs. Add to this interpretation the reduction of compound lotteries assumption—that is, the assumption maintaining that (f, α) is equivalent to its reduction f^a—to conclude that $g \succ f$ is sufficient for the reduction of (g, α) to be preferred over the reduction of (f, α) for all α in the aforementioned set of measures. This assertion is formalized by the belief consistency axiom. To state the axiom, we use the following notation and definition: Let h^p denote the constant act whose payoff is $h^p(s) = p$ for every $s \in S$, and let $\mathcal{A} := \{\alpha \in \Delta(S) \mid \forall f \in H, \forall p \in \Delta(X), f \succ h^p \Rightarrow \neg(h^p \succ f^\alpha)\}$. The set $\mathcal{A}$ has the interpretation of "distributions consistent with the preferences."

[13]We thank a referee for calling our attention to this lemma and providing its proof.

[14]For each act–probability pair $(f, \alpha) \in H \times \Delta(S)$, we denote by f^α the constant act defined by $f^\alpha(s) = \sum_{s' \in S} \alpha_{s'} f(s')$ for all $s \in S$.

A.5—Belief Consistency: *For all* $f, g \in H$, $g \succ f$ *implies* $g^{\alpha} \succ f^{\alpha}$ *for all* $\alpha \in \mathcal{A}$.

The necessity of this condition is implied by Theorem 1. Hence, taken together, axioms A.1–A.5 amount to the condition that to assess the merits of the alternative acts, each of the measures in $\bigcup_{U \in \mathcal{U}} \Pi^U$ combines with each of the utility functions in $\mathcal{U}$.

The next result is a representation theorem that totally separates beliefs from tastes. Specifically, it shows that a preference relation satisfies A.1–A.5 if and only if there is a nonempty set $\mathcal{U}$ of utility functions on X and a nonempty set $\mathcal{M}$ of probability measures on S such that, when presented with a choice between two acts, the decision maker prefers one act over another if and only if the former act yields higher expected utility according to every combination of a utility function and a probability measure in these sets.

For a set of functions $\mathcal{U}$ on X, we denote by $\langle \mathcal{U} \rangle$ the closure of the convex cone in $\mathbb{R}^{|X|}$ generated by all the functions in $\mathcal{U}$ and all the constant functions on X.

THEOREM 2: *Let* $\succ$ *be a binary relation on* H. *Then the following conditions are equivalent*:

(i) H *is* $\succ$*-bounded and* $\succ$ *satisfies* A.1–A.5.

(ii) *There exist nonempty closed sets* $\mathcal{U}$ *and* $\mathcal{M}$ *of real-valued functions on* X *and probability measures on* S, *respectively*, *such that*,

$$\sum_{s \in S} \pi(s) \sum_{x \in X} h^M(x, s) U(x) > \sum_{s \in S} \pi(s) \sum_{x \in X} h(x, s) U(x) > \sum_{s \in S} \pi(s) \sum_{x \in X} h^m(x, s) U(x)$$

for all $h \in H$ *and* $(\pi, U) \in \mathcal{M} \times \mathcal{U}$, *and for all* $h, h' \in H$, $h \succ h'$ *if and only if*

$$\sum_{s \in S} \pi(s) \sum_{x \in X} h(x, s) U(x) > \sum_{s \in S} \pi(s) \sum_{x \in X} h'(x, s) U(x) \quad \forall (\pi, U) \in \mathcal{M} \times \mathcal{U}. \tag{4}$$

Moreover, *if* $\mathcal{V}$ *and* $\mathcal{M}'$ *is another pair of sets of real-valued functions on* X *and probability measures on* S *that represent* $\succ$ *in the sense of* (4), *then* $\langle \mathcal{U} \rangle = \langle \mathcal{V} \rangle$ *and* $\mathrm{cl}(\mathrm{conv}(\mathcal{M})) = \mathrm{cl}(\mathrm{conv}(\mathcal{M}'))$, *where* $\mathrm{cl}(\mathrm{conv}(\mathcal{M}))$ *is the closure of the convex hull of* $\mathcal{M}$. *Also*, $\pi(s) > 0$ *for all* $s \in S$ *and* $\pi \in \mathcal{M}$.

3.2. *Knightian Uncertainty*

Consider the extension of the Anscombe–Aumann (1963) model to include incomplete preferences, and suppose that the incompleteness is entirely due to incomplete beliefs. Bewley (1986) dealt with this case, which is referred to as Knightian uncertainty.

The model of Knightian uncertainty requires a formal definition of complete tastes. To provide such a definition, we invoke the property of negative transitivity.[15] The next axiom requires that the restriction of the preference relation to constant acts exhibits negative transitivity, thereby implying complete tastes.

A.6—Negative Transitivity on Constant Acts: *The restriction of* $\succ$ *to the set of constant acts* H^c *is negatively transitive*.

Let $\succ^c$ denote the restriction of $\succ$ to the set of all constant acts H^c and define $\succsim^c$ on H^c as follows: for all $h^p, h^q \in H^c$, $h^p \succsim^c h^q$ if $\neg(h^q \succ h^p)$. Then A.6 implies that the weak preference relation $\succsim^c$ on H^c is complete, which is the assumption of Bewley (1986).

The next theorem is our version of Knightian uncertainty.

THEOREM 3: *Let* $\succ$ *be a binary relation on* H. *Then the following conditions are equivalent*:

(i) H *is* $\succ$*-bounded*, *and* $\succ$ *satisfies* A.1–A.4 *and* A.6.

(ii) *There exists a nonempty closed set* $\mathcal{M}$ *of probability measures on* S *and a real-valued*, *affine function* U *on* $\Delta(X)$ *such that*

$$\sum_{s\in S} U\big(h^M(s)\big)\pi(s) > \sum_{s\in S} U\big(h(s)\big)\pi(s) > \sum_{s\in S} U\big(h^m(s)\big)\pi(s)$$

for all $h \in H$ *and* $\pi \in \mathcal{M}$, *and for all* $h, h' \in H$,

$$h \succ h' \quad \Leftrightarrow \quad \sum_{s\in S} U\big(h(s)\big)\pi(s) \succ \sum_{s\in S} U\big(h'(s)\big)\pi(s) \quad \forall \pi \in \mathcal{M}. \tag{5}$$

Moreover, U *is unique up to positive linear transformation*, *the closed convex hull of* $\mathcal{M}$ *is unique*, *and for all* $\pi \in \mathcal{M}$, $\pi(s) > 0$ *for any* s.

3.3. *Complete Beliefs and Subjective Expected Multiutility Representation*

Consider next the dual case in which incompleteness of the decision maker's preferences is due solely to the incompleteness of his tastes. This situation was modeled in Ok, Ortoleva, and Riella (2012) using an axiom they called reduction.[16] We propose here an alternative formulation based on the idea of completeness of beliefs. First, we give a definition of *coherent beliefs*.

[15]A strict partial order $\succ$ on a set D is said to exhibit negative transitivity if for all $x, y, z \in D$, $\neg(x \succ y)$ and $\neg(y \succ z)$ imply $\neg(x \succ z)$.

[16]The reduction axiom of Ok, Ortoleva, and Riella (2012) requires that for every $h \in H$, there exists a probability measure μ on S such that $h^\mu \sim h$.

To define the notion of coherent beliefs, we use the following notation: For each event E, denote by pEq the act whose payoff is p for all $s \in E$ and q for all $s \in S - E$. Denote by $p\alpha q$ the constant act whose payoff in every state is $\alpha p + (1-\alpha)q$. A *bet* on an event E is the act pEq, whose payoffs satisfy $h^p \succ h^q$.

Suppose that the decision maker considers the constant act $p\alpha q$ preferable to the bet pEq. This preference is taken to mean that he believes α exceeds the likelihood of E. This belief is coherent if it holds for any other bet on E and the corresponding constant acts (that is, if $h^{p'} \succ h^{q'}$, then the constant act $p'\alpha q'$ is preferable to the bet $p'Eq'$). The same logic applies when the bet pEq is preferable to the constant act $p\alpha q$.[17] The formal definition follows.

DEFINITION 1: A preference relation $\succ$ on H exhibits *coherent beliefs* if for all events E and $p, q, p', q' \in \Delta(X)$ such that $h^p \succ h^q$ and $h^{p'} \succ h^{q'}$, $p\alpha q \succ pEq$ if and only if $p'\alpha q' \succ p'Eq'$, and $pEq \succ p\alpha q$ if and only if $p'Eq' \succ p'\alpha q'$.

It is noteworthy that the axiomatic structure of the preference relation depicted by A.1–A.4 implies that the decision maker's beliefs are coherent.

LEMMA 5: *Let $\succ$ be a nonempty binary relation on H satisfying A.1–A.4. If H is $\succ$-bounded, then $\succ$ exhibits coherent beliefs.*

The proof is an immediate implication of Theorem 1 and is omitted.

The idea of complete beliefs is captured by the following axiom.[18]

A.7—Complete Beliefs: *For all events E and $\alpha \in [0,1]$, and constant acts h^p and h^q such that $h^p \succ h^q$, either $h^p \alpha h^q \succ h^p E h^q$ or $h^p E h^q \succ h^p \alpha' h^q$ for every $\alpha > \alpha'$.*

A preference relation $\succ$ displays complete beliefs if it satisfies A.7. If the beliefs are complete, then the incompleteness of the preference relation on H is due entirely to the incompleteness of tastes.

The next theorem is the subjective expected multiutility version of the Anscombe–Aumann (1963) model corresponding to the situation in which the decision maker's beliefs are complete.[19]

[17]This idea, which we refer to as coherent beliefs, is a variation on an axiom, dubbed betting neutrality, of Grant and Polak (2006).

[18]Unlike the weak reduction of Ok, Ortoleva, and Riella (2012), neither complete beliefs nor complete tastes involve an existential clause.

[19]See Ok, Ortoleva, and Riella (2012, Theorem 4) for their version of this result.

THEOREM 4: *Let $\succ$ be a binary relation on H. Then the following conditions are equivalent*:

(i) *H is $\succ$-bounded, and $\succ$ satisfies A.1–A.4 and A.7.*

(ii) *There exists a nonempty closed set $\mathcal{U}$ of real-valued functions on X and a probability measure π on S such that*

$$\sum_{s\in S}\pi(s)\sum_{x\in X}h^M(x,s)U(x) > \sum_{s\in S}\pi(s)\sum_{x\in X}h(x,s)U(x) > \sum_{s\in S}\pi(s)\sum_{x\in X}h^m(x,s)U(x)$$

for all $h \in H$ and $U \in \mathcal{U}$, and for all $h, h' \in H$,

$$h \succ h' \quad \Leftrightarrow \quad \sum_{s\in S}\pi(s)\sum_{x\in X}h(x,s)U(x) > \sum_{s\in S}\pi(s)\sum_{x\in X}h'(x,s)U(x) \quad \forall U \in \mathcal{U}. \tag{6}$$

The probability measure π is unique and $\pi(s) > 0$ for all $s \in S$. Moreover, if $\mathcal{V}$ is another set of real-valued functions on X that represent $\succ$ in the sense of (6), *then $\langle\mathcal{V}\rangle = \langle\mathcal{U}\rangle$.*

REMARK 2: For every event E, the *upper probability of E* is $\pi^u(E) = \inf\{\alpha \in [0,1] \mid p^M\alpha p^m \succ p^M E p^m\}$ and the *lower probability of E* is $\pi^l(E) = \sup\{\alpha \in [0,1] \mid p^M E p^m \succ p^M \alpha p^m\}$. Lemma 5 asserts that the upper and lower probabilities are well defined. Theorem 4 implies that a preference relation $\succ$ satisfying A.1–A.4 displays complete beliefs if and only if $\pi^u(E) = \pi^l(E)$ for every E.

4. CONCLUDING REMARKS

4.1. *Weak Preferences: Definition and Representation*

Taking the strict preference relation $\succ$ as a primitive, it is customary to define weak preference relations as the negation of $\succ$. Formally, given a binary relation $\succ$ on H, define a binary relation $\succcurlyeq$ on H by $f \succcurlyeq g$ if $\neg(g \succ f)$.[20] If the strict preference relation $\succ$ is negatively transitive and irreflexive, then the weak preference relation is complete. According to this approach, it is impossible to distinguish noncomparability from indifference. We propose below a new concept of induced weak preferences, denoted $\succcurlyeq_{\mathrm{GK}}$, that makes it possible to make such a distinction.

[20] See, for example, Chateauneuf (1987) and Kreps (1988).

DEFINITION 2: For all $f, g \in H$, $f \succcurlyeq_{GK} g$ if $h \succ f$ implies $h \succ g$ for all $h \in H$.

Note that $\succ$ is not the asymmetric part of $\succcurlyeq_{GK}$. Moreover, if $\succ$ satisfies A.1–A.3, then the derived binary relation $\succcurlyeq_{GK}$ on H is a weak order (that is, transitive and reflexive) satisfying the Archimedean and independence axioms but is not necessarily complete. The indifference relation $\sim_{GK}$ (that is, the symmetric part of $\succcurlyeq_{GK}$) is an equivalence relation.[21] Karni (2011) showed that the weak preference relation in Definition 2 agrees with the customary definition if and only if $\succ$ is negatively transitive and $\succcurlyeq_{GK}$ is complete.[22]

It can be shown that the representations in Theorems 1, 2, 3, and 4 extend to the weak preference relation in Definition 2. Consider, for instance, the representation in Theorem 1. It can be shown that H is $\succ$-bounded and $\succ$ is nonempty satisfying A.1–A.4 if and only if for all $h, h' \in H$,

$$\begin{aligned} h \succcurlyeq_{GK} h' \quad \Leftrightarrow \quad & \sum_{s\in S} U\big(h(s)\big)\pi(s) \\ & \geq \sum_{s\in S} U\big(h'(s)\big)\pi(s) \quad \text{for all } (\pi, U) \in \Phi, \end{aligned}$$

where Φ is the set of probability–utility pairs that figure in Theorem 1. Similar extensions apply to Theorems 2, 3, and 4.

Ok, Ortoleva, and Riella (2012), introduced an axiom, dubbed weak reduction, asserting that for any act f, there exists $\alpha \in \Delta(S)$ such that $f^\alpha \succeq f$, where $f^\alpha = \sum_{s\in S} \alpha_s f^s$. For $\succcurlyeq_{GK}$, weak reduction and independence imply dominance. Suppose $g \succ f^s$ for every $s \in S$. By weak reduction there exists $\bar{\alpha} \in \Delta(S)$ such that $f^{\bar{\alpha}} \succcurlyeq_{GK} f$. Since $g \succ f^s$ for every $s \in S$, by the independence axiom, $g \succ f^{\bar{\alpha}}$. Thus, $g \succ f^{\bar{\alpha}} \succcurlyeq_{GK} f$. Hence, by definition of $\succcurlyeq_{GK}$, $g \succ f$.

4.2. *Related Literature*

Seidenfeld, Schervish, and Kadane (1995), Nau (2006), and Ok, Ortoleva, and Riella (2012) studied axiomatic theories of incomplete preferences involving the indeterminacy of both beliefs and tastes. All of these papers invoke

[21]Derived weak orders, close in spirit to Definition 2, based on a pseudo-transitive weak order appear in Chateauneuf (1987).

[22]The standard practice in decision theory is to take the weak preference relation as primitive and define the strict preference relation as its asymmetric part. Invoking the standard practice, Dubra (2011) showed that if the weak preference relation on $\Delta(X)$ is nontrivial (that is, $\succ \neq \emptyset$) and satisfies the independence axiom, then any two of the following three axioms, completeness, Archimedean, and mixture continuity, imply the third. Thus, a nontrivial, partial preorder satisfying independence must fail to satisfy one of the continuity axioms. Karni (2011) showed that a nontrivial preference relation $\succcurlyeq_{GK}$ may satisfy independence, Archimedean, and mixture continuity, and yet be incomplete.

the analytical framework of Anscombe and Aumann (1963). As in this paper, Nau (2006) assumed that the set of outcomes (that is, the union of the supports of the roulette lotteries) is finite and there are best and worst acts. Seidenfeld, Schervish, and Kadane (1995) considered a more general setting in which the consequences are (roulette) lotteries with finite or countably infinite supports, and rather than assuming the existence of best and worse elements in the choice set, they proved that the set of acts and the preference relation may be extended to include such elements. Ok, Ortoleva, and Riella (2012) assumed that the support of the roulette lotteries is compact (metric) space. They neither assumed nor proved the existence of best and worst acts.

With regard to the preference relation, as in this paper, Seidenfeld et al. invoked the strict preference relation as primitive. However, they defined an indifference relation and weak preference relation differently from the approach described in the preceding subsection. Nau (2006) and Ok, Ortoleva, and Riella (2012) took the weak preference relation as a primitive.

Seidenfeld et al. and Nau assumed that the preference relation exhibits state independence to obtain multiprior expected multiutility representations with state-dependent utility functions.[23] Since these studies sought a representation that entails a set of probability–utility pairs, in which the utility functions are state independent, they amended their models with additional conditions that strengthen the state-independence axiom. With their additional conditions, Seidenfeld, Schervish, and Kadane (1995) obtained a representation involving almost state-independent utilities; Nau (2006) obtained a representation by a set of probabilities and state-dependent utility function pairs that is the convex hull of a set of probabilities and state-independent utility pairs. Like the model in this paper, Nau's (2006) model entails a finite set of consequences, and a best and a worst act.[24] The main difference is the underlying axiomatic structure.

Neither Seidenfeld, Schervish, and Kadane (1995) nor Nau (2006) studied any of the special cases considered in Section 3. Ok, Ortoleva, and Riella (2012) introduced a new axiom, dubbed the "weak reduction axiom," and showed that a preference relation is continuous and satisfies independence and weak reduction if and only if it admits either a multiprior expected utility representation or a single prior expected multiutility representation. The model of Ok, Ortoleva, and Riella (2012) does not allow for incompleteness of both beliefs and tastes. Their result corresponds to the last two cases analyzed in Section 3. However, unlike in our model in which these cases correspond to

[23]In the absence of completeness, state independence is not enough to ensure that the representation involves only sets of probabilities and state-independent utilities. Indeed, Lemma 3 asserts that state independence is implied in our model by the presence of the dominance axiom.

[24]Nau (2006) provided an excellent discussion of Seidenfeld, Schervish, and Kadane (1995) and an explanation of why their extended preference relation is representable by sets of probabilities and almost state-independent utilities but not state-independent utilities.

specific axioms depicting the completeness of either beliefs or tastes, in Ok, Ortoleva, and Riella (2012) both cases are possible, as the weak reduction axiom does not specify which aspect of the preferences—tastes or beliefs—is complete and which is incomplete.

Replacing weak reduction with the dominance axiom in the setting of Ok, Ortoleva, and Riella (2012) does not lead to a state-independent representation. In other words, the dominance axiom applied to the weak preference relation $\succeq$ in the framework of Ok, Ortoleva, and Riella (2012), where $\succeq$ is assumed to satisfy independence and (strong) continuity, does not necessarily imply state independence. To see this, let $S = \{s, t\}$, and fix a constant act $h^p = (p, p)$ and a nonconstant act $f = (p, q)$. Let $f \succeq' h^p$ and suppose that $\succeq'$ is determined by the direction $f - h^p$. Observe that $\succeq'$ satisfies independence, continuity, and dominance, but not state independence (by definition, $q \succ'_t p$, but $\succ'_s$ is empty). Hence, this relation does not satisfy state independence. Notice that in this example, the interior of dominance cone is empty, and there are no best and worst elements in H. It is worth emphasizing that all existing axiomatizations of multiprior expected multiutility representations rely on the existence of best and worst acts. Whether axioms A.1–A.4 and the assumption that the dominance cone has a nonempty interior, without assuming the existence of best and worst elements, imply state independence is an open question.

5. PROOFS

Whenever suitable, we use the following convention. Although, in most of our results, a function U (in representing set $\mathcal{U}$) is defined on X, we refer to its natural extension to $\Delta(X)$ by U.

5.1. *Proof of Lemma 1*

(i) $\Rightarrow$ (ii). Let $B(\succ) := \{\lambda(f - h) \mid f \succ h \text{ and } f, h \in H \text{ and } \lambda > 0\}$. Here, $f - h \in R^{|X|\cdot|S|}$ is defined by $(f - h)(s) = f(s) - h(s) \in \mathbb{R}^{|X|}$ for all $s \in S$.

Each $f \in H$ is a point in $\mathbb{R}^{|X|\cdot|S|}$. Since for each state, the weights on consequences add up to 1, f can also be seen as a point in $\mathbb{R}^{(|X|-1)\cdot|S|}$. (For example, if $X = \{x_1, x_2, x_3\}$ and $S = \{s_1, s_2\}$, then $f = (\frac{1}{2}, \frac{1}{3}, \frac{1}{6}; \frac{1}{4}, 0, \frac{3}{4}) \in R^6$ corresponds to $(\frac{1}{2}, \frac{1}{3}; \frac{1}{4}, 0)$ in R^4.) For any act $f \in H$, the corresponding act in $R^{(|X|-1)\cdot|S|}$ is denoted by $\phi(f)$. Thus, $\phi : R^{|X|\cdot|S|} \to \mathbb{R}^{(|X|-1)\cdot|S|}$ is a one-to-one linear mapping. Define $\phi(B(\succ)) := \{\lambda\phi(f - h) \mid f \succ h \text{ and } f, h \in H \text{ and } \lambda > 0\}$.

CLAIM 1: *The set $\phi(B(\succ))$ is a convex and open cone in $R^{(|X|-1)\cdot|S|}$.*

PROOF: By the independence axiom, $\phi(B(\succ))$ is a convex cone. To see this, pick any $h_1, h_2 \in \phi(B(\succ))$ and $\alpha_1, \alpha_2 > 0$. We need to show that $\alpha_1 h_1 + \alpha_2 h_2$ belongs to $\phi(B(\succ))$.

By definition, $h_1, h_2 \in \phi(B(\succ))$ implies that $h_1 = \lambda_1\phi(f_1 - g_1)$ and $h_2 = \lambda_2\phi(f_2 - g_2)$ for $\lambda_1, \lambda_2 > 0$ and $f_1, g_1, f_2, g_2 \in H$ such that $f_1 \succ g_1$ and $f_2 \succ g_2$:

$$
\begin{aligned}
(7) \qquad \alpha_1 h_1 + \alpha_2 h_2 &= \alpha_1\lambda_1\phi(f_1 - g_1) + \alpha_2\lambda_2\phi(f_2 - g_2) \\
&= (\alpha_1\lambda_1 + \alpha_2\lambda_2) \\
&\quad \times \left(\left(\frac{\alpha_1\lambda_1}{\alpha_1\lambda_1 + \alpha_2\lambda_2}\phi(f_1) + \frac{\alpha_2\lambda_2}{\alpha_1\lambda_1 + \alpha_2\lambda_2}\phi(f_2)\right)\right. \\
&\quad \left. - \left(\frac{\alpha_1\lambda_1}{\alpha_1\lambda_1 + \alpha_2\lambda_2}\phi(g_1) + \frac{\alpha_2\lambda_2}{\alpha_1\lambda_1 + \alpha_2\lambda_2}\phi(g_2)\right)\right).
\end{aligned}
$$

Define $f := \frac{\alpha_1\lambda_1}{\alpha_1\lambda_1+\alpha_2\lambda_2} f_1 + \frac{\alpha_2\lambda_2}{\alpha_1\lambda_1+\alpha_2\lambda_2} f_2$ and $g := \frac{\alpha_1\lambda_1}{\alpha_1\lambda_1+\alpha_2\lambda_2} g_1 + \frac{\alpha_2\lambda_2}{\alpha_1\lambda_1+\alpha_2\lambda_2} g_2$. Then the independence axiom implies that $f \succ g$. Also, (7) implies $\alpha_1 h_1 + \alpha_2 h_2 = (\alpha_1\lambda_1 + \alpha_2\lambda_2)\phi(f - g)$. Therefore, $\alpha_1 h_1 + \alpha_2 h_2 \in \phi(B(\succ))$.

To show that $\phi(B(\succ))$ is open in $R^{(|X|-1)\cdot|S|}$, let $\bar{p} := (\frac{1}{|X|}, \frac{1}{|X|}, \dots, \frac{1}{|X|}) \in \Delta(X)$ and $\bar{h} := (\bar{p}, \bar{p}, \dots, \bar{p}) \in H$. Note that $\phi(B(\succ))$ is open in $R^{(|X|-1)\cdot|S|}$ if and only if $\phi(\bar{h} + B(\succ))$ is open in $R^{(|X|-1)\cdot|S|}$. We know that $\phi(\bar{h} + B(\succ)) = \{\phi(\bar{h}) + \lambda\phi(h - \bar{h}) \mid \lambda > 0, h \in H \text{ and } h \succ \bar{h}\}$.[25] Thus, to show $\phi(B(\succ))$ is open, it is enough to show that set $\{\phi(\bar{h}) + \lambda\phi(h - \bar{h}) \mid \lambda > 0, h \in H \text{ and } h \succ \bar{h}\}$ is open in $R^{(|X|-1)\cdot|S|}$. Since, the set $\{\phi(\bar{h}) + \lambda\phi(h - \bar{h}) \mid \lambda > 0, h \in H \text{ and } h \succ \bar{h}\}$ is convex, to show that this set is open, it is enough to show that each point of this set is an algebraic interior point. Now pick any $\phi(g) \in \{\phi(\bar{h}) + \lambda\phi(h - \bar{h}) \mid \lambda > 0, h \in H \text{ and } h \succ \bar{h}\}$ and any $d \in R^{(|X|-1)\cdot|S|}$. Then $g = \bar{h} + \lambda(h - \bar{h})$ for some $\lambda > 0$ and $h \in H$ such that $h \succ \bar{h}$. Pick small $\mu > 0$ so that $h_1 := (1-\mu)\bar{h} + \mu g \in H$ and $\phi(f_1) := (1-\mu)\phi(\bar{h}) + \mu(\phi(g) + d) \in \phi(H)$.

Since $h_1 \succ \bar{h}$, by the Archimedean axiom, there exists $\beta' > 0$ such that $(1 - \beta')h_1 + \beta' h^m \succ \bar{h}$. Specifically, $(1-\beta)h_1 + \beta f_1 \succ \bar{h}$ for all β such that $\beta \in (0, \beta')$. This implies that for all $\beta \in (0, \beta')$, $\phi(g) + \beta d \in \{\phi(\bar{h}) + \lambda\phi(h - \bar{h}) \mid \lambda > 0, h \in H \text{ and } h \succ \bar{h}\}$. Thus, $\phi(g)$ is an algebraic interior point of $\{\phi(\bar{h}) + \lambda\phi(h - \bar{h}) \mid \lambda > 0, h \in H \text{ and } h \succ \bar{h}\}$. *Q.E.D.*

It is easy to check that for any $f, g \in H$,

$$
(8) \qquad f - g \in B(\succ) \quad \text{if and only if} \quad \phi(f) - \phi(g) \in \phi\big(B(\succ)\big).
$$

[25]It is easy to show that $\{\phi(\bar{h}) + \lambda\phi(h - \bar{h}) \mid \lambda > 0, h \in H \text{ and } h \succ \bar{h}\} \subset \phi(\bar{h} + B(\succ))$. To show the opposite direction, suppose $\phi(g) \in \phi(\bar{h} + B(\succ))$. Then $g = \bar{h} + \lambda(f_1 - f_2)$ for $\lambda > 0$ and $f_1, f_2 \in H$ such that $f_1 \succ f_2$. For small enough $\mu > 0$, $\bar{h} + \mu(f_1 - f_2) \in H$ holds. Denote this act by h. Then, by the independence axiom, $h \succ \bar{h}$ and $g = \bar{h} + \frac{\lambda}{\mu}(h - \bar{h})$. Hence $\phi(g) \in \{\phi(\bar{h}) + \lambda\phi(h - \bar{h}) \mid \lambda > 0, h \in H \text{ and } h \succ \bar{h}\}$.

Since $\phi(B(\succ))$ is an open and convex cone in $R^{(|X|-1)\cdot|S|}$, we can find a supporting hyperplane at each boundary point of $\phi(B(\succ))$. Each such hyperplane has its normal vector, $u \in R^{(|X|-1)\cdot|S|}$. Define $w_u : H \to \mathbb{R}$ by $w_u(h) = u(\phi(h))$ for all $h \in H$ (that is, $w_u(h) = \sum_{s\in S} w_u(h(s), s)$, where for each $s \in S$, $w_u(h(s), s) = \sum_{x\in X-\{x'\}} u(x, s)h(x, s)$, and $x' \in X$ is not used in the domain of ϕ). The collection of the functions w_u that correspond to all these hyperplanes is denoted by $\mathcal{W}$. Then each element of $\mathcal{W}$ is affine in its first argument. Using (8), it is easy to verify that $\mathcal{W}$ represents $\succ$. If $B(\succ)$ is smooth, then each of the supporting hyperplanes is unique, and the closedness of $\mathcal{W}$ is easy to verify. If $B(\succ)$ is not smooth, then there may be boundary points that have multiple supporting hyperplanes. In this case, include all the functions that correspond to the vectors defining these hyperplanes in $\mathcal{W}$ to show that it is closed.

(ii) $\Rightarrow$ (i). Axioms A.1 and A.3 are easy to show. We show that representation (2) implies the Archimedean axiom A.2. For all $f \in H$ and $w \in \mathcal{W}$, denote $f \cdot w := \sum_{s\in S}\sum_{x\in X} f(x, s)w(x, s)$.

Let $f \succ g \succ h$, then, by the representation $f \cdot w > g \cdot w > h \cdot w$ for all $w \in \mathcal{W}$. For each w, define $\alpha_w := \inf\{\alpha \in (0, 1) \mid \alpha f \cdot w + (1-\alpha)h \cdot w > g \cdot w\}$. To show that the Archimedean axiom holds, it is enough to show that $\sup\{\alpha_w \mid w \in \mathcal{W}\} < 1$. Suppose not. Then there is a sequence $\{w_n\} \subset \mathcal{W}$ such that $\alpha_{w_n} \to 1$. But $\mathcal{W} \subset \mathbb{R}^{|X|\times|S|}$ is closed and can be normalized to be bounded. Hence, without loss of generality, $\mathcal{W}$ is a compact set. Therefore, there is a convergent subsequence of $\{w_n\}$. Suppose that $w_n \to w^*$, $w^* \in \mathcal{W}$. Since, α_w is a continuous function of w, we have $\alpha_{w^*} = 1$. This contradicts $\alpha_{w^*} < 1$. *Q.E.D.*

5.2. *Proof of Lemma 2*

Suppose $\mathcal{W}$ and $\mathcal{W}'$ are two sets of real-valued functions that represent $\succ$ in the sense of (2). Note that $D \subset \langle\widehat{\mathcal{W}'}\rangle \cap \langle\widehat{\mathcal{W}}\rangle$.

Suppose that $\langle\widehat{\mathcal{W}'}\rangle \neq \langle\widehat{\mathcal{W}}\rangle$. Without loss of generality, assume that there exists $w \in \langle\widehat{\mathcal{W}}\rangle - \langle\widehat{\mathcal{W}'}\rangle$. Since $\langle\widehat{\mathcal{W}'}\rangle$ is a closed and convex cone, there exists a hyperplane that strictly separates $\{w\}$ from $\langle\widehat{\mathcal{W}'}\rangle$. Let $\bar{h} \in \mathbb{R}^{|X|\cdot|S|}$ be the normal of the hyperplane. Then $\bar{h} \cdot w > \bar{h} \cdot w'$ for all $w' \in \langle\widehat{\mathcal{W}'}\rangle$. But $\langle\widehat{\mathcal{W}'}\rangle$ is a cone, hence $\bar{h} \cdot w > 0$. If $\bar{h} \cdot w' > 0$ for some $w' \in \langle\widehat{\mathcal{W}'}\rangle$, then $\lambda w' \in \langle\widehat{\mathcal{W}'}\rangle$ for all $\lambda \in \mathbb{R}_+$ and $\lambda\bar{h} \cdot w' > \bar{h} \cdot w$ for some $\lambda \in \mathbb{R}_+$, a contradiction. Hence,

$$\bar{h} \cdot w > 0 \geq \bar{h} \cdot w' \quad \text{for all } w' \in \langle\widehat{\mathcal{W}'}\rangle. \tag{9}$$

CLAIM 2: *For all* $s \in S$, $\sum_{x\in X} \bar{h}(x, s) = 0$.

PROOF: Suppose not. Then $\theta\bar{h} \cdot \delta_s > 0$ for some $\theta \in \mathbb{R}$ and $s \in S$. But $\theta\delta_s \in \langle\widehat{\mathcal{W}'}\rangle$, which contradicts (9). *Q.E.D.*

Let $\bar{h}(\cdot, s) = \bar{h}^+(\cdot, s) - \bar{h}^-(\cdot, s)$, where $\bar{h}^+(x, s) = \bar{h}(x, s)$ if $\bar{h}(x, s) > 0$ and $\bar{h}^+(x, s) = 0$ otherwise, and $\bar{h}^-(x, s) = -\bar{h}(x, s)$ if $\bar{h}(x, s) < 0$ and $\bar{h}^-(x, s) = 0$ otherwise. Then $\sum_{x\in X} \bar{h}^+(x, s) = \sum_{x\in X} \bar{h}^-(x, s) = c_s \geq 0$.

CLAIM 3: *For some $s \in S$, $c_s > 0$.*

PROOF: Suppose that $c_s = 0$ for all $s \in S$. Then $\bar{h}(\cdot, s) = 0$ for all $s \in S$, hence $\bar{h} \cdot w = 0$. This contradicts (9). *Q.E.D.*

Let $c_t = \max\{c_s \mid s \in S\}$. Define $p_t(x) = \bar{h}^+(x, t)/c_t$ and $q_t(x) = \bar{h}^-(x, t)/c_t$ for all $x \in X$. For all $s \in S - \{t\}$ such that $c_s > 0$, let $p_s(x) = \bar{h}^+(x, s)/c_s$ and $q_s(x) = \bar{h}^-(x, s)/c_s$ for all $x \in X - \{x^0\}$, and let $p_s(x^0) = 1 - \sum_{x\in X-\{x^0\}} p_s(x)$ and $q_s(x^0) = 1 - \sum_{x\in X-\{x^0\}} q_s(x)$. For s such that $c_s = 0$, let $p_s(x^0) = q_s(x^0) = 1$ and $p_s(x) = q_s(x) = 0$ for all $x \in X - \{x^0\}$.

Define $h_p, h_q \in H$ by $h_p(x, s) = p_s(x)$ and $h_q(x, s) = q_s(x)$ for all $(x, s) \in X \times S$.

CLAIM 4: *There exists $w \in \widehat{\mathcal{W}}$ that satisfies equation* (9).

PROOF: Since $w \in \langle\widehat{\mathcal{W}}\rangle$, there is sequence $\{\alpha_n w_n + (1 - \alpha_n)d_n\}$ such that $\lim_{n\to\infty}(\alpha_n w_n + (1 - \alpha_n)d_n) = w$, where w_n is in the cone spanned by $\widehat{\mathcal{W}}$ and d_n is in the cone spanned by D. Since $\bar{h} \cdot (\alpha_n w_n + (1 - \alpha_n)d_n) = \alpha_n \bar{h} \cdot w_n$, by the left inequality of (9), for large enough n, we have $\bar{h} \cdot w_n > 0$. We regard this w_n as w. *Q.E.D.*

For the $h_p$and h_q above, we have $h_p \cdot w > h_q \cdot w$ and $h_p \cdot w' \leq h_q \cdot w'$ for all $w' \in \mathcal{W}'$. The second inequality implies that for any $f \in H$,

$$(10) \qquad f \succ h_q \quad \text{implies} \quad f \succ h_p;$$

$h_p \cdot w > h_q \cdot w$ implies that there exists $\beta \in (0, 1)$ such that $h_p \cdot w > ((1-\beta)h_q + \beta h^M) \cdot w > h_q \cdot w$. This yields a contradiction to (10) since $(1 - \beta)h_q + \beta h^M \succ h_q$. *Q.E.D.*

5.3. *Proof of Theorem 1*

(i) $\Rightarrow$ (ii). Define an auxiliary binary relation $\succcurlyeq$ on H as follows: For all $f, g \in H$, $f \succcurlyeq g$ if $h \succ f$ implies $h \succ g$ for all $h \in H$. Let $B := \{\lambda(h' - h) \mid h' \succcurlyeq h, h', h \in H, \lambda \geq 0\}$. Then $\phi(B)$ is a closed convex cone with nonempty interior in $R^{(|X|-1)\cdot|S|}$. By Theorem V.9.8 in Dunford and Schwartz (1957), there is a dense set T in its boundary such that each point of T has a unique tangent. Let $\mathcal{W}^o$ be the set of linear functions on $R^{(|X|-1)\cdot|S|}$ defined by the collection of all the supporting hyperplanes corresponding to dense set T. Without loss of generality, we assume that each function in $\mathcal{W}^o$ has unit normal vector. It is easy to see that $\mathcal{W}^o$ represents $\succcurlyeq$. For the rest of the proof, we use the notation $w(f)$ to express $w(\phi(f))$ for functions $w \in \mathcal{W}^o$. With this convention, for all $h, f \in H$,

$$(11) \quad h \succcurlyeq f \quad \text{if and only if} \quad \sum_{s\in S} w\bigl(h(s), s\bigr) \geq \sum_{s\in S} w\bigl(f(s), s\bigr) \quad \text{for all } w \in \mathcal{W}^o.$$

For every $f \in H$, let $H^c(f)$ be the convex hull of $\{f^s \mid s \in S\}$. For all $\alpha \in \Delta(S)$, let $f^\alpha \in H^c(f)$ be the constant act defined by $f^\alpha = \sum_{s\in S} \alpha_s f^s$. Now A.3 implies that $g \succ f^s$ for every $s \in S$ if and only if $g \succ f^\alpha$ for every $\alpha \in \Delta(S)$. Hence, an equivalent statement of A.4 is as follows:

A.4′—Reduction Consistency: *For all $f, g \in H$, $g \succ f^\alpha$ for every $\alpha \in \Delta(S)$ implies $g \succ f$.*

Before presenting the main argument of the proof, we provide some useful facts.

CLAIM 5: *For all $f, g \in H$, if $g \succcurlyeq f^\alpha$ for all $\alpha \in \Delta(S)$, then $g \succcurlyeq f$.*

The proof is immediate application of A.4, the preceding argument, and the definition of $\succcurlyeq$. Henceforth, when we invoke axiom A.4, we use it in either the equivalent strict preference form A.4′ or the weak form given in Claim 5, as the need arises.

To state the next result, we invoke the following notation. For each $h \in H$ and $s \in S$, let $h_{-s}p$ be the act that is obtained by replacing the sth coordinate of h, $h(s)$, with p. Let h^p denote the constant act whose payoff is $h^p(s) = p$ for every $s \in S$.

CLAIM 6: *If $h^p \succcurlyeq h^q$, then $h^p \succcurlyeq h^p_{-s}q$ for all $s \in S$.*

PROOF: For any $\alpha \in \Delta(S)$, $(h^p_{-s}q)^\alpha$ is a convex combination of h^p and h^q. To be exact, $(h^p_{-s}q)^\alpha = (1 - \alpha_s)h^p + \alpha_s h^q$. By A.3 applied to $\succcurlyeq$, we have $h^p \succcurlyeq \alpha_s h^p + (1 - \alpha_s)h^q \succcurlyeq h^q$ (that is, $h^p \succcurlyeq (h^p_{-s}q)^\alpha$ for all $\alpha \in \Delta(S)$).[26] Hence, by Claim 5, $h^p \succcurlyeq h^p_{-s}q$. *Q.E.D.*

We now turn to the main argument. In particular, we show that the component functions $\{w(\cdot, s)\}_{s\in S}$ of each function $w \in \mathcal{W}^o$ are positive linear transformations of one another.

[26]For a proof that $\succcurlyeq$ satisfies independence, see Galaabaatar and Karni (2011).

LEMMA 6: *If $\hat{w} \in \mathcal{W}^o$, then for all nonnull $s, t \in S$, $\hat{w}(\cdot, s)$ and $\hat{w}(\cdot, t)$ are positive linear transformations of one another.*

PROOF: By way of negation, suppose that there exist s, t such that $\hat{w}(\cdot, s)$ and $\hat{w}(\cdot, t)$ are not positive linear transformations of one another. Then there are $p, q \in \Delta(X)$ such that $\hat{w}(p, s) > \hat{w}(q, s)$ and $\hat{w}(q, t) > \hat{w}(p, t)$. Without loss of generality, let p be a lottery such that $\hat{w}(h^p) > \hat{w}(h^q)$ and $p(x) > 0$ for all $x \in X$. Define $q(\lambda) = \lambda p + (1 - \lambda)q$ for $\lambda \in (0, 1)$. Then $\hat{w}(p, s) > \hat{w}(q(\lambda), s)$ and $\hat{w}(q(\lambda), t) > \hat{w}(p, t)$. Following Ok, Ortoleva, and Riella (2012), we use the following construction. Let $f_\lambda \in H$ be defined as follows: $f_\lambda(s') = p$ if $s' = s$, $f_\lambda(s') = q(\lambda)$ if $s' = t$, and, for $s' \neq s, t$, $f_\lambda(s') = p$ if $\hat{w}(p, s') \geq \hat{w}(q(\lambda), s')$ and $f_\lambda(s') = q(\lambda)$ otherwise.

Clearly, $\sum_{s\in S} \hat{w}(f_\lambda(s), s) > \sum_{s\in S} \hat{w}((f_\lambda)^\alpha(s), s)$ for all $\alpha \in \Delta(S)$. Since f_λ involves only p and $q(\lambda)$, $\{(f_\lambda)^\alpha \mid \alpha \in \Delta(S)\} = \{\alpha h^p + (1 - \alpha)h^{q(\lambda)} \mid \alpha \in [0, 1]\}$.

Since $\hat{w} \in \mathcal{W}^o$, there exists $g \in H$ such that $g \succcurlyeq h^p$, $\hat{w}(g) = \hat{w}(h^p)$, and $\hat{w}$ is the unique supporting hyperplane at g.

The cone $B = \{\alpha(f - g) \mid f, g \in H, f \succcurlyeq g, \alpha \geq 0\}$ defines an extension of the auxiliary relation $\succcurlyeq$ to the linear space generated by H. With slight abuse of notation we denote the extended relation by $\succcurlyeq$. The extended relation satisfies all the properties of the original auxiliary relation.

CLAIM 7: *There exists $\beta^*(\lambda) > 0$ such that $h^p + \beta^*(\lambda)(g - h^p) \succcurlyeq h^{q(\lambda)}$.*

PROOF: Suppose not. Then, for any $n \in \{1, 2, \ldots\}$, there exists $w_n \in \mathcal{W}^o$ such that $w_n(h^p + n(g - h^p)) < w_n(h^{q(\lambda)})$. Since w_n is linear, we can regard w_n as a vector and $w_n(f)$ as the inner product $w_n \cdot \phi(f)$. Hence, we have

$$(12) \qquad nw_n \cdot \phi\big((g) - h^p\big) < w_n \cdot \phi\big(h^{q(\lambda)} - h^p\big) \quad \text{for all } n.$$

Since $\|w_n\| = 1$, we can find convergent subsequence $\{w_{n_k}\}$. Without loss of generality we assume that $\{w_n\}$ itself is convergent and $w_n \to w^* \in \mathrm{cl}(\mathcal{W}^o)$. The right-hand side of inequality (12) converges to $w^* \cdot \phi(h^{q(\lambda)} - h^p)$. If $w^* \cdot \phi(g - h^p) > 0$, then the left-hand side of inequality (12) tends to $+\infty$ as $n \to \infty$—a contradiction. Hence, $w^*(g) = w^*(h^p)$ since $g \succcurlyeq h^p$. Also, $w_n(h^p) \leq w_n(h^p + n(g - h^p)) < w_n(h^{q(\lambda)})$ implies $w^*(h^p) \leq w^*(h^{q(\lambda)})$. Since $\hat{w}(h^p) > \hat{w}(h^{q(\lambda)})$, $\hat{w} \neq w^*$. This contradicts the uniqueness of the supporting hyperplane at $\phi(g) \in \phi(H)$. This completes the proof of the claim. *Q.E.D.*

Let $g_\lambda = h^p + \beta^*(\lambda)(g - h^p)$. Then $g_\lambda \succcurlyeq h^p$ and $g_\lambda \succcurlyeq h^{q(\lambda)}$. By choosing λ close to 1 and applying the independence axiom to the extended relation, we can find $\beta(\lambda) \in (0, 1)$ so that for such λ, g_λ is feasible (i.e., $g_\lambda(s) \in \Delta(X)$ for all $s \in S$). By virtue of being on the hyperplane defined by $\hat{w}$, $\sum_{s\in S} \hat{w}(g_\lambda(s), s) = \hat{w}(h^p)$. Since $g_\lambda \succcurlyeq h^p, h^{q(\lambda)}$, we have $g_\lambda \succcurlyeq (f_\lambda)^\alpha$ for all $\alpha \in \Delta(S)$. Hence, by Claim 5, $g_\lambda \succcurlyeq f_\lambda$. But $\sum_{s\in S} \hat{w}(f_\lambda(s), s) > \hat{w}(h^p) =$

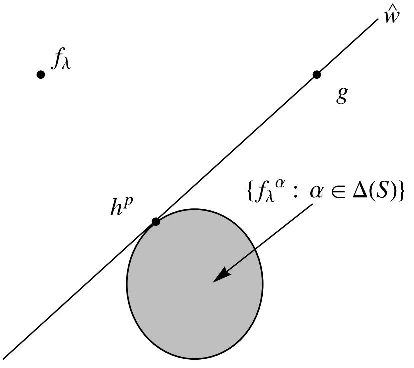

FIGURE 1.—The separating hyperplane.

$\sum_{s\in S} \hat{w}(g_\lambda(s), s)$, which is a contradiction (see Figure 1). This completes the proof of Lemma 6. *Q.E.D.*

The representation (3) is implied by the following arguments. First, by the standard argument: For each $w \in \mathcal{W}^o$, define $U^w(\cdot) = w(\cdot, 1)$ and for all $s \in S$, let $w(\cdot, s) = b_s^w U^w(\cdot) + a_s^w$, $b_s^w > 0$. Define $\pi^w(s) = b_s^w / \sum_{s'\in S} b_{s'}^w$ for all $s \in S$. Let $\mathcal{U}$ be the collection of distinct U^w and for each $U \in \mathcal{U}$, let $\Pi^U = \{\pi^w \mid \forall w$ such that $U^w = U\}$. Second, if there are kinks in B so that there is more than one supporting hyperplanes, then there is at least one w that can be expressed as a limit point of sequence $\{w_n\}$ from $\mathcal{W}^o$. Since any w_n has the property that each of its components is a positive linear transformation of the others, w has the same property. If we add all those w's to $\mathcal{W}^o$, then the new set of functions will represent $\succ$.

(ii) $\Rightarrow$ (i). Axioms A.1–A.3 are implied by Lemma 1. The $\succ$-boundedness of H and A.4 are immediate implications of the representation. The uniqueness result is implied by Lemma 1. *Q.E.D.*

5.4. *Proof of Lemma 4*

Suppose that $f, g \in H$ are such that $f(s) \succ g(s)$ for all $s \in S$. Define $h \in H$ by $h(s) = \frac{1}{|S|-1} \sum_{s'\neq s} f(s')$ for all $s \in S$. Observe that $\frac{1}{|S|}f + (1 - \frac{1}{|S|})h$ is a constant act. By A.3, for each s,

$$\frac{1}{|S|}f + \left(1 - \frac{1}{|S|}\right)h = \frac{1}{|S|}f(s) + \left(1 - \frac{1}{|S|}\right)h(s)$$
$$\succ \frac{1}{|S|}g(s) + \left(1 - \frac{1}{|S|}\right)h(s).$$

By A.4,

$$\frac{1}{|S|}f+\left(1-\frac{1}{|S|}\right)h \succ \frac{1}{|S|}g+\left(1-\frac{1}{|S|}\right)h.$$

Hence, by A.3, $f \succ g$. *Q.E.D.*

5.5. *Proof of Theorem 2*

(i) $\Rightarrow$ (ii). Suppose that $\succ$ on H satisfies A.1–A.5. Let $\mathcal{M} := \{\alpha \in \Delta(S) \mid f \succ h^p$ implies $\neg(h^p \succ f^\alpha)$ for any $p \in \Delta(X), f \in H\}$.

By A.5, $g \succ f$ implies that $g^\alpha \succ f^\alpha$ for all $\alpha \in \mathcal{M}$. By Theorem 1, $g^\alpha \succ f^\alpha$ for all $\alpha \in \mathcal{M}$ if and only if $U(g^\alpha) > U(f^\alpha)$ for all $U \in \mathcal{U}$ and $\alpha \in \mathcal{M}$. By the affinity of $U \in \mathcal{U}$, $U(g^\alpha) > U(f^\alpha)$ for all $U \in \mathcal{U}$ and $\alpha \in \mathcal{M}$ if and only if $\sum_{s\in S} U(g(s))\alpha(s) > \sum_{s\in S} U(f(s))\alpha(s)$ for all $(\alpha, U) \in \mathcal{M} \times \mathcal{U}$. Hence, $g \succ f$ implies $\sum_{s\in S} U(g(s))\alpha(s) > \sum_{s\in S} U(f(s))\alpha(s)$ for all $(\alpha, U) \in \mathcal{M} \times \mathcal{U}$.

To prove the inverse implication, suppose

$$\sum_{s\in S} U\big(g(s)\big)\alpha(s) > \sum_{s\in S} U\big(f(s)\big)\alpha(s)$$

for all $(\alpha, U) \in \mathcal{M} \times \mathcal{U}$. Theorem 1 implies that $g \succ f$ if and only if $\sum_{s\in S} U(g(s))\alpha(s) > \sum_{s\in S} U(f(s))\alpha(s)$ for all $(\alpha, U) \in \{(\alpha, U) \mid U \in \mathcal{U}, \alpha \in \Pi^U\}$. Since A.5 implies $\bigcup_{U\in\mathcal{U}} \Pi^U \subset \mathcal{M}$, we have $g \succ f$.

(ii) $\Rightarrow$ (i). This part is easy to check. To prove the uniqueness of the set of utility functions, we restrict attention to constant acts. Then we have $U(h^p) > U(h^q)$ for all $U \in \mathcal{U}$ if and only if $V(h^p) > V(h^q)$ for all $V \in \mathcal{V}$. By the proof of uniqueness result of Dubra, Maccheroni, and Ok (2004), we obtain $\langle\mathcal{U}\rangle = \langle\mathcal{V}\rangle$.

To prove the uniqueness of beliefs, suppose that each one of the pairs $(\mathcal{U}, \mathcal{M})$ and $(\mathcal{V}, \mathcal{M}')$ represents $\succ$. Assume $\mathrm{cl}(\mathrm{conv}(\mathcal{M})) \neq \mathrm{cl}(\mathrm{conv}(\mathcal{M}'))$. Then, without loss of generality, there exists $\pi \in \mathcal{M}$ such that $\pi \notin \mathrm{cl}(\mathrm{conv}(\mathcal{M}'))$. But $\pi \notin \mathrm{cl}(\mathrm{conv}(\mathcal{M}'))$ implies $\pi \notin \mathrm{cl}(\mathrm{cone}(\mathcal{M}'))$, where $\mathrm{cl}(\mathrm{cone}(\mathcal{M}'))$ is the closure of the convex cone generated by $\mathcal{M}'$.

Thus, there exists a hyperplane that strictly separates π and $\mathrm{cl}(\mathrm{cone}(\mathcal{M}'))$. In other words, there is a nonzero vector $a \in \mathbb{R}^{|S|}$ such that

$$\pi \cdot a > \pi' \cdot a \quad \text{for all } \pi' \in \mathrm{cl}\big(\mathrm{cone}(\mathcal{M}')\big). \tag{13}$$

Invoking the fact that $\mathrm{cl}(\mathrm{cone}(\mathcal{M}'))$ is a cone,

$$\pi \cdot a > 0 \geq \pi' \cdot a \quad \text{for all } \pi' \in \mathrm{cl}\big(\mathrm{cone}(\mathcal{M}')\big). \tag{14}$$

By equation (14), we have $\pi \cdot a > 0 \geq \pi' \cdot a$ for all $\pi' \in \mathcal{M}'$. Normalize $\mathcal{U}$ and $\mathcal{V}$ so that for any $U \in \mathcal{U} \cup \mathcal{V}$, $U(p^M) - U(p^m) = \max\{a_i \mid i = 1, 2, \dots, |S|\}$. Then for any $i = 1, \dots, |S|$, there exists $\hat{p}_i, \hat{q}_i \in \Delta(X)$ such that $a_i = U(\hat{p}_i) - U(\hat{q}_i)$.

Define acts $f := (\hat{p}_1, \hat{p}_2, \dots, \hat{p}_{|S|})$ and $g := (\hat{q}_1, \hat{q}_2, \dots, \hat{q}_{|S|})$. Then $0 \geq \pi' \cdot a$ for all $\pi' \in \mathcal{M}'$ implies $\sum_{s\in S} \pi'(s)V(g(s)) \geq \sum_{s\in S} \pi'(s)V(f(s))$ for all $V \in \mathcal{V}$ and $\pi' \in \mathcal{M}'$. Therefore, for any $h \in H$,

$$h \succ g \quad \text{then} \quad h \succ f. \tag{15}$$

But $\pi \cdot a > 0$ implies $\sum_{s\in S} \pi(s)U(f(s)) > \sum_{s\in S} \pi(s)U(g(s))$ for all $U \in \mathcal{U}$. Pick any $U^* \in \mathcal{U}$. Then there exists $\lambda \in (0, 1)$ such that $\sum_{s\in S} \pi(s)U^*(f(s)) > (1 - \lambda)\sum_{s\in S} \pi(s)U^*(g(s)) + \lambda \sum_{s\in S} \pi(s)U^*(p^M) > \sum_{s\in S} \pi(s)U^*(g(s))$. Since $(1 - \lambda)g + \lambda h^M \succ g$, the last inequality is a contradiction to (15). *Q.E.D.*

5.6. *Proof of Theorem 3*

(i) $\Rightarrow$ (ii). Recall that $\succ^c$ is the restriction of $\succ$ to the set of all constant acts H^c and that $\succsim^c$ on H^c is defined as follows: for all $h^p, h^q \in H^c$, $h^p \succsim^c h^q$ if $\neg(h^q \succ h^p)$. By A.6, $\succsim^c$ is complete.

By Theorem 1, for all $h^p, h^q \in H^c$, $h^p \succ^c h^q$ if and only if $\sum_{s\in X} p(x)U(x) > \sum_{s\in X} q(x)U(x)$ for all $(\pi, U) \in \Phi$. By Kreps (1988, Theorem (5.4)), all U in the above representation are positive affine transformations of one another. Pick one of them and denote it by $\bar{U}$. Define $\mathcal{M} := \{\pi \mid (\pi, U) \in \Phi \text{ for some } U\}$. Then, for all $h, g \in H$, $h \succ g$ if and only if $\sum_{s\in S} \pi(s)(\sum_{x\in X} h(x, s)\bar{U}(x)) > \sum_{s\in S} \pi(s)(\sum_{x\in X} g(x, s)\bar{U}(x))$ for all $\pi \in \mathcal{M}$.

The proof that (ii) $\Rightarrow$ (i) is straightforward. The uniqueness result is implied by the uniqueness of Theorem 2. *Q.E.D.*

5.7. *Proof of Theorem 4*

(i) $\Rightarrow$ (ii). First, we show that A.1–A.3 and A.7 assure a unique probability measure over S. Let $\pi^u(E) = \inf\{\alpha \in [0, 1] \mid p^M \alpha p^m \succ p^M E p^m\}$ and $\pi^l(E) = \sup\{\alpha \in [0, 1] \mid p^M E p^m \succ p^M \alpha p^m\}$.

CLAIM 8: *For any $E \subset S$, $\pi^u(E) = \pi^l(E)$.*

PROOF: Axiom A.3 implies that $\pi^u(E) \geq \pi^l(E)$. Suppose that $\pi^u(E) > \pi^l(E)$.[27] Then there exist α_1, α_2 such that $\pi^u(E) > \alpha_1 > \alpha_2 > \pi^l(E)$. Since $\pi^u(E) > \alpha_1$ implies $p^M \alpha_1 p^m \succ p^M E p^m$ does not hold, A.7 implies $p^M E p^m \succ p^M \alpha_2 p^m$, which is a contradiction to $\alpha_2 > \pi^l(E)$. Therefore, $\pi^u(E) = \pi^l(E)$. *Q.E.D.*

Define $\pi(E) := \pi^u(E) = \pi^l(E)$. Next, we show that π is a probability measure.

[27]To be exact, A.3 implies mixture monotonicity – that is, for all $f, g \in H$ and $0 \leq \alpha < \beta \leq 1$, $f \succ g$ implies that $\beta f + (1 - \beta)g \succ \alpha f + (1 - \alpha)g$ (see Kreps (1988, Lemma 5.6)). Mixture monotonicity implies that $\pi^u(E) \geq \pi^l(E)$.

CLAIM 9: *The set function* $\pi : 2^S \to [0, 1]$ *is a probability measure*.

PROOF: By definition, $\pi(S) = 1$. Since S is a finite set, it is enough to show that $\pi(E \cup \{s\}) = \pi(E) + \pi(s)$ for all $E \subseteq S$ and for all $s \notin E$.

First, we show that $\pi(E \cup \{s\}) \leq \pi(E) + \pi(s)$. Without loss of generality, assume that $\pi(E) + \pi(s) < 1$. Pick any $\varepsilon > 0$ such that $\pi(E) + \pi(s) + 2\varepsilon < 1$. Then there exist $\alpha_1, \alpha_2, \beta_1, \beta_2 \in [0, 1]$ such that $\pi(E) < \beta_1 < \alpha_1 < \pi(E) + \varepsilon$ and $\pi(s) < \beta_2 < \alpha_2 < \pi(s) + \varepsilon$.

If we can show that $p^M(\alpha_1 + \alpha_2)p^m \succ p^M(E \cup \{s\})p^m$, then we have $\pi(E \cup \{s\}) < \alpha_1 + \alpha_2 < \pi(E) + \pi(s) + 2\varepsilon$, which implies $\pi(E \cup \{s\}) \leq \pi(E) + \pi(s)$.[28] Suppose that $p^M(\alpha_1 + \alpha_2)p^m \succ p^M(E \cup \{s\})p^m$ does not hold. Then, by A.7, $p^M(\beta_1 + \beta_2)p^m \prec p^M(E \cup \{s\})p^m$.

We know that $p^M\beta_1 p^m \succ p^M E p^m$ and $p^M\beta_2 p^m \succ p^M\{s\}p^m$ imply that for all $w \in \mathcal{W}$,

$$\beta_1 \sum_{s\in S} w(p^M, s) + (1 - \beta_1) \sum_{s\in S} w(p^m, s) > \sum_{t\in E} w(p^M, t) + \sum_{t\notin E} w(p^m, t)$$

and

$$\beta_2 \sum_{s\in S} w(p^M, s) + (1 - \beta_2) \sum_{s\in S} w(p^m, s) > w(p^M, s) + \sum_{t\neq s} w(p^m, t).$$

Adding these two inequalities, we obtain that for all $w \in \mathcal{W}$,

$$(\beta_1 + \beta_2) \sum_{s\in S} w(p^M, s) + (1 - \beta_1 - \beta_2) \sum_{s\in S} w(p^m, s) + \sum_{s\in S} w(p^m, s) > w(p^M(E \cup \{s\})p^m) + \sum_{s\in S} w(p^m, s).$$

Hence, for all $w \in \mathcal{W}$,

$$(\beta_1 + \beta_2) \sum_{s\in S} w(p^M, s) + (1 - \beta_1 - \beta_2) \sum_{s\in S} w(p^m, s) > \sum_{s\in S} w(p^M(E \cup \{s\})p^m, s).$$

But this is obviously a contradiction of $p^M(\beta_1 + \beta_2)p^m \prec p^M(E \cup \{s\})p^m$. Thus, $\pi(E \cup \{s\}) \leq \pi(E) + \pi(s)$.

[28]Recall that, by Lemma 3, $h^M = (p^M, \dots, p^M)$ and $h^m = (p^m, \dots, p^m)$.

Suppose $\pi(E \cup \{s\}) < \pi(E) + \pi(s)$. Then there exists α such that $\pi(E \cup \{s\}) < \alpha < \pi(E) + \pi(s)$. Since $0 \leq \alpha - \pi(E) < \pi(s)$, we can find $\alpha_1 < \alpha$ such that $\alpha - \pi(E) < \alpha_1 < \pi(s)$. Thus, we have $\alpha - \alpha_1 \in (0, \pi(E))$ and $\alpha_1 < \pi(s)$. Therefore, by using the same argument above, we can have

$$\begin{aligned} &p^M\{s\}p^m \succ p^M\alpha_1 p^m \text{ and } p^M E p^m \succ p^M(\alpha - \alpha_1)p^m \\ &\quad\Rightarrow \quad p^M\big(E \cup \{s\}\big)p^m \succ p^M \alpha p^m. \end{aligned}$$

This is a contradiction to $\pi(E \cup \{s\}) < \alpha$. *Q.E.D.*

Now we enter the proof of Theorem 4. Suppose $\alpha > \pi(E)$. Then, by Lemma 1,

$$\begin{aligned} &p^M\alpha p^m \succ p^M E p^m \quad \text{if and only if} \\ &\quad \sum_{s\in S} w\big(p^M\alpha p^m, s\big) > \sum_{s\in S} w\big(p^M E p^m, s\big) \quad \forall w \in \mathcal{W}. \end{aligned} \tag{16}$$

Equation (16) implies that for all $w \in \mathcal{W}$,

$$\begin{aligned} &\alpha \sum_{s\in S} w\big(p^M, s\big) + (1-\alpha)\sum_{s\in S} w\big(p^m, s\big) \\ &\quad > \sum_{s\in E} w\big(p^M, s\big) + \sum_{s\notin E} w\big(p^m, s\big), \end{aligned}$$

which, in turn, implies that for all $w \in \mathcal{W}$,

$$\begin{aligned} &\alpha \sum_{s\notin E} w\big(p^M, s\big) + (1-\alpha)\sum_{s\in E} w\big(p^m, s\big) \\ &\quad > (1-\alpha)\sum_{s\in E} w\big(p^M, s\big) + \alpha\sum_{s\notin E} w\big(p^m, s\big). \end{aligned} \tag{17}$$

Equation (17) implies that for all $w \in \mathcal{W}$,

$$\frac{\alpha}{1-\alpha} > \frac{\sum_{s\in E} w(p^M, s) - \sum_{s\in E} w(p^m, s)}{\sum_{s\notin E} w(p^M, s) - \sum_{s\notin E} w(p^m, s)} \quad \forall \alpha > \pi(E).$$

Hence,

$$\frac{\pi(E)}{1-\pi(E)} \geq \frac{\sum_{s\in E} w(p^M, s) - \sum_{s\in E} w(p^m, s)}{\sum_{s\notin E} w(p^M, s) - \sum_{s\notin E} w(p^m, s)} \quad \forall w \in \mathcal{W}.$$

For all $\alpha < \pi(E)$, we can repeat the same argument. Therefore, we get for all $w \in \mathcal{W}$,

$$\frac{\alpha}{1-\alpha} < \frac{\sum_{s\in E} w(p^M, s) - \sum_{s\in E} w(p^m, s)}{\sum_{s\notin E} w(p^M, s) - \sum_{s\notin E} w(p^m, s)} \quad \forall \alpha < \pi(E).$$

Hence,

$$\frac{\pi(E)}{1-\pi(E)} \leq \frac{\sum_{s\in E} w(p^M, s) - \sum_{s\in E} w(p^m, s)}{\sum_{s\notin E} w(p^M, s) - \sum_{s\notin E} w(p^m, s)} \quad \forall w \in \mathcal{W}.$$

Thus, we conclude that

$$\frac{\pi(E)}{1-\pi(E)} = \frac{\sum_{s\in E} w(p^M, s) - \sum_{s\in E} w(p^m, s)}{\sum_{s\notin E} w(p^M, s) - \sum_{s\notin E} w(p^m, s)} \quad \forall w \in \mathcal{W}.$$

Lemma 5 implies that whenever $h^x \succ h^m$, $p^M \alpha p^m \succ p^M E p^m$ if and only if $\delta_x \alpha p^m \succ \delta_x E p^m$. Thus, for all $w \in \mathcal{W}$,

$$\text{(18)} \qquad \frac{\pi(E)}{1-\pi(E)} = \frac{\sum_{s\in E} w(p^M, s) - \sum_{s\in E} w(p^m, s)}{\sum_{s\notin E} w(p^M, s) - \sum_{s\notin E} w(p^m, s)} = \frac{\sum_{s\in E} w(\delta_x, s) - \sum_{s\in E} w(p^m, s)}{\sum_{s\notin E} w(\delta_x, s) - \sum_{s\notin E} w(p^m, s)}.$$

Let $S = \{s_1, s_2, \dots, s_n\}$ and $E = \{s_i\}$. By equation (18), we have for all $w \in \mathcal{W}$,

$$\text{(19)} \qquad \frac{1-\pi(s_i)}{\pi(s_i)} = \frac{\sum_{s\in S-\{s_i\}} w(\delta_x, s) - \sum_{s\in S-\{s_i\}} w(p^m, s)}{w(\delta_x, s_i) - w(p^m, s_i)}.$$

Hence,

$$\text{(20)} \qquad \frac{1}{\pi(s_i)} = \frac{\sum_{s\in S} w(\delta_x, s) - \sum_{s\in S} w(p^m, s)}{w(\delta_x, s_i) - w(p^m, s_i)}.$$

Thus,

$$\frac{\pi(s_i)}{\pi(s_j)} = \frac{w(\delta_x, s_i) - w(p^m, s_i)}{w(\delta_x, s_j) - w(p^m, s_j)} \quad \forall i, j \in \{1, \dots, n\}. \tag{21}$$

By taking $j = 1$, we get

$$w(\delta_x, s_i) = \frac{\pi(s_i)}{\pi(s_1)}\big(w(\delta_x, s_1) - w\big(p^m, s_1\big)\big) + w\big(p^m, s_i\big), \tag{22}$$

which implies

$$w(p, s_i) = \frac{\pi(s_i)}{\pi(s_1)} w(p, s_1) - \frac{\pi(s_i)}{\pi(s_1)} w\big(p^m, s_1\big) + w\big(p^m, s_i\big). \tag{23}$$

Suppose that $h, g \in H$. Then

$$\begin{aligned} &h \succ g \quad \text{if and only if} \\ &\quad \sum_s w\big(h(s), s\big) > \sum_s w\big(g(s), s\big) \quad \text{for all } w \in \mathcal{W}. \end{aligned}$$

By using equations (19)–(23), we can easily show that

$$\begin{aligned} &\sum_s w\big(h(s), s\big) > \sum_s w\big(g(s), s\big) \quad \text{for all } w \in \mathcal{W} \quad \text{if and only if} \\ &\quad \sum_i \pi(s_i) w\big(h(s_i), s_1\big) > \sum_i \pi(s_i) w\big(g(s_i), s_1\big) \quad \text{for all } w \in \mathcal{W}. \end{aligned}$$

Define $\mathcal{U} = \{w(\cdot, s_1) \mid w \in \mathcal{W}\}$. Then the last two equations imply

$$\begin{aligned} &h \succ g \quad \text{if and only if} \\ &\quad \sum_{s \in S} \pi(s) U\big(h(s)\big) > \sum_{s \in S} \pi(s) U\big(g(s)\big) \quad \text{for all } U \in \mathcal{U}. \end{aligned}$$

The proof of (ii) $\Rightarrow$ (i) is straightforward. The uniqueness follows from the uniqueness result in Dubra, Maccheroni, and Ok (2004) (by restricting $\succ$ to constant acts).

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Manuscript received October, 2010; final revision received March, 2012.