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Submanifolds of a Lorentzian Para-Sasakian Manifold

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Abstract. Recently, Matsumoto [1] introduced the notion of Lorentzian para-contact structure and studied its several properties. The object of the present paper is to study the submanifolds of Lorentzian para-Sasakian manifolds.

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1. Introduction

Let $\bar{M}$ be a real n-dimensional manifold of class C^∞ endowed with an endomorphism ϕ of the tangent boundle, a C^∞-vector field ξ which is called the structure vector field, a 1-form η and a Lorentzian metric g with signature $(-,+,+,+)$ satisfy

$$\phi^2 X = X + \eta(X)\xi, \qquad \eta(\xi) = -1, \tag{1.1}$$

$$g(\phi X, \phi Y) = g(X,Y) + \eta(X)\eta(Y), \qquad g(X,\xi) = \eta(X) \tag{1.2}$$

for any X, $Y \in T\bar{M}$, where $T\bar{M}$ is the tangent bundle of $\bar{M}$. Then such a structure (ϕ, ξ, η, g) is termed as Lorentzian para-contact structure and the manifold $\bar{M}$ with a Lorentzian para-contact structure is called a Lorentzian para-contact manifold [l].

Also, in a Lorentzian para-contact structure the following relations hold:

$$\phi\xi = 0, \qquad \eta\circ\phi = 0, \quad \operatorname{rank}(\phi) = n-1. \tag{1.3}$$

A Lorentzian para-contact manifold M is called a Lorentzian para-Sasakian (LP-Sasakian) manifold if

$$(\bar{\nabla}_X \phi)(Y) = g(X,Y)\xi + \eta(Y)X + 2\eta(X)\eta(Y)\xi, \tag{1.4}$$

and

$$\bar{\nabla}_X \xi = \phi X \tag{1.5}$$

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for any $X, Y \in T\bar{M}$, where $\bar{\nabla}$ is the Riemannian connection with respect to g. Again, if we put $\Omega(X,Y) = g(X, \phi Y)$, then Ω is a symmetric (0,2) tensor field [1]. Thus we have from (1.5)

$$\Omega(X,Y) = (\bar{\nabla}_X \eta)Y. \tag{1.6}$$

Also, from (1.4), it follows that

$$(\bar{\nabla}_Z \Omega)(X,Y) = g(X, (\bar{\nabla}_Z \phi)Y) = g((\bar{\nabla}_Z \phi)X, Y), \tag{1.7}$$

$$(\bar{\nabla}_Z \Omega)(X,Y) = g(Y,Z)\eta(X) + g(X,Z)\eta(Y) + 2\eta(X)\eta(Y)\eta(Z) \tag{1.8}$$

for any $X, Y, Z \in T\bar{M}$.

LP-Sasakian manifolds have also been studied by K. Matsumoto and Mihai [2], Mihai and Rosca [3] and Matsumoto, De and Shaikh [4]. Let M be a Riemannian submanifold of a semi-Riemannian manifold $\bar{M}$. Then the Gauss and Weingarten formulae are given by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X,Y), \tag{1.9}$$

$$\bar{\nabla}_X N = -A_N X + \nabla^{\perp}_X N \tag{1.10}$$

for each $X, Y \in TM$ and each $N \in T^{\perp}M$, where ∇ is the Levi-Civita connection on M, $\nabla^{\perp}$ is the normal connection on the normal bundle $T^{\perp}M$, h is the second fundamental form of M and A_N is the shape operator with respect to the normal section N. Then we know that

$$g(h(X,Y), N) = g(A_N X, Y) \tag{1.11}$$

for each $X, Y \in TM$ and $N \in T^{\perp}M$. We denote by the same symbol g both metrics on $\bar{M}$ and M.

Definition 1.1. *A submanifold M is said to be*
(i) totally geodesic in $\bar{M}$ if

$$h = 0 \quad \text{or equivalently} \quad A_N = 0 \tag{1.12}$$

for any $N \in T^{\perp}M$.
(ii) minimal in $\bar{M}$ if the mean curvature vector H satisfies

$$H \overset{\text{def}}{=} \frac{Tr(h)}{dimM} = 0 \tag{1.13}$$

and (iii) totally umbilical if

$$h(X,Y) = g(X,Y)H. \tag{1.14}$$

In Section 2 of this paper, we obtain some properties of submanifolds of an LP-Sasakian manifold. In the last section, we prove the main result concerning the non-existence of an anti-invariant distribution on the submanifold of an LP-Sasakian manifold which include the results of [5] as particular case. Sub-manifolds of an LP-Sasakian manifold have been studied by two of the present authors [5], Prasad [6], Kalpana and Guha [7] and others. Throughout this paper, we shall assume the following notation:

(a) M is a submanifold of an LP-Sasakian manifold $\bar{M}$,
(b) $\{\xi\}$ is the 1- dimensional distribution spanned by ξ,
(c) TM and $T^{\perp}M$ are the tangent and normal bundles of M, respectively, and
(d) $D^{\perp}$ is an anti- invariant distribution (i.e. $\phi D^{\perp} \subset T^{\perp}M$) of M such that $D^{\perp} \cap \{\xi\} = 0$.

2. Submanifolds of an LP-Sasakian manifold

We first prove the following lemma:

Lemma 2.1. *For a submanifold M of an LP-Sasakian manifold $\bar{M}$, we have*

$$\phi X = \nabla_X \xi + h(X, \xi), \quad \xi \in TM, \tag{2.1}$$

$$\phi X = -A_\xi X + \nabla^{\perp}_X \xi, \quad \xi \in T^{\perp}M, \tag{2.2}$$

$$\eta(A_N X) = 0, \quad \xi \in T^{\perp}M, \tag{2.3}$$

$$\eta(A_N X) = g(\phi X, N), \quad \xi \in TM \tag{2.4}$$

for $X \in TM$ and $N \in T^{\perp}M$.

Proof. From (1.5) and (1.9), we get (2.1). Also, from (1.5) and (1.10), we obtain (2.2). Again, in view of (1.2), (2.3) is obvious. Lastly, for $\xi \in TM$, we get

$$\eta(A_N X) = g(\xi, A_N X) = -g(\xi, \bar{\nabla}_X N) = g(\bar{\nabla}_X \xi, N) = g(\phi X, N),$$

where (1.2), (1.5) and (1.10) have been used. These complete the proof of our lemma. □

Theorem 2.1. *Let M be a submanifold of an LP-Sasakian manifold $\bar{M}$ such that the structure vector field ξ is tangent to M. Then M is invariant (i.e. $\phi TM \subset TM$) if and only if $h(X, \xi) = 0$, and M is anti-invariant (i.e. $\phi TM \subset T^{\perp}M$) if and only if $\nabla_X \xi = 0$.*

Since it is trivial from (2.1), we omit to prove our theorem.

Theorem 2.2. *If M is a totally umbilical submanifold of an LP-Sasakian manifold $\bar{M}$ such that the structure vector field ξ is tangent to M, then*

(i) M is necessarily minimal and consequently totally geodesic and
(ii) M is an invariant submanifold of $\bar{M}$ and $\nabla_X \xi \neq 0$.

Proof. Let M be totally umbilical. Using (l.2), (1.3) and (2.1) in (1.14), we get

$$0 = h(\xi, \xi) = g(\xi, \xi)H = -H.$$

Hence, in view of (1.13) and (1.14), we obtain (i).
The second part follows from Theorem 2.1 and the above (i). □

Theorem 2.3. *A submanifold M of an LP-Sasakian manifold $\bar{M}$ such that the structure vector field ξ is normal to M is anti-invariant in $\bar{M}$ if and only if $A_\xi X = 0$. Consequently, if M is totally geodesic, then it is anti-invariant.*

Proof. Since ξ is normal to M, by virtue of (1.10) and (2.2), we get

$$g(\phi X, Y) = -g(A_\xi X, Y) = -g(h(X,Y),\xi), \quad X, Y \in TM,$$

which provides the proof of our theorem. □

3. Non-existence of an anti-invariant distribution

First, we prove the following:

Lemma 3.1. *For a submanifold M of an LP-Sasakian manifold $\bar{M}$, we have*

$$(\bar{\nabla}_Z \Omega)(X,Y) = -g(A_{\phi Y} X, Z) - \Omega(X, \nabla_Z Y) - \Omega(X, h(Z,Y)) \tag{3.1}$$

for $Y \in D^\perp$, $X, Z \in TM$.

$$(\bar{\nabla}_Z \Omega)(X,Y) = -g(A_{\phi X} Y + A_{\phi Y} X, Z) \tag{3.2}$$

for $X, Y \in D^\perp$, $Z \in TM$.

Proof. Let $Y \in D^\perp$, $Z \in TM$. Then, by virtue of (1.11) and the fact $\phi Y \in T^\perp M$, we get

$$(\bar{\nabla}_Z \phi) Y = -A_{\phi Y} Z + \nabla^\perp_Z \phi Y - \phi(\bar{\nabla}_Z Y). \tag{3.3}$$

Using this equation in (1.7), we can easily derive (3.1).

Next, in the special case of $X \in D^\perp$, since $\phi X \in T^\perp M$, (3.1) in view of (1.5) and (1.11) yields (3.2). □

Lemma 3.2. *Let M be a submanifold of an LP-Sasakian manifold $\bar{M}$ and $D^\perp \perp \{\xi\}$. Then we get*

$$(\bar{\nabla}_Z \Omega)(X,X) = 0 \tag{3.4}$$

for $X \in D^\perp$ and $Z \in TM$. And consequently

$$A_{\phi X} X = 0 \tag{3.5}$$

for $X \in D^\perp$.

Proof. Since $D^\perp \perp \{\xi\}$, we have $\eta(X) = 0$ for any $X \in D^\perp$ and hence in view of (1.8), we get (3.4). Again (3.5) follows from (3.2) and (3.4). □

Theorem 3.1. *There does not exist any anti-invariant distribution $D^\perp$ on a submanifold M of an LP-Sasakian manifold $\bar{M}$ if ξ is tangent to M and $D^\perp \perp \{\xi\}$.*

Proof. Since $D^\perp \perp \{\xi\}$, we get $\eta(X) = 0$ for any $X \in D^\perp$. Thus, from (1.2), (2.4) and (3.5), we have

$$0 = \eta(A_{\phi X} X) = g(\phi X, \phi X) = g(X, X)$$

for any $X \in D^\perp$. This means $D^\perp = \{0\}$.
This proves the theorem. □

A submanifold M of an LP-Sasakian manifold $\bar{M}$ is said to be semi-invariant submanifold [8] if the following conditions are satisfied

(i) $TM = D \oplus D^{\perp} \oplus \{\xi\}$, where D, $D^{\perp}$ are orthogonal differentiable distributions on M and $\{\xi\}$ is the 1-dimensional distribution spanned by ξ,
(ii) The distribution D is invariant by ϕ, that is, $\phi D_x = D_x$ for each $x \in M$,
(iii) The distribution $D^{\perp}$ is anti-invariant under ϕ, that is $\phi D^{\perp} \subset T_x M^{\perp}$ for each $x \in M$.

If both the distribution D and $D^{\perp}$ are non-zero then the semi-invariant submanifold is called a proper semi-invariant submanifold.

Hence by virtue of Theorem 3.1, we have the following:

Corollary 3.1. *An LP-Sasakian manifold does not admit any proper semi-invariant submanifold.*

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References

[1] K. Matsumoto, On Lorentzian para contact manifolds, *Bull. Yamagata Univ. Nat. Sci.* **12** (1989), 151–156.
[2] K. Matsumoto and I. Mihai, On a certain transformation in a Lorentzian para-Sasakian manifold, *Tensor, N. S.* **47** (1988), 189–197.
[3] I. Mihai and R. Rosca, On Lorentzian P-Sasakian manifolds, *Classical Analysis, World Scientific Publi., Singapore* (1992), 155–169.
[4] K. Matsumoto, U. C. De and A. A. Shaikh, On Lorentzian Para-Sasakian manifolds, *Rendiconti del seminario Mathematico di Messina Serie II supplemento al n.* **3** (1999).
[5] U. C. De, A. A. Shaikh, Non-existence of proper semi- invariant submanifolds of a Lorentzian para-Sasakian manifold, *Bull. Malays. Math. Soc.(2)* **22** (1999), 179–183.
[6] B. Prasad, Semi-invariant submanifolds of a Lorentzian Para-Sasakian manifold, *Bull. Malays. Math. Soc. (2)* **21** (1998), 21–26.
[7] Kalpana and G. Guha, Semi-invariant submanifolds of a Lorentzian Para-Sasakian manifold, *Ganit, J. Bangladesh Math. Soc.* **13** (1993), 71–76.
[8] A. Bejancu and N. Papaghiuc, Semi-invariant submanifold of a Sasakian manifold, *An. Stiint. Univ. "Al. I. Cuza" Iasi* **27** (1981), 163–170.