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SUBMANIFOLDS OF A RIEMANNIAN MANIFOLD WITH SEMISYMMETRIC METRIC CONNECTIONS

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ABSTRACT. We derive the Gauss curvature equation and the Codazzi-Mainardi equation with respect to a semisymmetric metric connection on a Riemannian manifold and the induced one on a submanifold. We then generalize the theorema egregium of Gauss.

1. **Introduction.** K. Yano [6] proved that a Riemannian manifold admits a semisymmetric metric connection with vanishing curvature tensor if and only if the manifold is conformally flat. Later, T. Imai [3], [4] studied some properties of hypersurfaces of a Riemannian manifold with a semisymmetric metric connection, and also obtained the Gauss curvature equation and the Codazzi-Mainardi equation with respect to a semisymmetric metric connection on a Riemannian manifold and the induced one on a hypersurface. The object of this paper is to derive the above two equations with respect to a semisymmetric metric connection on an $(n + p)$-dimensional Riemannian manifold and the induced one on an n-dimensional submanifold, and also to generalize the theorema egregium of Gauss. The notation of [5] will be used for the most part.

2. **Gauss equation and Weingarten equation.** Let M be an n-dimensional Riemannian manifold isometrically imbedded in an $(n + p)$-dimensional Riemannian manifold M'. We denote by g the Riemannian metric tensor on M' as well as the induced one on M. Since M has codimension p we can locally choose p cross sections ξ_i, $i = 1, 2, \ldots, p$, of the normal bundle $T(M)^{\perp}$ of M in M' which are orthonormal at each point of M.

A linear connection $\mathring{\nabla}'$ on M' is called a semisymmetric metric connection if $\mathring{\nabla}'g = 0$ (metric) and the torsion tensor $\mathring{T}'$ of $\mathring{\nabla}'$ satisfies $\mathring{T}'(X', Y') = \pi(Y')X' - \pi(X')Y'$ (semisymmetric) for X', $Y' \in \mathfrak{X}(M')$, where π is a 1-form on M' [6].

We now assume that a semisymmetric metric connection $\mathring{\nabla}'$ is given on M' by

$$\mathring{\nabla}'_{X'} Y' = \nabla'_{X'} Y' + \pi(Y')X' - g(X', Y')P' \tag{1}$$

for X', $Y' \in \mathfrak{X}(M')$, where ∇' denotes the Riemannian connection with respect to g and P' a vector field on M' defined by $g(P', X') = \pi(X')$ for X'

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$\in \mathfrak{X}(M')$ [6]. On M we define a vector field P and real-valued functions λ_i, $i = 1, 2, \ldots, p$, by decomposing P' into its unique tangential and normal components, thus

$$P' = P + \sum_{i=1}^{p} \lambda_i \xi_i. \tag{2}$$

If we denote by ∇ the induced Riemannian connection on M from ∇' on M', then we have the Gauss equation (with respect to ∇'),

$$\nabla'_X Y = \nabla_X Y + \sum_{i=1}^{p} h_i(X, Y)\xi_i \tag{3}$$

for X, $Y \in \mathfrak{X}(M)$, where h_i are the second fundamental forms on M [5, pp. 10–21]. Let a connection $\mathring{\nabla}$ on M be induced from the semisymmetric metric connection $\mathring{\nabla}'$ on M' by the equation which may be called the Gauss equation with respect to $\mathring{\nabla}'$,

$$\mathring{\nabla}'_X Y = \mathring{\nabla}_X Y + \sum_{i=1}^{p} \mathring{h}_i(X, Y)\xi_i \tag{4}$$

for X, $Y \in \mathfrak{X}(M)$, where $\mathring{h}_i$ are tensors of type (0,2) on M.

From (1), using (2), (3) and (4), we obtain

$$\begin{aligned} \mathring{\nabla}_X Y + \sum_{i=1}^{p} \mathring{h}_i(X, Y)\xi_i &= \nabla_X Y + \sum_{i=1}^{p} h_i(X, Y)\xi_i \\ &\quad + \pi(Y)X - g(X, Y)\left(P + \sum_{i=1}^{p} \lambda_i \xi_i\right), \end{aligned} \tag{5}$$

from which we get

$$\mathring{\nabla}_X Y = \nabla_X Y + \pi(Y)X - g(X, Y)P \tag{6}$$

for X, $Y \in \mathfrak{X}(M)$, and we also have

$$\mathring{h}_i = h_i - \lambda_i g. \tag{7}$$

Using (6), we get $\mathring{\nabla}_X\{g(Y,Z)\} = (\mathring{\nabla}_X g)(Y,Z) + \nabla_X\{g(Y,Z)\}$, from which follows $(\mathring{\nabla}_X g)(Y,Z) = 0$ for X, Y, $Z \in \mathfrak{X}(M)$, i.e.,

$$\mathring{\nabla} g = 0, \tag{8}$$

and

$$\mathring{T}(X, Y) = T(X, Y) + \pi(Y)X - \pi(X)Y = \pi(Y)X - \pi(X)Y \tag{9}$$

for X, $Y \in \mathfrak{X}(M)$, where $\mathring{T}$ and T denote the torsion tensors of connections $\mathring{\nabla}$ and ∇, respectively. Then from (8) and (9) we have [4]

THEOREM 1. *The induced connection on a submanifold of a Riemannian manifold with a semisymmetric metric connection is also a semisymmetric metric connection.*

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The Weingarten equation (with respect to ∇') is given by

$$(10) \qquad \nabla'_X \xi_i = -A_i(X) + D_X \xi_i$$

for $X \in \mathfrak{X}(M)$, where A_i are tensors of type (1,1) on M and D is a (metric) connection in the normal bundle $T(M)^\perp$ with respect to the fibre metric induced from g [5, pp. 10–21].

Note that from (3) and (10) we have $h_i(X, Y) = g(Y, A_i(X))$, and thus we get

$$(11) \qquad h_i(X, Y) = g(X, A_i(Y)) = g(A_i(X), Y)$$

for $X, Y \in \mathfrak{X}(M)$ since h_i are symmetric. While from (1) and (2) we have $\mathring{\nabla}'_X \xi_i = \nabla'_X \xi_i + \lambda_i X$, which together with (10) implies $\mathring{\nabla}'_X \xi_i = -(A_i - \lambda_i I) \cdot (X) + D_X \xi_i$, where I is the identity tensor. Defining tensors $\mathring{A}_i$ of type (1,1) on M by $\mathring{A}_i = A_i - \lambda_i I$, we get a more concise expression,

$$(12) \qquad \mathring{\nabla}'_X \xi_i = -\mathring{A}_i(X) + D_X \xi_i$$

for $X \in \mathfrak{X}(M)$, which may be called the Weingarten equation with respect to $\mathring{\nabla}'$.

We obtain, using (7) and (11), the following result which will be used later [1], [5, pp. 10–21]:

LEMMA. *The induced linear transformations, also denoted by A_i and $\mathring{A}_i$, of the tangent space $T_m(M)$ at $m \in M$ defined by* (10) *and* (12) *satisfy, respectively, $h_i(X, Y) = g(A_i(X), Y)$ and $\mathring{h}_i(X, Y) = g(\mathring{A}_i(X), Y)$ for $X, Y \in T_m(M)$, and thus are symmetric with respect to g, i.e., $g(A_i(X), Y) = g(X, A_i(Y))$ and $g(\mathring{A}_i(X), Y) = g(X, \mathring{A}_i(Y))$ for $X, Y \in T_m(M)$.*

The mean curvature normal H of M (with respect to ∇) is given by $H = (1/n) \sum_{i=1}^p (\operatorname{trace} A_i)\xi_i$ [5, pp. 29–42]. We define similarly the mean curvature normal $\mathring{H}$ of M with respect to $\mathring{\nabla}$ by $\mathring{H} = (1/n) \sum_{i=1}^p (\operatorname{trace} \mathring{A}_i)\xi_i$. Let X_j, $j = 1, 2, \ldots, n$, be n orthonormal local vector fields on M. Then H and $\mathring{H}$ can be expressed as

$$(13) \quad H = \frac{1}{n} \sum_{i=1}^{p} \left\{ \sum_{j=1}^{n} h_i(X_j, X_j) \right\} \xi_i, \qquad \mathring{H} = \frac{1}{n} \sum_{i=1}^{p} \left\{ \sum_{j=1}^{n} \mathring{h}_i(X_j, X_j) \right\} \xi_i.$$

If $h_i = k_i g$, where k_i are real-valued functions on M, then M is said to be totally umbilical (with respect to ∇). Similarly, if $\mathring{h}_i = k_i g$, then M is said to be totally umbilical with respect to $\mathring{\nabla}$ [4], [7, pp. 91–93].

We get from (2), (7) and (13) the following results [4]:

THEOREM 2. *The mean curvature normal of M and that of M with respect to the semisymmetric metric connection $\mathring{\nabla}$ coincide if and only if the vector field P' is tangent to M.*

THEOREM 3. *A submanifold M of a Riemannian manifold M' is totally umbilical if and only if it is totally umbilical with respect to the semisymmetric metric connection $\mathring{\nabla}$.*

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3. **Gauss curvature equation and Codazzi-Mainardi equation.** We denote by $R'(X', Y')Z' = \nabla'_{X'}\nabla'_{Y'}Z' - \nabla'_{Y'}\nabla'_{X'}Z' - \nabla'_{[X',Y']}Z'$ for $X', Y', Z' \in \mathfrak{X}(M')$ and $R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$ for $X, Y, Z \in \mathfrak{X}(M)$ the curvature tensors of ∇' and ∇, respectively. Then the Gauss curvature equation (with respect to ∇' and ∇) is given by

$$\begin{aligned} R'(W,Z,X,Y) &= R(W,Z,X,Y) \\ &\quad + \sum_{i=1}^{p} \{h_i(X,Z)h_i(Y,W) - h_i(Y,Z)h_i(X,W)\} \end{aligned} \tag{14}$$

for $X, Y, Z, W \in \mathfrak{X}(M)$, where

$$R'(W',Z',X',Y') = g(R'(X',Y')Z', W')$$

for $X', Y', Z', W' \in \mathfrak{X}(M')$ and

$$R(W,Z,X,Y) = g(R(X,Y)Z, W)$$

for $X, Y, Z, W \in \mathfrak{X}(M)$ are, respectively, the Riemann-Christoffel curvature tensors of M' and M (with respect to ∇' and ∇), and the Codazzi-Mainardi equation (with respect to ∇' and ∇) is given by

$$\begin{aligned} R'(\xi_i,Z,X,Y) &= (\nabla_X h_i)(Y,Z) - (\nabla_Y h_i)(X,Z) \\ &\quad + \sum_{j=1}^{p} g(\{h_j(Y,Z)D_X\xi_j - h_j(X,Z)D_Y\xi_j\}, \xi_i) \end{aligned} \tag{15}$$

for $X, Y, Z \in \mathfrak{X}(M)$ [1], [5, pp. 22–29].

Next we shall find the Gauss curvature equation and the Codazzi-Mainardi equation with respect to the semisymmetric metric connections $\mathring{\nabla}'$ and $\mathring{\nabla}$. The curvature tensors of $\mathring{\nabla}'$ and $\mathring{\nabla}$ are defined, respectively, by

$$\mathring{R}'(X',Y')Z' = \mathring{\nabla}'_{X'}\mathring{\nabla}'_{Y'}Z' - \mathring{\nabla}'_{Y'}\mathring{\nabla}'_{X'}Z' - \mathring{\nabla}'_{[X',Y']}Z'$$

for $X', Y', Z' \in \mathfrak{X}(M')$ and $\mathring{R}(X,Y)Z = \mathring{\nabla}_X\mathring{\nabla}_Y Z - \mathring{\nabla}_Y\mathring{\nabla}_X Z - \mathring{\nabla}_{[X,Y]}Z$ for $X, Y, Z \in \mathfrak{X}(M)$. Then from (4), (9) and (12) we get

$$\begin{aligned} \mathring{R}'(X,Y)Z &= \mathring{R}(X,Y)Z + \sum_{i=1}^{p} \{\mathring{h}_i(X,Z)\mathring{A}_i(Y) - \mathring{h}_i(Y,Z)\mathring{A}_i(X)\} \\ &\quad + \sum_{i=1}^{p} \{(\mathring{\nabla}_X\mathring{h}_i)(Y,Z) - (\mathring{\nabla}_Y\mathring{h}_i)(X,Z) \\ &\qquad\qquad + \mathring{h}_i(\pi(Y)X - \pi(X)Y, Z)\}\xi_i \\ &\quad + \sum_{i=1}^{p} \{\mathring{h}_i(Y,Z)D_X\xi_i - \mathring{h}_i(X,Z)D_Y\xi_i\} \end{aligned} \tag{16}$$

for $X, Y, Z \in \mathfrak{X}(M)$.

We define the Riemann-Christoffel curvature tensors of M' and M with respect to $\mathring{\nabla}'$ and $\mathring{\nabla}$, respectively, by $\mathring{R}'(W',Z',X',Y') = g(\mathring{R}'(X',Y')Z', W')$ and $\mathring{R}(W,Z,X,Y) = g(\mathring{R}(X,Y)Z, W)$ for $X', Y', Z', W' \in \mathfrak{X}(M')$ and $X, Y, Z, W \in \mathfrak{X}(M)$. Then from (16) and the Lemma, we obtain the Gauss curvature equation with respect to $\mathring{\nabla}'$ and $\mathring{\nabla}$ [4]:

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$$\mathring{R}'(W,Z,X,Y) = \mathring{R}(W,Z,X,Y) + \sum_{i=1}^{p} \{\mathring{h}_1(X,Z)\mathring{h}_i(Y,W) - \mathring{h}_i(Y,Z)\mathring{h}_i(X,W)\} \tag{17}$$

for X, Y, Z, $W \in \mathfrak{X}(M)$. We also have from (16) the Codazzi-Mainardi equation with respect to $\mathring{\nabla}'$ and $\mathring{\nabla}$ [4]:

$$\mathring{R}'(\xi_i,Z,X,Y) = (\mathring{\nabla}_X\mathring{h}_i)(Y,Z) - (\mathring{\nabla}_Y\mathring{h}_i)(X,Z) + \mathring{h}_i(\pi(Y)X - \pi(X)Y,Z) + \sum_{j=1}^{p} g(\{\mathring{h}_j(Y,Z)D_X\xi_j - \mathring{h}_j(X,Z)D_Y\xi_j\},\xi_i) \tag{18}$$

for X, Y, $Z \in \mathfrak{X}(M)$.

Now we suppose the Riemannian manifold M' is conformally flat and that the submanifold M is totally umbilical, then we can assume $\mathring{R}' = 0$ [6], and we also have $\mathring{h}_i = k_i g$, since M is totally umbilical with respect to $\mathring{\nabla}$ by Theorem 3. Then from (17) we get

$$\mathring{R}(W,Z,X,Y) = \{g(Y,Z)g(X,W) - g(X,Z)g(Y,W)\}\sum_{i=1}^{p}(k_i)^2 \tag{19}$$

for X, Y, Z, $W \in \mathfrak{X}(M)$, which implies that M is also conformally flat ($n > 3$) [3]. Thus we have [4]

THEOREM 4. *A totally umbilical submanifold in a conformally flat Riemannian manifold is conformally flat.*

4. **Theorema egregium.** We obtain a generalization of the theorema egregium of Gauss with respect to semisymmetric metric connection by the method of N. Hicks [1].

From the Gauss curvature equation (17) and the Lemma, we get

$$\mathring{R}'(X,Y,X,Y) = \mathring{R}(X,Y,X,Y) + \sum_{i=1}^{p}\{g(\mathring{A}_i(X),Y)^2 - g(\mathring{A}_i(X),X)g(\mathring{A}_i(Y),Y)\}$$

for X, $Y \in \mathfrak{X}(M)$. Therefore we have

THEOREM 5. *Let $\mathcal{P}$ be a 2-dimensional subspace of $T_m(M)$, and let $\mathring{R}'(\mathcal{P})$ and $\mathring{R}(\mathcal{P})$ be, respectively, the sectional curvatures of $\mathcal{P}$ in M' and M with respect to the semisymmetric connections $\mathring{\nabla}'$ and $\mathring{\nabla}$. If X and Y form an orthonormal base of $\mathcal{P}$, then*

$$\mathring{R}'(\mathcal{P}) = \mathring{R}(\mathcal{P}) + \sum_{i=1}^{p}\{g(\mathring{A}_i(X),Y)^2 - g(\mathring{A}_i(X),X)g(\mathring{A}_i(Y),Y)\}. \tag{20}$$

As immediate consequences of Theorem 5 we get [6]

COROLLARY 1. *If* dim $M' = 3$ *and M is a surface in M', then the determinant of $\mathring{A}_i$ (where there is now only one such map) is independent of $\mathring{A}_i$ but depends only on the Riemannian metric tensor g and the semisymmetric metric connections $\mathring{\nabla}'$ and $\mathring{\nabla}$.*

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COROLLARY 2. *If M' is a conformally flat Riemannian manifold of dimension* 3 *and M is a surface in M', then there exists a semisymmetric metric connection* $\mathring{\nabla}$ *on M for which* $\det \mathring{A}_i$ *is an intrinsic invariant of M, and, when P' is tangent to M,* $\det \mathring{A}_i$ $(= \mathring{R}(\mathcal{P}))$ *is equal to* $\det A_i$ *which is the Gaussian curvature of M.*

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