Submanifolds of Complex Space Forms Admitting an Almost Contact Metric Compound Structure (*) (**).

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Dedicated to professor Eulyong Pak on his 60th birthday

Summary. – *Real hypersurfaces of an almost Hermitian manifold naturally admit an almost contact metric structure and the $(f, g, u, v, w, \lambda, \mu, \nu)$-structure is defined on submanifolds of codimension 3 of an almost Hermitian manifold. We study the so-called semi-invariant submanifolds of a complex space form with almost contact metric compound structure which is a general notion of $(f, g, u, v, w, \lambda, \mu, \nu)$-structure.*

0. – Introduction.

K. YANO and one of the present authors [11] have studied the notion of $(f, g, u, v, w, \lambda, \mu, \nu)$-structure induced in a submanifold M of codimension 3 in an almost Hermitian manifold, and studied conditions for such a structure to define an almost contact metric structure in M.

By the way, Y. TASHIRO and I.-B. KIM [11] have generalized the notion of $(f, g, u, v, w, \lambda, \mu, \nu)$-structure recently by defining the so-called metric compound structure in a submanifold of an almost Hermitian manifold.

On the other hand, the present authors [3] studied a submanifold of codimension 3 of a complex projective space admitting an almost contact metric structure. The purpose of the present paper is to devote in generalizing the intrinsic character of a submanifold of codimension 3 of a complex space form. Our main result appears in § 5, in which, by the method of Riemannian fibre bundles, we prove that an m-dimensional complete semi-invariant submanifold M of a complex projective space CP^m admitting an almost contact metric compound structure is globally isometric to $M^c_{p,q}(a, b) = \tilde{\pi}(S^{2p+1}(a) \times S^{2q+1}(b))$, where $\tilde{\pi}$ is a natural projection of a $(2m+1)$-dimensional unit sphere S^{2m+1} onto a complex projective space CP^m defined by the Hopf-fibration, (p, q) is some of $(n-1)/2$ and $a^2 + b^2 = 1$.

(*) Entrata in Redazione il 1 settembre 1984.
(**) This research was partially supported by Korean Science and Engineering Foundation Grant.
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In determining the submanifold, we quote the following theorem.

THEOREM A [9]. – *$M^c_{p,q}(a, b)$ are only complete hypersurface of a complex projective space in which the second fundamental form A commutes with the fundamental tensor f of the submersion π compatible with $\tilde{\pi}$.*

Manifolds, submanifolds, geometric objects and mappings discussed in this paper are assumed to be differentiable and of C^∞. We use throughout this paper the systems of indices as follows:

$$\varkappa,\ \mu,\ \nu,\ \lambda = 1, 2, \dots, 2m+1; \qquad \alpha, \beta, \gamma, \delta = 1, 2, \dots, n+1\,,$$
$$A, B, C, D = 1, 2, \dots, 2m; \qquad h, i, j, k = 1, 2, \dots, n\,,$$
$$w, x, y, z = 1^*, 2^*, \dots, p^*\,, \qquad n + p = 2m\,.$$

The summation convention will be used with respect to those systems of indices.

The authors would like to express here their sincere gratitude to Professor J. S. PAK who gave them many valuable suggestions to improve the paper.

1. – Preliminaries.

Let $\tilde{M}$ be a $2m$-dimensional almost Hermitian manifold covered by a system of coordinate neighborhoods $\{\tilde{U}; x^A\}$ and (F, G) the almost Hermitian structure, where F is the almost complex structure tensor and G the almost Hermitian metric tensor of $\tilde{M}$. We denote by $F_B{}^A$ and G_{CB} components of F and G respectively. Then we have

$$F_B{}^A F_C{}^B = -\,\delta_C{}^A\,, \qquad F_C{}^D F_B{}^E G_{DE} = G_{CB}\,, \tag{1.1}$$

$\delta_C{}^A$ being the Kronecker delta.

If we put the covariant components of F as

$$F_{CB} = F_C{}^A G_{BA}\,, \tag{1.2}$$

then F_{CB} is skew-symmetric with respect to the indices C and B.

Let M be an n-dimensional Riemannian manifold covered by a system of coordinate neighborhoods $\{U; x^h\}$ and immersed isometrically in $\tilde{M}$ by the immersion $i\colon M \to \tilde{M}$. We identify $i(M)$ with M itself and represent the immersion locally by

$$x^A = x^A(x^h)\,. \tag{1.3}$$

We now put $B_i{}^A = \partial_i x^A$ $(\partial_i = \partial/\partial x^i)$. Then $B_i{}^A$ are n linearly independent vectors of $\tilde{M}$ tangent to M. And denote by $C_x{}^A$ mutually orthogonal unit normal vector field of M. Then we have $G_{CB}B_i{}^B C_x{}^C = 0$ and the metric tensor of the normal bundles of M is given by $g_{zy} = G_{CB}C_z{}^C C_y{}^B = \delta_{zy}$. Therefore, vector fields $B_i{}^A$ and $C_x{}^A$ span the tangent space $T_p(\tilde{M})$ of $\tilde{M}$ at every point P of M. The metric tensor g of M induced from that of $\tilde{M}$ is given by

$$g_{ij} = G_{CB}B_j{}^C B_i{}^B \tag{1.4}$$

since the immersion is isometric.

The transforms of the tangent vectors $B_j{}^A$ and the normal vectors $C_x{}^A$ to M by F are expressed in the form

$$F_B{}^A B_j{}^B = f_j{}^h B_h{}^A + f_j{}^x C_x{}^A\,, \tag{1.5}$$

$$F_B{}^A C_x{}^B = -f_x{}^h B_h{}^A + f_x{}^y C_y{}^A\,, \tag{1.6}$$

where $f_j{}^h$ are components of a tensor field of type (1, 1), $f_j{}^x$ those of 1-form for each fixed x, $f_x{}^h$ vector field associated with $f_j{}^x$ given by $f_x{}^h = f_j{}^y g^{jh} g_{yx}$, $f_x{}^y$ function for fixed indices x and y. Putting $f_{ji} = f_j{}^h g_{hi}$, $f_{jx} = f_j{}^y g_{yx}$, $f_{xj} = f_x{}^h g_{hj}$ and $f_{xy} = f_x{}^z g_{zy}$, we can easily find

$$f_{ji} = -f_{ij}\,, \qquad f_{jx} = f_{xj}\,, \qquad f_{xy} = -f_{yx}\,. \tag{1.7}$$

Applying F to (1.5) and (1.6) respectively and using (1.1) and these expressions, we have

$$f_j{}^t f_t{}^h = -\delta_j^h + f_j{}^x f_x{}^h\,, \tag{1.8}$$

$$f_j{}^t f_t{}^y - f_j{}^x f_x{}^y = 0\,, \qquad f_x{}^t f_t{}^i + f_x{}^y f_y{}^i = 0\,, \tag{1.9}$$

$$f_y{}^z f_z{}^x = -\delta_y^x + f_y{}^t f_t{}^x\,. \tag{1.10}$$

The second equation of (1.1) and (1.4) imply

$$f_j{}^t f_i{}^s g_{ts} = g_{ji} - f_j{}^x f_{ix}\,. \tag{1.11}$$

Now, removing the almost Hermitian ambient manifold $\tilde{M}$, we suppose that an n-dimensional Riemannian manifold M admits a metric tensor g_{ji}, a tensor field $f_j{}^h$ of type (1, 1), p vector fields $f_x{}^h$, p 1-forms $f_j{}^x$ and $p(p-1)/2$ scalar fields f_{xy} satisfying the relationships (1.8)~(1.11). Such a set $(f_j{}^h, g_{ji}, f_x{}^h, f_x{}^y)$ is said to be a *metric compound structure* on M.

If we put

$$F = \begin{pmatrix} f_i{}^h & -f_x{}^h \\ f_{xi} & f_{xy} \end{pmatrix} \quad \text{and} \quad G = \begin{pmatrix} g_{ji} & 0 \\ 0 & \delta_{xy} \end{pmatrix}, \tag{1.12}$$

then the set (F, G) defines an almost Hermitian structure in the product manifold $M\times R^p$ of the manifold M with a p-dimensional Euclidean space R^p.

We suppose that M admits an almost contact metric compound structure. Then we have

$$f_j{}^t f_t{}^h = -\delta_j^h + P_j P^h , \tag{1.13}$$

$$f_j{}^t P_t = 0 , \quad f_j{}^h P^j = 0 , \tag{1.14}$$

$$P_i P^i = 1 \tag{1.15}$$

and

$$f_j{}^t f_i{}^s g_{ts} = g_{ji} - P_j P_i , \tag{1.16}$$

where P_j is a 1-form and P^h vector field associated with P_j given by $P^h = g^{hj} P_j$ on M.

In this case we know that the dimension n of M is odd and the rank of $(f_j{}^i)$ is equal to $n-1$.

Comparing (1.11) and (1.16), we have

$$f_j{}^x f_{ix} = P_j P_i . \tag{1.17}$$

This equation shows that the product of the matrix $(f_j{}^x)$ with its transpose is of rank 1 and hence the matrix $(f_j{}^x)$ by itself is of rank 1.

Therefore, we may put

$$f_j{}^x = \nu^x P_j , \tag{1.18}$$

where ν^x are certain scalar fields for each x.

Since $f_j{}^x f_x{}^j = P_j P^j = 1$, we have

$$\nu_x \nu^x = 1 \tag{1.19}$$

and hence (1.9) and (1.10) are reduced respectively to

$$f_x{}^y \nu^x = 0 , \quad \nu_y f_x{}^y = 0 \tag{1.20}$$

and

$$f_y{}^z f_z{}^x = -\delta_y^x + \nu_y \nu^x . \tag{1.21}$$

The equations (1.19)~(1.21) form an almost contact metric structure on R^p at every point of M, and consequently we see that the dimension p of R^p is odd.

Conversely, assuming that an almost contact metric structure $(f_y{}^x, g_x, \nu^x)$ on R^p is admitted, we can prove that the metric compound strucure $(f_j{}^h)\ g_{ji},\ f_x{}^h,\ f_y{}^x)$ induces an almost contact metric structure $(f_j{}^h, g_{ji}, P^h)$ on M.

Thus we have

THEOREM 1.1 ([11]). – *Let* $(f_j{}^h, g_{ji}, f_x{}^h, f_y{}^x)$ *be a metric compound structure on* M. *In order that* $f_j{}^h$ *and* g_{ji} *constitute an almost contact metric structure* $(f_j{}^h, g_{ji}, P^h)$ *on* M, *it is necessary and sufficient that* $f_y{}^x$ *and* g_{yx} *constitute an almost contact metric structure* $(f_y{}^x, g_{yx}, \nu^x)$ *on* R^p *at every point of* M.

From above discussions we also have

THEOREM 1.2 ([11]). – *In order for a metric compound structure* $(f_j{}^h, g_{ji}, f_x{}^h, f_y{}^x)$ *to be almost contact metric structure, it is necessary and sufficient that the matrix* $(f_x{}^h)$ *is of rank* 1, *that is, the* p *vector fields* $f_x{}^h$ *are all parallel to one another.*

A metric compound structure admitting an almost contact metric structure is said to be an *almost contact metric compound structure* on M.

2. – Submanifolds of codimension p of an almost Hermitian manifold.

In this section we assume that n-dimensional submanifold M of codimension p of an almost Hermitian manifold $\tilde{M}$ admits an almost contact metric compound structure $(f_j{}^h, g_{ji}, f_x{}^h, f_y{}^x)$ and consequently $(f_j{}^h, g_{ji}, P^h)$ defines an almost contact metric structure. So, (1.13)~(1.16) are valid.

The vector field N^A defined by

$$N^A = \nu^x C_x{}^A \tag{2.1}$$

is unit normal to M because $G_{CB}\, C_x{}^C\, C_y{}^B = \delta_{xy}$ and $\nu_x \nu^x = 1$.

If we transform the tangent vectors $B_i{}^A$ and the unit normal vector N^A by F, then we have

$$F_B{}^A B_i{}^B = f_i{}^h B_h{}^A + P_i N^A\,, \tag{2.2}$$

$$F_B{}^A N^B = -P^h B_h{}^A \tag{2.3}$$

respectively because of (1.18), (1.20) and (2.1).

It is well known that the submanifold M of an almost Hermitian manifold satisfying (2.2) and (2.3) is semi-invariant with respect to N^A and we call N^A the distinguished normal to M ([1], [10]).

Now, we take N^A as $C_{1^*}{}^A$. Then we have from (2.1) that $\nu^{1^*} = 1$ and $\nu^{(x)} = 0$, where here and in the sequel, (x) runs over the range $\{2^*, \ldots, p^*\}$. For the convenience of notation, we write C^A in stead of $C_{1^*}{}^A$. Then we can represent (2.2) and (2.3) respectively as follows:

$$F_B{}^A B_i{}^B = f_i{}^h B_h{}^A + P_i C^A , \tag{2.4}$$

$$F_B{}^A C^B = - P^h B_h{}^A . \tag{2.5}$$

Taking account of (1.18), (1.20) and the fact that $\nu^{1^*} = 0$ and $\nu^{(x)} = 0$, we find from (1.6)

$$F_B{}^A C_{(x)}{}^B = f_{(x)}{}^{(y)} C_{(y)}{}^A . \tag{2.6}$$

Then, by applying F to (2.6), it follows

$$f_{(x)}{}^{(y)} f_{(y)}{}^{(z)} = - \delta_{(x)}{}^{(z)} . \tag{2.7}$$

Denoting by ∇_j the operator of van der Waerden-Bortolotti covariant differentiation with respect to the fundamental tensor g_{ji}, we have the equations of Gauss for M

$$\nabla_j B_i{}^A = A_{ji} C^A + A_{ji}{}^{(x)} C_{(x)}{}^A , \tag{2.8}$$

where A_{ji} and $A_{ji}{}^{(x)}$ are the second fundamental tensors with respect to normal vector fields C^A and $C_{(x)}{}^A$ respectively, and those of Weingarten

$$\nabla_j C^A = - A_j{}^h B_h{}^A + l_j{}^{(x)} C_{(x)}{}^A , \tag{2.9}$$

$$\nabla_j C_{(x)}{}^A = - A_j{}^h{}_{(x)} B_h{}^A - l_{j(x)} C^A + l_{j(x)}{}^{(y)} C_{(y)}{}^A , \tag{2.10}$$

where $A_j{}^h = g^{hi} A_{ji}$, $A_j{}^h{}_{(x)} = g^{hi} g_{(y)(x)} A_{ji}{}^{(y)} = A_{ji(x)} g^{hi}$, $l_j{}^{(x)}$ and $l_{j(x)}{}^{(y)}$ are the third fundamental tensors, $l_{j(x)} = l_j{}^{(y)} g_{(y)(x)}$.

Putting $l_{j(x)(y)} = l_{j(x)}{}^{(z)} g_{(z)(y)}$, we can easily verify $l_{j(x)(y)} = - l_{j(y)(x)}$ since $C_{(x)}{}^A$ are mutually orthogonal.

We now assume that the ambient manifold $\tilde{M}$ is a Kaehlerian manifold, that is, $\tilde{\nabla} F = 0$, where $\tilde{\nabla}$ is a covariant differentiation in $\tilde{M}$.

Differentiating (2.4)~(2.6) covariantly and using (2.8)~(2.10) and these equations, we can easily find

$$\nabla_j f_i{}^h = - A_{ji} P^h + A_j{}^h P_i , \tag{2.11}$$

$$\nabla_j P_i = - A_{jt} f_i{}^t , \quad \nabla_j P^h = A_j{}^t f_t{}^h , \tag{2.12}$$

$$A_{ji}{}^{(x)} f_{(x)}{}^{(y)} = A_{jt}{}^{(y)} f_i{}^t + l_j{}^{(y)} P_i , \tag{2.13}$$

$$A_{jt}{}^{(x)} P^t = - l_j{}^{(y)} f_{(y)}{}^{(x)} , \tag{2.14}$$

$$\nabla_j f_{(y)}{}^{(x)} = l_{j(y)}{}^{(z)} f_{(z)}{}^{(x)} - f_{(y)}{}^{(z)} l_{j(z)}{}^{(x)} . \tag{2.15}$$

If we transvect (2.13) with $f_{(y)}{}^{(z)}$ and take account of (2.7), then we obtain

$$A_{ji}{}^{(z)} = -A_{jt}{}^{(y)} f_i{}^t f_{(y)}{}^{(z)} - l_j{}^{(y)} P_i f_{(y)}{}^{(z)}\,, \tag{2.16}$$

from which

$$A^{(z)} = -P^t l_t{}^{(y)} f_{(y)}{}^{(z)}\,, \tag{2.17}$$

where we have put $A^{(z)} = g^{ji} A_{ji}{}^{(z)}$.

Also, transvecting (2.14) with $f_{(x)}{}^{(z)}$ and using (2.7), we find

$$l_j{}^{(z)} = A_{jt}{}^{(x)} P^t f_{(x)}{}^{(z)}\,. \tag{2.18}$$

The equations of Gauss for M in a Kaehlerian manifold $\tilde{M}$ are given by

$$K_{kji}{}^h = K_{DCB}{}^A B_k{}^D B_j{}^C B_i{}^B B^h{}_A + A_k{}^h A_{ji} - A_j{}^h A_{ki} + A_k{}^h{}_{(x)} A_{ji}{}^{(x)} - A_j{}^h{}_{(x)} A_{ki}{}^{(x)}\,, \tag{2.19}$$

where $K_{DCB}{}^A$ and $K_{kji}{}^h$ are the Riemann-Christoffel curvature tensors of $\tilde{M}$ and M respectively, and we have put $B^h{}_A = B_i{}^B g^{hi} G_{AB}$.

We now assume that the ambient manifold $\tilde{M}$ is a Kaehlerian manifold of constant holomorphic sectional curvature and hence its curvature tensor has the form

$$K_{DCB}{}^A = \frac{c}{4}(\delta_D{}^A g_{CB} - \delta_C{}^A g_{DB} + F_D{}^A F_{CB} - F_C{}^A F_{DB} - 2F_{DC} F_B{}^A)\,. \tag{2.20}$$

Substituting this into (2.19) and using (1.4) and (2.4), we can see that

$$\begin{aligned} K_{kji}{}^h = \frac{c}{4}(\delta_k{}^h g_{ji} - \delta_j{}^h g_{ki} + f_k{}^h f_{ji} - f_j{}^h f_{ki} - 2f_{kj} f_i{}^h) + A_k{}^h A_{ji} - A_j{}^h A_{ki} \\ + A_k{}^h{}_{(x)} A_{ji}{}^{(x)} - A_j{}^h{}_{(x)} A_{ki}{}^{(x)}\,. \end{aligned} \tag{2.21}$$

By taking account of (2.4), (2.5), (2.6), (2.9), (2.10) and (2.20), we have the equations of Codazzi:

$$\nabla_k A_{ji} - \nabla_j A_{ki} - l_{k(x)} A_{ji}{}^{(x)} + l_{j(x)} A_{ki}{}^{(x)} = \frac{c}{4}(P_k f_{ji} - P_j f_{ki} - 2P_i f_{kj})\,, \tag{2.22}$$

$$\nabla_k A_{ji}{}^{(x)} - \nabla_j A_{ki}{}^{(x)} + l_k{}^{(x)} A_{ji} - l_j{}^{(x)} A_{ki} + l_{k(y)}{}^{(x)} A_{ji}{}^{(y)} - l_{j(y)}{}^{(x)} A_{ki}{}^{(y)} = 0\,, \tag{2.23}$$

and those of Ricci are given by

$$\nabla_j l_i{}^{(x)} - \nabla_i l_j{}^{(x)} + A_j{}^t A_{ti}{}^{(x)} - A_i{}^t A_{jt}{}^{(x)} + l_{j(y)}{}^{(x)} l_i{}^{(y)} - l_{i(y)}{}^{(x)} l_j{}^{(y)} = 0\,, \tag{2.24}$$

$$\begin{aligned} \nabla_j l_{i(x)}{}^{(y)} - \nabla_i l_{j(x)}{}^{(y)} + A_j{}^t{}_{(x)} A_{it}{}^{(y)} - A_i{}^t{}_{(x)} A_{jt}{}^{(y)} + l_{j(x)} l_i{}^{(y)} - l_{i(x)} l_j{}^{(y)} \\ + l_{j(z)}{}^{(y)} l_{i(x)}{}^{(z)} - l_{i(z)}{}^{(y)} l_{j(x)}{}^{(z)} = \frac{c}{2} f_{ij} f_{(x)}{}^{(y)}\,. \end{aligned} \tag{2.25}$$

3. – Submanifolds of a complex space form admitting an almost contact metric compound structure.

In this section, we assume that the metric compound structure induced on an n-dimensional submanifold M of codimension p of a Kaehlerian manifold of constant holomorphic sectional curvature c, that is, which is also called a complex space form, defines an almost contact metric structure. And consequently $(f_j{}^h, g_{ji}, P^h)$ defines an almost contact metric structure on M.

We now suppose that the second fundamental tensors and the structure tensor $f_j{}^h$ commute each other, that is,

$$A_j{}^t f_t{}^h - f_j{}^t A_t{}^h = 0\,, \qquad A_j{}^t{}_{(x)} f_t{}^h - f_j{}^t A_t{}^h{}_{(x)} = 0\,,$$

or, equivalently

$$A_{jt} f_i{}^t + A_{it} f_j{}^t = 0\,, \tag{3.1}$$

$$A_{jt}{}^{(x)} f_i{}^t + A_{it}{}^{(x)} f_j{}^t = 0 \tag{3.2}$$

respectively.

Transvecting (3.1) with $f_k{}^i$ and using (1.13), we get

$$A_{jt}(-\delta_k^t + P_k P^t) + A_{it} f_k{}^i f_j{}^t = 0\,,$$

from which, taking the skew-symmetric part in j and k

$$(A_{jt}P^t)P_k - (A_{kt}P^t)P_j = 0\,,$$

which shows that

$$A_{jt}P^t = \alpha P_j\,, \tag{3.3}$$

α being a scalar field given by $\alpha = A_{ji}P^jP^i$.

If we take the symmetric part of (2.16) in j and i and use (3.2), then we find

$$2A_{ji}{}^{(z)} = -(l_j{}^{(x)}P_i + l_i{}^{(x)}P_j) f_{(x)}{}^{(z)}\,. \tag{3.4}$$

Transvection P^i gives

$$l_j{}^{(z)} = (l_t{}^{(z)}P^t)P_j \tag{3.5}$$

because of (2.7) and (2.14). Therefore, (3.4) reduces to

$$A_{ji}{}^{(z)} = A^{(z)}P_jP_i \tag{3.6}$$

with the aid of (2.17).

Differentiating (3.3) covariantly and making use of (2.12), we get

$$(\nabla_k A_{jt})P^t - A_j{}^t A_{ks} f_t{}^s = (\nabla_k \alpha)P_j - \alpha A_{kt} f_j{}^t ,$$

from which, taking the skew-symmetric part with respect to the indices k and j and utilizing (2.22), (3.5) and (3.6),

$$\frac{c}{2} f_{jk} + 2A_j{}^t A_{ts} f_k{}^s = (\nabla_k \alpha)P_j - (\nabla_j \alpha)P_k + 2\alpha A_{kt} f_j{}^t \tag{3.7}$$

with the help of (1.14), (1.15) and (3.1).

If we transvect (3.7) with P^j and use (1.14), (1.15) and (3.3), then we have

$$\nabla_k \alpha = \beta P_k , \tag{3.8}$$

where $\beta = P^t \nabla_t \alpha$. Thus, (3.7) becomes

$$\frac{c}{4} f_{jk} + A_j{}^t A_{ts} f_k{}^s = \alpha A_{jt} f_k{}^t .$$

Transvection $f_i{}^k$ yields

$$A_{jt} A_i{}^t = \alpha A_{ji} + \frac{c}{4}(g_{ji} - P_j P_i)$$

with the aid of (1.13) and (3.3).

Thus we have

LEMMA 3.1. – *Let M be a semi-invariant submanifold of a complex space form with the almost contact metric compound structure. If the second fundamental tensors of M are commutative with the structure tensor f induced on M, then an eigenpolynomial of a second fundamental tensor is given by*

$$A_{jt} A_i{}^t = \alpha A_{ji} + \frac{c}{4}(g_{ji} - P_j P_i) , \tag{3.9}$$

where α is a certain scalar field on M.

We now assume that the ambient manifold is a complex projective space CP^m of real dimension $2m$. Then (3.9) reduces to

$$A_{jt} A_i{}^t = \alpha A_{ji} + g_{ji} - P_j P_i . \tag{3.10}$$

Differentiating (3.8) covariantly, it follows

$$\nabla_j \nabla_k \alpha = (\nabla_j \beta)P_k + \beta \nabla_j P_k ,$$

from which, taking the skew-symmetric part in j and k,

$$(\nabla_j\beta)P_k-(\nabla_k\beta)P_j+\beta(\nabla_jP_k-\nabla_kP_j)=0\,,$$

or, using (2.12) and (3.1),

$$(\nabla_j\beta)P_k-(\nabla_k\beta)P_j+2\beta A_{kt}f_j{}^t=0\,. \tag{3.11}$$

Transvecting this with P^k, we find

$$\nabla_j\beta=\gamma P_j$$

for some scalar field γ. Thus (3.11) becomes $\beta A_{kt}f_j{}^t=0$. This yields

$$\beta(A_{kj}-\alpha P_kP_j)=0\,. \tag{3.12}$$

Let $M_\beta=\{P\in M:\beta(P)\neq 0\}$. Then M_β is an open subset of M and $A_{kj}=\alpha P_kP_j$ on M_β. But it can never occur by considering (3.10). Thus M_β is empty and hence $\beta=0$ on M. Therefore α is a constant.

We now compute the covariant derivative of a second fundamental tensor A_{ji} with respect to the distinguished normal C^A which will be useful in § 5.

Differentiating (3.10) covariantly, we get

$$\begin{aligned}(\nabla_kA_{jt})A_i{}^t+A_j{}^t\nabla_kA_{it}-\alpha\nabla_kA_{ji}&=-(\nabla_kP_j)P_i-P_j\nabla_kP_i\\&=A_{kt}f_j{}^tP_i+A_{kt}f_i{}^tP_j\end{aligned} \tag{3.13}$$

because of (2.12), from which, taking the skew-symmetric part with respect to the indices k and j,

$$A_j{}^t\nabla_kA_{it}-A_k{}^t\nabla_jA_{it}=2A_{kt}f_j{}^tP_i+\alpha(P_kf_{ji}-P_jf_{ki})$$

with the aid of (2.22) with $c=4$, (3.1), (3.3), (3.5) and (3.6). If we exchange the indices k and i in the above equation, then we get

$$A_j{}^t\nabla_iA_{kt}-A_i{}^t\nabla_jA_{kt}=2A_{it}f_j{}^tP_k+\alpha(P_if_{jk}-P_jf_{ik})\,.$$

Substituting (2.22) with (3.5) and (3.6) into this, we find

$$A_j{}^t\nabla_kA_{it}-A_i{}^t\nabla_kA_{jt}=(A_{it}f_k{}^t)P_j-(A_{jt}f_k{}^t)P_i+\alpha(P_jf_{ik}-P_if_{jk})\,.$$

Adding (3.13) and above equation, we obtain

$$2A_j{}^t\nabla_kA_{it}-\alpha\nabla_kA_{ji}=2A_{kt}f_j{}^tP_i+\alpha(P_jf_{ik}-P_if_{jk}) \tag{3.14}$$

with the aid of (3.1). Differentiating (3.3) covariantly and making use of (2.12), (3.1) and (3.10), we have

$$(\nabla_k A_{jt})P^t = \alpha f_{jk} . \tag{3.15}$$

Transvecting (3.14) with $A_h{}^j$ and using (3.3), (3.10) and (3.15), we get

$$\alpha A_j{}^t \nabla_k A_{it} + 2\nabla_k A_{ji} = (\alpha^2 + 2) f_{ik} P_j - 2 f_{kj} P_i - \alpha A_{jt} f_k{}^t P_i . \tag{3.16}$$

Then we have from (3.14) and (3.16) that

$$\nabla_k A_{ji} = f_{ik} P_j + f_{jk} P_i .$$

Thus, we have

LEMMA 3.2. – *Let M be an $n(>1)$-dimensional semi-invariant submanifold with the distinguished normal C^A of a complex projective space CP^m admitting an almost contact metric compound structure. If the second fundamental forms are commutative with the structure tensor f induced on M, then we have*

$$\nabla_k A_{ji} = f_{ik} P_j + f_{jk} P_i . \tag{3.17}$$

4. – Submanifolds of an even-dimensional Euclidean space admitting an almost contact metric compound structure.

In this section we assume that the metric compound structure $(f_j{}^h, g_{ji}, f_j{}^x, f_y{}^x)$ induced on a submanifold M of an even-dimensional Euclidean space E^{2m} defines an almost contact metric structure $(f_j{}^h, g_{ji}, P^h)$ and the second fundamental tensors of M commute with f, that is, (3.1) and (3.2) hold. Then (2.21) $\sim$ (2.25) with $c = 0$ are valid because the ambient manifold is Euclidean. As is already shown in § 3 under the assumption (3.1) and (3.2) hold, we get

$$l_j{}^{(x)} = (l_t{}^{(x)} P^t) P_j \quad \text{and} \quad A_{ji}{}^{(x)} = A^{(x)} P_j P_i .$$

Substituting these equations into (2.22) with $c = 0$, we find

$$\nabla_k A_{ji} - \nabla_j A_{ki} = 0 . \tag{4.1}$$

Also, (3.9) reduces to

$$A_{jt} A_i{}^t = \alpha A_{ji} \tag{4.2}$$

because of $c = 0$.

Differentiating (3.8) covariantly, it follows that

$$\nabla_j \nabla_k \alpha = (\nabla_j \beta) P_k + \beta \nabla_j P_k ,$$

from which, taking the skew-symmetric part in j and k,

$$(\nabla_j \beta) P_k - (\nabla_k \beta) P_j + \beta(\nabla_j P_k - \nabla_k P_j) = 0 ,$$

or, using (2.12) and (3.1)

$$(\nabla_j \beta) P_k - (\nabla_k \beta) P_j + 2\beta A_{kt} f_j{}^t = 0 . \tag{4.3}$$

Transvecting P^k gives

$$\nabla_j \beta = \gamma P_j ,$$

γ being a certain scalar field. Thus (4.3) reduces to $\beta A_{kt} f_j{}^t = 0$. Transvecting this with $f_i{}^j$, we have

$$\beta(A_{ki} - \alpha P_k P_i) = 0 \tag{4.4}$$

with the aid of (1.13) and (3.3).

We now assume that M is locally irreducible.

Let M_t be a subset of M such that $M_t = \{P \in M\colon \beta(P) \neq 0\}$. Then M_t is an open subset of M and $A_{ji} = \alpha P_j P_i$ on M_t. By considering (2.12), P_j is parallel on M_t. It contradicts the fact that M is locally irreducible. Consequently M_t is a void set and hence β is identically zero on M. Therefore we can see from (3.8) that α is constant.

Differentiating (4.2) covariantly, we get

$$(\nabla_k A_{jt}) A_i{}^t + A_j{}^t \nabla_k A_{it} = \alpha \nabla_k A_{ji} , \tag{4.5}$$

from which, by taking the skew-symmetric part in k and j and using (4.1),

$$A_j{}^t \nabla_k A_{it} - A_k{}^t \nabla_j A_{it} = 0 . \tag{4.6}$$

Exchanging the indices k and i in (4.6), we can write down

$$A_j{}^t \nabla_i A_{kt} - A_i{}^t \nabla_j A_{kt} = 0 . \tag{4.7}$$

Remebering (4.1), we get

$$A_j{}^t \nabla_k A_{it} - A_i{}^t \nabla_k A_{jt} = 0 . \tag{4.8}$$

Adding two equations (4.5) and (4.8), we have

$$2A_j{}^t\nabla_k A_{it} = \alpha\nabla_k A_{ji}\,. \tag{4.9}$$

Transvecting this with $A_h{}^j$, we find

$$\alpha A_h{}^t\nabla_k A_{it} = 0 \tag{4.10}$$

with the aid of (4.2). Since α is constant, we think of two cases whether α is zero or not. If α is zero, $A_{ji} = 0$ on M because of (4.2). And consequently P_j is parallel along M, which is a contradiction. Then, α is a nonzero constant and hence we obtain from (4.10)

$$A_h{}^t\nabla_k A_{it} = 0\,.$$

After all, this reduces to

$$\nabla_k A_{ji} = 0 \tag{4.11}$$

with the aid of (4.2) and $\alpha \neq 0$. Since M is locally irreducible, we have

$$A_{ji} = \varrho g_{ji} \tag{4.12}$$

for a certain nonzero constant ϱ. Also, it is easily proved from (4.2) and (4.12) that $\varrho = \alpha$. Thus, (4.12) becomes

$$A_{ji} = \alpha g_{ji}\,. \tag{4.13}$$

Substitution this into (2.12) gives

$$\nabla_j P_i = \alpha f_{ji}\,. \tag{4.14}$$

On the other hand, (2.18) and (3.6) yield

$$l_j{}^{(x)} = A^{(y)} f_{(y)}{}^{(x)} P_j\,. \tag{4.15}$$

Substituting this into (2.24) and using (3.6) and (4.13), we get

$$\nabla_j(A^{(y)} f_{(y)}{}^{(x)} P_i) - \nabla_i(A^{(y)} f_{(y)}{}^{(x)} P_j) + A^{(z)} f_{(z)}{}^{(y)}(l_{j(y)}{}^{(x)} P_i - l_{i(y)}{}^{(x)} P_j) = 0\,,$$

or, using (2.15) and (4.13),

$$\{(\nabla_j A^{(y)}) P_i - (\nabla_i A^{(y)}) P_j\} f_{(y)}{}^{(x)} + A^{(y)}(l_{j(y)}{}^{(z)} f_{(z)}{}^{(x)} P_i - l_{i(y)}{}^{(z)} f_{(z)}{}^{(x)} P_j) + 2\alpha f_{ji} A^{(y)} f_{(y)}{}^{(x)} = 0\,.$$

Transvection with $f_{(x)}{}^{(w)}$ gives

$$(4.16)\qquad (\nabla_j A^{(w)})P_i - (\nabla_i A^{(w)})P_j + A^{(y)}(l_{j(y)}{}^{(w)}P_i - l_{i(y)}{}^{(w)}P_j) + 2\alpha f_{ji}A^{(w)} = 0$$

with the aid of (2.7). Transvecting this with P^i and using (1.15), we obtain

$$\nabla_j A^{(w)} = Q^{(w)}P_j + R^{(w)}P_j - A^{(y)}\, l_{j(y)}{}^{(w)}\,,$$

where we have put

$$Q^{(w)} = P^t\nabla_t A^{(w)}\,, \qquad R^{(w)} = A^{(y)} l_{t(y)}{}^{(w)} P^t\,.$$

Thus, (4.16) reduces to

$$\alpha f_{ji} A^{(w)} = 0\,.$$

This means that

$$(4.17)\qquad A^{(w)} = 0\,.$$

Therefore, we can see that M is totally umbilical by means of (4.13) and $A_{ji}{}^{(x)} = 0$.
Summing up these facts, we have

THEOREM 4.1. – *Let M be a locally irreducible complete n-dimensional semi-invariant submanifold of an even-dimensional Euclidean space E^{2m} with almost contact metric compound structure. If the second fundamental tensors commute with the structure tensor f, then M is an n-dimensional sphere S^n.*

We now assume that the normal vectors $C_{(x)}{}^A$ are parallel in the subnormal bundle spanned by $C_{(x)}{}^A$, that is, $l_{j(x)} = 0$ and $l_{j(x)}{}^{(y)} = 0$, and M does not admit a cosympletic structure. Then we can easily find that M is contained as a real hypersurface of an $(n+1)$-dimensional Euclidean space $E^{n+1} \subset E^{2m}$ by virtue of $A_{ji}{}^{(x)} = 0$ induced from (2.17) and (3.6).

On the other hand, the scalar field α defined by (3.3) is proved to be a nonzero constant by the similar method used in Theorem 3.1 by considering that M does not admit a cosymplectic structure. Also, we can prove that A_{ji} is parallel. Therefore M has two constant principal curvature 0 and α. Moreover, their multiplicities are constant. So the distributions $D_0 = \{X\colon AX = 0\}$ and $D_\alpha = \{X\colon AX = \alpha X\}$ are parallel, completely integrable, totally geodesic in M and totally umbilical in E^{m+1}. Thus we have

THEOREM 4.2. – *Let M be an n-dimensional complete semi-invariant submanifold without cosymplectic structure of an even-dimensional Euclidean space E^{2m} admitting* *an almost contact metric compound structure. If the second fundamental tensors and the structure tensor commute and the normal vectors $C_{(x)}{}^A$ are parallel in the subnormal bundle spanned by $C_{(x)}{}^A$, then M is a product of a sphere and a plane $S^r \times E^{n-r}$ $(0 < r < n)$.*

COROLLARY 4.3. – *Let M be an n-dimensional complete minimal semi-invariant submanifold of an even-dimensional Euclidean space E^{2m} admitting an almost contact metric compound structure. If the same assumption as that of Theorem 3.2 is satisfied, then M is an n-dimensional plane E^n.*

5. – Submersion $\tilde{\pi}$: $S^{2m+1} \to CP^m$ and immersion i: $M \to CP^m$.

In this section, we assume that M is an $n(>1)$-dimensional submanifold of a complex projective space CP^m. As is well known, the unit sphere S^{2m+1} is a principal circle bundle over a complex projective space CP^m, which is characterized by the Hopf-fibration $\tilde{\pi}$: $S^{2m+1} \to CP^m$. We consider a Riemannian submersion π: $\bar{M} \to M$ compatible with $\tilde{\pi}$: $S^{2m+1} \to CP^m$, $\bar{M}$ being $\tilde{\pi}^{-1}(M)$. If we speak more precisely, π: $\bar{M} \to M$ is a Riemannian submersion with totally geodesic such that the following diagram commute:

$$\begin{tikzcd} \bar{M} \arrow[r, "\tilde{i}"] \arrow[d, "\pi"'] & S^{2m+1} \arrow[d, "\tilde{\pi}"] \\ M \arrow[r, "i"] & CP^m \end{tikzcd}$$

where $\tilde{i}$: $\bar{M} \to S^{2m+1}$ and i: $M \to CP^m$ are isometric immersions. Let S^{2m+1} be covered by a system of coordinate neighborhoods $\{\hat{U} : y^\varkappa\}$ such that $\tilde{\pi}(\hat{U}) = \tilde{U}$ are coordinate neighborhoods of CP^m with local coordinate system (x^A). We then represent the projection $\tilde{\pi}$: $S^{2m+1} \to CP^m$ locally by

$$x^A = x^A(y^\varkappa) \tag{5.1}$$

and we put

$$E_\varkappa{}^A = \partial_\varkappa x^A\,, \qquad \partial_\varkappa = \partial/\partial y^\varkappa\,, \tag{5.2}$$

where the matrix $(E_\varkappa{}^A)$ has the maximal rank $2m$.

Let $\xi^\varkappa$ be the components of the unit Sasakian structure vector ξ defined on S^{2m+1}. Since ξ is the vertical vector with respect to each fibre $\tilde{\pi}^{-1}(P)$, $\forall P \in CP^m$, $\{E_\varkappa{}^A, \xi_\varkappa\}$ constitutes a local coframe in S^{2m+1}, where we have put $\xi_\varkappa = g_{\varkappa\mu}\xi^\mu$ and

$g_{\varkappa\mu}$ denotes the fundamental metric tensor of S^{2m+1}. We denote by $\{E^{\varkappa}{}_A, \xi^{\varkappa}\}$ the frame corresponding this coframe. Then we get

$$E_{\varkappa}{}^A E^{\varkappa}{}_B = \delta_B{}^A , \quad E_{\varkappa}{}^A \xi^{\varkappa} = 0 , \quad \xi_{\varkappa} E^{\varkappa}{}_A = 0 . \tag{5.3}$$

We now take coordinate neighborhoods $\{\bar{U}:y^{\alpha}\}$ of $\bar{M}$ such that $\pi(\bar{U}) = U$ are coordinate neighborhoods of M with local coordinate system (x^h). Let the isometric immersion $\tilde{i}$ and i are locally expressed by $y^{\varkappa} = y^{\varkappa}(y^{\alpha})$ and $x^A = x^A(x^h)$ respectively. The commutativity of the preceding diagram implies

$$x^A(y^{\varkappa}(y^{\alpha})) = x^A(x^h(y^{\alpha})) ,$$

where π is expressed locally by $x^h = x^h(y^{\alpha})$. Which induces

$$B_h{}^A E_{\alpha}{}^h = E_{\varkappa}{}^A B_{\alpha}{}^{\varkappa} , \tag{5.4}$$

where $B_{\alpha}{}^{\varkappa} = \partial_{\alpha} y^{\varkappa}$ and $E_{\alpha}{}^h = \partial_{\alpha} x^h$.

For each point $P \in M$, we can choose the mutually orthogonal unit normal vector fields $C_x{}^A$ defined in a neighborhood U of P such that $\{B_i{}^A, C_x{}^A\}$ generates the tangent space of CP^m at $i(P)$. Let $\bar{P}$ be an arbitrary point of the fibre $\pi^{-1}(P)$ over P, then the horizontal lifts $C_x{}^{\varkappa}$ of $C_x{}^A$ are mutually orthogonal unit normal to $\bar{M}$ defined in the tubular neighborhood of $\bar{P}$ over U because of (5.4).

Taking account of this fact, (5.3) and (5.4), we find

$$\xi^{\varkappa} = \xi^{\alpha} B_{\alpha}{}^{\varkappa} \tag{5.5}$$

and

$$\xi^{\alpha} E_{\alpha}{}^h = 0 , \tag{5.6}$$

where ξ^{α} is a vector field on $\bar{M}$. Then (4.5) implies

$$\xi^{\alpha} \xi_{\alpha} = 1 \tag{5.7}$$

because of $\xi_{\varkappa}\xi^{\varkappa} = 1$, where $\xi_{\alpha} = \xi^{\beta} g_{\beta\alpha}$, $g_{\beta\alpha}$ being the fundamental metric tensor of $\bar{M}$ induced from $g_{\varkappa\mu}$ in such a way that $g_{\beta\alpha} = g_{\varkappa\mu} B_{\beta}{}^{\varkappa} B_{\alpha}{}^{\mu}$. Therefore, $\{E_{\alpha}{}^h, \xi_{\alpha}\}$ forms a local coframe in $\bar{M}$ corresponding $\{E_{\varkappa}{}^A, \xi_{\varkappa}\}$ in S^{2m+1}. Denoting by $\{E^{\alpha}{}_h, \xi^{\alpha}\}$ the frame corresponding this coframe, we have

$$E_{\alpha}{}^h E^{\alpha}{}_k = \delta_k{}^h , \quad \xi_{\alpha} E^{\alpha}{}_h = 0 , \quad \xi^{\alpha} E_{\alpha}{}^h = 0 . \tag{5.8}$$

Then, (5.4) and (5.8) imply that

$$E^{\varkappa}{}_A B_h{}^A = B_{\alpha}{}^{\varkappa} E^{\alpha}{}_h . \tag{5.9}$$

Since the metric tensors $g_{\varkappa\mu}$ and $g_{\beta\alpha}$ are both invariant with respect to the submersion $\tilde{\pi}$ and π respectively, the van der Waerden-Bortolotti covariant derivatives of $E_\lambda{}^A$, $E^\lambda{}_A$ and $E_\alpha{}^h$, $E^\alpha{}_h$ are given by

$$(5.10)\qquad \begin{cases} D_\mu E_\lambda{}^A = h_B{}^A(E_\mu{}^B\xi_\lambda + E_\lambda{}^B\xi_\mu)\,, \\ D_\mu E^\lambda{}_A = h_{BA}E_\mu{}^B\xi^\lambda - h_A{}^B\xi_\mu E^\lambda{}_B\,, \end{cases}$$

$$(5.11)\qquad \begin{cases} \overline{\nabla}_\beta E_\alpha{}^h = h_j{}^h(E_\beta{}^j\xi_\alpha + E_\alpha{}^j\xi_\beta)\,, \\ \overline{\nabla}_\beta E^\alpha{}_h = h_{jh}E_\beta{}^j\xi^\alpha - h_h{}^j\xi_\beta E^\alpha{}_j \end{cases}$$

respectively, where D_μ and $\overline{\nabla}_\beta$ are the operators of the covariant differentiation of S^{2m+1} and $\overline{M}$ respectively, $h_B{}^A = g^{AC}h_{BC}$, $h_j{}^h = g^{hi}h_{ji}$, h_{BC} and h_{ji} are the structure tensors induced from the submersions $\tilde{\pi}$ and π respectively (see ISHIHARA-KONISHI [7]).

On the other hand, the equations of Gauss for $\overline{M}$ are given by

$$(5.12)\qquad \overline{\nabla}_\beta B_\alpha{}^\varkappa = A_{\beta\alpha}C^\varkappa + A_{\beta\alpha}{}^{(x)}C_{(x)}{}^\varkappa\,,$$

where $A_{\beta\alpha}$ and $A_{\beta\alpha}{}^{(x)}$ are the second fundamental tensors with respect to the normals $C^\varkappa = C^A E^\varkappa{}_A$ and $C_{(x)}{}^\varkappa = C_{(x)}{}^A E^\varkappa{}_A$ respectively, and those of Weingarten by

$$(5.13)\qquad \overline{\nabla}_\beta C^\varkappa = -A_\beta{}^\alpha B_\alpha{}^\varkappa + l_\beta{}^{(x)}C_{(x)}{}^\varkappa\,,$$

$$(5.14)\qquad \overline{\nabla}_\beta C_{(x)}{}^\varkappa = -A_\beta{}^\alpha{}_{(x)}B_\alpha{}^\varkappa - l_{\beta(x)}C^\varkappa + l_{\beta(x)}{}^{(y)}C_{(y)}{}^\varkappa\,,$$

where $A_\beta{}^\alpha = g^{\gamma\alpha}A_{\beta\gamma}$, $A_\beta{}^\alpha{}_{(x)} = g^{\gamma\alpha}g_{(y)(x)}A_{\beta\gamma}{}^{(y)} = g^{\alpha\gamma}A_{\beta\gamma(x)}$, $l_\beta{}^{(x)}$ and $l_{\beta(x)}{}^{(y)}$ the third fundamental tensors and $l_{\beta(x)} = l_\beta{}^{(y)}g_{(y)(x)}$.

On the other hand, (5.4) and (5.9) imply that $\nabla_j = E^\alpha{}_j\overline{\nabla}_\alpha$. We now put $\tilde{F}_\mu{}^\lambda = D_\mu\xi^\lambda$. From the definition of a Sasakian structure it follows that

$$(5.15)\qquad \tilde{F}_\mu{}^\lambda\tilde{F}_\varkappa{}^\mu = -\delta_\varkappa^\lambda + \xi_\varkappa\xi^\lambda\,,\quad \tilde{F}_\mu{}^\lambda\xi^\mu = 0\,,\quad \xi_\lambda\tilde{F}_\mu{}^\lambda = 0\,,\quad \tilde{F}_{\lambda\mu} + \tilde{F}_{\mu\lambda} = 0$$

and

$$(5.16)\qquad D_\mu\tilde{F}_\lambda{}^\varkappa = -g_{\mu\lambda}\xi^\varkappa + \delta_\mu^\varkappa\xi_\lambda\,,\quad D_\mu\xi^\lambda = \tilde{F}_\mu{}^\lambda\,,$$

where we have put $\tilde{F}_{\mu\lambda} = \tilde{F}_\mu{}^\varkappa g_{\varkappa\lambda}$. Denoting by L the lie differentiation with respect to ξ, we find

$$(5.17)\qquad L\tilde{F}_\varkappa{}^\mu = 0$$

because of (5.16). Putting

$$(5.18)\qquad F_B{}^A = \tilde{F}_\mu{}^\varkappa E^\mu{}_B E_\varkappa{}^A\,,$$

we can see that $F_B{}^A$ defines a global tensor field of the same type as that of $\tilde{F}_\mu{}^\lambda$ because of (5.17), $LE^\lambda{}_A = 0$ and $LE_\lambda{}^A = 0$ (see [7]).

Differentiating $\xi^\varkappa E_\varkappa{}^A = 0$ covariantly along S^{2m+1} and using (5.10), (5.16) and (5.18), we find

$$F_B{}^A = -h_B{}^A, \tag{5.19}$$

which implies

$$F_B{}^A F_C{}^B = -\delta_C{}^A \tag{5.20}$$

with the aid of (5.4) and (5.15).

Differentiating (5.18) covariantly along CP^m, and using (5.4) and (5.10), we find

$$\tilde{\nabla}_C F_B{}^A = 0, \tag{5.21}$$

where $\tilde{\nabla}$ denotes the projection of D given by $\tilde{\nabla}_A = E^\varkappa{}_A D_\varkappa$. Therefore, the base space CP^m for S^{2m+1} admits a Kaehler structure $(F_B{}^A, G_{BC})$ represented by the structure tensor $h_B{}^A$ of the submersion $\tilde{\pi}$: $S^{2m+1} \to CP^m$ defined by the Hopf-fibration.

On the other hand, by taking account of the co-Gauss equation for the submersion $\tilde{\pi}$: $S^{2m+1} \to CP^m$ and (5.19), we can see that the base space CP^m is a Kaehlerian manifold of constant holomorphic sectional curvature 4 given by (2.20).

As to transforms of $B_\alpha{}^\mu$ and $C_x{}^\mu$ by $\tilde{F}_\mu{}^\varkappa$, we have

$$\begin{cases} \tilde{F}_\mu{}^\varkappa B_\alpha{}^\mu = f_\alpha{}^\beta B_\beta{}^\varkappa + f_\alpha{}^x C_x{}^\varkappa, \\ \tilde{F}_\mu{}^\varkappa C_x{}^\mu = -f_x{}^\beta B_\beta{}^\varkappa + \varphi_x{}^y C_y{}^\varkappa \end{cases} \tag{5.22}$$

where $f_\alpha{}^\beta$ is a tensor field of type (1, 1), $f_\beta{}^x$ 1-form for fixed x, $f_x{}^\beta$ a vector field associated with $f_\beta{}^x$ defined by $f_\beta{}^x = f_y{}^\alpha g_{\alpha\beta} g^{yx}$ and $\varphi_x{}^y$ a scalar field for fixed x and y on $\bar{M}$.

Now we suppose that n-dimensional submanifold M of CP^m is semi-invariant with respect to the distinguished normal C^A. Then we can have the algebraic relationships (1.13)~(1.16) and (2.7) and the structure equations (2.11)~(2.15).

If we make use of (2.4), (2.5), (5.4), (5.9) and (5.22), then we obtain

$$f_j{}^h = f_\beta{}^\alpha E^\beta{}_j E_\alpha{}^h, \quad P_j = f_\beta{}^{1^*} E^\beta{}_j, \quad P^i = f_{1^*}{}^\beta E_\beta{}^i, \quad \varphi_{(x)}{}^{(y)} = f_{(x)}{}^{(y)}. \tag{5.23}$$

Thus, (5.22) reduces to

$$\begin{cases} \tilde{F}_\mu{}^\varkappa B_\alpha{}^\mu = f_\alpha{}^\beta B_\beta{}^\varkappa + P_\alpha C^\varkappa, \\ \tilde{F}_\mu{}^\varkappa C^\mu = -P^\alpha B_\alpha{}^\varkappa, \\ \tilde{F}_\mu{}^\varkappa C_{(x)}{}^\mu = f_{(x)}{}^{(y)} C_{(y)}{}^\varkappa, \end{cases} \tag{5.24}$$

where we have put $f_\beta{}^{1^*} = P_\beta$ and $f_{1^*}{}^\beta = P^\beta$.

Applying $\tilde{F}$ to (5.24) and using (5.15) and these expressions, we easily find

$$f_\alpha{}^\gamma f_\gamma{}^\beta = -\delta_\alpha{}^\beta + \xi_\alpha \xi^\beta + P_\alpha P^\beta , \tag{5.25}$$

$$f_\alpha{}^\beta P_\beta = 0 , \quad f_\alpha{}^\beta P^\alpha = 0 , \tag{5.26}$$

$$P_\alpha P^\alpha = 1 , \tag{5.27}$$

$$P_\alpha \xi^\alpha = 0 , \quad P^\alpha \xi_\alpha = 0 , \tag{5.28}$$

$$\xi_\alpha f_\beta{}^\alpha = 0 , \quad f_\alpha{}^\beta \xi^\alpha = 0 , \tag{5.29}$$

$$f_{(x)}{}^{(y)} f_{(y)}{}^{(z)} = -\delta_{(x)}{}^{(z)} . \tag{5.30}$$

If we apply the operator of the covariant differentiation $\overline{\nabla}_\gamma = B_\gamma{}^\varkappa D_\varkappa$ to (5.24) and using (5.5), (5.12), (5.13), (5.14) and (5.16), then we have

$$\overline{\nabla}_\beta f_\alpha{}^\gamma = -g_{\beta\alpha}\xi^\gamma + \delta_\beta{}^\gamma \xi_\alpha - A_{\beta\alpha} P^\gamma + A_\beta{}^\gamma P_\alpha , \tag{5.31}$$

$$\overline{\nabla}_\beta P_\alpha = -A_{\beta\gamma} f_\alpha{}^\gamma , \quad \overline{\nabla}_\beta P^\alpha = A_\beta{}^\gamma f_\gamma{}^\alpha , \tag{5.32}$$

$$A_{\beta\alpha}{}^{(y)} f_{(y)}{}^{(x)} = A_{\beta\gamma}{}^{(x)} f_\alpha{}^\gamma + l_\beta{}^{(x)} P_\alpha , \tag{5.33}$$

$$A_{\beta\gamma}{}^{(x)} P^\gamma = -l_\beta{}^{(y)} f_{(y)}{}^{(x)} , \tag{5.34}$$

$$\overline{\nabla}_\beta f_{(y)}{}^{(x)} = l_{\beta(y)}{}^{(z)} f_{(z)}{}^{(x)} - l_{\beta(z)}{}^{(x)} f_{(y)}{}^{(z)} . \tag{5.35}$$

Differentiating (5.5) covariantly along $\overline{M}$ and utilizing (5.12), (5.16) and the first equation of (5.24), we find

$$\overline{\nabla}_\beta \xi^\alpha = f_\beta{}^\alpha , \tag{5.36}$$

$$A_{\beta\alpha}\xi^\alpha = P_\beta , \tag{5.37}$$

$$A_{\beta\alpha}{}^{(x)} \xi^\alpha = 0 . \tag{5.38}$$

On the other hand, by differentiating (5.6) covariantly and taking account of (5.11), the first relationship of (5.23) and (5.36), it follows that

$$f_j{}^h = -h_j{}^h . \tag{5.39}$$

If we apply the operator $\nabla_j = B_j{}^A \tilde{\nabla}_A = E^\alpha{}_j \overline{\nabla}_\alpha = B_j{}^B E^\varkappa{}_B D_\varkappa$ to (5.4) and use (2.8), (5.10), (5.11), (5.12), (5.19) and (5.39), then we get

$$\begin{aligned}(A_{ji} C^A + A_{ji}{}^{(x)} C_{(x)}{}^A) E_\alpha{}^i + B_i{}^A E^\beta{}_j \{-f_k{}^i (E_\beta{}^k \xi_\alpha + \xi_\beta E_\alpha{}^k)\} \\ = -F_C{}^A (E_\mu{}^C \xi_\varkappa + E_\varkappa{}^C \xi_\mu) B_j{}^B E^\mu{}_B B_\alpha{}^\varkappa + (A_{\beta\alpha} C^\varkappa + A_{\beta\alpha}{}^{(x)} C_{(x)}{}^\varkappa) E_\varkappa{}^A E^\beta{}_j ,\end{aligned}$$

or using (5.3), (5.8) and the first equation of (5.24),

$$A_{ji}E_\alpha{}^i = A_{\beta\alpha}E^\beta{}_j - P_j\xi_\alpha ,$$
$$A_{\beta\alpha}{}^{(x)}E^\beta{}_j = A_{ji}{}^{(x)}E_\alpha{}^i .$$

Transvecting these equations with $E_\gamma{}^j$ respectively and using (5.8), (5.23) and (5.37), we have

$$A_{\gamma\beta} = A_{ji}E_\gamma{}^j E_\beta{}^i + P_\gamma\xi_\beta + P_\beta\xi_\gamma , \tag{5.40}$$
$$A_{\gamma\beta}{}^{(x)} = A_{ji}{}^{(x)}E_\gamma{}^j E_\beta{}^i . \tag{5.41}$$

Since the ambient manifold S^{2m+1} for $\bar{M}$ is a space of constant curvature 1, the equations of Gauss for $\bar{M}$ are given by

$$K_{\delta\gamma\beta}{}^\alpha = \delta_\varepsilon{}^\alpha g_{\gamma\beta} - \delta_\gamma{}^\alpha g_{\delta\beta} + A_\delta{}^\alpha A_{\gamma\beta} - A_\gamma{}^\alpha A_{\delta\beta} + A_\delta{}^{\alpha(x)}A_{\gamma\beta(x)} - A_\gamma{}^{\alpha(x)}A_{\delta\beta(x)} , \tag{5.42}$$

where $K_{\delta\gamma\beta}{}^\alpha$ is the Riemann-Christoffel curvature tensor of $\bar{M}$, those of Codazzi by

$$\bar{\nabla}_\gamma A_{\beta\alpha} - \bar{\nabla}_\beta A_{\gamma\alpha} - l_{\gamma(x)}A_{\beta\alpha}{}^{(x)} + l_{\beta(x)}A_{\gamma\alpha}{}^{(x)} = 0 , \tag{5.43}$$
$$\bar{\nabla}_\gamma A_{\beta\alpha}{}^{(x)} - \bar{\nabla}_\beta A_{\gamma\alpha}{}^{(x)} + l_\gamma{}^{(x)}A_{\beta\alpha} - l_\beta{}^{(x)}A_{\gamma\alpha} + l_{\gamma(y)}{}^{(x)}A_{\beta\alpha}{}^{(y)} - l_{\beta(y)}{}^{(x)}A_{\gamma\alpha}{}^{(y)} = 0 , \tag{5.44}$$

and those of Ricci by

$$\bar{\nabla}_\beta l_\alpha{}^{(x)} - \bar{\nabla}_\alpha l_\beta{}^{(x)} + A_\beta{}^\gamma A_{\gamma\alpha}{}^{(x)} - A_\alpha{}^\gamma A_{\beta\gamma}{}^{(x)} + l_{\beta(y)}{}^{(x)}l_\alpha{}^{(y)} - l_{\alpha(y)}{}^{(x)}l_\beta{}^{(y)} = 0 , \tag{5.45}$$
$$\begin{aligned}\bar{\nabla}_\beta l_{\alpha(x)}{}^{(y)} - \bar{\nabla}_\alpha l_{\beta(x)}{}^{(y)} + A_\beta{}^\gamma{}_{(x)}A_{\alpha\gamma}{}^{(y)} - A_\alpha{}^\gamma{}_{(x)}A_{\beta\gamma}{}^{(y)} + l_{\beta(x)}l_\alpha{}^{(y)} - l_{\alpha(x)}l_\beta{}^{(y)} \\ + l_{\beta(z)}{}^{(y)}l_{\alpha(x)}{}^{(z)} - l_{\alpha(z)}{}^{(y)}l_{\beta(x)}{}^{(z)} = 0 .\end{aligned} \tag{5.46}$$

We now assume that the second fundamental tensor of the base space M for $\bar{M}$ commute with the structure tensor $f_j{}^h$ of the submersion π, that is, (3.1) and (3.2) hold. Then we can easily verify that the second fundamental tensors of the total space $\bar{M}$ also commute with $f_\beta{}^\alpha$ because of (5.23), (5.26), (5.29), (5.40) and (5.41), that is,

$$A_\beta{}^\gamma f_\gamma{}^\alpha - f_\beta{}^\gamma A_\gamma{}^\alpha = 0 , \quad A_\beta{}^\gamma{}_{(x)}f_\gamma{}^\alpha - f_\beta{}^\gamma A_\gamma{}^\alpha{}_{(x)} = 0 ,$$

or, equivalently

$$A_{\beta\gamma}\, f_\alpha{}^\gamma + A_{\alpha\gamma}f_\beta{}^\gamma = 0 , \tag{5.47}$$
$$A_{\beta\gamma}{}^{(x)}\, f_\alpha{}^\gamma + A_{\alpha\gamma}{}^{(x)}f_\beta{}^\gamma = 0 . \tag{5.48}$$

Transvecting (5.40) with P^γ and taking account of (3.3), (5.23), (5.27) and (5.28), we get

$$A_{\beta\gamma}P^\gamma = \alpha P_\beta + \xi_\beta \,. \tag{5.49}$$

If we substitute (3.6) into (5.41) and make use of (5.23), we then have

$$A_{\beta\alpha}{}^{(x)} = A^{(x)}P_\beta P_\alpha \,, \tag{5.50}$$

which implies that the mean curvatures of M and $\overline{M}$ are the same with the aid of (5.27).

On the other hand, transvection (5.34) with $f_{(x)}{}^{(z)}$ yields

$$l_\beta{}^{(z)} = A^{(x)}f_{(x)}{}^{(z)}P_\beta \tag{5.51}$$

with the aid of (5.30) and (5.50).

We first prove

LEMMA 5.1. – *Let M be an $n(>1)$-dimensional semi-invariant submanifold with distinguished normal C^A of a complex projective space CP^m of real dimension $2m$. If the second fundamental tensors of M are commutative with the structure tensor of the submersion π, then we have*

$$A_{\beta\alpha}A_\gamma{}^\alpha = \alpha A_{\beta\gamma} + g_{\beta\gamma} \,. \tag{5.52}$$

PROOF. – Transvecting (5.40) with $A_\alpha{}^\gamma = A_j{}^iE_\alpha{}^jE^\gamma{}_i + P_\alpha\xi^\gamma + P^\gamma\xi_\alpha$ and taking account of (3.3), (5.8), (5.23), (5.27) and (5.28), we obtain

$$A_{\beta\gamma}A_\alpha{}^\gamma = A_{jt}A_i{}^tE_\beta{}^jE_\alpha{}^i + \alpha(P_\beta\xi_\alpha + P_\alpha\xi_\beta) + P_\beta P_\alpha + \xi_\beta\xi_\alpha \,,$$

or, using (3.9) with $c = 4$,

$$A_{\beta\gamma}A_\alpha{}^\gamma = \alpha A_{\beta\alpha} + g_{\beta\alpha}$$

with the aid of (5.8), (5.23) and (5.40). Thus, the lemma is proved.

Next, we prove

LEMMA 5.2. – *Under the same assumptions as those stated in Lemma 5.1, we have*

$$A_{\beta\alpha}{}^{(x)} = 0 \quad \text{and} \quad A_{ji}{}^{(x)} = 0 \,. \tag{5.53}$$

PROOF. – Differentiating (5.50) covariantly and using (5.32), we get

$$\overline{\nabla}_\gamma A_{\beta\alpha}{}^{(x)} = (\overline{\nabla}_\gamma A^{(x)})P_\beta P_\alpha - A^{(x)}(A_{\gamma\varepsilon}f_\beta{}^\varepsilon P_\alpha + A_{\gamma\varepsilon}f_\alpha{}^\varepsilon P_\beta) \,,$$

from which, taking the skew-symmetric part with respect to the indices γ and β and using (5.44),

$$l_\beta{}^{(x)}A_{\gamma\alpha}-l_\gamma{}^{(x)}A_{\beta\alpha}+l_{\beta(y)}{}^{(x)}A_{\gamma\alpha}{}^{(y)}-l_{\gamma(y)}{}^{(x)}A_{\beta\alpha}{}^{(y)}$$
$$=\{(\overline{\nabla}_\gamma A^{(x)})P_\beta-(\overline{\nabla}_\beta A^{(x)})P_\gamma\}P_\alpha-A^{(x)}(2A_{\gamma\varepsilon}f_\beta{}^\varepsilon P_\alpha+A_{\gamma\varepsilon}f_\alpha{}^\varepsilon P_\beta-A_{\beta\varepsilon}f_\alpha{}^\varepsilon P_\gamma)$$

because of (5.47). Substituting (5.50) and (5.51) into this equation, we find

$$\begin{aligned}(5.54)\quad & A^{(y)}(f_{(y)}{}^{(x)}P_\beta A_{\gamma\alpha}-f_{(y)}{}^{(x)}P_\gamma A_{\beta\alpha}+l_{\beta(y)}{}^{(x)}P_\gamma P_\alpha-l_{\gamma(y)}{}^{(x)}P_\beta P_\alpha)\\ &=\{(\overline{\nabla}_\gamma A^{(x)})P_\beta-(\overline{\nabla}_\beta A^{(x)})P_\gamma\}P_\alpha-A^{(x)}(2A_{\gamma\varepsilon}f_\beta{}^\varepsilon P_\alpha+A_{\gamma\varepsilon}f_\alpha{}^\varepsilon P_\beta-A_{\beta\varepsilon}f_\alpha{}^\varepsilon P_\gamma)\,.\end{aligned}$$

Transvection $P^\beta P^\alpha$ gives

$$\overline{\nabla}_\gamma A^{(x)}=A^{(y)}(f_{(y)}{}^{(x)}\xi_\gamma-l_{\gamma(y)}{}^{(x)})+L^{(x)}P_\gamma$$

with the aid of (5.26), (5.27), (5.28) and (5.29), where we have put $L^{(x)}=A^{(y)}l_{\beta(y)}{}^{(x)}P^\beta+$ $+P^\beta\overline{\nabla}_\beta A^{(x)}$. Hence (5.54) reduces to

$$A^{(y)}f_{(y)}{}^{(x)}(P_\beta A_{\gamma\alpha}-P_\gamma A_{\beta\alpha})$$
$$=A^{(y)}f_{(y)}{}^{(x)}(\xi_\gamma P_\beta-\xi_\beta P_\gamma)P_\alpha-A^{(x)}(2A_{\gamma\varepsilon}f_\beta{}^\varepsilon P_\alpha+A_{\gamma\varepsilon}f_\alpha{}^\varepsilon P_\beta-A_{\beta\varepsilon}f_\alpha{}^\varepsilon P_\gamma)\,.$$

Transvection P^α yields

$$(5.55)\qquad A^{(x)}A_{\gamma\varepsilon}f_\beta{}^\varepsilon=0$$

because of (5.26), (5.37) and (5.49).

Transvecting this with $f_\alpha{}^\beta$ and using (5.25), we find

$$A^{(x)}(-A_{\gamma\alpha}+P_\gamma\xi_\alpha+P_\alpha\xi_\gamma+\alpha P_\gamma P_\alpha)=0$$

because of (5.27) and (5.49). If we transvect this with $g^{\gamma\alpha}$ and make use of (5.27) and (5.28), then we have

$$(5.56)\qquad A^{(x)}(A-\alpha)=0\,,$$

where $A=g^{\beta\alpha}A_{\beta\alpha}$. By computing the square of norm of (5.55), we obtain $A^{(x)}=0$ with the aid of (5.56) and $n>1$. Therefore, it follows that $A_{ji}{}^{(x)}=0$ and $A_{\beta\alpha}{}^{(x)}=0$ because of (3.6) and (5.50). Thus, Lemma 5.2 is completely proved.

If the normal vectors $C_{(x)}{}^A$ are parallel in the subnormal bundle spanned by $C_{(x)}{}^A$, we can easily prove from (2.25) and Lemma 5.2 that M is a real hypersurface of a complex projective space CP^m. Therefore, by Theorem A in § 0, we have

THEOREM 5.3. – *Let M be an $n(>1)$-dimensional semi-invariant submanifold with distinguished normal C^A of CP^m. If the second fundamental tensors of M are commutative with the structure tensor of the submersion π and the normal vectors $C_{(x)}{}^A$ are parallel in the subnormal bundle spanned by $C_{(x)}{}^A$, then M is the model space $M^c_{p,q}(a, b)$, where (p, q) is some portion of $(n-1)/2$ and $a^2 + b^2 = 1$.*

LEMMA 5.4. – *Under the same assumptions as those stated in Lemma 5.1, we obtain*

$$\overline{\nabla}_\gamma A_{\beta\alpha} = 0 . \tag{5.57}$$

PROOF. – Applying the operator $\nabla_k = E_k{}^\gamma \overline{\nabla}_\gamma$ to both sides of (5.40), we have

$$\begin{aligned} E_k{}^\gamma \overline{\nabla}_\gamma A_{\beta\alpha} = (\nabla_k A_{ji}) E_\beta{}^j E_\alpha{}^i + A_{ji} E^\gamma{}_k (\overline{\nabla}_\gamma E_\beta{}^j) E_\alpha{}^i + A_{ji} E_\beta{}^j E^\gamma{}_k \overline{\nabla}_\gamma E_\alpha{}^i \\ + E^\gamma{}_k (\overline{\nabla}_\gamma P_\beta) \xi_\alpha + P_\beta E^\gamma{}_k \overline{\nabla}_\gamma \xi_\alpha + (E^\gamma{}_k \overline{\nabla}_\gamma P_\alpha) \xi_\beta + P_\alpha E^\gamma{}_k \overline{\nabla}_\gamma \xi_\beta . \end{aligned}$$

Substituting (5.11) with $h_j{}^h = -f_j{}^h$, (5.32) and (5.36) into this equation, we get

$$E^\gamma{}_k \overline{\nabla}_\gamma A_{\beta\alpha} = (\nabla_k A_{ji} + P_i f_{kj} + P_j f_{ki}) E_\beta{}^j E_\alpha{}^i - (A_{kt} f_i{}^t + A_{it} f_k{}^t)(E_\beta{}^i \xi_\alpha + E_\alpha{}^i \xi_\beta)$$

because of (5.23) and (5.40), from which, using (3.1) and (3.17),

$$E^\gamma{}_k \overline{\nabla}_\gamma A_{\beta\alpha} = 0 . \tag{5.58}$$

On the other hand, by Lemma 5.2, we can have from (5.43)

$$\overline{\nabla}_\gamma A_{\beta\alpha} - \overline{\nabla}_\beta A_{\gamma\alpha} = 0 . \tag{5.59}$$

Transvecting (5.58) with $E_\delta{}^k$, we get

$$\overline{\nabla}_\delta A_{\beta\alpha} = (\xi^\gamma \overline{\nabla}_\gamma A_{\beta\alpha}) \xi_\delta .$$

Differentiating (5.37) and making use of (5.32), (5.37), (5.47) and (5.59), we have $\overline{\nabla}_\varepsilon A_{\beta\alpha} = 0$. Therefore Lemma 5.4 is proved.

We consider the identity:

$$\tfrac{1}{2} \Delta (A_{\beta\alpha}{}^x A^{\beta\alpha}{}_x) = (\overline{\nabla}^\gamma \overline{\nabla}_\gamma A_{\beta\alpha}{}^x) A^{\beta\alpha}{}_x + \|\overline{\nabla}_\gamma A_{\beta\alpha}{}^x\|^2 ,$$

where $\Delta = g^{\gamma\beta} \overline{\nabla}_\gamma \overline{\nabla}_\beta$ and $A_{\beta\alpha}{}^{1^*} = A_{\beta\alpha_{1^*}} = A_{\beta\alpha}$.

From this identity we can see that the second fundamental tensors $A_{\beta\alpha}{}^x$ are parallel because of (5.53) and (5.57). Thus the first normal space $N_1(\overline{P})$ defined to be the orthogonal complement of $\{C_x{}^\varkappa \in T^\perp_{\overline{p}}(\overline{M}) : A_{C_x{}^\varkappa} = 0\}$ in $T^\perp_{\overline{p}}(\overline{M})$ is invariant under parallel translation with respect to the connection in the normal bundle and of constant dimension 1, where $A_{C_x{}^\varkappa}$ are the second fundamental tensors associated with $C_x{}^\varkappa$ and $T^\perp_{\overline{p}}(\overline{M})$ is the normal space at $\overline{p} \in \overline{M}$. Thus, by the reduction theorem ([2]), we conclude the total space $\bar{M}$ for M is contained in an $(n+2)$-dimensional unit sphere S^{n+2} $(\subset S^{2m+1})$ and consequently the base space M is contained as a hypersurface of a complex projective space $CP^{(n+1)/2}$ of real dimension $n+1$ (see [2]). And hence the diagram in the beginning in § 5 reduces to

$$\begin{tikzcd} \bar{M} \arrow[r, "\tilde{i}"] \arrow[d, "\pi"'] \arrow[dr, bend left] & S^{n+2} \subset S^{2m+1} \arrow[d, "\tilde{\pi}"] \\ M \arrow[r, "i"] & CP^{(n+1)/2} \subset CP^m \end{tikzcd}$$

Therefore, taking account of Theorem A in § 0, we have

THEOREM 5.5. – *Let M be an $n(>1)$-dimensional complete semi-invariant submanifold with the distinguished normal C^A of a complex projective space CP^m of real dimension $2m$. If the second fundamental tensors are commutative with the structure tensor of the submersion π, then M is the model space $M^c_{p,q}(a, b)$, where (p, q) is some portion of $(n-1)/2$ and $a^2 + b^2 = 1$.*

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