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SUBMANIFOLDS AND SPECIAL STRUCTURES ON THE OCTONIANS

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0. Introduction

Geometries associated to the exceptional groups and "exceptional" representations of classical groups often display interesting features closely related to (but distinctly different from) the more familiar features of the classical groups. This paper centers on the geometries in $\mathbf{E}^7$ and $\mathbf{E}^8$ whose groups of symmetries are $G_2 \subseteq SO(7)$ and $\mathrm{Spin}(7) \subseteq SO(8)$. Both of these groups are related to the octonians (sometimes called Cayley numbers) and may be defined in terms of octonionic multiplication. In particular, G_2, the compact exceptional group of (real) dimension 14, is the subgroup of algebra automorphisms of $\mathbf{O}$ (the octonians) and $\mathrm{Spin}(7) \subseteq SO(8)$ may be defined as the subgroup of $GL_{\mathbf{R}}(\mathbf{O})$ generated by right multiplication by unit octonians which are purely imaginary.

The geometry of the algebra $\mathbf{O}$ is closely related to the complex numbers. In §1, we develop some of the properties of $\mathbf{O}$ that we need for later sections. (Our presentation is essentially borrowed from Appendix A of [12], but any mistakes are, of course, due to the author.) A particularly interesting property is described as follows: If we let $\operatorname{Im}\mathbf{O} \subseteq \mathbf{O}$ be the hyperplane (through 0) orthogonal to $1 \in \mathbf{O}$, and we let $S^6 \subseteq \operatorname{Im}\mathbf{O}$ be the space of unit vectors, then right multiplication by $u \in S^6$ induces a linear transformation $R_u\colon \mathbf{O} \to \mathbf{O}$ which is orthogonal and satisfies $(R_u)^2 = -1$. Thus, associated to each $u \in S^6$ is a complex structure on $\mathbf{O}$ (induced by $J = R_u$) which is compatible with the natural inner product on $\mathbf{O}$. We denote by $\mathbf{O}_u$ the Hermitian vector space whose underlying real vector space (with inner product) is $\mathbf{O}$ and whose complex structure is given by R_u.

Classically, this observation was used to define an almost complex structure on S^6 as follows: If $u \in S^6$, then R_u preserves the 2-plane spanned by 1 and u and therefore preserves its orthogonal 6-plane, which may be identified with $T_uS^6 \subseteq \operatorname{Im}\mathbf{O}$ after translation to the origin. Thus R_u induces a complex

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structure on T_uS^6 for each $u \in S^6$. This almost complex structure is not integrable (even locally) to a complex structure (see below). In [Ca], Calabi noticed that for *any* oriented $M^6 \subseteq \operatorname{Im}\mathbf{O}$, $R_{N(x)}$ induces a complex structure on T_xM^6 (where $N(x)$ is the oriented unit normal). Thus every oriented $M^6 \subseteq \operatorname{Im}\mathbf{O}$ inherits an almost complex structure. Moreover, M^6 inherits a metric from $\operatorname{Im}\mathbf{O}$, so we actually have a $U(3)$-structure on M^6. (Calabi calls these structures "almost Hermitian." He also proves that such M^6 possess a canonical $SU(3)$-substructure but we will not need this.) Calabi shows that the second fundamental form II of M decomposes with respect to the $U(3)$-structure into a piece $\mathrm{II}^{1,1}$ of type (1,1) and a piece $\mathrm{II}^{0,2}$ of type (0,2). He then shows that the almost complex structure of M is integrable if and only if $\mathrm{II}^{1,1} = 0$ and that the canonical 2-form of the $U(3)$-structure, say Ω, is closed if and only if $\mathrm{II}^{0,2} = 0$ and $\operatorname{tr}_{\mathrm{I}}\mathrm{II}^{1,1} = 0$. From this it follows that the $U(3)$-structure on M^6 is Kähler if and only if $\mathrm{II} \equiv 0$, so that M^6 is a hyperplane (or a union of pieces of hyperplanes). Calabi then constructs nontrivial examples of $M^6 \subseteq \operatorname{Im}\mathbf{O}$ for which the almost complex structure is integrable. His examples are of the form $S \times \mathbf{R}^4 \subseteq \operatorname{Im}\mathbf{O}$, where $S \subseteq \mathbf{R}^3$ is a minimal surface, $\mathbf{R}^3 \subseteq \operatorname{Im}\mathbf{O}$ is an *associative* 3-plane, and $\mathbf{R}^4 = (\mathbf{R}^3)^\perp$. Calabi leaves open the problem of determining whether or not there are nontrivial $M^6 \subseteq \operatorname{Im}\mathbf{O}$ for which the canonical 2-form is closed.

In [10], Gray generalized Calabi's construction somewhat by considering hypersurfaces in N^7 where T_xN^7 has a vector cross product modeled on $\operatorname{Im}\mathbf{O} \simeq \mathbf{R}^7$. In the case $N = \operatorname{Im}\mathbf{O}$, Gray observes that the canonical 2-form Ω on $M^6 \subseteq \operatorname{Im}\mathbf{O}$ is always co-closed, i.e., $\delta\Omega = 0$ (or equivalently $d\Omega^2 = 0$).

In the present paper, after some preliminary work establishing the structure equations of $\mathrm{Spin}(7) \subseteq SO(8)$, we study oriented manifolds $M^6 \subseteq \mathbf{O}$. As is pointed out in [12], every oriented 6-plane in $\mathbf{O} \simeq \mathbf{R}^8$ is a complex three-plane in $\mathbf{O}_u$ for a unique $u \in S^6$. Thus, every oriented six-manifold in $\mathbf{O}$ inherits a natural $U(3)$-structure generalizing the case where $M^6 \subseteq \operatorname{Im}\mathbf{O}$. In this case, we decompose the second fundamental form II of M into *three* pieces and prove the analogues of Calabi's theorems concerning when the $U(3)$-structure is complex integrable and when $d\Omega = 0$. In particular, we show that the induced $U(3)$-structure on $M^6 \subseteq \mathbf{O}$ is Kähler if and only if M^6 is a complex hypersurface in $\mathbf{O}_u$ for some fixed $u \in S^6$. We then go further in the study of those $M^6 \subseteq \mathbf{O}$ for which the $U(3)$-structure is complex integrable but which are not Kähler. We show that such M^6 are foliated by 4-planes in $\mathbf{O}$ in a unique way. We refer to this foliation as the *asymptotic ruling* of M^6. Using the moving frame, we prove that if the asymptotic ruling is parallel then M^6 is the product of a fixed 4-plane in $\mathbf{O}$ with a minimal surface in the orthogonal 4-plane. In particular, we show that Calabi's examples are exactly the M^6 with parallel asymptotic ruling which lie in the hyperplane $\operatorname{Im}\mathbf{O} \subseteq \mathbf{O}$. We then use Cartan's theory of differential systems in involution to show that the analytic non-Kähler but complex $M^6 \subseteq \mathbf{O}$ "depend" on 12 analytic functions of 1 (real) variable. (For a more precise statement, see §3).

We observe, as did Gray, that the canonical 2-form on $M^6 \subseteq \mathbf{O}$ is *always* co-closed. Finally, we show that any $M^6 \subseteq \mathbf{O}$ for which the canonical 2-form Ω is closed is necessarily Kähler (and therefore must be a complex hypersurface in $\mathbf{O}_u$ for some fixed $u \in S^6$). In particular, such $M^6 \subseteq \operatorname{Im}\mathbf{O}$ must be hyperplanes. This recovers a result of Gray (see [10]).

In the final section of the paper, we study the "complex curves" in S^6, i.e., those maps $\phi\colon M^2 \to S^6$ where M^2 is a Riemann surface and $d\phi$ is complex linear with respect to the almost complex structure on S^6 induced by the inclusion $S^6 \subseteq \operatorname{Im}\mathbf{O}$. This study is motivated by the fact that the cone on such a complex curve gives a 3-fold in $\operatorname{Im}\mathbf{O}$ which is *associative* in the sense of [12]. Such cones are absolutely mass minimizing and their singular structure reflects the singular structure of general associative varieties in $\mathbf{R}^7 \subseteq \operatorname{Im}\mathbf{O}$. We first prove that the almost complex structure on S^6 determines the metric structure of S^6 so that any invariant of the *local* almost complex structure is also a metric invariant (for a more precise statement, see Proposition 4.1 and its proof). (This is the compact-form analogue of Cartan's characterization of the split form of G_2 as the pseudo-group of a certain differential system on a five manifold.) This justifies our use of the metric structure on S^6 to study the almost complex structure of S^6.

Since the generalized Cauchy-Riemann equations for local mappings of Riemann surfaces into an *almost* complex manifold form a determined elliptic system (which is first order, quasi-linear) we expect the local theory of complex curves in S^6 to be analogous to the local theory of complex curves in $\mathbf{C}^3$. (In the analytic category, this is certainly the case.) Along these lines, we develop a Frenet formalism for complex cuves in S^6 analogous to that developed for complex curves in $\mathbf{CP}^3$. We define the first, second and the third fundamental forms of $\phi\colon M^2 \to S^6$ as holomorphic sections of line bundles over M^2. In particular, the third fundamental form III, analogous to the torsion of a space curve, plays a crucial role. The assumption that $\mathrm{III} \neq 0$ places severe restrictions on the divisors of the three fundamental forms (see [11] for terminology concerning Riemann surfaces). We are able to prove, for example, that if $M^2 = \mathbf{P}^1$, then $\mathrm{III} \not\equiv 0$ is impossible. It seems likely that for fixed genus g, the space of complex curves $\phi\colon M^2 \to S^6$ (where M^2 has genus g) with $\mathrm{III} \not\equiv 0$ is finite dimensional, but we have not proven this.

Turning to those curves with $\mathrm{III} \equiv 0$, we show that these curves are the integrals of a holomorphic differential system on the complex 5-quadric. We then use a normal form (due to Cartan in [6]) for this holomorphic system to construct generically 1-1 complex curves $\phi\colon M^2 \to S^6$ with $\mathrm{III} \equiv 0$ for any Riemann surface M^2 so that the ramification divisor of ϕ has arbitrarily large degree. The author would like to express his gratitude to Phillip Griffiths for explaining the technical aspects of line bundles over M^2 used in this construction (see the proof of Theorem 4.10). This shows, in a sense, that the compact curves of torsion zero ($\mathrm{III} \equiv 0$) are "more general" than those with torsion nonzero. This should be contrasted with the situation in $\mathbf{CP}^3$, for example.

Throughout this paper, we assume that the reader is familiar with the theory of moving frames. For notation concerning almost complex and complex manifolds, the reader should consult [8] or [15]. We make one extension of their terminology: If M is almost comlex and $\pi\colon B \to M$ is bundle over M, we speak of a form on B as being of type (p, q) if it is locally a linear combination (with coefficients in $C^\infty(B)$) of pullbacks under π^* of forms of type (p, q) on M. This will cause no problem except in the case that B is also almost complex and π_* is not complex linear. In this case, we are careful to distinguish the two so that no confusion can arise (hopefully). For notions concerning Riemann surfaces, we have used [11] as the basic reference.

Finally, the author would like to acknowledge (gratefully) conversations and inspiration from Eugenio Calabi, Phillip Griffiths, and Reese Harvey.

1. The octonians and Spin(7)

In this section, we give a brief description of the octonian algebra $\mathbf{O}$ and derive a few of its properties. We then go on to define the group $\mathrm{Spin}(7) \subseteq SO(8)$ by octonian multiplication and to derive its Lie algebra and structure equations in a form suitable for our differential geometric investigations in the following sections. For more details on $\mathbf{O}$ and Spin(7), the reader is encouraged to consult Appendix A in [12] and the classical references listed in its bibliography.

An inner product algebra over $\mathbf{R}$ is a vector space $\mathbf{A}$ over $\mathbf{R}$ which possesses a nondegenerate inner product $\langle\,,\rangle\colon \mathbf{A} \times \mathbf{A} \to \mathbf{R}$ and a multiplication $\mathbf{A} \times \mathbf{A} \to \mathbf{A}$ with unit $1 \in \mathbf{A}$ so that for all $x, y \in \mathbf{A}$

$$\langle xy, xy\rangle = \langle x, x\rangle\langle y, y\rangle. \tag{1.1}$$

For convenience's sake we will identify $\mathbf{R}$ with the 1-dimensional subalgebra of $\mathbf{A}$ generated by $1 \in \mathbf{A}$. By (1.1), we have $\langle 1,1\rangle = \langle 1,1\rangle^2$. If $\langle 1,1\rangle = 0$, then

$\langle x, x\rangle = 0$ for all $x \in \mathbf{A}$, contradicting the nondegeneracy assumption. Hence $\langle 1, 1\rangle = 1$. We define the orthogonal compliment of 1 to be $\operatorname{Im}\mathbf{A} \subseteq \mathbf{A}$. It is a proper subspace and we have $\mathbf{A} = \mathbf{R} \oplus \operatorname{Im}\mathbf{A}$. Give $x \in \mathbf{A}$, we define $\bar{x} \in \mathbf{A}$,

$$\bar{x} = 2\langle x, 1\rangle - x. \tag{1.2}$$

We denote $\langle x, 1\rangle$ by $\operatorname{Re} x$ and $(x - \operatorname{Re} x)$ by $\operatorname{Im} x$. Clearly $x \in \operatorname{Im}\mathbf{A}$ if and only if $x = -\bar{x}$ or $x = \operatorname{Im} x$ or $\operatorname{Re} x = 0$.

If we polarize (1.1) in the x-variable, we get the identity

$$\langle xy, zy\rangle = \langle x, z\rangle\langle y, y\rangle. \tag{1.3}$$

If we expand $\langle x(1 + w), y(1 + w)\rangle$ in two ways and compare terms, we find

$$\langle xw, y\rangle = \langle x, y(2\langle w, 1\rangle - w)\rangle$$

or

$$\langle xw, y\rangle = \langle x, y\bar{w}\rangle \tag{1.4}$$

for all $x, y, w \in \mathbf{A}$. In the same way, we get

$$\langle wx, y\rangle = \langle x, \bar{w}y\rangle. \tag{1.4'}$$

Using (1.4) and (1.4′) repeatedly, we get

$$\begin{aligned}\langle w, \bar{y}\bar{x}\rangle &= \langle yw, \bar{x}\rangle = \langle y, \bar{x}\bar{w}\rangle = \langle xy, \bar{w}\rangle \\ &= \langle w(xy), 1\rangle = \langle w, \overline{(xy)}\rangle\end{aligned}$$

for all $x, y, w \in \mathbf{A}$. It follows that

$$\overline{(xy)} = \bar{y}\bar{x}. \tag{1.5}$$

From (1.5), we conclude that $x\bar{x}$ is real for all $x \in A$. but then $\langle x, x\rangle = \langle x\bar{x}, 1\rangle = x\bar{x}$.

$$x\bar{x} = \langle x, x\rangle = \bar{x}x. \tag{1.6}$$

Polarizing (1.6) we get

$$\langle x, y\rangle = \tfrac{1}{2}(x\bar{y} + y\bar{x}). \tag{1.7}$$

We also compute

$$\langle (xw)\bar{w}, y\rangle = \langle xw, yw\rangle = \langle x, y\rangle\langle w, w\rangle = \langle x\langle w, w\rangle, y\rangle$$

so

$$(xw)\bar{w} = x(w\bar{w}) \quad (= x\langle w, w\rangle). \tag{1.8}$$

by subtracting $2(xw)\langle w, 1\rangle$ from both sides of (1.8), we get

$$(xw)w = xw^2 \tag{1.9}$$

in spite of the fact that we have not assumed that **A** is associative. In a similar manner, we get

$$\bar{w}(wx) = (\bar{w}w)x, \tag{1.8$'$}$$

$$w(wx) = w^2x. \tag{1.9$'$}$$

By polarizing (1.8) and (1.8′), we get the identities

$$(xu)\bar{v} + (xv)\bar{u} = 2x\langle u, v\rangle, \tag{1.10}$$

$$u(\bar{v}x) + v(\bar{u}x) = 2x\langle u, v\rangle. \tag{1.11}$$

In particular, if $\langle u, v\rangle = 0$, then $(xu)\bar{v} = -(xv)\bar{u}$ and $u(\bar{v}x) = -v(\bar{u}x)$. We may use these facts to prove the following lemma (see [12]).

Lemma 1.1. *If* $\mathbf{B} \subseteq \mathbf{A}$ *is an inner product subalgebra and* $u \in \mathbf{A}$ *is orthogonal to* **B**, *then* $\mathbf{B}u \perp \mathbf{B}$ *and* $\mathbf{B} \oplus \mathbf{B}u$ *is a subalgebra of* **A** *which satisfies*

$$(a + bu)(c + du) = \big(ac - \langle u, u\rangle\bar{d}b\big) + \big(da + b\bar{c}\big)u. \tag{1.12}$$

This lemma allows us to start with $\mathbf{B} = \mathbf{R}$ and "build up" to **A** by successively adding on orthogonal subspaces. Using this technique, one can show that if we assume that $\langle\,,\rangle$ is positive definite, then there are only four inner product algebras over **R**, namely **R**, **C**, **H** (the quaternions) and **O** (the octonians).

Explicitly, we may regard **O** as the vector space $\mathbf{H} \oplus \mathbf{H}$. If we write 1 for $(1,0) \in \mathbf{O}$ and ε for $(0,1) \in \mathbf{O}$, the above lemma shows that the multiplication in **O** must be given by

$$(a + b\varepsilon), (c + d\varepsilon) = (ac - \bar{d}b) + (da + b\bar{c})\varepsilon \tag{1.13}$$

where the inner product satisfies

$$\langle(a + b\varepsilon), (a + b\varepsilon)\rangle = a\bar{a} + b\bar{b} \tag{1.4}$$

whenever $a, b, c, d \in \mathbf{H}$.

We let $S^6 = \{u \in \operatorname{Im}\mathbf{O} \mid \langle u, u\rangle = 1\}$. The elements of S^6 are called the *imaginary units* of **O**. For any $u \in S^6$, we have $u = -\bar{u}$, so $u^2 = -u\bar{u} = -\langle u, u\rangle = -1$. We may use u to define a map $J_u\colon \mathbf{O} \to \mathbf{O}$ given by

$$J_u(x) = xu. \tag{1.15}$$

The identity (1.9) shows that $J_u^2(x) = (xu)u = xu^2 = -x$, so J_u defines a complex structure on **O**. We write $\mathbf{O}_u$ to denote **O** endowed with the complex structure J_u. If $u \neq v$, then clearly $J_u \neq J_v$, so we actually have a six-sphere of *distinct* complex structures on **O**. However, because S^6 is connected, we see that the orientation of **O** induced by the natural orientation of $\mathbf{O}_u$ as a complex vector space does not depend on u. We refer to this orientation as the natural orientation of **O**.

Using (1.3), we see that if $u \in S$, then

$$\langle J_u(x), J_u(y)\rangle = \langle xu, yu\rangle = \langle x, y\rangle\langle u, u\rangle = \langle x, y\rangle$$

so J_u is an isometry for each $u \in S$. Moreover, it follows that $\mathbf{O}_u$ is endowed with a natural Hermitian structure with respect to the inner product $\langle\, ,\rangle$. We denote the group of complex linear transformations of $\mathbf{O}_u$ by $GL(\mathbf{O}_u)$ and the special unitary transformations of $\mathbf{O}_u$ with its Hermitian metric by $SU(\mathbf{O}_u)$.

We let Spin(7) $\subseteq SO(8)$ denote the subgroup generated by the set $\{J_u \mid u \in S^6\} \subseteq SO(8)$. It is known (see [12]) that Spin(7) is a connected, simply connected, compact Lie group of real dimension 21. Its center is $\{\pm I_8\} \simeq \mathbf{Z}/2$ and Spin(7)$/\{\pm I_8\}$ is isomorphic to $SO(7)$, a simple group. We want to make explicit the structure equations of Spin(7) as a subgroup of $SO(8)$ in such a way that its relationship with the complex structures J_u is made clear.

Let $u \in S^6$ be an imaginary unit which is orthogonal to $\varepsilon \in \mathbf{O}$. For each $\lambda \in \mathbf{R}$, $(\cos\lambda\varepsilon + \sin\lambda u)$ is an imaginary unit. Hence $J_\varepsilon \circ J_{(\cos\lambda\varepsilon+\sin\lambda u)} = -\cos\lambda I + \sin\lambda J_\varepsilon \circ J_u$ is an element of Spin(7). We easily compute that $J_\varepsilon \circ J_u + J_u \circ J_\varepsilon = 0$ by using (1.10). Thus $(J_\varepsilon \circ J_u)^2 = J_\varepsilon \circ J_u \circ J_\varepsilon \circ J_u = -J_\varepsilon^2 \circ J_u^2 = -I$. It follows that

$$\exp(\lambda J_\varepsilon \circ J_u) = \cos\lambda I + \sin\lambda J_\varepsilon \circ J_u. \tag{1.16}$$

Thus, if spin(7) $\subseteq so(8)$ is the Lie algebra of Spin(7), we see that $J_\varepsilon \circ J_u \in$ spin(7) for all $u \in S^6$ with $\langle u, \varepsilon\rangle = 0$. Since spin(7) is a vector space, we see that $L \subseteq$ spin(7) where

$$L = \{J_\varepsilon \circ J_w \mid w \in \operatorname{Im}\mathbf{O}, \langle \varepsilon, w\rangle = 0\}. \tag{1.17}$$

Note that $\dim_{\mathbf{R}} L = 6$.

To go further, we will choose a basis and exhibit L as a vector space of matrices. In order to do this, let j and k be orthogonal imaginary units in $\mathbf{H}$.[1] We define the *standard basis of* $\mathbf{C} \otimes_{\mathbf{R}} \mathbf{O}$, $(N, E, \overline{N}, \overline{E}) = (N, E_1, E_2, E_3, \overline{N}, \overline{E}_1, \overline{E}_2, \overline{E}_3)$ as follows: We set $N = \frac{1}{2}(1 - i\varepsilon)$, $\overline{N} = \frac{1}{2}(1 + i\varepsilon)$ and

$$\begin{aligned} E_1 &= jN, & \overline{E}_1 &= j\overline{N}, \\ E_2 &= kN, & \overline{E}_2 &= k\overline{N}, \\ E_3 &= (kj)N, & \overline{E}_3 &= (kj)\overline{N}. \end{aligned} \tag{1.18}$$

(Note that conjugation in $\mathbf{C} \otimes_{\mathbf{R}} \mathbf{O}$ occurs *only* in the $\mathbf{C}$-factor.) By using the formulae in (1.13) and some elementary calculation, we see that if we set

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[1] In all that follows, we *never* use i to denote a quaternion or an octonian. For us, $i \in \mathbf{C}$ and $\mathbf{C} \not\subset \mathbf{H}$.

$w = 2\mathbf{R}e(a^1E_1 + a^2E_2 + a^3E_3)$ where $a_i \in \mathbf{C}$ and $\mathbf{R}e\colon \mathbf{C} \otimes_{\mathbf{R}} \mathbf{O} \to \mathbf{O}$ is the real projection, then $w \in \operatorname{Im}\mathbf{O}$, $\langle \varepsilon, w \rangle = 0$ and

$$\begin{aligned} & J_\varepsilon \circ J_w(N, E_1, E_2, E_3) \\ & = (\bar{N}, \bar{E}_1, \bar{E}_2, \bar{E}_3) \begin{pmatrix} 0 & i\bar{a}^1 & i\bar{a}^2 & i\bar{a}^3 \\ -i\bar{a}^1 & 0 & ia^3 & -ia^2 \\ -i\bar{a}^2 & -ia^3 & 0 & ia^1 \\ -i\bar{a}^3 & ia^2 & -ia^1 & 0 \end{pmatrix}. \end{aligned} \tag{1.19}$$

To simplify matters, if $a = (a^i)$ is any column vector of height 3 (with complex entries), we define $[a]$ to be the 3×3 skew symmetric matrix

$$[a] = \begin{pmatrix} 0 & a^3 & -a^2 \\ -a^3 & 0 & a^1 \\ a^2 & -a^1 & 0 \end{pmatrix}. \tag{1.20}$$

Note that $[a]$ is the matrix of the linear transformation from $\mathbf{C}^3$ to $\mathbf{C}^3$ determined by cross product with $a \in \mathbf{C}^3$. We will eventually need the following identities for $a, b \in \mathbf{C}^3$ and $A \in M_{3\times 3}(\mathbf{C})$.

$$\begin{aligned} & [a]b + [b]a = 0, \\ & [Aa] = (\operatorname{tr} A)[a] - {}^tA[a] - [a]A, \\ & [a][b] = b\,{}^ta - {}^tab I_3 \end{aligned} \tag{1.21}$$

(I_3 is the 3×3 identity matrix).

We may now rewrite (1.19) in the more compact form

$$J_\varepsilon \circ J_w(N, E) = (\bar{N}, \bar{E}) \begin{pmatrix} 0 & i\,{}^t\bar{a} \\ -i\bar{a} & [ia] \end{pmatrix} \tag{1.19$'$}$$

where $w = 2\,\mathbf{R}e(aE)$ (row by column multiplication is understood). It follows that, expressed in the full basis $(N, E, \bar{N}, \bar{E})$ we have

$$J_\varepsilon \circ J_w(N, E, \bar{N}, \bar{E}) = (N, E, \bar{N}, \bar{E}) \begin{pmatrix} 0 & 0 & 0 & -i\,{}^ta \\ 0 & 0 & ia & [-i\bar{a}] \\ 0 & i\,{}^t\bar{a} & 0 & 0 \\ -i\bar{a} & [ia] & 0 & 0 \end{pmatrix}. \tag{1.19$''$}$$

Thus, imbedding $\operatorname{End}(\mathbf{O}) \hookrightarrow M_{8\times 8}(\mathbf{C})$, the space of 8×8 complex matrices, via the standard basis, we get

$$L = \left\{ \begin{pmatrix} 0 & 0 & 0 & -{}^ta \\ 0 & 0 & a & [\bar{a}] \\ 0 & -{}^t\bar{a} & 0 & 0 \\ \bar{a} & [a] & 0 & 0 \end{pmatrix} \,\middle|\, a \in \mathbf{C}^3 = M_{3\times 1}(\mathbf{C}) \right\}. \tag{1.22}$$

An easy computation, using (1.21), then shows

$$[L, L] = \left\{ \begin{pmatrix} \kappa & 0 \\ 0 & \bar{\kappa} \end{pmatrix} \middle| \begin{array}{l} \kappa + {}^t\bar{\kappa} = 0 \text{ and } \operatorname{tr} \kappa = 0 \\ \text{and } \kappa \in M_{4\times 4}(\mathbf{C}) \end{array} \right\}. \tag{1.23}$$

Since $L \subseteq \operatorname{spin}(7)$, $[L, L] \subseteq \operatorname{spin}(7)$, and $L \cap [L, L] = 0$; and since

$$\begin{aligned} \dim_{\mathbf{R}} L &= 6, \\ \dim_{\mathbf{R}} [L, L] &= 15, \\ \dim_{\mathbf{R}} \operatorname{spin}(7) &= 21, \end{aligned} \tag{1.24}$$

we conclude that

$$\operatorname{spin}(7) = L \oplus [L, L]. \tag{1.25}$$

Finally, note that $[L, L] = su(\mathbf{O}_\varepsilon)$, the Lie algebra of $SU(\mathbf{O}_\varepsilon)$. If we note that

$$gl(\mathbf{O}_\varepsilon) \cap \operatorname{spin}(7) = su(\mathbf{O}_\varepsilon) \tag{1.26}$$

and that Spin(7) is connected, we deduce that

$$GL(\mathbf{O}_\varepsilon) \cap \operatorname{Spin}(7) = SU(\mathbf{O}_\varepsilon). \tag{1.27}$$

We record our main result so far:

Proposition 1.1. *Extend the elements of* Spin(7) $\subseteq$ End(**O**) *complex linearly so that* Spin(7) $\subseteq$ End($\mathbf{C} \otimes_{\mathbf{R}} \mathbf{O}$). *If we use the standard basis* $(N, E, \bar{N}, \bar{E})$ *of* $\mathbf{C} \otimes_{\mathbf{R}} \mathbf{O}$ *to represent* $\operatorname{End}_{\mathbf{C}}(\mathbf{C} \otimes_{\mathbf{R}} \mathbf{O})$ *as the* 8×8 *complex valued matrices, then*

$$\operatorname{spin}(7) = \left\{ \begin{pmatrix} ic & -{}^t\bar{b} & 0 & -{}^ta \\ b & d & a & [\bar{a}] \\ 0 & -{}^t\bar{a} & -ic & -{}^tb \\ \bar{a} & [a] & \bar{b} & \bar{d} \end{pmatrix} \middle| \begin{array}{l} a, b \in M_{3\times 1}(\mathbf{C}), \\ c \in \mathbf{R}, d \in M_{3\times 3}(\mathbf{C}), \\ d + {}^t\bar{d} = 0, \\ \operatorname{tr} d + ic = 0. \end{array} \right\}.$$

As we will see below, Spin(7) actually satisfies $GL(\mathbf{O}_u) \cap \operatorname{Spin}(7) = SU(\mathbf{O}_u)$ for all $u \in S^6$.

For $x, y \in \mathbf{O}$, we define $x \times y$ by the formula

$$x \times y = \tfrac{1}{2}(\bar{y}x - \bar{x}y). \tag{1.28}$$

$x \times y$ is called the cross product of x and y. Clearly $x \times y \in \operatorname{Im} \mathbf{O}$. We have the useful identities

$$\langle x, y \rangle = 0 \Rightarrow x \times y = \bar{y}x = -\bar{x}y, \tag{1.29}$$

$$E_i \times \bar{N} = N \times \bar{E}_i = 0. \tag{1.30}$$

For each $u \in S^6$, we let $r_u \colon \operatorname{Im} \mathbf{O} \to \operatorname{Im} \mathbf{O}$ be defined by $r_u(x) = \bar{u}(xu) = (\bar{u}x)u$ (this last association formula follows easily from (1.8) and (1.8′)). Using the Moufang identities (see Appendix B of [12]), one can verify that there exists a homomorphism $\chi\colon \mathrm{Spin}(7) \to SO(7) \subseteq GL_{\mathbf{R}}(\mathrm{Im}\,\mathbf{O})$ which satisfies $\chi(J_u) = r_u$ (existence is the only doubtful point; uniqueness is clear). Furthermore, we have the following equivariance: For $g \in \mathrm{Spin}(7)$ and $x, y \in \mathbf{O}$

$$g(x) \times g(y) = \chi(g)(x \times y). \tag{1.31}$$

A basis $(n, f, \bar{n}, \bar{f})$ of $\mathbf{C} \otimes_{\mathbf{R}} \mathbf{O}$ is said to be *admissible* if there exists $g \in \mathrm{Spin}(7) \subseteq M_{8\times 8}(\mathbf{C})$ so that

$$(n, f, \bar{n}, \bar{f}) = (N, E, \bar{N}, \bar{E})g. \tag{1.32}$$

The space of admissible bases forms a manifold diffeomorphic to Spin(7). In fact, we may use (1.32) as a definition of the quantities n, f_i, etc. as $\mathbf{C} \otimes_{\mathbf{R}} \mathbf{O}$-valued functions on Spin(7). Using (1.30) and (1.31), we have the following formulae for any admissible frame:

$$f \times \bar{n} = \bar{n} \times \bar{f} = 0. \tag{1.33}$$

Now, differentiating (1.32), we get

$$\begin{aligned} d(n, f, \bar{n}, \bar{f}) &= (N, E, \bar{N}, \bar{E})dg \\ &= (n, f, \bar{n}, \bar{f})g^{-1}dg \\ &= (n, f, \bar{n}, \bar{f})\phi, \end{aligned}$$

where $\phi = g^{-1}dg$ is the canonical spin(7)-valued left-invariant 1-form on Spin(7). Consulting Proposition 1.1, we get

Proposition 1.2 (*The First Structure Equations*). *There exist left-invariant* 1-*forms on* Spin(7): ρ *with values in* $\mathbf{R}$; θ, $\mathfrak{h}$ *with values in* $M_{3\times 1}(\mathbf{C})$; *and* κ *with values in* 3 × 3 *skew-Hermitian matrices satisfying*

$$\operatorname{tr}\kappa + i\rho = 0, \tag{1.34}$$

$$\begin{aligned} d(n, f, \bar{n}, \bar{f}) &= (n, f, \bar{n}, \bar{f}) \begin{pmatrix} i\rho & -{}^t\bar{\mathfrak{h}} & 0 & -{}^t\theta \\ \mathfrak{h} & \kappa & \theta & [\bar{\theta}\,] \\ 0 & -{}^t\bar{\theta} & -i\rho & -{}^t\mathfrak{h} \\ \bar{\theta} & [\theta] & \bar{\mathfrak{h}} & \bar{\kappa} \end{pmatrix} \\ &= (n, f, \bar{n}, \bar{f})\phi, \end{aligned} \tag{1.35}$$

where ϕ *satisfies*

$$d\phi = -\phi \wedge \phi. \tag{1.36}$$

Remark. Note that in terms of $\mathbf{R}$-valued 1-forms κ has 9 components which are independent and whose linear combinations include ρ; $\mathfrak{h}$ has 6 components; and θ has 6 components making a total of 21 independent 1-forms (as expected). Moreover, in working with the structure equations (1.36) the following bracket identities will be extremely useful. If α and β are 1-forms with values in $M_{3\times 1}(\mathbf{C})$ and γ is a 1-form with values in $M_{3\times 3}(\mathbf{C})$, we have

$$[a] \wedge \beta = [\beta] \wedge \alpha, \tag{1.36a}$$

$$[\gamma \wedge \alpha] = (\operatorname{tr} \gamma) \wedge [a] - {}^t\gamma \wedge [a] + [a] \wedge \gamma, \tag{1.36b}$$

$$[a] \wedge [\beta] = {}^t\beta \wedge \alpha I_3 - \beta \wedge {}^t\alpha. \tag{1.36c}$$

For our work in later sections, we will need the identities

$${}^t\alpha \wedge [\alpha] \wedge \alpha = -6\alpha^1 \wedge \alpha^2 \wedge \alpha^3, \tag{1.36d}$$

$$[M\alpha] \wedge \alpha = \tfrac{1}{2}(\operatorname{tr} M - {}^tM)[\alpha] \wedge \alpha, \tag{1.36e}$$

where M is an 3×3 matrix of 0-forms. From these last two follows the useful identity

$${}^t\alpha \wedge [M\alpha] \wedge \alpha = -2 \operatorname{tr} M\alpha^1 \wedge \alpha^2 \wedge \alpha^3. \tag{1.36f}$$

To complete this section, we develop the moving frame equations for $\mathbf{O}$ with its standard Spin(7)-structure. We let $\mathcal{F} = \mathbf{O} \times \operatorname{Spin}(7)$ and let $x\colon \mathcal{F} \to \mathbf{O}$ denote projection onto the first factor. All functions and forms on Spin(7) will be regarded as functions and forms on $\mathcal{F}$ via pullback by projection on the second factor. For our purposes, it will be more useful to think of $\mathcal{F}$ as the space of pairs $(y; (n, f, \bar{n}, \bar{f}))$ consisting of a base point $y \in \mathbf{O}$ and an admissible basis $(n, f, \bar{n}, \bar{f})$ at that point. Since we have essentially identified Spin(7) with the space of admissible bases, this should cause no problem.

We let $(N^*, E^*, \bar{N}^*, \bar{E}^4)$ denote the dual basis of $(N, E, \bar{N}, \bar{E})$ in $(\mathbf{C} \otimes_{\mathbf{R}} \mathbf{O})^*$. Thus we have the identity

$$NN^*(y) + EE^*(y) + \bar{N}\bar{N}^*(y) + \bar{E}\bar{E}^*(y) = y \tag{1.37}$$

for all $y \in \mathbf{O}$ (note that E^* is a *column* of 1-forms of height 3). It follows that

$$dx = (N, E, \bar{N}, \bar{E}) \begin{pmatrix} x^*(N^*) \\ x^*(E^*) \\ x^*(\bar{N}^*) \\ x^*(\bar{E}^*) \end{pmatrix}. \tag{1.38}$$

If we set

$$\begin{pmatrix} \nu \\ \omega \\ \bar{\nu} \\ \bar{\omega} \end{pmatrix} = g^{-1} \begin{pmatrix} x^*(N^*) \\ x^*(E^*) \\ x^*(\bar{N}^*) \\ x^*(\bar{E}^*) \end{pmatrix} = \psi, \tag{1.39}$$

we get

Proposition 1.3 (*The Second Structure Equations*).

$$dx = (n, f, \bar{n}, \bar{f}) \begin{pmatrix} \nu \\ \omega \\ \bar{\nu} \\ \bar{\omega} \end{pmatrix} = (n, f, \bar{n}, \bar{f})\psi, \tag{1.40}$$

$$d\psi = -\phi \wedge \psi. \tag{1.41}$$

The geometric interpretation of these equations is the standard one in the theory of moving frames (see [3]). We will make extensive use of these equations to study submanifolds in **O**.

2. Spin(7) geometry in O and Im O

In this section, we investigate some of the special properties of **O** with its Spin(7)-structure. We begin with the geometry of the oriented 2-planes in **O**.

Let $\tilde{G}(2, \mathbf{O})$ denote the Grassmannian of oriented 2-planes in **O**. It is known that $\tilde{G}(2, \mathbf{O})$ is a manifold of dimension 12 (over the reals) and is connected and simply connected (see [14]). Spin(7) acts on **O** and therefore has a natural induced action on $\tilde{G}(2, \mathbf{O})$. We may even define a map ξ: Spin(7) → $\tilde{G}(2, \mathbf{O})$ as follows:

First, we imbed $\tilde{G}(2, \mathbf{O}) \hookrightarrow \Lambda^2_{\mathbf{R}}\mathbf{O}$ via the Plücker imbedding: If $\beta \in \tilde{G}(2, \mathbf{O})$ is an oriented 2-plane and $x, y \in \beta$ form an oriented orthonormal pair, then we identify β with $x \wedge y \in \Lambda^2_{\mathbf{R}}\mathbf{O}$. Second, if $g \in \mathrm{Spin}(7)$ is given, we let $(n, f, \bar{n}, \bar{f}) = (N, E, \bar{N}, \bar{E})g$ be the associated admissible basis. Because $g \in SO(8)$, $n = \frac{1}{2}(a - ib)$ where $(a, b) \in \mathbf{O} \times \mathbf{O}$ is an orthonormal pair. We then define

$$\xi(g) = a \wedge b = -2in \wedge \bar{n}. \tag{2.1}$$

Proposition 2.1. *The mapping* ξ: Spin(7) → $\tilde{G}(2, \mathbf{O})$ *is surjective and makes* Spin(7) *into a principal right* $U(3)$*-bundle over* $\tilde{G}(2, \mathbf{O})$. *Thus*

$$\mathrm{Spin}(7)/U(3) \simeq \tilde{G}(2, \mathbf{O}).$$

Proof. We compute the differential of ξ as

$$d\xi = -2i(f\mathfrak{h} + \bar{f}\bar{\theta}) \wedge n - 2in \wedge (f\theta + \bar{f}\bar{\mathfrak{h}}). \tag{2.2}$$

It follows that ξ has rank 12 at every $g \in \mathrm{Spin}(7)$. Because Spin(7) and $\tilde{G}(2, \mathbf{O})$ are compact and $\dim_{\mathbf{R}} \tilde{G}(2, \mathbf{O}) = 12$ surjectivity follows. For $g, h \in \mathrm{Spin}(7)$, we obviously have the formula

$$\xi(gh) = \Lambda^2 h(\xi(g)), \tag{2.3}$$

where $\Lambda^2 h$: $\Lambda^2_{\mathbf{R}}\mathbf{O} \to \Lambda^2_{\mathbf{R}}\mathbf{O}$ is the second exterior power of $h \colon \mathbf{O} \to \mathbf{O}$. It follows that the fibers of ξ are the left cosets in Spin(7) of the stabilizer of any $\beta \in \tilde{G}(2, \mathbf{O})$, say $H \subseteq$ Spin(7). The homotopy sequence of the fibration $H \to$ Spin(7) $\to \tilde{G}(2, \mathbf{O})$ plus the fact that $\tilde{G}(2, \mathbf{O})$ and Spin(7) are connected and simply connected shows that H is connected and its Lie algebra is defined by the equations $\boldsymbol{\theta} = \mathfrak{h} = 0$ (by (2.2)). This implies that $H = U(3)$ by inspection. q.e.d.

It is well known (see [7]) that the Grassmannian of oriented 2-planes in any Euclidean vector space has a natural complex structure. For our purposes, it is more convenient to take the conjugate complex structure to the one used by Chern. (By our conventions, the Gauss map of an oriented minimal surface in $\mathbf{E}^n$ is *holomorphic.*) We describe the complex structure on $\tilde{G}(2, \mathbf{O})$ by saying that a complex valued 1-form α on $\tilde{G}(2, \mathbf{O})$ is of type (1,0) if and only if $\xi^*(\alpha)$ is a linear combination of the forms $\{\mathfrak{h}^1, \mathfrak{h}^2, \mathfrak{h}^3, \bar{\theta}^1, \bar{\theta}^2, \bar{\theta}^3\}$. Examination of the structure equations

$$(1.36') \qquad \begin{aligned} d\mathfrak{h} &= -\mathfrak{h} \wedge i\rho - \kappa \wedge \mathfrak{h} - [\bar{\boldsymbol{\theta}}\ ] \wedge \bar{\boldsymbol{\theta}}, \\ d\bar{\boldsymbol{\theta}} &= -\bar{\boldsymbol{\theta}} \wedge i\rho - [\boldsymbol{\theta}] \wedge \mathfrak{h} - \bar{\kappa} \wedge \bar{\boldsymbol{\theta}} \end{aligned}$$

shows that this is a well-defined concept and that the almost complex structure defined above is actually integrable.

A special feature of $\mathbf{O}$ is the cross product (1.28). Because the cross product is alternating ($x \times y = -y \times x$) it follows that it induces a well-defined map $\Lambda^2\mathbf{O} \to \operatorname{Im}\mathbf{O}$. If x, $y \in \mathbf{O}$ form an orthonormal pair, (1.29) implies that

$$\langle x \times y, x \times y \rangle = \langle \bar{y}x, \bar{y}x \rangle = \langle \bar{y}, \bar{y} \rangle \langle x, x \rangle = 1$$

so $x \times y \in S^6$. Moreover the identities

$$(2.4) \qquad \begin{aligned} x(y \times x) &= y = J_{y \times x}(x), \\ y(y \times x) &= -x = J_{y \times x}(y) \end{aligned}$$

follow from (1.29) when x and y are orthonormal, *showing that the 2-plane $\alpha = x \wedge y$ is a complex line in $\mathbf{O}_{y \times x}$.* Thus, we have a map $\gamma \colon \tilde{G}(2, \mathbf{O}) \to S^6$ defined by

$$(2.5) \qquad \gamma(a \wedge b) = b \times a = -a \times b$$

when $a, b \in \mathbf{O}$ are orthonormal. This map has the property that, for $\alpha \in \tilde{G}(2, \mathbf{O})$, $\gamma(\alpha)$ is the unique imaginary unit so that α is a complex line in $\mathbf{O}_{\gamma(\alpha)}$. In particular, γ is *surjective* and $\gamma^{-1}(u)$ is canonically identified with $\mathbf{CP}^3_u$, the projectivization of $\mathbf{O}_u \simeq \mathbf{C}^4$.

We have the formula

$$(2.6) \qquad \gamma \circ \xi(g) = 2i(n \times \bar{n})$$

where $(n, f, \bar{n}, \bar{f}) = (N, E, \bar{N}, \bar{E})g$. Using (2.4) we get the identity

$$n(2in \times \bar{n}) = in. \tag{2.7}$$

Proposition 2.2. *For any admissible basis* $(n, f, \bar{n}, \bar{f})$, (n, f_1, f_2, f_3) *is a unitary basis*[2] *of* $\mathbf{O}_{2in\times\bar{n}}$. *The mapping* $\gamma \circ \xi$: Spin(7) $\to S^6$ *is surjective and gives* Spin(7) *the structure of a principal right* $SU(4)$*-bundle over* S^6. *In fact* $(\gamma \circ \xi)^{-1}(u)$ *corresponds to the space of special unitary bases of* $\mathbf{O}_u$ *with its canonical Hermitian structure.*

Proof. If we differentiate (2.7) and compare coefficients of θ, we get, using (1.33) that

$$f(2in \times \bar{n}) = if. \tag{2.8}$$

It follows from (2.7) and (2.8) that n, f_1, f_2, and f_3 are (1,0) vectors in $\mathbf{C} \otimes_{\mathbf{R}} \mathbf{O}_u$ where $u = 2in \times \bar{n}$. Since Spin(7) $\subseteq SO(8)$ and since N, E_1, E_2, E_3 are orthogonal and Hermitian orthogonal, it follows that n, f_1, f_2, f_3 must also form a *unitary basis* of $\mathbf{O}_u$. The surjectivity of $\gamma \circ \xi$ is clear since each map separately is known to be surjective. Computing the differential of $\gamma \circ \xi$, we get

$$d(\gamma \circ \xi) = 2i(dn \times \bar{n} + n \times d\bar{n}) = 2i\big((n \times f)\theta + (\bar{f} \times \bar{n})\bar{\theta}\ \big), \tag{2.9}$$

where we have used (1.35) and (1.33). It follows that the fibers of $\gamma \circ \xi$ are (unions of) the leaves of the foliation determined by the real and imaginary components of θ, and are therefore codimension 6. In fact, the remaining Lie algebra when we set $\theta = 0$ is clearly $su(4) \subseteq$ spin(7), so the leaves are the left cosets of $SU(4)$ in Spin(7). Again, because Spin(7) and S^6 are connected and simply connected, it follows that the fibers of $\gamma \circ \xi$ must be connected. We conclude that

$$\mathrm{Spin}(7)/SU(4) \simeq S^6. \tag{2.10}$$

The equivariance of $\gamma \circ \xi$ is easily seen to be

$$\gamma \circ \xi(gh) = \chi(h)(\gamma \circ \xi(g)). \tag{2.11}$$

The above remarks all combine to show that if $g \in (\gamma \circ \xi)^{-1}(u)$,n then $(n, f, \bar{n}, \bar{f}) = (N, E, \bar{N}, \bar{E})g$ is a special unitary basis of $\mathbf{O}_u$. q.e.d.

These remarks have an interesting consequence for $\tilde{G}(6, \mathbf{O})$, the Grassmannian of oriented 6-planes in $\mathbf{O}$. Using the metric and the natural orientation of $\mathbf{O}$, we may associate to each oriented six-plane $\zeta \in \tilde{G}(6, \mathbf{O})$ its oriented orthogonal 2-plane $\zeta^{\perp} \in \tilde{G}(2, \mathbf{O})$. Since $\zeta^{\perp}$ is a complex line in $\mathbf{O}_{\gamma(\zeta^{\perp})}$ and

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[2] Recall that if V^m is a real vector space with inner product and orthogonal complex structure J, then $\Lambda^{1,0}(V) = \{v \in \mathbf{C} \otimes_{\mathbf{R}} V \mid Jv = iv\}$, and a unitary basis of V is really a complex basis $\{e_1, \cdots, e_m\}$ of $\Lambda^{1,0}(V)$ which satisfies $\langle e_i, \bar{e}_j \rangle = \frac{1}{2}\delta_{ij}$.

because J_u is orthogonal for all $u \in S^6$, it follows that ζ is a complex three-plane in $\mathbf{O}_{\gamma(\zeta^\perp)}$. We refer to the complex structure induced on ζ in this way as the canonical complex structure of ζ. Since ζ also inherits a metric from $\mathbf{O}$, we see that ζ has a natural Hermitian structure. Referring to the structure equations (1.35) we see that if $(n, f, \bar{n}, \bar{f})$ and $(n', f', \bar{n}', \bar{f}')$ are two admissible bases with $-2in \wedge \bar{n} = -2in' \wedge \bar{n}' = \zeta^\perp \in \tilde{G}(2, \mathbf{O})$, then there exists a unitary matrix A which is 3×3 so that

$$n' = (\det A)^{-1} n, \quad f' = fA. \tag{2.12}$$

It follows that we have a canonical identification

$$\zeta^\perp \simeq \Lambda^3_{\mathbf{C}} \zeta^*, \tag{2.13}$$

a fact we will use later.

We will also have occasion to study the geometry of $\operatorname{Im}\mathbf{O}$ under a slightly smaller group than Spin(7). We get $G_2 \subseteq \text{Spin}(7)$ be the subgroup which leaves $1 \in \mathbf{O}$ fixed. Thus G_2 is a compact subgroup of Spin(7). If we define p: Spin(7) $\to S^7 \subseteq \mathbf{O}$ by setting $p(g) = n + \bar{n}$ where $(n, f, \bar{n}, \bar{f}) = (N, E, \bar{N}, \bar{E})g$, then clearly $p^{-1}(1) = G_2$. Computing the differential of p, we get

$$dp = i(n - \bar{n})\rho + f(\mathfrak{h} + \theta) + \bar{f}(\bar{\mathfrak{h}} + \bar{\theta}\ ). \tag{2.14}$$

It follows that p has rank 7 and gives Spin(7) the structure of a G_2-bundle over S^7. The connectedness and simple-connectedness of Spin(7) and S^7 shows that G_2 must be connected and that the Lie algebra of G_2 is obtained from that of Spin(7) by setting $\rho = \mathfrak{h} + \theta = 0$.

For $g \in G_2$, we say an admissible basis of $\mathbf{C} \otimes_{\mathbf{R}} \mathbf{O}$, $(n, f, \bar{n}, \bar{f}) = (N, E, \bar{N}, \bar{E})g$ is G_2-admissible. Since $n + \bar{n} \equiv 1$ for such bases, we remove this information and set $u = i(n - \bar{n})$. We then have the following proposition whose proof is an easy computation and is omitted.

Proposition 2.3 (*The Structure Equations of* G_2). *The map* $u \colon G_2 \to S^6$ *makes* G_2 *into a principal right* $SU(3)$*-bundle over* S^6. *In fact, we have the structure equations*

$$du = f(-2i\theta) + \bar{f}(2i\bar{\theta}\ ), \tag{2.15}$$

$$df = u(-i\,{}^t\bar{\theta}\ ) + f\kappa + \bar{f}\,[\theta], \tag{2.16}$$

$$d\theta = -\kappa \wedge \theta + [\bar{\theta}\ ] \wedge \bar{\theta}, \tag{2.17}$$

$$d\kappa = -\kappa \wedge \kappa + 3\theta \wedge {}^t\bar{\theta} - {}^t\theta \wedge \bar{\theta} I_3. \tag{2.18}$$

It follows that S^6 *possesses a unique nonintegrable almost complex structure so that a complex-valued* 1*-form* $\alpha \in \Omega^1_{\mathbf{C}}(S^6)$ *is of type* (1,0) *if and only if* $u^*(\alpha)$ *is a linear combination of* $\{\theta^1, \theta^2, \theta^3\}$.

Remark. The existence of an almost complex structure on S^6 will also follow from the next section.

Finally, we will need to study the structure of the Grassmannian of oriented 2-planes in $\mathrm{Im}\,\mathbf{O}$, $\tilde{G}(2, \mathrm{Im}\,\mathbf{O})$. We define the map $\eta\colon G_2 \to \tilde{G}(2, \mathrm{Im}\,\mathbf{O})$ by

$$\eta(G) = -2if_1 \wedge \tilde{f}_1. \tag{2.19}$$

One easily verifies that η gives G_2 the structure of a principal right $U(2)$-bundle over $\tilde{G}(2, \mathrm{Im}\,\mathbf{O})$. Since $\eta(g)$ is a complex line in $\mathbf{O}_{2in \times \bar{n}}$, and since $n + \bar{n} \equiv 1$, we easily compute that $2in \times \bar{n} = i(n - \bar{n}) = u$ so $\eta(g)$ is a complex line in $\mathbf{O}_u$. The structure equations (2.15) and (2.16) then show that $\eta(g)$ is a complex line in T_uS^6 with the canonical almost complex structure of Proposition 2.3. It follows that there exists a unique map $\eta\colon \tilde{G}(2, \mathrm{Im}\,\mathbf{O}) \to S^6$ satisfying $u = \pi \circ \eta$. Unfortunately, π_* is not complex linear on the tangent spaces, so it is *not* a map of almost complex manifolds. The following proposition displays the structure of this map vis à vis the almost complex structures of $\tilde{G}(2, \mathrm{Im}\,\mathbf{O})$ and S^6. It will be used extensively in §4.

Proposition 2.4. *The natural complex structure on* $\tilde{G}(2, \mathbf{O})$ *is described as follows: If* α *is a compact* 1*-form on* $\tilde{G}(2, \mathrm{Im}\,\mathbf{O})$, *then it is of type* (1,0) *if and only if* $\eta^*(\alpha)$ *is a linear combination of* $\{\kappa_1^2, \kappa_1^3, \bar{\theta}^1, \theta^2, \theta^3\}$. *Moreover, the holomorphic tangent bundle of* $\tilde{G}(2, \mathrm{Im}\,\mathbf{O})$ *has a natural* G_2*-invariant splitting into complex subbundles* L_0, L_+, L_-, *where* $L_0 = \ker \pi_*$, L_+ *is the space of vectors on which* π_* *is complex linear, and* L_- *is the space of vectors on which* π_* *is complex anti-linear. If we get* $\mathcal{L}_0 \oplus \mathcal{L}_+ \oplus \mathcal{L}_- = \Omega^{1,0}(\tilde{G}(2, \mathrm{Im}\,\mathbf{O}))$ *be the splitting dual to* $L_0 \oplus L_+ \oplus L_- = T^{1,0}\tilde{G}(2, \mathrm{Im}\,\mathbf{O})$ *then we have the characterizations*

$$\mathcal{L}_0 = \left\{\alpha \in \Omega^1_{\mathbf{C}} \mid \eta^*(\alpha) \equiv 0 \bmod\left(\kappa_1^2, \kappa_1^3\right)\right\}, \tag{2.20}$$

$$\mathcal{L}_+ = \left\{\alpha \in \Omega^1_{\mathbf{C}} \mid \eta^*(\alpha) \equiv 0 \bmod(\theta^2, \theta^3)\right\}, \tag{2.21}$$

$$\mathcal{L}_- = \left\{\alpha \in \Omega^1_{\mathbf{C}} \mid \eta^*(\alpha) \equiv 0 \bmod \bar{\theta}^1\right\}, \tag{2.22}$$

where we have written $\Omega^1_{\mathbf{C}}$ *for* $\Omega^1_{\mathbf{C}}(\tilde{G}(2, \mathrm{Im}\,\mathbf{O}))$. *Finally, the natural map* $\mathbf{CP}TS^6 \to \tilde{G}(2, \mathrm{Im}\,\mathbf{O})$ (*which arises by simply regarding a complex line* $\lambda \subseteq T_uS^6$ *as an oriented* 2*-plane in* $\mathrm{Im}\,\mathbf{O}$) *is a diffeomorphism and we have a commutative diagram*

$$\begin{tikzcd}
\mathbf{CP}TS^6 \arrow[rr, "\sim"] \arrow[dr, "\beta"'] & & \tilde{G}(2, \mathrm{Im}\,\mathbf{O}) \arrow[dl, "\pi"] \\
& S^6 &
\end{tikzcd}$$

where β *is the base point projection.*

Proof. These are all elementary calculations using the structure equations and will be left to the reader.

3. Oriented 6-manifolds in O

Let M^6 be an abstract oriented 6-manifold with a smooth differentiable structure. Let X: $M^6 \to \mathbf{O}$ be a smooth immersion of M^6 into $\mathbf{O}$. We say that an admissible frame $(y;\, n,\, f,\, \bar{n},\, \bar{f}) \in \mathfrak{F}$ is *adapted* at $p \in M$ if $X(p) = y$ and if (f_1, f_2, f_3) is a (1,0) basis of $X_*(T_pM^6)$ with its induced orientation from M^6 and complex structure induced from right multiplication by $2in \times \bar{n}$. We let $\mathfrak{F}_X(M)$ denote the space of pairs $(p, (y;\, n,\, f,\, \bar{n},\, \bar{f}))$, $p \in M^6$, $(y;\, n,\, f,\, \bar{n},\, \bar{f}) \in \mathfrak{F}$ where $(y;\, n,\, f,\, \bar{n},\, \bar{f})$ is adapted at p. We call $\mathfrak{F}_X(M)$ *the adapted frame bundle of the immersion* X: $M^6 \to \mathbf{O}$. We have a commutative diagram:

$$\begin{tikzcd}
\mathfrak{F}(M) \arrow[r, "\tilde{X}"] \arrow[d, "p"] & \mathfrak{F} \arrow[d, "x"] \\
M \arrow[r, "X"] & \mathbf{O}
\end{tikzcd}$$

We see that p: $\mathfrak{F}_X(M) \to M$ is a right $U(3)$-bundle over M which may be regarded as a subbundle of the $GL(6, \mathbf{R})$ bundle of the tangential frames of M. We simply refer to this G-structure as the $U(3)$-structure on M induced by the immersion X: $M \to \mathbf{O}$. The reader should be aware that other authors have called such structures "almost hermitian".

The forms on $\mathfrak{F}$ pullback under $\tilde{X}^*$ to give forms on $\mathfrak{F}_X(M)$ which we continue to denote by the same letters. The following basic theorem follows immediately from the theory of moving frames and the structure equations of $\mathbf{O}$ (see §1, (1.35), (1.36), (1.40), (1.41)).

Theorem 3.1. *Let X: $M^6 \to \mathbf{O}$ be an oriented immersion and let p: $\mathfrak{F}_X(M) \to M$ be the adapted frame bundle. Then M inherits a $U(3)$-structure where $\mathfrak{F}_X(M)$ is the bundle of unitary frames and whose features are described as follows*:

(i) $\nu = \bar{\nu} = 0$ *on* $\mathfrak{F}_X(M)$.

(ii) *A form* $\alpha \in \Omega^1_{\mathbf{C}}(M)$ *is of type* (1,0) *if and only if* $p^*(\alpha) \equiv 0 \mod(\omega^1, \omega^2, \omega^3)$.

(iii) *A canonical* 2-*form*, Ω, *of type* (1,1) *is associated to the* $U(3)$-*structure and is characterized by the condition* $p^*(\Omega) = (i/2)\,{}^t\omega \wedge \bar{\omega}$.

(iv) *The metric g on M induced by X from* $\mathbf{O}$ *satisfies* $p^*(g) = {}^t\omega \circ \bar{\omega}$.

(v) *The structure equations hold*:

$$dx = f\omega + \bar{f}\bar{\omega}, \tag{3.1}$$

$$dn = ni\rho + f\,\mathfrak{h} + \bar{f}\bar{\theta}, \tag{3.2}$$

$$df = -n\,{}^t\mathfrak{h} + f\kappa - \bar{n}\,{}^t\bar{\theta} + \bar{f}\,[\theta], \tag{3.3}$$

(*and the equations gotten from these by conjugation*).

We omit the proof.

Of course, a $U(3)$-structure has many invariants and those $U(3)$-structures which satisfy extra conditions are of particular interest. Among these, the most important for us will be the following: A $U(3)$-structure on M will be said to be

(i) *complex* if the underlying almost complex structure is integrable to a complex structure (by the Newlander-Nirenberg theorem, this is equivalent to the condition $d\alpha \equiv 0 \bmod \Omega^{1,0}(M)$ for all $\alpha \in \Omega^{1,0}(M)$; see [7]);

(ii) *symplectic* if the canonical two-form Ω is closed;

(iii) *co-symplectic* if Ω is co-closed, i.e., $\delta\Omega = 0$ (this is equivalent to either of the conditions $d\Omega^2 = 0$ or $d * \Omega = 0$);

(iv) *Kähler* if it is both complex and symplectic;

(v) *co-Kähler* if it is both complex and co-symplectic.

Note that symplectic implies co-symplectic, but not conversely (see below). Complex $U(3)$-structures are often called "Hermitian".[3]

Our analysis of $U(3)$-structures induced by oriented immersions X: $M^6 \to \mathbf{O}$ begins with the second fundamental form. If we differentiate the equation $\nu = 0$ on $\mathfrak{F}_X(M)$, the structure equations (1.41) give

$$ {}^t\bar{\mathfrak{h}} \wedge \omega + {}^t\theta \wedge \bar{\omega} = 0. \tag{3.4}$$

Applying Cartan's Lemma, we conclude that there exist 3×3 matrices of functions, A, B, C on $\mathfrak{F}_X(M)$ (with complex values) satisfying

$$ A = {}^tA, \quad C = {}^tC, \tag{3.5}$$

$$ \begin{pmatrix} \mathfrak{h} \\ \theta \end{pmatrix} = \begin{pmatrix} \bar{B} & \bar{A} \\ {}^tB & \bar{C} \end{pmatrix} \begin{pmatrix} \omega \\ \bar{\omega} \end{pmatrix}. \tag{3.6}$$

Using these formulae, we easily compute the second fundamental form of X: $M^6 \to \mathbf{O}$ as an Euclidean immersion as

$$ \mathrm{II} = -2\,\mathrm{Re}\left\{ \left({}^t\bar{\mathfrak{h}} \circ \omega + {}^t\theta \circ \bar{\omega} \right) n \right\}. \tag{3.7}$$

Classically, one views II as a linear map II: $S^2(TM) \to NM$ where TM is the tangent bundle of the immersion X. Using the almost complex structure on M and the orientation of the 2-plane bundle N_xM, we have canonical splittings

$$ \begin{aligned} \mathbf{C} \otimes_{\mathbf{R}} S^2(TM) &= S^{2,0}_{\mathbf{C}}(M) \oplus S^{1,1}_{\mathbf{C}}(M) \oplus S^{0,2}_{\mathbf{C}}(M), \\ \mathbf{C} \otimes_{\mathbf{R}} NM &= N^{1,0}M \oplus N^{0,1}M, \end{aligned} $$

where the bundles on the right are complex vector bundles over M. For example, $S^{2,0}_{\mathbf{C},q}(M)$ for $q \in M$ is spanned by products of the form $e_1 \circ e_2$ where

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[3] The reader should be aware that other terminology has been used for these concepts. Compare [13], [2] and [10].

e_1 and e_2 are (1,0) vectors in $T_{\mathbf{C},q}M$. If we extend II complex linearly to a map $\mathbf{C} \otimes_{\mathbf{R}} S^2(TM) \to \mathbf{C} \otimes_{\mathbf{R}} NM$, and split it into components via the above splittings, we see that II has three independent pieces, the rest being determined by symmetry and reality of II. These components are $\mathrm{II}^{2,0}$: $\mathrm{S}^{2,0}(\mathrm{M}) \to (M) \to N^{1,0}M$ given on $\mathfrak{F}_X(M)$ by

$$II^{2,0} = \left(-{}^t\omega \circ A\omega\right)n, \tag{3.7a}$$

$\mathrm{II}^{1,1}$: $S^{1,1}_{\mathbf{C}}(M) \to N^{1,0}M$ given by

$$\mathrm{II}^{1,1} = \left(-{}^t\bar{\omega} \circ {}^tB\omega - {}^t\omega \circ B\bar{\omega}\right)n, \tag{3.7b}$$

and $\mathrm{II}^{0,2}$: $S^{0,2}_{\mathbf{C}}(M) \to N^{1,0}M$ given by

$$\mathrm{II}^{0,2} = -\left({}^t\bar{\omega} \circ \bar{C}\bar{\omega}\right)n. \tag{3.7c}$$

From this, one easily computes the trace of II with respect to the first fundamental form $\mathrm{I} = {}^t\omega \circ \bar{\omega}$ as

$$H = \tfrac{1}{6}\mathrm{tr}_{\mathrm{I}}\,\mathrm{II} = -\tfrac{1}{3}\left(\mathrm{tr}\,Bn + \mathrm{tn}\,\bar{B}\bar{n}\right). \tag{3.8}$$

H is often called the mean curvature vector of the immersion *X*. The above discussion gives us a geometric interpretation of the components of II with respect to the *U*(3)-structure. We will now relate these components to the special conditions discussed above for *U*(3)-structures.

Theorem 3.2. *Let* X: $M^6 \to \mathbf{O}$ *be an immersion of the oriented manifold* M^6. *The induced* $U(3)$*-structure is complex if and only if* $B = 0$.

Proof. By Theorem 3.1 and the Newlander-Nirenberg theorem, it suffices to show that the condition $B = 0$ is equivalent to the condition $d\omega^i \equiv 0 \bmod\{\omega^1, \omega^2, \omega^3\}$ for $i = 1, 2, 3$ (note that these are equations on $\mathfrak{F}_X(M)$).

We compute by (1.41) and (3.6) that

$$\begin{aligned} d\omega &= -\kappa \wedge \omega - [\bar{\theta}\ ] \wedge \bar{\omega} \\ &\equiv -[\bar{\theta}\ ] \wedge \bar{\omega} \quad \mathrm{mod}\{\omega^1, \omega^2, \omega^3\} \\ &\equiv -[{}^t\bar{B}\bar{\omega}] \wedge \bar{\omega} \quad \mathrm{mod}\{\omega^1, \omega^2, \omega^3\}. \end{aligned}$$

If $B \equiv 0$, then we obviously have $d\omega \equiv 0 \bmod\{\omega^1, \omega^2, \omega^3\}$ so one direction is done. Conversely, if $d\omega \equiv 0$, then we must have $[{}^t\bar{B}\bar{\omega}] \wedge \bar{\omega} = 0$. If we set ${}^t\bar{B}\bar{\omega} = \beta = (\beta^i)$ where the β^i are 1-forms, this equation becomes the equations

$$\beta^i \wedge \bar{\omega}^j = \beta^j \wedge \bar{\omega}^i$$

for all *i*, *j*. Since $\bar{\omega}^1 \wedge \bar{\omega}^2 \wedge \bar{\omega}^3 \neq 0$, this easily implies $\beta^i = 0$ and hence $B = 0$.

Theorem 3.3. *Let X: $M^6 \to \mathbf{O}$ be an immersion of an oriented manifold M^6. The induced $U(3)$-structure is symplectic if and only if $C = 0$ and* $\operatorname{tr} B = 0$.

Proof. Since p: $\mathcal{F}_X(M) \to M$ is a submersion, we have $d\Omega = 0$ if and only if $dp^*\Omega = 0$. We compute using (1.41) and (3.6)

$$\begin{aligned} dp^*\Omega &= (i/2)\big({}^t d\omega \wedge \overline{\omega} - {}^t\omega \wedge d\overline{\omega}\big) \\ &= (i/2)\big(-{}^t\overline{\omega} \wedge [\bar{\theta}\ ] \wedge \overline{\omega} + {}^t\omega \wedge [\theta] \wedge \omega\big) \\ &= (i/2)\big({}^t\omega \wedge [{}^tB\omega] \wedge \omega - {}^t\overline{\omega} \wedge [{}^t\overline{B}\overline{\omega}] \wedge \overline{\omega}\big) \\ &\quad + (i/2)\big({}^t\omega \wedge [\overline{C}\overline{\omega}] \wedge \omega - {}^t\overline{\omega}[C\omega] \wedge \overline{\omega}\big) \\ &= -\operatorname{Im}\big((\operatorname{tr} B)\omega^1 \wedge \omega^2 \wedge \omega^3\big) \\ &\quad + (i/2)\big({}^t\omega \wedge [\overline{C}\overline{\omega}] \wedge \omega - {}^t\overline{\omega} \wedge [C\omega] \wedge \overline{\omega}\big). \end{aligned}$$

Separating the forms out by type we see that $d\Omega = 0$ if and only if $\operatorname{tr} B = 0$ *and* ${}^t\omega \wedge [\overline{C}\overline{\omega}]\omega = 0$ (by (1.36a)) which clearly implies $C = 0$ since the ω^i and the $\overline{\omega}^i$ are independent.

Theorem 3.4. *Let X: $M^6 \to \mathbf{O}$ be an immersion of an oriented manifold M^6. The induced $U(3)$-structure is always co-symplectic.*

Proof. Using the formula for $d\Omega$ developed in the last proof, we compute

$$d\Omega^2 = -\tfrac{1}{2}{}^t\omega \wedge \overline{\omega} \wedge \big({}^t\omega \wedge [\overline{C}\overline{\omega}] \wedge \omega - {}^t\overline{\omega} \wedge [C\omega] \wedge \overline{\omega}\big).$$

Separating the equation by type we see that $d\Omega^2 = 0$ if and only if

$${}^t\omega \wedge \overline{\omega} \wedge \big({}^t\overline{\omega}[C\omega] \wedge \overline{\omega}\big) = 0$$

or

$${}^t\omega \wedge \overline{\omega} \wedge \big({}^t\overline{\omega}[\overline{\omega}]C\omega\big) = 0$$

which is equivalent to

$$\overline{\omega}^1 \wedge \overline{\omega}^2 \wedge \overline{\omega}^3 \wedge \big({}^t\omega \wedge C\omega\big) = 0.$$

Since C is symmetric, we have ${}^t\omega \wedge C\omega = 0$. Hence $d\Omega^2 = 0$ is an identity. By our previous remarks, we see that this is equivalent to the co-symplectic condition.

Theorem 3.5. *Let X: $M^6 \to \mathbf{O}$ be an immersion of a connected oriented manifold M^6. The induced $U(3)$-structure is Kählerian if and only if $X(M^6)$ is a complex hypersurface in $\mathbf{O}_u$ for some fixed $u \in S^6$.*

Proof. By Theorems 3.3 and 3.2 we see that M^6 is Kähler if and only if $B = C = 0$. By (3.6) we see that this is equivalent to $\theta = 0$ on $\mathcal{F}_X(M)$. For any $(q, (y; n, f, \bar{n}, \bar{f})) \in \mathcal{F}_X(M)$, we know that $X_*(T_qM)$ is complex with respect to the complex structure J_u where $u = 2in \times \bar{n}$, by Proposition 2.2. Equation 2.9 then shows that $u = 2in \times \bar{n}$ is locally constant on $\mathcal{F}_X(M)$ since $\theta = 0$.

Because M is connected, $\mathcal{F}_X(M)$ is connected as well. Thus $u = 2in \times \bar{n}$ is a constant so that $X_*(T_qM)$ is a complex 3-plane in $\mathbf{O}_u$ for all $q \in M$. Thus $X(M) \subseteq \mathbf{O}_u$ is a complex manifold.

Corollary 3.6. *If X: $M^6 \to \mathbf{O}$ is an oriented, connected immersion so that the image $X(M)$ lies in a hyperplane and the induced $U(3)$-structure is Kähler, then $X(M)$ lies in a 6-plane.*

Proof. Since $X(M) \subseteq \mathbf{O}_u$ is a complex hypersurface and since any 7-plane in $\mathbf{O}_u$ contains a unique $\mathbf{C}^3$, it follows that $X(M) \subseteq \mathbf{C}^3 \subseteq \mathbf{O}_u$.

Historical Remarks. Theorems 3.2 and 3.3 as well as Corollary 3.6 were derived by Calabi under the assumption that $X(M) \subseteq \operatorname{Im} \mathbf{O}$. Also, compare Fukami and Ishihara [9]. Theorems 3.2, 3.3 and 3.4 as well as Corollary 3.6 were derived by Gray in [10], though his terminology is much different. Gray also proves that if $M^6 \subseteq \operatorname{Im} \mathbf{O}$ and $d\Omega = 0$, then M^6 is flat (compare Theorem 3.13 below, which implies Gray's result). In addition, Gray considers other combinations of conditions on the A, B, and C. We will not discuss these.

Further Remarks. The above theorems are not complete in the sense that we do not yet know that there exist *any* immersions X: $M^6 \to \mathbf{O}$ whose induced $U(3)$-structure is complex but not Kähler or which is symplectic but not Kähler. In [2], Calabi shows how to construct immersions X: $M^6 \to \operatorname{Im} \mathbf{O}$ which are complex (but not Kähler) starting with an arbitrary minimal surface $S \subseteq \mathbf{R}^3 \subseteq \operatorname{Im} \mathbf{O}$ (where $\mathbf{R}^3 \subseteq \operatorname{Im} \mathbf{O}$ is an associative 3-plane) and letting $M^6 = S \times (\mathbf{R}^3)^{\perp}$ with X just the natural inclusion X: $S \times (\mathbf{R}^3) \subseteq \operatorname{Im} \mathbf{O}$. Since minimal surfaces in $\mathbf{R}^3$ depend on 2 arbitrary functions of 1 variable (in Cartan's sense, see [4]), this gives a class of complex (but not Kähler) immersions depending on 2 arbitrary functions of 1 variable.

Since the complex and symplectic conditions represent *overdetermined* systems of partial differential equations for the immersing function X: $M^6 \to \mathbf{O}$, and moreover, since these equations arise naturally in the moving frame context, we will apply the theory of differential systems to these existence problems. We start with a proposition about complex immersions.

Proposition 3.7. *Let X: $M^6 \to \mathbf{O}$ be an immersion of an oriented manifold into $\mathbf{O}$. If the induced $U(3)$-structure is complex, then the rank of C is at most 1. Moreover, if $U \subseteq M$ is the open set where $C \neq 0$ and $U \neq \varnothing$ then there exist functions a, c on $\mathcal{F}_X(U)$ with values in $M_{1\times 3}(\mathbf{C})$ which are well defined up to sign and which satisfy*

$$C = {}^t c \cdot c, \tag{3.9}$$

$$A = \tfrac{1}{2}({}^t a \cdot c + {}^t c \cdot a). \tag{3.10}$$

(*Note that the right hand sides are 3×3 symmetric complex matrices so this makes sense.*)

Remark. A point $q \in M$ where $C = 0$ will be called a Kähler-umbilic (or K-umbilic). Thus, Theorem 3.5 says that if $V \subseteq M$ is an open subset of a complex $X\colon M \hookrightarrow \mathbf{O}$ consisting only of K-umbilics, then $X(V)$ is actually Kähler and a complex hypersurface in $\mathbf{O}_u$ for some u.

Proof. Suppose $X\colon M^6 \to \mathbf{O}$ is complex. Then by Theorem 3.2, we have $B = 0$, so

$$\bar{\mathfrak{h}} = A\omega, \quad \bar{\theta} = C\omega. \tag{3.11}$$

Differentiating the first equation and using (1.36) we get

$$dA \wedge \omega + A \wedge d\omega = d\bar{\mathfrak{h}} = -[\theta] \wedge \theta + \bar{\mathfrak{h}} \wedge i\rho - \bar{\kappa} \wedge \bar{\mathfrak{h}}. \tag{3.12}$$

In (3.12), the only term of type (0,2) is $[\theta] \wedge \theta$. Since the forms $\{\omega^i, \bar{\omega}^i \mid i = 1, 2, 3\}$ are independent, we get

$$[\theta] \wedge \theta = 0. \tag{3.13}$$

This is equivalent to the equations $\theta^i \wedge \theta^j = 0$ for all i, j. Thus the θ^i are all multiplies of a single form. Since $\theta = \overline{C\omega}$ and the $\bar{\omega}^i$ are independent, it follows that C has rank 1 or 0. Since $C = {}^tC$, it follows that there exists a $M_{1\times 3}(\mathbf{C})$-valued function c on $\mathfrak{F}_X(M)$ uniquely defined up to sign satisfying $C = {}^tcc$.

The case $C \equiv 0$ is covered by Theorem 3.5, so let us assume that $C \not\equiv 0$ and restrict attention to the open subset where $C \neq 0$, say $U \subseteq M$. By passing to a double cover of $\mathfrak{F}_X(U)$, we may choose c smoothly (see the remark at the end of the proof). Differentiating the second equation of (3.11) we get

$$dC \wedge \omega + Cd\omega = d\bar{\theta} = -\bar{\theta} \wedge i\rho - [\theta] \wedge \mathfrak{h} - \bar{\kappa} \wedge \bar{\theta}. \tag{3.14}$$

In (3.14), the only term of type (0,2) is $[\theta] \wedge \mathfrak{h}$. Thus

$$[\theta] \wedge \mathfrak{h} = [\bar{\theta}\ ] \wedge \bar{\mathfrak{h}} = [C\omega] \wedge A\omega = 0. \tag{3.15}$$

Elementary linear algebra using (1.21) then establishes the result that there exists a unique a on $\mathfrak{F}_X(U)$ with values in $M_{1\times 3}(\mathbf{C})$ satisfying (3.10). q.e.d.

Remarks. For application to Theorems 3.8–3.12, let us carry these calculations a little further. If we substitute $C = {}^tcc$ and $A = \frac{1}{2}({}^tac + {}^tca)$ into (3.12) and (3.14) respectively, we may collect and cancel terms to rearrange these equations in the forms

$$ {}^t\sigma \wedge c\omega + {}^tc(\sigma \wedge \omega) = 0, \tag{3.14$'$}$$

$$ {}^t\tau \wedge c\omega + {}^tc(\tau \wedge \omega) + {}^t\sigma \wedge a\omega + {}^ta\sigma \wedge \omega = 0, \tag{3.12$'$}$$

where we have set

$$\sigma = dc - c(\kappa + (i/2)\rho), \tag{3.16}$$

$$\tau = da - a(\kappa - (3i/2)\rho) - \tfrac{1}{2}c\big(a[\bar{\omega}]\,{}^tc\big). \tag{3.17}$$

Applying linear algebra and Cartan's lemma, we conclude from (3.12′) and (3.14′) that there exist $M_{1\times 3}(\mathbf{C})$ valued functions r, s on $\mathcal{F}_X(U)$ (uniquely defined) so that

$$\sigma = {}^t\omega\left({}^tcs + \tfrac{1}{2}{}^tsc\right), \tag{3.18}$$

$$\tau = {}^t\omega\left({}^tcr + \tfrac{1}{2}{}^trc + {}^tas + \tfrac{1}{2}{}^tsa\right). \tag{3.19}$$

The presence of the $\frac{1}{2}$ factor in (3.16) and (3.17) shows that c and a change sign if they are transported around a generator of $\pi_1(U(3)) \simeq \mathbf{Z}$ in the fibers of $p\colon \mathcal{F}_X(U) \to U$. Thus c and a represent "spinor" quantities (rather than tensor quantities) on M. Equations (3.16–3.19) may then be regarded as expressing the fact that s is the covariant derivative of c and r is the covariant derivative of a. This explains why we must double cover $\mathcal{F}_X(U)$ in order to get c and a well-defined.

Using this last proposition, we see that for a complex immersion $X\colon M^6 \to \mathbf{O}$ which is free of Kähler-umbilics, the formulas (3.7) simplify to

$$\begin{aligned} \mathrm{II}^{2,0} &= -(a\omega)\circ(c\omega)n, \\ \mathrm{II}^{1,1} &= 0, \\ \mathrm{II}^{0,2} &= -(\overline{c\omega})\circ(\overline{c\omega})n. \end{aligned} \tag{3.20a,b,c}$$

With this in mind, we define the *asymptotic subbundle* of the immersion $X\colon M^6 \to \mathbf{O}$ by

$$\mathcal{A}(M) = \{v \in TM \mid c\omega(v) = 0\} \tag{3.21}$$

and the *bi-asymptotic subbundle* by

$$\mathcal{B}(M) = \{v \in TM \mid c\omega(V) = a\omega(v) = 0\}. \tag{3.22}$$

Note that because $\mathrm{II}^{2,0}$ and $\mathrm{II}^{0,2}$ are well defined on M, $\mathcal{A}(M)$ and $\mathcal{B}(M)$ are well defined. $\mathcal{B}(M)$ need not have constant rank since $a\omega \wedge c\omega$ can vanish along a subvariety (or be identically zero, for that matter). However, $\mathcal{B}(M)$ has constant rank on a dense open set in M. Note also that $\mathcal{A}(M) \subseteq TM$ is a complex subbundle of complex rank 2, while $\mathcal{B}(M) \subseteq \mathcal{A}(M)$ may have either complex rank 1 or 2.

Theorem 3.8. *$\mathcal{A}(M)$ is an integrable holomorphic subbundle of TM. The image of each leaf of the associated holomorphic foliation under the immersion $X\colon M^6 \to \mathbf{O}$ is (an open subset of) a real 4-plane in $\mathbf{O}$. On the open set where $\mathcal{B}(M)$ has constant rank, it, too, is an integrable holomorphic subbundle of TM. If* $\operatorname{rk}\mathcal{B}(M) = 1$, *then the leaves of the associated holomorphic foliation map under X to 2-planes is $\mathbf{O}$.*

Proof. Let $\mathcal{F}_X^{(1)}(M) \subseteq \mathcal{F}_X(M)$ be the subbundle defined by the condition that $\{f_2, f_3\}$ gives a (1,0) basis of $X_*(\mathcal{Q}(M))$, i.e., f_2 and f_3 span the asymptotic subspaces in $X(M)$. $\mathcal{F}_X^{(1)}(M)$ is clearly a $U(1) \times U(2)$ bundle over M. We restrict all of our forms on $\mathcal{F}_X(M)$ to $\mathcal{F}_X^{(1)}(M)$. By definition, $c\omega \wedge \omega^1 = 0$ so $c = (c_1, 0, 0)$ for some complex valued function c_1 on $\mathcal{F}_X^{(1)}(M)$, $c_1 \neq 0$, and $c\omega = c_1\omega^1$. If we write $s = (s_1, s_2, s_3)$, the equations (3.16) and (3.14) combine to give

$$(3.23)\quad (dc_1, 0, 0) = c_1\big(\kappa_1^1 + \tfrac{i}{2}\rho + s_1\omega^1 + \tfrac{1}{2}s\omega, \kappa_2^1 + s_2\omega^1, \kappa_3^1 + s_3\omega^1\big).$$

In particular, we get

$$(3.23')\qquad \kappa_2^1 = -s_2\omega^1, \quad \kappa_3^1 = -s_3\omega^1.$$

Also, (3.11) reads

$$(3.24)\qquad \bar\theta^1 = c_1^2\omega^1, \quad \bar\theta^2 = \bar\theta^3 = 0.$$

Using (1.41) and (3.23) we compute

$$(3.25)\qquad d\big(c_1\omega^1\big) = \big(\tfrac{i}{2}\rho - \tfrac{1}{2}s\omega\big) \wedge c_1\omega^1.$$

It follows that $c_1\omega^1 = c\omega$ is well defined on M up to a complex multiple of modulus 1 and that its annihilator $\mathcal{Q}(M)$ is a holomorphic integrable subbundle of TM. Of course, the leaves are characterized by the condition $\omega^1 = 0$.

If we regard $\tilde G(4, \mathbf{O})$ as imbedded in $\Lambda^4_{\mathbf{R}}\mathbf{O}$ by the Plücker imbedding, then the function $-4n \wedge \bar n \wedge f_1 \wedge \bar f_1$: $\mathcal{F}_X^{(1)}(M) \to \tilde G(4, \mathbf{O}) \subseteq \Lambda^4_{\mathbf{R}}\mathbf{O}$ assigns to each adapted frame the 4-plane which is orthogonal to $X_*(\mathcal{Q}_q(M))$ where $q \in M$ is the base of the frame. We may compute the differential of this function as

$$(3.26)\qquad \begin{aligned} d\big(-4n \wedge \bar n \wedge f_1 \wedge \bar f_1\big) = {} & -4\big(f_2\bar a_2\bar c_1\bar\omega^1 + f_3\bar a_3\bar c_1\bar\omega^1\big) \wedge \bar n \wedge f_1 \wedge \bar f_1 \\ & -4n \wedge \big(\bar f_2 a_2 c_1\omega^1 + \bar f_3 a_3 c_1\omega^1\big) \wedge f_1 \wedge \bar f_1 \\ & -4n \wedge \bar n \wedge \big(f_2\bar s_2\bar\omega^1 + f_3\bar s_3\bar\omega^1\big) \wedge \bar f_1 \\ & -4n \wedge \bar n \wedge f_1 \wedge \big(\bar f_2 s_2\omega^1 + \bar f_3 s_3\omega^1\big). \end{aligned}$$

It follows that on an integral of $\omega^1 = 0$, $d(-4n \wedge \bar n \wedge f_1 \wedge \bar f_1) = 0$ so that the normal 4-plane field to the image of each leaf of $\omega^1 = 0$ is constant. It follows that the image of each leaf under X is (an open subset of) a 4-plane in $\mathbf{O}$.

We now turn to $\mathcal{B}(M) \subseteq \mathcal{Q}(M)$. If $a\omega \wedge c\omega \equiv 0$, then $\mathcal{B}(M) = \mathcal{Q}(M)$ so there is nothing to prove. Hence we assume $a\omega \wedge c\omega \not\equiv 0$ and restrict to the open set where $a\omega \wedge c\omega \neq 0$. We define $\mathcal{F}_X^{(2)}(M) \subseteq \mathcal{F}_X^{(1)}(M)$ to be the subbundle defined by the extra condition that f_3 gives a (1,0) basis of $X_*(\mathcal{B}(M))$. $\mathcal{F}_X^{(2)}(M)$ is a $U(1) \times U(1) \times U(1)$-bundle over M. We restrict all of our forms to $\mathcal{F}_X^{(2)}(M)$. By definition, the span of $\{c\omega, a\omega\}$ is the same as the span of $\{\omega^1, \omega^2\}$. Thus, there exist complex functions a_1, a_2 on $\mathcal{F}_X^{(2)}(M)$ with $a_2 \neq 0$

satisfying $a = (a_1, a_2, 0)$. Examining (3.17) and (3.19), we get the analogue of (3.23′)

$$\kappa_3^2 = -s_3\omega^2 - (c_1 r_3/a_2)\omega^1. \tag{3.27}$$

Also, using (3.16–3.19), we compute that

$$d\,c\omega \equiv d\,a\omega \equiv 0 \quad \operatorname{mod}\{c\omega, a\omega\} = \{\omega^1, \omega^2\}. \tag{3.28}$$

Thus the bundle $\mathfrak{B}$ is holomorphic and integrable. The map $(-2if_3 \wedge \bar{f}_3)$: $\mathfrak{F}_X^{(2)}(M) \to \tilde{G}(2, \mathbf{O})$ assigns to each element of $\mathfrak{F}_X^{(2)}(M)$ the two-plane $X_*(\mathfrak{B}_q(M))$ where $q \in M$ is the base of the frame. Its differential is

$$\begin{aligned} d\big(-2if_3 \wedge \bar{f}_3\big) &= 2i\big(f_1 s_3\omega^1 + f_2\big(s_3\omega^2 + (c_1 r_3/a_2)\omega^1\big)\big) \wedge \bar{f}_3 \\ &\quad + 2if_3 \wedge \big(\bar{f}_1\bar{s}_3\bar{\omega}^1 + \bar{f}_2\big(\bar{s}_3\bar{\omega}^2 + (\bar{c}_1\bar{r}_3)/\bar{a}_2\big)\bar{\omega}^2\big) \\ &\quad -2i\big(\bar{f}_2 c_1\omega^1\big) \wedge \bar{f}_3 - 2i\big(f_2\bar{c}_1\bar{\omega}^1\big) \wedge f_3. \end{aligned} \tag{3.29}$$

It follows that along the leaves of $\omega^1 = \omega^2 = 0$, the tangent plane of the image in $\mathbf{O}$ is parallel, hence the image is (an open subset of) a 2-plane in $\mathbf{O}$.

Remarks. In view of this result, we will refer to the holomorphic foliation associated to $\mathfrak{A}(M)$ as the *asymptotic ruling* of M. We say that X: $M^6 \to \mathbf{O}$ is *asymptotically degenerate* if $\mathfrak{B}(M) \equiv \mathfrak{A}(M)$ and we say that the immersion is *asymptotically parallel* if the ruling is parallel in $\mathbf{O}$, i.e., the images of the leaves form a parallel family of 4-planes in $\mathbf{O}$. Calabi's examples are asymptotically parallel, so this family cannot be empty. Referring to (3.26), we see that

(i) X: $M^6 \to \mathbf{O}$ is asymptotically degenerate if and only if $a\omega \wedge c\omega = 0$,

(ii) X: $M^6 \to \mathbf{O}$ is asymptotically parallel if and only if $a\omega \wedge c\omega = s\omega \wedge c\omega = 0$.

The notion of *bi-asymptotic ruling* for complex, non-Kähler, asymptotically nondegenerate immersions is clear. For such an immersion, the bi-asymptotic ruling cannot be absolutely parallel because of the presence of the terms involving $c_1\omega^1$ in (3.29). More directly, this is not possible because if each of the planes $X_*(T_qM)$ contained a common complex line then they would all be complex with respect to a fixed J_u (i.e., the one which makes the common 2-plane complex) so the immersion would have to be Kähler. The correct notion of *bi-asymptotically parallel* is that the lines in each asymptotic leaf are parallel. By (3.29), we have

(iii) X: $M^6 \to \mathbf{O}$ is bi-asymptotically parallel if and only if $a\omega \wedge c\omega \neq 0$ and $s\omega \wedge a\omega \wedge c\omega = 0$.

We want to introduce one more special class of complex, non K-umbilic immersions X: $M^6 \to \mathbf{O}$.

For *any* oriented immersion $X\colon M^6 \to \mathbf{O}$, we define a map $\xi_X\colon M^6 \to \tilde{G}(2, \mathbf{O})$ where we take the oriented normal:

$$\xi_X(q) = N_q M \in \tilde{G}(2, \mathbf{O}). \tag{3.30}$$

We have

Proposition 3.9. *The map $\xi_X\colon M^6 \to \tilde{G}(2, \mathbf{O})$ is anti-holomorphic with respect to the natural complex structure on $\tilde{G}(2, \mathbf{O})$ and the almost complex structure on M^6 if and only if the immersion is Kähler. It is holomorphic if and only if $A = B = 0$. In particular, any such immersion where $X(M)$ is not a* 6*-plane is complex, asymptotically degenerate, and non-K-umbilic.*

Proof. By (2.2) and the discussion following, the forms $\{\mathfrak{h}^i, \bar{\theta}^j\}$ generate the pullbacks of the (1,0) forms on $\tilde{G}(2, \mathbf{O})$ under the canonical map $\mathfrak{F} \to \tilde{G}(2, \mathbf{O})$ which sends $(y, n, f, \bar{n}, \bar{f})$ to the oriented 2-plane spanned by n. It follows from (3.30) that ξ_X is anti-holomorphic if and only if $\bar{\mathfrak{h}} \equiv \theta \equiv 0$ $\operatorname{mod}\{\omega^1, \omega^2, \omega^3\}$ holds on $\mathfrak{F}_X(M)$. But this is clearly equivalent to $B = C = 0$ in (3.6), and by Theorems 3.2 and 3.3 this is equivalent to Kähler.

To continue, ξ_X is holomorphic if and only if $\mathfrak{h} \equiv \bar{\theta} \equiv 0 \operatorname{mod}\{\omega^1, \omega^2, \omega^3\}$, which is equivalent to $A = B = 0$. q.e.d.

Thus, our last special class of complex immersions is given by

(iv) $X\colon M^6 \to \mathbf{O}$ has holomorphic normal Gauss map if and only if $a\omega = 0$ (and the immersion is complex).

We now proceed to investigate the existence and "generality" of these various types of complex immersions in the analytic category. For this, we will use the theory of exterior differential systems and the Cartan-Kähler Theorem. For more details on the methods used, the reader should consult [1].

Theorem 3.10. *Let $\alpha\colon \mathbf{R} \to \mathbf{O}$ be an analytic immersion and let $\beta\colon \mathbf{R} \to \tilde{G}(2, \mathbf{O})$ be an analytic immersion satisfying the two conditions*

(i) *$\alpha'(t)$ is orthogonal to $\beta(t)$ for all $t \in \mathbf{R}$.*

(ii) *$\gamma \circ \beta\colon \mathbf{R} \to S^6$ is an immersion.*

Then there exists a unique connected analytic immersion $X\colon M^6 \to \mathbf{O}$ which is complex, so that $\alpha(\mathbf{R}) \subseteq X(M^6)$, and so that $\beta(t)$ is orthogonal to $X(M^6)$ at $\alpha(t) \in X(M^6)$.

Remarks. *From now until Theorem* 3.12, *we assume all data are analytic* and do not mention this point again.

If $X\colon M^6 \to \mathbf{O}$ is a non-Kähler, complex immersion, we may select $\tilde{\alpha}\colon \mathbf{R} \to M^6$ to be an immersion transverse to the asymptotic ruling, set $\alpha = X \circ \tilde{\alpha}\colon \mathbf{R} \to \mathbf{O}$ and let $\beta(t)\colon \mathbf{R} \to \tilde{G}(2, \mathbf{O})$ be given by $\beta(t) = n_{\tilde{\alpha}(t)} M$. The fact that $\tilde{\alpha}$ is transverse to the asymptotic ruling implies that $\gamma \circ \beta = (2in \times \bar{n}) \circ \beta$ is an immersion so the hypotheses are fulfilled. According to Theorem 3.10, *the pair (α, β) determine $X(M^6)$ completely.* Intuitively, a single generic curve in

$X(M^6)$ together with the knowledge of its normal along the curve completely determines $X(M^6)$, or at least, the connected component which contains the curve.

We may use this theorem to determine the "generality" of the complex, non-Kähler immersions $X\colon M^6 \to \mathbf{O}$. Fix a three-plane, $\mathbf{R}^3 \subseteq \mathbf{O}$ (since Spin acts transitively on $G(3, \mathbf{O})$, it does not matter which one). The unparametrized curves in $\mathbf{R}^3$ "depend on 2 functions of 1-variable." Choosing a 2-plane field along such a curve which is *normal* to the curve along the curve requires 10 functions of 1-variable since $\dim G(2, 7) = 10$. The genericity assumption (ii) in Theorem 3.10 only removes a small set of such choices. Thus, we can specify the essential (α, β) information using 12 functions of 1-variable. This gives a class of complex, non-Kähler submanifolds in $\mathbf{O}$ depending on 12 functions of 1-variable. One might expect, naïvely, that the "generic" complex, non-Kähler submanifold intersects $\mathbf{R}^3$ in a curve (by transversality). Thus, one might guess that the complex, non-Kähler submanifolds of $\mathbf{O}$ depend on 12 functions of 1-variable. We will show that this is the case in the proof below.

This is in contrast to the case of complex, *Kähler* submanifolds of $\mathbf{O}$. By Theorem 3.5, these are (up to constants) the same as complex hypersurfaces in $\mathbf{C}^4$. Locally, these depend on 1 holomorphic function of 3 complex variables (or equivalently, 2 real functions of 3 real variables). This is one of those cases where the "degenerate" solutions of a system of PDE form a larger class than the "generic" solutions.

Proof of Theorem 3.10. Let $\Xi = \mathfrak{F} \times M_{1\times 3}(\mathbf{C}) \times (M_{1\times 3}(\mathbf{C}) - \{(0)\})$ and let $a\colon \Xi \to M_{1\times 3}(\mathbf{C})$ and $c\colon \Xi \to M_{1\times 3}(\mathbf{C}) - \{(0)\}$ be the projections onto the second and third factors respectively.

We let I be the Pfaffian system on Ξ generated by the forms $\nu, \bar{\nu}$, the components of $\theta - {}^t\bar{c}\bar{c}\bar{\omega}$ and $\bar{\theta} - {}^tcc\omega$, and the components of $\bar{\mathfrak{h}} - \frac{1}{2}({}^tac + {}^tca)\omega$ and $\mathfrak{h} - \frac{1}{2}({}^t\bar{a}\bar{c} + {}^t\bar{c}\bar{a})\bar{\omega}$. Since I is invariant under conjugation, it may be regarded as the complexification of a real Pfaffian system of rank $2 + 6 + 6 = 14$.

Any complex, non-Kähler immersion $X\colon M^6 \to \mathbf{O}$ gives rise, by Proposition 3.7, to an immersion of $\tilde{\mathfrak{F}}_X(M)$, the spin double cover of $\mathfrak{F}_X(M)$ into Ξ, say $\tilde{X}\colon \tilde{\mathfrak{F}}_X(M) \to \Xi$ which is an integral of I and on which, the fifteen components of $\tilde{X}^*(\omega)$ and $\tilde{X}^*(\kappa)$ are independent.

Conversely, from the theory of moving frames, we see that any integral $Y\colon N^{15} \to \Xi$ of I on which $Y^*(\omega)$ and $Y^*(\kappa)$ have fifteen independent components may be regarded as the restriction of some $\tilde{X}\colon \tilde{\mathfrak{F}}_X(M) \to \Xi$ to an open subset of $\tilde{\mathfrak{F}}_X(M)$ for some complex, non-Kähler immersion $X\colon M^6 \to \mathbf{O}$ for some M.

We first prove that I is involutive. We easily compute

$$(3.31\text{a})\quad d\nu \equiv d\bar{\nu} \equiv 0 \quad \operatorname{mod} I,$$

$$(3.31\text{b})\quad d(\bar{\theta} - {}^t c c\omega) \equiv -{}^t\sigma \wedge c\omega - {}^t c(\sigma \wedge \omega) \quad \operatorname{mod} I,$$

$$(3.31\bar{\text{b}})\quad d(\theta - {}^t\bar{c}\bar{c}\bar{\omega}) \equiv -{}^t\bar{\sigma} \wedge \bar{c}\bar{\omega} - {}^t\bar{c}(\bar{\sigma} \wedge \bar{\omega}) \quad \operatorname{mod} I,$$

$$(3.31\text{c})\quad d\big(\bar{\mathfrak{h}} - \tfrac{1}{2}({}^t ac + {}^t ca)\omega\big)$$
$$\equiv -\tfrac{1}{2}\big({}^t\tau \wedge c\omega + {}^t c\tau \wedge \omega + {}^t\sigma \wedge a\omega + {}^t a\sigma \wedge \omega\big) \quad \operatorname{mod} I,$$

$$(3.31\bar{\text{c}})\quad d\big(\mathfrak{h} - \tfrac{1}{2}({}^t\bar{a}\bar{c} + {}^t\bar{c}\bar{a})\bar{\omega}\big)$$
$$\equiv -\tfrac{1}{2}\big({}^t\bar{\tau} \wedge \bar{c}\bar{\omega} + {}^t\bar{c}\bar{\tau} \wedge \bar{\omega} + {}^t\bar{\sigma} \wedge \bar{a}\bar{\omega} + {}^t\bar{a}\bar{\sigma} \wedge \bar{\omega}\big) \quad \operatorname{mod} I,$$

where σ and τ are the forms defined by (3.16) and (3.17) (now, of course, we regard a and c as independent functions on Ξ).

If we now let $v \in T_x\xi$ be any tangent vector which annihilates I and which satisfies $c\omega(v) \neq 0$, we see from (3.31) that the reduced characters of Cartan, s_i', satisfy

$$(3.32)\qquad s_1' = 12, \quad s_\alpha' = 0 \quad \text{for } \alpha > 1.$$

On the other hand, the formulae for the integral elements at a point $\chi \in \Xi$ are given by (3.18) and (3.19). Thus the integral elements depend on 12 parameters at a point (six each from r and s). Cartan's test is satisfied and the system is involutive.

It follows from the Cartan-Kähler Theorem that any integral curve of I on which $c\omega \neq 0$ has a unique extension to a 15 dimensional integral on which ω and κ have 15 independent components. (Note that $\omega = 0$ defines the Cauchy characteristics of the integral.) Moreover, the 15 dimensional integrals on which ω and κ are independent depend on $s_1' = 12$ functions of 1-variable.

To prove Theorem 3.10, let $\alpha\colon \mathbf{R} \to \mathbf{O}$ and $\beta\colon \mathbf{R} \to \tilde{G}(2, \mathbf{O})$ be given. Select a framing $\hat{\alpha}\colon \mathbf{R} \to \mathfrak{F}$ so that

$$\hat{\alpha}(t) = \big(\alpha(t); n(t), f(t), \bar{n}(t), \bar{f}(t)\big),$$

where $-2in \wedge \bar{n} = \beta$. Then we have

$$\hat{\alpha}^*(\nu) = \hat{\alpha}^*(\bar{\nu}) = 0,$$

since $\alpha'(t) \perp \beta(t)$. Moreover, there clearly exist $c, a\colon \mathbf{R} \to M_{1\times 3}(\mathbf{C})$ so that

$$\hat{\alpha}^*(\bar{\theta} - {}^t c c\omega) = 0,$$
$$\hat{\alpha}^*\big(\bar{\mathfrak{h}} - \tfrac{1}{2}({}^t ac + {}^t ca)\omega\big) = 0,$$

and we may use these to define a map $\check{\alpha}\colon \mathbf{R} \to \Xi$ which is an integral of I. By (2.9), the hypothesis (ii) in the theorem guarantees that $\check{\alpha}^*(c\omega) \neq 0$. By the above discussion there is a unique extension to a 15 dimensional integral. Two different choices of framing for $\hat{\alpha}$ differ by a Cauchy characteristic motion so they rise to the same 15 dimensional integral.

Theorem 3.11. *The class of asymptotically degenerate, non-Kähler, complex six-manifolds in* $\mathbf{O}$ *depends on* 8 *functions of one variable. The subclass of those with holomorphic normal Gauss map depends on* 6 *functions of* 1*-variable.*

Proof. Let $\Xi = \mathfrak{F} \times \mathbf{C} \times (M_{1\times 3}(\mathbf{C}) - \{(0)\})$ and let $\lambda \colon \Xi \to \mathbf{C}$ and $c \colon \Xi \to (M_{1\times 3}(\mathbf{C}) - \{(0)\})$ be the projections on the second and third factors respectively.

Let I be the system on Ξ generated by $\{\nu, \bar{\nu}, \theta - {}^t\overline{cc\omega}, \bar{\theta} - {}^tcc\omega, \bar{\mathfrak{h}} - \lambda\bar{\theta}, \mathfrak{h} - \bar{\lambda}\theta\}$. I is clearly the complexification of a real Pfaffian system of rank 14. If $M^6 \subseteq \mathbf{O}$ is an asymptotically degenerate, non-Kähler complex submanifold, then Proposition 3.7 and the remarks following Theorem 3.8 show that there is a canonical imbedding of $\tilde{\mathfrak{F}}_X(M^6)$ (where $X \colon M^6 \hookrightarrow \mathbf{O}$ is inclusion) into Ξ as an integral of I satisfying the independence condition that ω and κ restrict to $\mathfrak{F}_X(M)$ so that their fifteen components remain independent. Conversely any integral of I satisfying the independence condition is (an open subset of) some $\tilde{\mathfrak{F}}_X(M)$ for some asymptotically degenerate, non-Kähler complex submanifold $\mathbf{O}$. We now study I. Elementary calculation then shows that we have the following structure equations and their conjugates.

$$dv \equiv 0, \tag{3.32a}$$

$$d(\bar{\theta} - {}^tcc\omega) \equiv -({}^t\sigma \wedge c\omega + {}^tc\sigma \wedge \omega,) \qquad \text{mod } I, \tag{3.32b}$$

$$d(\bar{\mathfrak{h}} - \lambda\bar{\theta}\ ) \equiv -(d\lambda + 2i\rho\lambda)\,{}^tcc\omega, \tag{3.32c}$$

where σ is defined by (3.16). Again, if we select a vector $v \in T_x\Xi$ which annihilates I and on which $c\omega(v) \neq 0$, the integral element that it spans has Cartan character $s_1' = 8$. Since $14 + 8 = 22$ is the dimension of the Cartan system of I we see that $s_\alpha' = 0$ for $\alpha > 1$. Now the formula for the integral elements at a point is given by (3.18) and

$$d\lambda + 2i\rho\lambda = \mu(c\omega), \tag{3.33}$$

where $\mu \in \mathbf{C}$ is arbitrary (as is $s \in M_{1\times 3}(\mathbf{C})$). Thus the integral elements at a point depend on $s_1' = 8$ parameters so Cartan's test is satisfied. It follows that the system is involutive and that the general 15 dimensional integral satisfying the independence condition depends on 8 functions of 1-variable.

The second part of the theorem follows immediately by restricting I and the structure equations to $\{\lambda \equiv 0\} \subseteq \Xi$. This system is now clearly involutive with $s_1' = 6$. Proposition 3.9 then shows that these integrals project to $\mathbf{O}$ to be complex, non-Kähler six-manifolds $M^6 \subseteq \mathbf{O}$ with holomorphic normal Gauss map. Thus they depend on 6 functions of 1-variable.

Remarks. It is not difficult to show that the information to be specified in terms of a curve $\alpha: \mathbf{R} \to \mathbf{O}$ and a normal plane field $\beta: \mathbf{R} \to \tilde{G}(2, \mathbf{O})$ in order that the associated complex, non-Kähler $M^6 \subseteq \mathbf{O}$ be asymptotically degenerate or have holomorphic normal Gauss map is much the same as in Theorem 3.11. However, in order to have asymptotic degeneracy, β must satisfy a system of 4 (ordinary) differential equations and in order to have holomorphic normal Gauss map, β must satisfy 2 more (ordinary) differential equations. These differential equations are Spin(7) invariant of course and may be interpreted as stating that β is an integral of certain differential systems on $\tilde{G}(2, \mathbf{O})$ or on a first prolongation space of $\tilde{G}(2, \mathbf{O})$.

The Monge characteristics of the Pfaffian systems in Theorems 3.10 and 3.11 project to be the asymptotic rulings of admissible integrals and therefore depend only on constants. By using the integration techniques which Cartan developed in [5] for systems of this kind, we see that an essential use of the Cartan-Kähler Theorem only occurs in the extension of the one dimensional integral to a two dimensional integral. The remaining extensions along the asymptotic rulings and the frame directions can be done by ordinary differential equations alone. Thus the essential partial differential equations required is a system of nonlinear elliptic partial differential equations for functions of two variables whose principal symbol is the same as the symbol of the Cauchy-Riemann equations for a complex curve in $\mathbf{C}^6$, $\mathbf{C}^4$, and $\mathbf{C}^3$. This leads us to suspect that there may be a method of generating the solutions of these equations starting with the given data as respectively 6, 4, or 3 holomorphic functions of 1-variable. This would be analogy with the Weierstrass formulas for minimal surfaces in $\mathbf{R}^3$ in terms of one holomorphic function of one-variable. We do not yet know whether such formulas exist for the above problems.

Two problems remain along these lines. One is the problem of determining the generality of the bi-asymptotically parallel complex, non-Kähler six-manifolds in $\mathbf{O}$. We leave this as (a rather involved) exercise for the interested reader. The other problem is to determine the generality of the asymptotically parallel, complex, non-Kähler six-manifolds in $\mathbf{O}$. While we could set up the relevant differential system and show that these depend on 4 functions of 1-variable, a more direct approach is possible. In fact, we can describe these completely.

First, we describe a special feature of the Spin(7) geometry of $\mathbf{O}$. We already know that Spin(7) acts transitively on $\tilde{G}(2, \mathbf{O})$ and it is not difficult to verify that Spin(7) acts transitively on $\tilde{G}(3, \mathbf{O})$. However, Spin(7) does *not* act transitively on $\tilde{G}(4, \mathbf{O})$. In fact, the orbit structure is quite interesting. One particular orbit has been studied extensively by Harvey-Lawson [12]. We may describe it as follows: We define a map η: Spin(7) → $\tilde{G}(4, \mathbf{O})$ by

$$\eta(G) = -4f_2 \wedge \bar{f}_2 \wedge f_3 \wedge \bar{f}_3. \tag{3.34}$$

This map has the equivariance $\eta(gh) = \Lambda^4 h(\eta(g))$ and, computing the differential of η, using (1.36), we see that η has rank 12. The image η(Spin(7)) is a compact 12-manifold in $\tilde{G}(4, \mathbf{O})$. Harvey and Lawson show that the 4-planes in η(Spin(7)) are characterized by the condition that each of these 4-planes is a complex 2-plane with respect to the complex structure on **O** induced by any of its sub 2-planes. The negative of η(Spin(7)), gotten by reversing the orientation on the planes in η(Spin(7)) is another 12 dimensional orbit. Harvey and Lawson show that $\tilde{G}(4, \mathbf{O}) - \{\eta(\mathrm{Spin}(7))\} \cup \{-\eta(\mathrm{Spin}(7))\}$ is foliated smoothly by 15 dimensional orbits of Spin(7). $-\eta$(Spin(7)) is the manifold of *Cayley 4-planes* in **O** in Harvey and Lawson's terminology. In view of this, we will refer to the elements of η(Spin(7)) by the epithet "*anti-Cayley 4-planes.*" The concerned reader will be pleased to know that we will not use this terminology any further than the next theorem and the remark following. Also, we now disable the analytic assumption.

Theorem 3.12. *Suppose that $M^6 \subseteq \mathbf{O}$ is a complex, non-Kähler, asymptotically parallel submanifold of* **O**. *Let $\mathbf{O} = P^4 \oplus Q^4$ be the orthogonal direct sum so that the rulings of M^6 are parallel to Q^4. Both P^4 and Q^4 are anti-Cayley planes with the orientation compatible with the rulings of M. Moreover, the orthogonal projection $\mathbf{O} \to P^4$ induces a map $M^6 \to P^4$ whose image is an oriented minimal surface in P^4.*

Conversely, if we start with an anti-Cayley splitting $\mathbf{O} = P^4 \oplus Q^4$ which is orthogonal and let $S \subseteq P^4$ be a surface, then $S \times Q^4 \subseteq \mathbf{O}$ will be complex if and only if S is minimal. Moreover, if S is minimal (and is not a complex curve in P^4 for some one of P^4's complex structures) then $S \times Q^4$ is a complex, non-Kähler, asymptotically parallel submanifold of **O**.

Remarks. A specialized version of this theorem was proved by Calabi [2]. In order to see how his theorem relates to ours, we give a brief discussion of his result. If $Q^4 \subseteq \mathbf{O}$ is an anti-Cayley subspace and moreover $1 \in Q^4$, then one can show that Q^4 is actually a subalgebra of **O** isomorphic to the quaternions. In particular $\mathrm{Im}\, Q^4 = Q^4 \cap \mathrm{Im}\, \mathbf{O}$ is an "associative" 3-plane in $\mathrm{Im}\, \mathbf{O}$. Calabi showed that if $A^3 \subseteq \mathrm{Im}\, \mathbf{O}$ is any associative 3-plane and $S \subseteq A^3$ is a surface, then $S \times (A^3)^\perp \subseteq \mathrm{Im}\, \mathbf{O}$ is a complex submanifold if and only if S is minimal. (In this formula, $(A^3)^\perp$ is the orthogonal 4-plane in $\mathrm{Im}\, \mathbf{O}$, not all of **O**.)

Clearly our theorem implies Calabi's and shows that, up to a rigid Spin(7) motion of **O**, Calabi's examples are exactly those asymptotically parallel complex, non-Kähler submanifolds which happen to lie in a hyperplane in **O**.

As Calabi points out in his examples, the complex structure on $S \times Q^4$ is *not* the product structure unless $S \subseteq P^4$ is a complex curve in P^4 with respect to one of its canonical complex structures. In this case, of course, $S \times Q^4$ is actually a Kähler submanifold of $\mathbf{O}$. (In Calabi's examples the condition was that S not be a plane.)

Proof. First suppose that M^6 is as in the theorem's hypotheses. Let $\mathfrak{F}_X^{(1)}(M)$ be the reduced frame bundle with f_1 orthogonal to the rulings (we let X: $M^6 \to \mathbf{O}$ simply be inclusion) as in the proof of Theorem 3.8. It follows that $P^4 \equiv -4n \wedge \bar{n} \wedge f_1 \wedge \bar{f}_1$ and $Q^4 \equiv -4f_2 \wedge f_2 \wedge f_3 \wedge f_3$. We have already seen that the asymptotically parallel assumption implies, by (3.26) that

$$a_2 = a_3 = s_2 = s_3 = 0. \tag{3.35}$$

From this, we conclude, using (3.11) and (3.23′), that

$$\theta^2 = \theta^3 = \mathfrak{h}^2 = \mathfrak{h}^3 = \kappa_1^2 = \kappa_3^2 = 0, \tag{3.36}$$

while

$$\bar{\mathfrak{h}}^1 = a_1 c_1 \omega^1, \quad \bar{\theta}^1 = c_1^2 \omega^1. \tag{3.37}$$

Considering the basic structure equation $dx = f_i \omega^i + \bar{f}_i \bar{\omega}^i$, we see that if we project onto P^4 orthogonally to Q^4 by e: $\mathbf{O} \to P^4$, we get

$$d(e \circ x) = f_1 \omega^1 + \bar{f}_1 \bar{\omega}^1. \tag{3.38}$$

Thus $e \circ x$ has rank 2 and n is normal to the image while f_1 is tangential. By restricting to a leaf of $\omega^2 = \omega^3 = 0$ (the annihilator of the fiber foliation of $e \circ x$), we see that we get the adapted frame bundle of the image surface in P^4, say S. In particular, f_1 is a (1,0) vector for the natural complex structure on S as a surface (oriented) in P^4 and ω^1 is a (1,0) form. By the structure equations (3.3) and the formulas (3.36) and (3.37) we compute

$$df_1 = -n a_1 c_1 \omega^1 - \bar{n} c_1^2 \omega^1 + f_1 \kappa_1^1. \tag{3.39}$$

Since $df_1 \wedge f_1 \equiv 0 \bmod \Omega^{1,0}(S)\ (= \{\omega^1\})$, we see that the tangential Gauss map $S \to \tilde{G}(2, P^4)$ (which associates to each point in S the oriented tangent plane $-2if_1 \wedge \bar{f}_1$) is holomorphic. It is well known that this is equivalent to the property that S is a minimal surface in P^4. (Warning: remember that the complex structure that we use on $\tilde{G}(2, \mathbf{R}^N)$ is conjugate to the one used by Chern in [7].)

Conversely, let $\mathbf{O} = P^4 \oplus Q^4$ be an anti-Cayley splitting and let $S \subseteq P^4$ be an oriented surface. Let $M^6 = S \times Q^4$. Let $\mathfrak{F}^{(1)}(M) \subset \mathfrak{F}$ be the bundle over M consisting of pairs $(x; (n, f, \bar{n}, \bar{f}))$ so that $x \in M$, $-2in \wedge \bar{n}$ is the oriented normal to $T_x M$, $-2if_1 \wedge \bar{f}_1$ is the oriented tangent to S, and $-4f_2 \wedge \bar{f}_2 \wedge f_3 \wedge \bar{f}_3 \equiv Q^4$, as an oriented plane. This bundle exists (and has fiber $U(1) \times U(2)$)

because of our assumption that P^4 and Q^4 are anti-Cayley. If we differentiate the equation $-4f_1 \wedge \bar{f}_2 \wedge f_3 \wedge \bar{f}_3 \equiv Q^4$ we immediately get

$$\theta^2 = \theta^3 = \mathfrak{h}^2 = \mathfrak{h}^3 = \kappa_1^2 = \kappa_1^3 = 0. \tag{3.40}$$

(Twelve relations should have been expected anyway since $\dim \eta(\mathrm{Spin}(7)) = 12$.) This simplifies the structure equations on n and f_1 to

$$d(n, f_1, \bar{n}, \bar{f}_1) = (n, f, \bar{n}, \bar{f}_1)\begin{pmatrix} i\rho & -\bar{\mathfrak{h}}^1 & 0 & -\theta^1 \\ \mathfrak{h}^1 & \kappa_1^1 & \theta^1 & 0 \\ 0 & -\bar{\theta}^1 & -i\rho & -\mathfrak{h}^1 \\ \bar{\theta}^1 & 0 & \bar{\mathfrak{h}}^1 & \bar{\kappa}_1^1 \end{pmatrix}. \tag{3.41}$$

Since $\nu = 0$, $d\nu = 0$, so (1.36) implies

$$-\bar{\mathfrak{h}}^1 \wedge \omega^1 - \theta^1 \wedge \bar{\omega}^1 = 0. \tag{3.42}$$

So Cartan's lemma implies that there exist a, b, c, so that

$$(\bar{\mathfrak{h}}^1 \quad \bar{\theta}^1) = \begin{pmatrix} a & b \\ c & \bar{b} \end{pmatrix}\begin{pmatrix} \omega^1 \\ \bar{\omega}^1 \end{pmatrix}. \tag{3.43}$$

Clearly the components A, B, C on M^6 are gotten from a, b, c by multiplying each of these scalars by the 3×3 matrix with a 2 in the upper left-hand corner and zeros elsewhere. Therefore M^6 is complex if and only if $b = 0$. The equations (3.41) and (3.43) combine to give

$$f_1 \wedge df_1 \equiv \bar{\omega}^1(bn + \bar{b}\bar{n}) \wedge f_1 \quad \mathrm{mod}\{\omega^1\} \tag{3.44}$$

so we see that S is minimal if and only if $b = 0$ (if and only if M^6 is complex).

Similarly, M is Kähler if and only if $b = c = 0$ (by Theorem 3.5) and this is equivalent to the condition $dn \wedge n \wedge f = df \wedge n \wedge f = 0$. This last differential condition is satisfied if and only if the change of frame along any connected component of S is complex linear. In other words, S is a union of complex curves (where each piece may be complex under a different complex structure on P^4).

Finally, if $c \neq 0$ but $b = 0$, M^6 is complex and the Q^4-ruling is clearly the asymptotic ruling of M. q.e.d.

For our final result of this section we turn to the study of symplectic immersions. The scarcity of examples other than the Kähler case is explained by the following improvement of Theorem 3.3.

Theorem 3.13. *Any immersion X: $M^6 \to \mathbf{O}$ whose induced $U(3)$-structure is symplectic is also Kähler.*

Proof. Assume that X: $M^6 \to \mathbf{O}$ induces a symplectic $U(3)$-structure on M. Let $\mathfrak{F}_X(M)$ be the adapted frame bundle. By Theorem 3.3 we know that $C = 0$

and $\operatorname{tr} B = 0$. In particular

$$\theta = {}^tB\omega. \tag{3.45}$$

If we differentiate this relation and use (1.36), we get

$$-\kappa \wedge \theta + \theta \wedge i\rho - [\bar{\theta}\ ]\bar{\mathfrak{h}} = {}^t dB\omega + {}^tB\bigl(-\kappa \wedge \omega - [\bar{\theta}\ ] \wedge \bar{\omega}\bigr). \tag{3.46}$$

Since $\bar{\mathfrak{h}} = A\omega + B\bar{\omega}$, when we compare in (0,2) parts of both sides of (3.46) we find

$$[{}^tB\bar{\omega}] \wedge B\bar{\omega} = {}^tB[{}^t\bar{B}\bar{\omega}] \wedge \bar{\omega}. \tag{3.46$'$}$$

Since $\operatorname{tr} B = 0$, we may use the identities (1.21) and (1.36e) to rewrite this equation in the forms

$$-\bar{B}[\bar{\omega}] \wedge B\bar{\omega} - [\bar{\omega}] \wedge {}^t\bar{B}B\bar{\omega} = -\tfrac{1}{2}{}^tB\bar{B}[\bar{\omega}] \wedge \bar{\omega},$$

$$\bar{B}[B\bar{\omega}] \wedge \bar{\omega} + [{}^t\bar{B}B\bar{\omega}] \wedge \bar{\omega} = \tfrac{1}{2}{}^tB\bar{B}[\bar{\omega}] \wedge \bar{\omega},$$

$$(-\bar{B}{}^tB + \operatorname{tr} {}^t\bar{B}B - {}^tB\bar{B})[\bar{\omega}] \wedge \bar{\omega} = {}^tB\bar{B}[\bar{\omega}] \wedge \bar{\omega},$$

since the $\bar{\omega}^i$ are independent, it follows that

$$2{}^tB\bar{B} + \bar{B}{}^tB = \operatorname{tr} {}^t\bar{B}B\, I_3. \tag{3.47}$$

If $B \equiv 0$, we are done, so we assume $B \not\equiv 0$ and restrict our attention to a neighborhood of a point where $B \neq 0$. Since (3.47) is invariant under conjugation by a unitary matrix, we may put B in upper triangular form and compute using the condition $\operatorname{tr} B = 0$. We find that B must be of the form

$$B = e^f U^{-1} T U, \tag{3.48}$$

where f is a complex function, U is a 3×3 unitary matrix, T is the constant matrix

$$T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \beta^2 \end{pmatrix}, \tag{3.49}$$

and β is a nontrivial cube root of unity: $\beta^2 + \beta + 1 = 0$. In particular, the form $\mathrm{II}^{1,1}$ is Hermitian, so we may choose a unitary frame field which diagonalizes it (and hence B as well).

Thus, let $(n, f, \bar{n}, \bar{f})$ be such a frame field on our neighborhood and pull down all the forms on $\mathcal{F}_X(M)$. We now have

$$B = e^f T. \tag{3.50}$$

We now return to the equation $\theta = {}^tB\omega = B\omega$ armed with this new information. We have

$$d\theta = dB\omega + Bd\omega$$

using (1.36) and simplifying, we get

$$-\kappa \wedge \theta + \theta \wedge i\rho - [\bar{\theta}\ ] \wedge \bar{\mathfrak{h}} = dB \wedge \omega - B\big(\kappa \wedge \omega + [\bar{\theta}\ ] \wedge \bar{\omega}\big),$$

or

$$\big(dB - B\kappa + \kappa B - i\rho B + [\bar{B}\bar{\omega}]A\big) \wedge \omega = 0,$$

or

$$\big(dfT - T\kappa + \kappa T + i\rho T - [\bar{T}\bar{\omega}]e^{\bar{f}-f}A\big) \wedge \omega = 0. \tag{3.51}$$

In particular, by Cartan's lemma, we see that all of the entries in the 3×3 matrix in the parentheses are multiples of $\{\omega^1, \omega^2, \omega^3\}$. Checking the terms on the diagonal and using the fact that the ω^i and $\bar{\omega}^i$ are independent, we immediately see that A must be diagonal and that $df + i\rho \equiv 0 \operatorname{mod}\{\omega^1, \omega^2, \omega^3\}$. Examining the diagonal terms more closely, we see that $df + i\rho = 0$. If we differentiate this last result we get

$$d(i\rho) = {}^t\bar{\mathfrak{h}} \wedge \mathfrak{h} + {}^t\theta \wedge \bar{\theta} = 0, \tag{3.52}$$

and this implies that even the diagonal terms of A must be zero, so $A = 0$. Equation 3.51 now simplifies to

$$(T\kappa - \kappa T) \wedge \omega = 0, \tag{3.53}$$

and this implies that κ is diagonal. Using the structure equation for κ, we get

$$\begin{aligned} d\kappa &= -\kappa \wedge \kappa + \mathfrak{h} \wedge {}^t\bar{\mathfrak{h}} + \theta \wedge {}^t\bar{\theta} - [\bar{\theta}\ ] \wedge [\theta] \\ &= B\bar{\omega} \wedge {}^t\omega\bar{B} + 2B\omega \wedge {}^t\bar{\omega}\bar{B} - {}^t\omega \wedge B\bar{B}\bar{\omega}. \end{aligned}$$

However, this last expression is never diagonal while $B \neq 0$. Thus, we have a contradiction and $B \neq 0$ while $C = \operatorname{tr} B = 0$ is impossible.

4. The complex curves in S^6

In this section, we turn to a different aspect of the geometry of the octonians. We have already seen that $S^6 \subseteq \operatorname{Im}\mathbf{O}$ is endowed with an almost complex structure. Clearly the subgroup of the Spin(7) transformations which leaves S^6 invariant and preserves its orientation must fix both 1 and $0 \in \mathbf{O}$. It follows that this group is G_2. We have seen that the function $u\colon G_2 \to S^6$ and the functions $f_i\colon G_2 \to \mathbf{C} \otimes_{\mathbf{R}} \operatorname{Im}\mathbf{O}$ allow us to regard G_2 as the bundle of special unitary frames in S^6 and that left multiplications in G_2 act as the special unitary transformations of S^6. Since this action is simply transitive on the special unitary frames, the general theory tells us that if $U \subseteq S^6$ is connected and $\phi\colon U \to S^6$ is a special unitary map, then ϕ is the restriction to U of the action on all of S^6 induced by a left multiplication in G_2.

A submanifold $M^k \subseteq S^6$ (or an immersion $\phi\colon M^k \to S^6$) will be said to be *almost complex* if the tangent space T_xM^k (or the image $\phi_*(T_xM^k)$) is a complex subspace of T_xS^6 (or $T_{\phi(x)}S^6$) for all $x \in M^k$.

Proposition 4.1. *There is no $M^4 \subseteq S^6$ which is almost complex. Moreover, any (smooth) map $\phi\colon U \to S^6$ (where $U \subseteq S^6$ is open and connected) whose differential is complex linear at each point of U is either a constant map or the restriction to U of a G_2-action $S^6 \to S^6$.*

Proof. Suppose that $M^4 \subseteq S^6$ is almost complex. Let $\mathcal{F}(M^4) \subseteq G_2$ be the space of frames $(u; f)$ so that $u \in M^4$ and $\{f_2, f_3, \bar{f}_2, \bar{f}_3\}$ spans T_uM^4. $\mathcal{F}(M^4)$ is a $U(2)$-bundle over M. Since $du = f(-2i\theta) + \bar{f}(2i\bar{\theta})$, we see that $\theta^1 = 0$ on $\mathcal{F}(M^4)$ and that $\theta^2 \wedge \theta^3 \wedge \bar{\theta}^2 \wedge \bar{\theta}^3$ $(\neq 0)$ descends to be a well-defined volume form on M^4. By (2.17), we have

$$0 = d\theta^1 = -\kappa^1_2 \wedge \theta^2 - \kappa^1_3 \wedge \theta^3 - 2\bar{\theta}^2 \wedge \bar{\theta}^3.$$

It follows that $d\theta^1 \wedge \theta^2 \wedge \theta^3 = -2\theta^2 \wedge \theta^3 \wedge \bar{\theta}^2 \wedge \bar{\theta}^3 = 0$, which is a contradiction.

Now suppose that $\phi\colon U \to S^6$ has complex linear differential where $U \subseteq S^6$ is open and connected. Let $U' \subseteq U$ be the open set where ϕ has maximal rank. The complex rank of ϕ on U' cannot be 1 or 2 since in that case either the fibers of ϕ or the image of ϕ (locally) would be almost complex 4-manifolds in S^6, which we already know to be impossible. If the rank of ϕ on U' is 0, then $\phi\colon U \to S^6$ is the constant map. Hence let us assume that the rank of ϕ (over $\mathbf{C}$) is 3 on U'. Then ϕ is locally a diffeomorphism. We are going to show that ϕ must actually be a special unitary transformation on U', then we will be done.

Choose a special unitary frame field $\{f_1, f_2, f_3\}$ on a neighborhood of $u \in U'$, and choose another special unitary frame field $\{g_1, g_2, g_3\}$ on a neighborhood of $\phi(u)$ in S^6. Let $\{\omega^i\}$ be the forms dual to $\{f_i\}$ and let $\{\eta^i\}$ be the forms dual to $\{g_i\}$. By (2.17), we have

$$d\omega \equiv [\bar{\omega}] \wedge \bar{\omega} \quad \operatorname{mod} \Omega^{1,0}(S^6),$$
$$d\eta \equiv [\bar{\eta}\,] \wedge \bar{\eta} \quad \operatorname{mod} \Omega^{1,0}(S^6).$$

If $\phi_*(f) = gA^{-1}$ where A is a 3×3 complex matrix with $\det A \neq 0$, we dualize and get

$$\phi^*(\eta) = A\omega,$$

(note that we are using the complex linear assumption to ensure that forms of type (1,0) are preserved). In view of the formulae for $d\omega$ and $d\eta$ mod $\Omega^{1,0}(S^6)$, we see that this implies

$$d(A\omega) \equiv A[\bar{\omega}] \wedge \bar{\omega} \equiv [\overline{A\omega}] \wedge \overline{A\omega} \quad \operatorname{mod} \Omega^{1,0}(S^6).$$

Thus, we have $A[\overline{\omega}] \wedge \overline{\omega} = [\overline{A}\overline{\omega}] \wedge \overline{A}\overline{\omega}$. If we use the identity ${}^tM[M\alpha] \wedge M\alpha = (\det M)[\alpha] \wedge \alpha$, and the fact that $\overline{\omega}^1 \wedge \overline{\omega}^2 \wedge \overline{\omega}^3 \neq 0$, we immediately get

$${}^t\overline{A}A = \det \overline{A} I_3$$

which clearly implies that $\det \overline{A} = 1$, so A is special unitary. q.e.d.

The above proposition shows that any invariants of the almost complex structure of S^6 are also special unitary invariants of S^6, so it is natural to use the $SU(3)$-structure on S^6 to study questions about the complex structure.

One of the most interesting features of the almost complex structure on S^6 is the presence of "complex curves" on S^6. These are defined as follows: Let M^2 be a connected Riemann surface. A map $\phi: M^2 \to S^6$ will be called a complex curve if ϕ has complex linear differential at each point and ϕ is not the constant map.

One of the reasons for studying such objects is that Harvey and Lawson [12] have shown that the cone on $\phi(M^2)$ is absolutely mass minimizing in $\operatorname{Im}\mathbf{O} = \mathbf{R}^7$. In fact, this cone is *associative* in their sense. Conversely, if $C^3 \subseteq \operatorname{Im}\mathbf{O}$ is an associative cone with vertex at $0 \in \operatorname{Im}\mathbf{O}$, then $C^3 \cap S^6$ is a complex curve at its smooth points. In the usual techniques for studying singular minimal submanifolds, it is important to be able to understand the cones which are minimal. Thus the study of complex curves in S^6 is intimately related to the structure of singularities of associative submanifolds of $\operatorname{Im}\mathbf{O}$ (see [12]).

We will develop a theory of complex curves in S^6 which is analogous to the Frenet formulas for a real curve in Euclidean 3-space. Let $\phi: M^2 \to S^6$ be a complex curve (we always assume that M^2 is connected). We let $x: \mathfrak{F}_\phi \to M^2$ and $T_\phi \to M^2$ be the pull back bundles of $G_2 \to S^6$ and $T^{1,0}(S^6) \to S^6$ respectively. In formulas, we have

$$\mathfrak{F}_\phi = \{(x, g) \in M^2 \times G_2 \mid \phi(x) = u(g)\},$$
$$T_\phi = \{(x, v) \in M^2 \times T^{1,0}(S^6) \mid v \in T^{1,0}_{\phi(x)}(S^6)\}.$$

Of course, since $T^{1,0}(S^6)$ has a special unitary structure with G_2 as its unitary frame bundle, it follows that $\mathfrak{F}_\phi$ is the special unitary frame bundle of T_ϕ. Moreover, the natural map $\mathfrak{F}_\phi \to G_2$ pulls back both κ and θ to be well-defined forms on $\mathfrak{F}_\phi$ which we continue to denote by κ and θ (since we will now work on $\mathfrak{F}_\phi$ until Theorem 4.7, this should cause no confusion). Also, for functions and sections whose domain is in M^2, we will often work on $\mathfrak{F}_\phi$ and pull these quantities up from M^2 via x^* without comment. For example, any section $s: M^2 \to T_\phi$ can be written in the form $s = f_i s^i$ where the f_i are actually maps $f_i: \mathfrak{F}_\phi \to T_\phi$ and s_i are functions on $\mathfrak{F}_\phi$. Using this convention, the pull back of κ induces a connection on T_ϕ which is compatible with its special Hermitian structure. Namely $\nabla\colon \Gamma(T_\phi) \to \Gamma(T_\phi \otimes T^*_{\mathbb{C}}M^2)$ is given by

$$\nabla\left(f_i s^i\right) = f_i \otimes \left(ds^i + \kappa^i_j s^j\right).$$

Since we are working over a Riemann surface, it is well known that there is a unique holomorphic structure on T_ϕ so that ∇ is compatible with the holomorphic structure (see [15]). We suppose that T_ϕ is given this holomorphic structure and refer to T_ϕ hereafter as a holomorphic, special Hermitian vector bundle over M^2 of rank 3.

Another thing to notice is that $\{\theta^1, \theta^2, \theta^3\}$ are semi-basic with respect to $x\colon \mathcal{F}_\phi \to M^2$. Moreover, they are of type (1,0) since $\phi\colon M^2 \to S^6$ has complex linear differential.

Lemma 4.2. *If we set* $\mathrm{I} = f_i \otimes \theta^i$, *then* I *is a well-defined section of* $T_\phi \otimes (T')^*$ (*where* $(T')^* = \Lambda^{1,0}M^2$ *as a holomorphic line bundle*). *Moreover* I *is a nonzero holomorphic section of this bundle.*

Proof. That I is well defined is clear. Moreover, I has values in $T_\phi \otimes (T')^*$ by definition. It remains to show that I is holomorphic and that $\mathrm{I} \not\equiv 0$. Choose a uniformizing parameter z on a neighborhood of $x_0 \in M$. In a neighborhood of $x^{-1}(x_0) \subseteq \mathcal{F}_\phi$, there exist functions a^i so that $\theta^i = a^i dz$. It follows that $\theta^i \wedge \theta^j = 0$, so we have $d\theta^i = -\kappa^i_j \wedge \theta^j$. This translates to $(da^i + \kappa^i_j a^j) \wedge dz = 0$ so there exist b^i so that

$$da^i + \kappa^i_j a^j = b^i dz.$$

Thus, when we compute $\bar\partial \mathrm{I} \in \Gamma(T_\phi \otimes (T')^* \otimes (T')^*)$, we get

$$\begin{aligned} \bar\partial \mathrm{I} &= \bar\partial\left(f_i \otimes \theta^i\right) = \pi^{0,1} \circ \left(\nabla\left(f_i a^i\right) \otimes dz\right) \\ &= f_i \otimes dz \otimes \pi^{0,1}\left(da^i + \kappa^i_j a^i\right) \\ &= f_i \otimes dz \otimes \pi^{0,1}\left(b^i dz\right) \\ &= 0, \end{aligned}$$

so I is holomorphic. If $\mathrm{I} \equiv 0$, then by our definitions $\phi\colon M^2 \to S^6$ has rank 0 at every point and hence must be a constant map, contradicting our assumptions.

Remark. It is clear that I is the section of $T_\phi \otimes (T')^*$ which represents the "evaluation map" $\phi_*(T') \to T_\phi$.

Since $\mathrm{I} \not\equiv 0$, we see that there exists a holomorphic line bundle $\tau \subseteq T_\phi$ so that I is a nonzero section of $\tau \otimes (T')^*$. We let R be the ramification divisor of I. That is,

$$R = \sum_{p\,:\,\mathrm{I}(p)=0} \operatorname{ord}_p(\mathrm{I}) \cdot p.$$

R is obviously effective, and we have (see [11])

$$\tau = T' \otimes [R].$$

In particular, we have

$$\deg \tau = \deg T' + \deg R \geqslant \deg T' = \chi(M).$$

Now we adapt frames in accordance with the general theory. We let $\mathcal{F}_\phi^{(1)} \subseteq \mathcal{F}_\phi$ be the subbundle of pairs (x, g) where $f_3(g) \in \tau_x$. Then $\mathcal{F}_\phi^{(1)}$ is a $U(2)$-bundle over M. The canonical connection on τ is described as follows: If $s\colon M \to \tau$ is a section, then $s = f_3 s^3$ for some s^3 well-defined on $\mathcal{F}_\phi^{(1)}$. Then

$$\nabla s = f_3 \otimes \left(ds^3 + \kappa_3^3 s^3\right).$$

Similarly, the quotient bundle $N_\phi = T_\phi/\tau$ has a natural holomorphic Hermitian structure. Let us let (f_1), $(f_2)\colon \mathcal{F}_\phi^{(1)} \to N_\phi$ be the functions $f_1, f_2\colon \mathcal{F}_\phi^{(1)} \to T_\phi$ followed by the projection $T_\phi \to N_\phi$. If $s\colon M \to N_\phi$ is any section, then $s = (f_1)s^1 + (f_2)s^2$ for s^1 and s^2 on $\mathcal{F}_\phi^{(1)}$ and we have

$$\nabla s = (f_1) \otimes \left(ds^1 + \kappa_1^1 s^1 + \kappa_2^1 s^2\right) + (f_2) \otimes \left(ds^2 + \kappa_1^2 s^1 + \kappa_2^2 s^2\right).$$

Note that since I has values in $\tau \otimes (T')^*$, we must have $\theta^1 = \theta^2 = 0$ on $\mathcal{F}_\phi^{(1)}$ so that $\mathrm{I} = f_3 \otimes \theta^3$. If we differentiate these two equations using (2.17) we get

$$d\theta^1 = -\kappa_3^1 \wedge \theta^3 = 0, \quad d\theta^2 = -\kappa_3^2 \wedge \theta^3 = 0.$$

It follows that κ_3^1 and κ_3^2 are of type (1,0).

Lemma 4.3. *Let* $\mathrm{II} = (f_1) \otimes f^3 \otimes \kappa_3^1 + (f_2) \otimes f^3 \otimes \kappa_3^2$, *where* f^3 *is the dual of* f_3 *(so* $f^3\colon \mathcal{F}_\phi^{(1)} \to \tau^*$*). Then* II *is a holomorphic section of* $N_\phi \otimes \tau^* \otimes (T')^*$.

We omit the proof. It is similar to that of Lemma (4.2). II is the analogue of the first curvature of the map $\phi\colon M^2 \to S^6$. The following lemma shows that this intuition is correct.

Suppose $\mathrm{II} = 0$, then we must have $\kappa_3^1 = \kappa_3^2 = 0$ on $\mathcal{F}_\phi^{(1)}$. But then the structure equations (2.15–2.16) show that $d(-2iu \wedge f_3 \wedge \bar{f}_3) \equiv 0$ so that u always lies in the 3-plane $\xi^3 = -2iu \wedge f_3 \wedge \bar{f}_3$ which is fixed. We have just proven

Lemma 4.4. *If* $\mathrm{II} = 0$, *then* $\phi(M) \subseteq S^2 = \xi^3 \cap S^6$ *where* ξ^3 *is a fixed three dimensional subspace of* $\operatorname{Im} \mathbf{O}$.

Remark. The three planes of the form $-2iu \wedge f_3 \wedge \bar{f}_3$ in $\operatorname{Im}\mathbf{O}$ are the *associative* planes in $\operatorname{Im}\mathbf{O}$. There is a (real) 8 parameter family of them in $\tilde{G}(3, \operatorname{Im}\mathbf{O})$.

From now on, let us assume that $\mathrm{II} \not\equiv 0$. Let F be the *flexor divisor* of II. That is

$$F = \sum_{p\colon \mathrm{II}(p) = 0} \operatorname{ord}_p(\mathrm{II}) \cdot p.$$

F is effective and we have a result analogous to the one for I: There exists a holomorphic line bundle $\nu \subseteq N_\phi$ so that II is a section of $\nu \otimes \tau^* \otimes (T')^*$, so

$$\nu = [F] \otimes \tau \otimes T' = [F] \otimes [R] \otimes T' \otimes T'.$$

In particular,

$$\deg \nu = 2 \deg T' + \deg F + \deg R \geqslant 2 \deg T' = 2\chi(M).$$

We set $\beta = N_\phi/\nu$ and note that β inherits a holomorphic Hermitian structure. Moreover, we may adapt frames further $\mathcal{F}_\phi^{(2)} \subseteq \mathcal{F}_\phi^{(1)}$ so that for each $(x, g) \in \mathcal{F}_\phi^{(2)}$, we have $(f_2)(g) \in \nu_x$.

Then $\mathcal{F}_\phi^{(2)}$ is a $U(1) \times U(1)$-bundle over M. A section of ν is of the form $s = (f_2)s^2$ and we have the formula

$$\nabla s = (f_2) \otimes \left(ds^2 + \kappa_2^2 s^2\right).$$

We let $((f_1))$: $\mathcal{F}_\phi^{(2)} \to \beta$ be the reduction of (f_1) mod ν. Then a section σ: $M^2 \to \beta$ is of the form $\sigma = ((f_1))\sigma^1$ and we have

$$\nabla \sigma = ((f_1)) \otimes \left(d\sigma^1 + \kappa_1^1 \sigma^1\right).$$

Since II is a section of $\nu \otimes \tau^* \otimes (T')^*$, on $\mathcal{F}_\phi^{(2)}$ we must have $\mathrm{II} = (f_2) \otimes f^3 \otimes \kappa_3^2$ and $\kappa_3^1 = 0$. Differentiating this, we get

$$d\kappa_3^1 = -\kappa_2^1 \wedge \kappa_3^2 = 0.$$

Since $\kappa_3^2 \neq 0$ and is of type (1,0) (vanishing only at isolated points) we see that κ_2^1 is of type (1,0).

Let (f^2): $\mathcal{F}_\phi^{(2)} \to \nu^*$ be the obvious dual map.

Lemma 4.5. *Let* $\mathrm{III} = ((f_1)) \otimes (f^2) \otimes \kappa_2^1$, *then* III *is a holomorphic section of* $\beta \otimes \nu^* \otimes (T')^*$.

(Proof omitted.)

We say that the curve has *null-torsion* if $\mathrm{III} \equiv 0$. Since there are no almost complex $M^4 \subseteq S^6$, it is clear that this condition will not have as simple a counterpart as the case of curves with zero torsion in $\mathbf{CP}^3$. Another difference between curves in $\mathbf{CP}^3$ and S^6 is that S^6 has an $SU(3)$-structure rather than just a $U(3)$-structure as $\mathbf{CP}^3$ does. Thus the holomorphic, metric isomorphism $\Lambda^3 T^{1,0} S^6 \simeq \mathbf{C}$ implies

$$\tau \otimes \nu \otimes \beta \simeq \mathbf{C}$$

canonically.

If $\mathrm{III} \not\equiv 0$, we define the *planar divisor* by

$$P = \sum_{p:\,\mathrm{III}(p)=0} \operatorname{ord}_p(\mathrm{III}) \cdot p.$$

In this case, we have

$$\beta = [P] \otimes \nu \otimes T'.$$

Theorem 4.6. *Let $M^2 = \mathbf{P}^1$, then any complex curve $\phi\colon M^2 \to S^6$ either has image in an S^2 $(= \xi^3 \cap S^6)$ or has null-torsion.*

Proof. If $\mathrm{II} = 0$, then $\phi(M^2) \subseteq S^2$ $(= \xi^3 \cap S^6)$, so assume that $\mathrm{II} \neq 0$. We must show that $\mathrm{III} \equiv 0$. If not, we have, for $R, F, P \geqslant 0$,

$$\beta = [P] \otimes \nu \otimes T', \quad \nu = [F] \otimes \tau \otimes T', \quad \tau = [R] \otimes T',$$

which implies, since $\tau \otimes \nu \otimes \beta$ is trivial, that

$$(T')^6 \otimes [3R + 2F + P] \simeq \mathbf{C}$$

thus $\deg T' \leqslant 0$, but $\deg T' = 2$ when $M = \mathbf{P}^1$.

Remarks. The computation in this theorem actually shows that if M^2 has genus g, then any complex curve $\phi\colon M^2 \to S^6$ with nonnull-torsion must satisfy

$$12(g-1) = 3\deg R + 2\deg F + \deg P,$$

where each of the divisors R, F, and P are effective. (More precisely, the effective divisor $3R + 2F + P$ is linearly equivalent to six times the canonical divisor.) This puts severe restrictions on the bundles τ, ν, and β. For example, if $g = 1$, so that M^2 is an elliptic curve, then a complex curve $\phi\colon M^2 \to S^6$ with $\mathrm{III} \neq 0$ must satisfy $R = F = P = 0$, so that $\tau = T'$, $\nu = (T')^2$, $\beta = (T')^3$.

By analogy with the situation of curves in $\mathbf{CP}^3$, one might expect that once the degrees of τ, ν, and β are fixed, the space of complex curves $\phi\colon M^2 \to S^6$ is finite dimensional. We do not know if this is the case.

We will now show that the complex curves with null-torsion display a much greater variety. In fact, we will show that every Riemann surface M has an infinite family of complex curves $\phi\colon M^2 \to S^6$ with no bound on the degree of the ramification divisor R.

We do this in several steps. First, we transform the problem of studying null-torsion complex curves in S^6 to a problem in the holomorphic category.

If $\phi\colon M^2 \to S^6$ is a complex curve with $\mathrm{II} \neq 0$, we can define the *binormal mapping* $b_\phi\colon M^2 \to \tilde{G}(2, \mathrm{Im}\,\mathbf{O})$ by letting $b_\phi(x)$ be the oriented plane $-2if_1 \wedge \bar{f}_1$ where $(f_i) \in \mathfrak{F}_\phi^{(2)}$ is an adapted frame at $x \in M$. It is easily seen that b_ϕ is well defined and is a lifting of ϕ:

$$\begin{tikzcd}
 & \tilde{G}(2, \mathrm{Im}\,\mathbf{O}) \arrow[d, "\pi"] \\
M^2 \arrow[ur, "b_\phi"] \arrow[r, "\phi"] & S^6
\end{tikzcd}$$

Theorem 4.7. *Let* $\phi\colon M^2 \to S^6$ *be a complex curve with* $\mathrm{II} \neq 0$. *Then the binormal mapping* $b_\phi\colon M^2 \to \tilde{G}(2, \mathrm{Im}\,\mathbf{O})$ *is holomorphic* (*with respect to the natural complex structure on* $\tilde{G}(2, \mathrm{Im}\,\mathbf{O})$) *if and only if* ϕ *satisfies* $\mathrm{III} = 0$. *Moreover, in this case* b_ϕ *is an integral of the holomorphic differential system* $\mathfrak{L} = \{\alpha \in \Omega^1_{\mathbb{C}}\tilde{G}(2, \mathrm{Im}\,\mathbf{O}) \mid \eta^*(\alpha) \equiv 0 \bmod \kappa^2_1, \kappa^3_1, \bar{\theta}^1\}$. *Conversely, any nonconstant holomorphic curve* $b\colon M^2 \to \tilde{G}(2, \mathrm{Im}\,\mathbf{O})$ *which is an integral of* $\mathfrak{L}$ *has the property that* $\phi = \pi \circ b\colon M^2 \to S^6$ *is a complex curve with either* $\mathrm{II} = 0$ *or* $\mathrm{II} \neq 0, \mathrm{III} = 0$ *and* $b = b_\phi$.

Proof. By Proposition 2.4 and the commutative diagram

$$\begin{tikzcd}
\mathfrak{F}_\phi^{(2)} \arrow[r] \arrow[d, "x"] & G_2 \arrow[d, "\eta"] \\
M^2 \arrow[r, "b_\phi"] & \tilde{G}(2, \mathrm{Im}\,\mathbf{O})
\end{tikzcd}$$

we see that b_ϕ is holomorphic if and only if the forms $\{\kappa^2_1, \kappa^3_1, \bar{\theta}^1, \theta^2, \theta^3\}$ restrict to $\mathfrak{F}_\phi^{(2)}$ to be of type (1,0). Since we already have $\kappa^3_1 = \theta^1 = \theta^2 = 0$, and since θ^3 is certainly of type (1,0), we see that the only further condition required is that κ^2_1 be of type (1,0). Thus b_ϕ is holomorphic if and only if $\kappa^1_2 = 0$, i.e., $\mathrm{III} = 0$.

The differential system $\mathfrak{L}$ is of type (1,0) by definition. When we compute the structure equations, we get

$$\left.\begin{aligned} d\kappa^2_1 &\equiv 3\theta^2 \wedge \bar{\theta}^1 \\ d\kappa^3_1 &\equiv 3\theta^3 \wedge \bar{\theta}^1 \end{aligned}\right\} \quad \mathrm{mod}\{\kappa^2_1, \kappa^3_1\},$$

$$d\bar{\theta}^1 \equiv -2\theta^2 \wedge \theta^3 \quad \mathrm{mod}\,\{\kappa^2_1, \kappa^3_1, \bar{\theta}^1\}.$$

Since $d\mathfrak{L} \bmod \mathfrak{L}$ consists of forms of type (2,0), we conclude that $\mathfrak{L}$ is locally generated by holomorphic 1-forms and is therefore holomorphic (see below for a more explicit description). By the argument above, if $\phi\colon M^2 \to S^6$ satisfies $\mathrm{II} \neq 0$, $\mathrm{III} = 0$, then $b_\phi^*(\mathfrak{L}) = 0$, so $b_\phi\colon M^2 \to \tilde{G}(2, \mathrm{Im}\,\mathbf{O})$ is an integral of $\mathfrak{L}$.

Conversely, suppose that $b\colon M^2 \to \tilde{G}(2, \mathrm{Im}\,\mathbf{O})$ is a nonconstant holomorphic curve which is an integral of $\mathfrak{L}$. By Proposition 2.4 (or directly from the structure equations), we see that $\mathfrak{L}^\perp = L_+ \subseteq T^{1,0}\tilde{G}(2, \mathrm{Im}\,\mathbf{O})$ has the property that the differential of $\pi\colon \tilde{G}(2, \mathrm{Im}\,\mathbf{O}) \to S^6$ is complex linear and injective when restricted to L_+. Thus $\phi = \pi \circ b\colon M^2 \to S^6$ is complex and ramifies only when b does (in particular, ϕ is not a constant map). We now easily verify that if we adapt frames along ϕ so that f_3 is tangent to ϕ and f_1 spans b, then the resulting $U(1) \times U(1)$-bundle $\mathfrak{F}_b \subseteq M \times G_2$ satisfies $\kappa^2_1 = \kappa^3_1 = \bar{\theta}^1 = 0$ (because b is an integral of $\mathfrak{L}$) and $\theta^2 = 0$ (because f_2 is tangent to ϕ). If $\kappa_3^2 = 0$ on $\tilde{\mathfrak{F}}_b$, then we have already seen that the three-plane $-2iu \wedge f_3 \wedge \bar{f}_3 = \xi^3$ is constant on $\tilde{\mathfrak{F}}_b$ and $\phi(M) \subseteq \xi^3 \cap S^6 = S^2$. If $\kappa_3^2 \neq 0$, then $\tilde{\mathfrak{F}}_b = \mathfrak{F}_\phi^{(2)}$ so that $b = b_\phi$ as desired.

Corollary 4.8. *If M^2 is compact and ϕ: $M^2 \to S^6$ is a complex curve with null-torsion, then ϕ is algebraic. In particular, it is real analytic.*

Proof. By Theorem 4.7, such curves are of the form $\phi = \pi \circ b_\phi$ where b_ϕ: $M^2 \to \tilde{G}(2, \operatorname{Im}\mathbf{O})$ is a holomorphic curve. Since we have a natural imbedding $\tilde{G}(2, \operatorname{Im}\mathbf{O}) \subseteq \mathbf{CP}^6$ (see below) as a nonsingular five-quadric, the curve b_ϕ: $M^2 \to \mathbf{CP}^6$ is algebraic. Finally, the projection η: $\tilde{G}(2, \operatorname{Im}\mathbf{O}) \to S^6$ is clearly algebraic. q.e.d.

In order to construct examples, it will be necessary to study the differential system $\mathfrak{L}$ more closely. This differential system was discovered by Cartan and Engel in connection with their early work on the exceptional group G_2. We will now give a brief exposition of this theory.

First, as is well known, the manifold $\tilde{G}(2, \operatorname{Im}\mathbf{O})$ may be interpreted as a submanifold of the projectivization of $\mathbf{C} \otimes_{\mathbf{R}} \operatorname{Im}\mathbf{O}$. Explicitly, if $x \wedge y \in \tilde{G}(2, \operatorname{Im}\mathbf{O})$ where $x, y \in \operatorname{Im}\mathbf{O}$ form an orthonormal pair, then we identify $x \wedge y$ with the complex line in $\mathbf{C} \otimes \operatorname{Im}\mathbf{O}$ spanned by $x - iy$. Extending the real inner product on $\mathbf{O}$ complex linearly to a complex inner product on $\mathbf{C} \otimes \operatorname{Im}\mathbf{O}$ (which we still denote by $\langle\, ,\rangle$), we see that $\langle x - iy, x - iy\rangle = (\langle x, x\rangle - \langle y, y\rangle) - 2i\langle x, y\rangle = 0$ when $\{x, y\}$ form an orthonormal pair. It follows that the above map $x \wedge y \to (x - iy)\mathbf{C}$ imbeds $G(2, \operatorname{Im}\mathbf{O}) \subseteq \mathbf{CP}^6$ as the five-quadric of null-lines (under the inner product $\langle\, ,\rangle$) in $\mathbf{C} \otimes_{\mathbf{R}} \operatorname{Im}\mathbf{O}$. With this identification, we may now write the map η: $G_2 \to \tilde{G}(2, \operatorname{Im}\mathbf{O})$ as $\eta = (f_1\mathbf{C}) \in \mathbf{CP}^6$. Since $d\eta \equiv df_1 \bmod f_1$, we see from the structure equations that this imbedding is holomorphic.

Second, we need the fact that G_2 is defined algebraically as the group of algebra automorphisms of $\mathbf{O}$ (see [12]). If we extend the inner product and multiplication of $\mathbf{O}$ complex linearly to $\mathbf{C} \otimes_{\mathbf{R}} \mathbf{O} = \mathbf{O}_{\mathbf{C}}$, then $\mathbf{O}_{\mathbf{C}}$ is a complex inner product algebra which contains $\mathbf{O} \subseteq \mathbf{O}_{\mathbf{C}}$ as the subalgebra invariant under *complex conjugation* (not $\mathbf{O}_{\mathbf{C}}$-conjugation). The group of (complex) automorphisms of $\mathbf{O}_{\mathbf{C}}$ is a 14-dimensional complex Lie group which we denote by $G_2(\mathbf{C})$. We have $G_2 \subseteq G_2(\mathbf{C})$ as the subgroup which commutes with complex conjugation (or equivalently, which preserves $\mathbf{O} \subseteq \mathbf{O}_{\mathbf{C}}$). If we define $(z(h), f(h), g(h)) = (\varepsilon, E, \bar{E})h$ where $h \in G_2(\mathbf{C})$ and $(\varepsilon, E, \bar{E})$ is as defined in (1.18), then z, f_i, and g_i are vector valued functions on $G_2(\mathbf{C})$ with values in $\operatorname{Im}\mathbf{O}_{\mathbf{C}} = \mathbf{C} \otimes_{\mathbf{R}} \operatorname{Im}\mathbf{O}$ and we easily verify the multiplication table and structure equations of $G_2(\mathbf{C})$ given by

$$\begin{array}{c|ccc} & z & f & g \\ \hline z & -1 & -if & ig \\ {}^tf & i\,{}^tf & [-{}^tg] & -nI_3 \\ {}^tg & -i\,{}^tg & -nI_3 & [-{}^tf] \end{array} \qquad \left(n = \tfrac{1}{2}(1 - iz)\right),$$

$$d(z\,f\,g) = (z\,f\,g)\begin{pmatrix} 0 & -i\,{}^t\eta & i\,{}^t\theta \\ -2i\theta & \kappa & [\eta] \\ 2i\eta & [\theta] & -{}^t\kappa \end{pmatrix}$$

where θ, η, and κ are left-invariant 1-forms on $G_2(\mathbf{C})$ with values in $M_{3\times 1}(\mathbf{C})$, $M_{3\times 1}(\mathbf{C})$, and $sl(3, \mathbf{C})$ respectively. If we restrict these functions and forms to $G_2 \subseteq G_2(\mathbf{C})$, we get $\eta = \bar{\theta}$, ${}^t\kappa + \bar{\kappa} = 0$ and $z = \bar{z}$, $g = \bar{f}$, so that these equations reduce to our known structure equations for G_2. The map $[f_1]$: $G_2(\mathbf{C}) \to \mathbf{CP}^6$ which sends $h \in G_2(\mathbf{C})$ to the line in $\mathbf{O_C}$ spanned by $f_1(h)$ has image $\tilde{G}(2, \operatorname{Im}\mathbf{O}) \subseteq \mathbf{CP}^6$. We may see this as follows: By the above multiplication table, $f_1^2 \equiv 0$ so

$$0 = \langle f_1^2, f_1^2 \rangle = (\langle f_1, f_1 \rangle)^2 = 0,$$

so f_1 spans a null-vector of $\langle\, , \rangle$. By the structure equations

$$d[f_1] \equiv -iz\eta^1 + f_2\kappa_1^2 + f_3\kappa_1^3 - g_2\theta^3 + g_3\theta^2 \mod f_1$$

so $[f_1]$: $G_2(\mathbf{C}) \to \tilde{G}(2, \operatorname{Im}\mathbf{O}) \subseteq \mathbf{CP}^6$ has rank 5 and is therefore surjective. Thus $G_2(\mathbf{C})$ acts as a group of bi-holomorphic transformations of $\tilde{G}(2, \operatorname{Im}\mathbf{O})$. More is true: $G_2(\mathbf{C})$ preserves the system $\mathcal{L}$. This follows immediately from the facts that $G_2(\mathbf{C})$ preserves a differential system on $\tilde{G}(2, \operatorname{Im}\mathbf{O})$ whose pull backs to $G_2(\mathbf{C})$ are linear combinations of $\{\eta^1, \kappa_1^2, \kappa_1^3\}$ (this in turn is obvious from the structure equations) and that when we restrict the forms $\{\eta^1\, \kappa_1^2, \kappa_1^3\}$ to G_2, we get $\{\theta^1, \kappa_1^2, \kappa_1^3\}$.

More algebraically, we can define $L_+ \subseteq T\tilde{G}(2, \operatorname{Im}\mathbf{O})$ as follows: If $v \in TG_2(\mathbf{C})$ satisfies $\eta^1(v) = \kappa_1^2(v) = \kappa_1^3(v) = 0$, then $d[f_1](v) \in L_+$, but we also have

$$d[f_1](v) \equiv -g_2\theta^3(v) + g_3\theta^2(v) \mod f_1.$$

The multiplication table shows that the three-plane $f_1 \wedge g_2 \wedge g_3$ is exactly the kernel of right multiplication by f_1 in $\operatorname{Im}\mathbf{O_C}$. Thus $L_+ = g_2 \wedge g_3 \operatorname{mod} f_1 \subseteq T\tilde{G}(2, \operatorname{Im}\mathbf{O})$ shows that L_+ (and hence $\mathcal{L}$) is an algebraically defined subbundle of $T\tilde{G}(2, \operatorname{Im}\mathbf{O})$ (in the holomorphic category) via complex octonionic multiplication. Since $G_2(\mathbf{C})$ acts as algebra automorphisms of $\mathbf{O_C}$, this gives us another proof that $G_2(\mathbf{C})$ leaves $\mathcal{L}$ invariant. (Cartan in [6], proves a striking converse: The pseudo-group of bi-holomorphic transformations of $\tilde{G}(2, \operatorname{Im} \mathbf{O})$ which preserve $\mathfrak{L}$ is *exactly* the pseudo-group generated by the action of $G_2(\mathbf{C})$. This is the complex analogue of Proposition 4.1, but it is much harder to prove. We refer the interested reader to [6].)

From our point of view, it will now be necessary to display $\mathfrak{L}$ explicitly on an affine coordinate chart on $\tilde{G}(2, \operatorname{Im} \mathbf{O})$ in order to construct integrals. The affine coordinate pieces are described as follows: Let $z \in \tilde{G}(2, \operatorname{Im} \mathbf{O})$ be given and let $\mathbf{P}^5_{z_0} \subseteq \mathbf{P}^6$ be the tangent projective to $\tilde{G}(2, \operatorname{Im} \mathbf{O})$ at z_0. It is easy to see that $\mathbf{P}^5_{z_0} \cap \tilde{G}(2, \operatorname{Im} \mathbf{O}) = V^4_{z_0}$ is a singular 4-quadric in $\mathbf{P}^5_{z_0}$. We let $A^5_{z_0} = \tilde{G}(2, \operatorname{Im} \mathbf{O}) - V_{z_0}$. It is easy to see that $A^5_{z_0} \simeq \mathbf{C}^5$ analytically. The following theorem is due to É. Cartan and we only sketch a proof below. See [6] for details.

Theorem 4.9. *There exist coordinates* (ζ, w, w_1, w_2, z): $A^5_{z_0} \to \mathbf{C}^5$ *which are bi-rational and so that the differential system* $\mathfrak{L}$ *restricted to* $A^5_{z_0}$ *has a holomorphic basis of the form*

$$dw - w_1 d\zeta, \quad dw_1 - w_2 d\zeta, \quad dz - (w_2)^2 d\zeta.$$

Sketch of proof. Let $S^5 \subseteq G_2(\mathbf{C})$ be the subgroup which is connected and which satisfies $0 = \kappa^1_1 = \kappa^1_2 = \kappa^2_3 = \kappa^2_2 = \kappa^2_3 = \kappa^3_2 = \theta^1 = \eta^2 = \eta^3$ when these forms are restricted to S^5. Then S^5 has complex dimension 5 and the remaining forms $\{\theta^2, \theta^3, \eta^1, \kappa^2_1, \kappa^3_1\}$ form a basis for the holomorphic left invariant forms. The remaining structure equations satisfy

$$d\kappa^2_1 = 3\theta^2 \wedge \eta^1, \quad d\kappa^3_1 = 3\theta^3 \wedge \eta^1, d\eta^1 = -2\theta^2 \wedge \theta^3, \quad d\theta^2 = d\theta^3 = 0.$$

Using first, θ^2 and θ^3, then η^1, and then κ^2_1 and κ^3_1, we see that there exist unique coordinates x_2, x_3, y, z_2, z_3 on S^5 (centered at the identity) satisfying

$$\theta^2 = dx_2, \quad \theta^3 = dx_3, \quad \eta^1 = dy - x_2 dx_3 + x_3 dx_2,$$

$$\kappa^2_1 = dz_2 + 3x_2 dy - \tfrac{3}{2}(x_2)^2 dx_3, \quad \kappa^3_1 = dz_3 + 3x_3 dy + \tfrac{3}{2}(x_3)^2 dx_2.$$

One can verify that the function $[f_1]$: $G_2(\mathbf{C}) \to \tilde{G}(2, \operatorname{Im} \mathbf{O})$ restricts to S^5 to give a bi-rational map

$$\mathbf{C}^5 \simeq S^5 \leftrightarrow A^5_{\bar{E}_1}$$

(with respect to the natural algebraic structure on $\tilde{G}(2, \operatorname{Im} \mathbf{O})$ and that on $\mathbf{C}^5$ induced by the $\{x_2, x_3, y, z_2, z_3\}$ coordinates).

If we now set

$$\zeta = x_2, \quad w_2 = -2x_3, \quad w_1 = y - x_2 x_3,$$

$$w = \tfrac{1}{3}z_2 + x_2 y - \tfrac{1}{2}(x_2)^2 x_3,$$

$$z = -\tfrac{2}{3}z_3 + (x_3)^2 x_2,$$

then the coordinates (ζ, w, w_1, w_2, z) are clearly bi-rationally equivalent to (x_2, x_3, y, z_2, z_3). Moreover, elementary calculation shows that the forms listed in the theorem span $\mathcal{L}$. q.e.d.

An immediate corollary of this theorem is that the integrals of $\mathcal{L}$ can be written (locally) in the form

$$w = f(\zeta), \quad w_1 = f'(\zeta) \quad (= df/d\zeta),$$
$$w_2 = f''(\zeta), \quad z = \int (f''(\zeta))^2 \, d\zeta,$$

where f is an arbitrary holomorphic function of ζ.

With this in mind, we now prove:

Theorem 4.10. *Given any Riemann surface M and any integer r, there exists a complex curve ϕ: $M \to S^6$ with* $\mathrm{II} \neq 0$, $\mathrm{III} \equiv 0$ *and with* $\deg R \geqslant r$ *where R is the ramification divisor of ϕ.*

Proof. Let M have genus g, and let f be a meromorphic function on M with a single pole of order m at p_0 and *simple* zeros $p_1, \cdots, p_m$. Thus the divisor of f is of the form $-mp_0 + p_1 + \cdots + p_m$ (where the p_α are necessarily distinct). Consider the differential df. Its divisor is of the form $(df) = -(m+1)p_0 + D$ where D is an effective divisor, $D = \Sigma_i a_i q_i$ $(a_i > 0)$ and $\deg D = 2g + m - 1 = \Sigma_i a_i$. (Note that $D = 0$ implies $M = \mathbf{P}^1$ and $m = 1$.) Let $\mathcal{L}(Np_0 - 6D)$ be the (finite dimensional) vector space of meromorphic functions on M with a pole of order at most N at p_0 and a zero divisor effectively containing $6D$. By Riemann-Roch, for N sufficiently large we have

$$l(Np_0 - 6D) = \dim_{\mathbf{C}} \mathcal{L}(Np_0 - 6D) = N - C \qquad (N \gg 0),$$

where C is a constant depending on the genus of M and the degree of D. For $h \in \mathcal{L}(Np_0 - 6D)$, the ratio dh/df represents an element of $\mathcal{L}((N-m)p_0 - D')$ where $D' = \Sigma_i (5a_i - 1) q_i \geqslant 0$. (If $D = 0$, then we set $D' = 0$.) Furthermore, $d^2h/df^2 = d(dh/df)/df$ represents an element of $\mathcal{L}((N - 2m)p_0 - D'')$ where $D'' = \Sigma_i (4a_i - 2) q_i$. Now consider the differential

$$\omega(h) = \left(\frac{d^2 h}{df^2} \right)^2 df.$$

This differential has only one pole (at p_0) so it has no residues. Let $\{\gamma_s \mid s = 1, \cdots, 2g\}$ be a basis of $H_1(M, \mathbf{R})$ and consider the quadratic forms Q_s on $\mathcal{L}(Np_0 - 6D)$ defined by

$$Q_s(h) = \int_{\gamma_s} \omega(h) = \langle \omega(h), \gamma \rangle.$$

The necessary and sufficient condition that $\omega(h)$ be expressible in the form $d\tilde{h} = \omega(h)$ is that $Q_s(h) = 0$ for all s. We let

$$H_N = \{h \in \mathcal{L}(Np_0 - 6D) \mid Q_s(h) = 0 \text{ for all } s\}.$$

For definiteness, let us set

$$\tilde{h}(p) = \int_{p_1}^{p} \omega(h)$$

for all $h \in H_N$. It follows that the curve $\phi \colon M - \{p_0\} \to \mathbf{C}^5$ given by

$$\zeta = f, \quad w = h, \quad w_1 = dh/df,$$
$$w_2 = d^2h/df^2, \quad z = \tilde{h}$$

is an integral of

$$dw - w_1 d\zeta, \quad dw_1 - w_2 d\zeta, \quad dz - (w_2)^2 d\zeta,$$

and it ramifies exactly over the divisor D. (Clearly this is the largest ramification since this is where df vanishes. Our choice is such that dh vanishes at least on $6D$, $d(dh/df)$ vanishes at least on D'', $d(d^2h/df^2)$ vanishes at least on $D''' = \Sigma_i(4a_i - 3)q_i \geqslant D$, while $d\tilde{h}$ clearly vanishes over D.)

By definition of H_N, its codimension in $\mathcal{L}(Np_0 - 6D)$ is *at most* $2g$, so we get $\dim H_N \geqslant N - C - 2g$. In particular, for $N \gg 0$, $H_N \neq (0)$. Finally, again by Riemann-Roch, the map $\mathcal{L}(Np_0 - 6D) \to \mathbf{C}^m$ given by $h \mapsto (h(p_1), \cdots, h(p_m))$ is *surjective* for sufficiently large N. Thus, for large enough N, we may assume that we can choose $h \in H_N$ so that $h(p_\alpha) \neq h(p_\beta)$ for $\alpha \neq \beta$. For such $\phi \colon M - \{p_0\} \to \mathbf{C}^5$, the curve is generically 1-1 and hence does not multiply cover its image. Since the functions f, h, dh/df, d^2h/df^2, $\tilde{h}$ have only one pole at p_0, it follows that they are algebraically related, so that $\phi(M - \{p_0\})$ is an algebraic curve in $\mathbf{C}^5$. Composing this with the bi-rational mapping $\mathbf{C}^5 \to \tilde{G}(2, \operatorname{Im}\mathbf{O}) = Q_5$ of Theorem 4.9 gives a map $\Phi \colon M^2 \to \tilde{G}(2, \operatorname{Im}\mathbf{O})$ which is generically 1-1, ramifies at least over D (it may also ramify over p_0), and is an integral of $\mathcal{L}$. It follows that the projection $\tilde{\phi} \colon M^2 \to \tilde{G}(2, \operatorname{Im}\mathbf{O}) \to S^6$ is a complex curve in S^6 which ramifies over D (and $\{p_0\}$ possibly). We now want to show that $\tilde{\phi} \colon M^2 \to S^6$ is locally 1-1. If $\mathrm{II} \neq 0$, then this is no problem since then we have $b_{\tilde{\phi}} = \Phi$ so if $\tilde{\phi}$ were a ramified covering of its image, Φ would be also (but we know it is not). Thus, we need only consider the case where $\mathrm{II} = 0$. If $\mathrm{II} = 0$, by the proof of Theorem 4.7, we see that when we adapt frames for the map $\Phi \colon M^2 \to \tilde{G}(2, \operatorname{Im}\mathbf{O})$ so that $\theta^2 = 0$, then we must also have $\kappa_3^2 = 0$. Thus Φ has a local lifting to G_2 as an integral of the system

$$\kappa_1^2 = \kappa_1^3 = \kappa_3^2 = \theta^1 = \theta^2 = 0$$

(plus conjugates). This system is completely integrable on G_2 and its 10 parameter family of integrals drops to $\tilde{G}(2, \operatorname{Im} \mathbf{O})$ as a 10 parameter family of linear $\mathbf{P}^1$'s in $\tilde{G}(2, \operatorname{Im} \mathbf{O})$. Thus $\mathrm{II} \equiv 0$ implies that $\Phi\colon M \to \tilde{G}(2, \operatorname{Im} \mathbf{O})$ is a generically 1-1 covering of a $\mathbf{P}^1$, i.e. $M = \mathbf{P}^1$ and Φ is unramified. Thus, if we make $m > 0$, $\tilde{\phi}\colon M^2 \to S^6$ is a generically 1-1 complex curve in S^6 of genus g and ramification at least $\deg D = 2g + m - 1$. Clearly we can make the ramification divisor R have degree greater than r by simply choosing m sufficiently large.

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