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SUBMANIFOLDS WITH HARMONIC MEAN CURVATURE VECTOR FIELD IN CONTACT 3-MANIFOLDS

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Abstract. Biharmonic or polyharmonic curves and surfaces in 3-dimensional contact manifolds are investigated.

Introduction. This paper concerns curves and surfaces in 3-dimensional contact manifolds whose mean curvature vector field is in the kernel of certain elliptic differential operators.

First we study submanifolds whose mean curvature vector field is in the kernel of the Laplacian (submanifolds with harmonic mean curvature vector fields). The study of such submanifolds is inspired by a conjecture of Bang-yen Chen [14]:

Harmonicity of the mean curvature vector field implies harmonicity of the immersion.

The harmonicity equation $\Delta\mathbb{H} = 0$ for the mean curvature vector field $\mathbb{H}$ of an immersed submanifold $\mathbf{x} : M^m \to \mathbb{E}^n$ in Euclidean n-space is equivalent to the *biharmonicity* of the immersion: $\Delta\Delta\mathbf{x} = 0$, since $\Delta\mathbf{x} = -m\mathbb{H}$. A submanifold $\mathbf{x} : M \to \mathbb{E}^n$ is said to be a *biharmonic submanifold* if $\Delta\mathbb{H} = 0$.

In 1985, Chen proved the nonexistence of proper biharmonic surfaces in Euclidean 3-space. The above conjecture by Chen is still open. Some partial positive answers have been obtained by several authors [16]–[19], [25]–[27].

The biharmonicity equation is a special case of the following condition:

$$\Delta\mathbb{H} = \lambda\mathbb{H}, \qquad \lambda \in \mathbb{R}.$$

That is, the mean curvature vector field is an eigenfunction of the Laplacian.

The study of Euclidean submanifolds with $\Delta\mathbb{H} = \lambda\mathbb{H}$ was initiated by Chen in 1988 (see [14]). It is known that submanifolds in $\mathbb{E}^n$ satisfying $\Delta\mathbb{H} = \lambda\mathbb{H}$ are either biharmonic ($\lambda = 0$), of 1-type or null 2-type. In particular all surfaces in $\mathbb{E}^3$ with $\Delta\mathbb{H} = \lambda\mathbb{H}$ are of constant mean curvature. Moreover a surface in $\mathbb{E}^3$ satisfies $\Delta\mathbb{H} = \lambda\mathbb{H}$ if and only if it is minimal, an open portion of a totally umbilical sphere or an open portion of a circular cylinder.

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F. Defever [17] showed that hypersurfaces satisfying $\Delta\mathbb{H} = \lambda\mathbb{H}$ are of constant mean curvature. Note that Chen [12], [13] studied spacelike submanifolds with $\Delta\mathbb{H} = \lambda\mathbb{H}$ in Minkowski space, hyperbolic space or de Sitter space. M. Barros and O. J. Garay showed that Hopf cylinders in S^3 with $\Delta\mathbb{H} = \lambda\mathbb{H}$ are Hopf cylinders over circles in the 2-sphere S^2. A. Ferrández, P. Lucas and M. A. Meroño [24] studied such submanifolds in anti de Sitter 3-space H^3_1.

In nonconstant curvature ambient spaces, results on biharmonic submanifolds are very few.

Recently, T. Sasahara [37]–[38] studied Legendre surfaces in the Sasakian space form $\mathbb{R}^5(-3)$ satisfying $\Delta\mathbb{H} = \lambda\mathbb{H}$. Moreover he introduced the notion of "φ-position vector field" and "φ-mean curvature vector field" for submanifolds in the Sasakian space form $\mathbb{R}^{2n+1}(-3)$. He investigated submanifolds in $\mathbb{R}^{2n+1}(-3)$ whose φ-mean curvature vector field $\mathbb{H}_\varphi$ satisfies $\Delta\mathbb{H}_\varphi = \lambda\mathbb{H}_\varphi$. In particular he classified curves and surfaces in $\mathbb{R}^3(-3)$ with $\Delta\mathbb{H}_\varphi = \lambda\mathbb{H}_\varphi$. Since both $\mathbb{R}^{2n+1}(-3)$ and S^{2n+1} are typical examples of Sasakian space forms, it seems interesting to study biharmonic submanifolds in general Sasakian space forms.

Based on these observations, in the first part of this paper, we shall study harmonicity of mean curvature vector fields of curves and surfaces in 3-dimensional Sasakian space forms. Several results for 3-dimensional sphere S^3 due to the Spanish research group (Barros, Garay, Ferrández, Lucas and Meroño) will be generalised to 3-dimensional Sasakian space forms.

Next, in the second part, we shall study another "biharmonicity" suggested by J. Eells and J. H. Sampson [23]. A smooth map $\phi : M \to N$ between Riemannian manifolds is said to be *biharmonic* (or *polyharmonic* of order 2) if its bitension field $\mathcal{T}_2(\phi)$ vanishes. In [9], "biharmonic" curves and surfaces in S^3 are classified. We shall classify Legendre curves and Hopf cylinders in 3-dimensional Sasakian space forms which are biharmonic in this sense.

In particular we shall prove the existence of nonminimal biharmonic Hopf cylinders in Sasakian space forms of holomorphic sectional curvature greater than 1 (Berger spheres).

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PART I

1. Preliminaries

1.1. *Contact manifolds.* We begin by recalling fundamental notions of contact Riemannian geometry from [7].

Let M be a $(2n+1)$-manifold. A one-form η is called a *contact form* on M if $(d\eta)^n \wedge \eta \neq 0$. A $(2n+1)$-manifold M together with a contact form is called a *contact manifold*. The *contact distribution* D of (M,η) is defined by

$$D = \{X \in TM \mid \eta(X) = 0\}.$$

On a contact manifold (M,η), there exists a unique vector field ξ such that

$$\eta(\xi) = 1, \qquad d\eta(\xi,\cdot) = 0.$$

This vector field ξ is called the *Reeb vector field* or *characteristic vector field* of (M,η).

Moreover there exists an endomorphism field φ and a Riemannian metric g on M such that

$$\varphi^2 = -I + \eta \otimes \xi, \qquad \eta(\xi) = 1, \tag{1}$$

$$g(\varphi X, \varphi Y) = g(X,Y) - \eta(X)\eta(Y), \qquad g(\xi,\cdot) = \eta, \tag{2}$$

$$d\eta(X,Y) = 2g(X,\varphi Y) \tag{3}$$

for all vector fields X, Y on M. On a contact manifold $(M,\eta;\xi,\varphi)$, there exists a Riemannian metric g satisfying (2). Such a metric g is called a *compatible metric* of M. A contact manifold (M,η) together with structure tensors (ξ,φ,g) is called a *contact Riemannian manifold*.

Proposition 1.1. *Let $(M,\eta;\xi,\varphi,g)$ be a contact Riemannian manifold. Then ξ is a Killing vector field if and only if*

$$\nabla_X \xi = -\varphi X, \qquad X \in \mathfrak{X}(M). \tag{4}$$

Here ∇ is the Levi-Civita connection of (M,g).

Definition 1.1. A contact Riemannian manifold (M,η,ξ,φ,g) is said to be a *Sasaki manifold* if

$$(\nabla_X \varphi)Y = g(X,Y)\xi - \eta(Y)X, \qquad X,Y \in \mathfrak{X}(M). \tag{5}$$

Note that on a Sasaki manifold, ξ is a Killing vector field.

Let $(M,\eta;\xi,\varphi,g)$ be a contact Riemannian manifold. A tangent plane at a point of M is said to be a *holomorphic plane* if it is invariant under φ. The sectional curvature of a holomorphic plane is called *holomorphic sectional curvature*. If the sectional curvature function of M is constant on all holomorphic planes in TM, then M is said to be of *constant holomorphic sectional curvature*. Complete and connected Sasaki manifolds of constant holomorphic sectional curvature are called *Sasakian space forms*. Let us denote by R the Riemannian curvature tensor field of the metric g which is defined by

$$R(X,Y) := \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}, \qquad X,Y \in \mathfrak{X}(M).$$

When $(M,\eta;\xi,\varphi,g)$ is a Sasakian space form of constant holomorphic sectional curvature c, then R is described by the following formula:

$$\begin{aligned}R(X,Y)Z = &\frac{c+3}{4}\{g(Y,Z)X - g(Z,X)Y\}\\&+\frac{c-1}{4}\{\eta(Z)\eta(X)Y - \eta(Y)\eta(Z)X\\&+g(Z,X)\eta(Y)\xi - g(Y,Z)\eta(X)\xi\\&-g(Y,\varphi Z)\varphi X - g(Z,\varphi X)\varphi Y + 2g(X,\varphi Y)\varphi Z\}.\end{aligned}$$

Note that even if the holomorphic sectional curvature is negative, a Sasakian space form is *not* negatively curved. In fact, the sectional curvature of plane sections containing ξ is 1 on any Sasaki manifold.

It is known that every 3-dimensional Sasakian space form is realised as a Lie group together with a left invariant Sasaki structure. More precisely the following is known (cf. [6]):

PROPOSITION 1.2. *A simply connected* 3*-dimensional Sasakian space form of constant holomorphic sectional curvature is isomorphic to one of the following*:

(1) *special unitary group* $\mathrm{SU}(2)$;
(2) *Heisenberg group* $\mathbb{R}^3(-3)$;
(3) *the universal covering group of the special linear group* $\mathrm{SL}_2(\mathbb{R})$

together with the canonical left invariant Sasaki structure. In particular a simply connected Sasakian space form of constant holomorphic sectional curvature 1 *is* $\mathrm{SU}(2)$ *with biinvariant metric of constant curvature* 1 (*hence isometric to the unit* 3*-sphere* S^3).

1.2. *Boothby–Wang fibration.* Let $(M^{2n+1},\eta;\xi,\varphi,g)$ be a contact Riemannian manifold. Then M is said to be *regular* if ξ generates a one-parameter group K of isometries on M such that the action of K on M is simply transitive. Note that if M is regular, then both φ and η are automatically K-invariant, i.e. $\pounds_\xi\varphi = 0$ and $\pounds_\xi\eta = 0$. The Killing vector field ξ induces a regular one-dimensional Riemannian foliation on M. We denote by $\overline{M} := M/K$ the orbit space (the space of all leaves) of a regular contact Riemannian manifold M under the K-action.

Let $\overline{X}_{\bar{p}}$ be a tangent vector of the orbit space $\overline{M}$ at $\bar{p} = \pi(p)$. Then there exists a tangent vector $\overline{X}^*_p$ of M at p which is orthogonal to ξ such that $\pi_{*p}\overline{X}^*_p = \overline{X}_{\bar{p}}$. The tangent vector $\overline{X}^*_p$ is called the *horizontal lift* of $\overline{X}_{\bar{p}}$ to M at p. The horizontal lift operation $* : \overline{X}_{\bar{p}} \mapsto \overline{X}^*_p$ is naturally extended to vector fields.

The contact structure on M induces an almost Hermitian structure on the orbit space $\overline{M}$:

$$J\overline{X} = \pi_*(\varphi\overline{X}^*), \qquad \overline{X} \in \mathfrak{X}(\overline{M}). \tag{6}$$

Denote by $\overline{\nabla}$ the Levi-Civita connection of $\overline{M}$. Then the fundamental equations for Riemannian submersions due to O'Neill [33] imply the following results.

PROPOSITION 1.3 ([32]). *Let M be a regular contact Riemannian manifold. Then for any $\overline{X}, \overline{Y} \in \mathfrak{X}(\overline{M})$,*

$$\nabla_{\overline{X}^*}\overline{Y}^* = (\overline{\nabla}_{\overline{X}}\overline{Y})^* - g(\overline{X}^*, \varphi\overline{Y}^*)\xi. \tag{7}$$

PROPOSITION 1.4 ([32]). *Sasakian space forms are regular Sasaki manifolds. The orbit space of a Sasakian space form of constant holomorphic sectional curvature c is a complex space form of constant holomorphic sectional curvature $c + 3$.*

W. M. Boothby and H. C. Wang [8] proved that if M is a compact regular contact manifold, then the natural projection $\pi : M \to \overline{M}$ defines a principal circle bundle over a symplectic manifold $\overline{M}$ and the symplectic form Ω of $\overline{M}$ determines an integral cocycle. Furthermore the contact form η gives a connection form of this circle bundle and satisfies $\pi^*\Omega = d\eta$. The fibering $\pi : M \to \overline{M}$ is called the *Boothby–Wang fibering* of a regular compact contact manifold M. Based on this result, we call the fibering $\pi : M \to \overline{M}$ of a regular contact Riemannian manifold M the "Boothby–Wang fibering" of M even if M is noncompact.

The unit sphere S^{2n+1} is a typical example of a regular compact Sasaki manifold. For S^{2n+1}, the Boothby–Wang fibering coincides with the *Hopf fibering* $S^{2n+1} \to \mathbb{C}P^n$.

In the 3-dimensional case, the Boothby–Wang fiberings of Sasakian space forms have the following matrix group models [6]:

$$\pi : \mathrm{SU}(2) \to S^2(c) = \mathrm{SU}(2)/\mathrm{U}(1),$$
$$\pi : \mathbb{R}^3(-3) \to \mathbf{C} = \mathbb{R}^3(-3)/\mathbb{R},$$
$$\pi : \mathrm{SL}_2\mathbb{R} \to H^2(c) = \mathrm{SL}_2(\mathbb{R})/\mathrm{SO}(2).$$

Here $S^2(c)$ and $H^2(c)$ are sphere and hyperbolic space of curvature c, respectively.

1.3. *Hopf cylinders.* Now we shall restrict our attention to 3-dimensional regular contact Riemannian manifolds M.

Let $\bar{\gamma}$ be a curve parametrised by arc length in $\overline{M}$ with curvature $\bar{\kappa}$. Take the inverse image $S_{\bar{\gamma}} := \pi^{-1}\{\bar{\gamma}\}$ of $\bar{\gamma}$ in M^3.

Here we compute the fundamental quantities of $S_{\bar{\gamma}}$. Denote by $\overline{P} = (\overline{\mathbf{p}}_1, \overline{\mathbf{p}}_2)$ the Frenet frame field of $\bar{\gamma}$. By using the complex structure J of $\overline{M}^2$, $\overline{\mathbf{p}}_2$ is given by

$$\overline{\mathbf{p}}_2 = J\overline{\mathbf{p}}_1,$$

Then the Frenet–Serret formula for $\bar{\gamma}$ is

$$\overline{\nabla}_{\bar{\gamma}'}P = P\begin{pmatrix}\bar{\kappa} & 0\\ 0 & -\bar{\kappa}\end{pmatrix}.$$

Here the function $\bar{\kappa}$ is the (signed) curvature of $\bar{\gamma}$.

Let $\mathbf{t} = (\overline{\mathbf{p}}_1)^*$ be the horizontal lift of $\overline{\mathbf{p}}_1$ with respect to the Boothby–Wang fibering. Then $(\mathbf{t}, \xi)$ gives an orthonormal frame field of S. We choose a unit normal vector field $\mathbf{n}$ to be $\mathbf{n} = (\overline{\mathbf{p}}_2)^*$. Since $\bar{\mathbf{p}}_2$ is defined by $\overline{\mathbf{p}}_2 = J\overline{\mathbf{p}}_1$, we have $\mathbf{n} = \varphi\mathbf{t}$. In fact,

$$(\overline{\mathbf{p}}_2)^* = (J\overline{\mathbf{p}}_1)^* = \varphi(\overline{\mathbf{p}}_1)^* = \varphi\mathbf{t}.$$

Let us denote by ∇^S the Levi-Civita connection of S. The *second fundamental form II* derived from $\mathbf{n}$ is defined by the *Gauß formula*

$$\nabla_X Y = \nabla^S_X Y + \mathit{II}(X,Y)\mathbf{n}, \qquad X, Y \in \mathfrak{X}(S). \tag{8}$$

By (7),

$$\nabla_{\mathbf{t}}\mathbf{t} = (\overline{\nabla}_{\overline{\mathbf{p}}_1}\overline{\mathbf{p}}_1)^* - g(\mathbf{t}, \varphi\mathbf{t})\xi = (\bar{\kappa}\circ\pi)\mathbf{n}.$$

Hence $\nabla^S_{\mathbf{t}}\mathbf{t} = 0$. Since ξ is Killing, we have $\nabla^S_{\mathbf{t}}\xi = \nabla^S_{\xi}\xi = 0$. Thus $S_{\bar{\gamma}}$ is flat. The second fundamental form II is described as

$$\mathit{II}(\mathbf{t},\mathbf{t}) = \bar{\kappa}\circ\pi, \qquad \mathit{II}(\mathbf{t},\xi) = -1, \qquad \mathit{II}(\xi,\xi) = 0.$$

The mean curvature is $H = (\bar{\kappa}\circ\pi)/2$ and the mean curvature vector field $\mathbb{H}$ is $\mathbb{H} = H\mathbf{n}$.

In case $M = S^3$, $S_{\bar{\gamma}}$ is called the *Hopf cylinder*. In particular if $\bar{\gamma}$ is closed, then $S_{\bar{\gamma}}$ is a flat torus in S^3 and is called the *Hopf torus* over $\bar{\gamma}$ (H. B. Lawson, cf. [31], [35]). The Hopf torus over a geodesic in $S^2(4)$ coincides with the Clifford minimal torus. We call the flat surface $S_{\bar{\gamma}}$ in a regular contact Riemannian manifold M a *Hopf cylinder* over the curve $\bar{\gamma}$ in $\overline{M}$.

1.4. *Curves in Riemannian 3-manifolds.* Let (M, g) be a Riemannian manifold and $\gamma = \gamma(s) : I \to M$ a curve in M parametrised by arclength. We regard γ as a 1-dimensional Riemannian manifold with respect to the metric induced by g.

We recall the following definition (cf. [2]).

Definition 1.2. If $\gamma(s)$ is a unit speed curve in a Riemannian 3-manifold (M^3, g), we say that γ is a *Frenet curve* if there exists an orthonomal frame field $P = (\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3)$ along γ and two nonnegative functions κ and τ such that P satisfies the following *Frenet–Serret formula*:

$$\nabla_{\gamma'}P = P\begin{pmatrix}0 & -\kappa & 0\\ \kappa & 0 & -\tau\\ 0 & \tau & 0\end{pmatrix}, \qquad \mathbf{p}_1 = \gamma'(s).$$

The functions κ and τ are called the *curvature* and *torsion* of γ respectively.

Geodesics can be regarded as Frenet curves with $\kappa = 0$. A curve with constant curvature and zero torsion is called a (*Riemannian*) *circle*. A *helix* is a curve whose curvature and torsion are constants. Riemannian circles are regarded as degenerate helices. Helices which are not circles are frequently called *proper helices*.

Note that, in general ambient space (M^3, g), geodesics may have nonvanishing torsion. In fact, as we shall see later, Legendre geodesics in a Sasakian 3-manifold have constant torsion 1.

The Frenet–Serret formula for γ implies that the mean curvature vector field $\mathbb{H}$ of a Frenet curve γ is given by

$$\mathbb{H} = \nabla_{\gamma'}\gamma' = \kappa \mathbf{p}_2.$$

Let us denote by $\varDelta$ the Laplace operator acting on the space $\varGamma(\gamma^*TM)$ of all smooth sections of the vector bundle

$$\gamma^*TM := \bigcup_{s\in I} T_{\gamma(s)}M$$

over I. Then $\varDelta$ is given explicitly by

$$\varDelta = -\nabla_{\gamma'}\nabla_{\gamma'}.$$

LEMMA 1.1. *The mean curvature vector field* $\mathbb{H}$ *of a Frenet curve* γ *is harmonic in* γ^*TM $(\varDelta\mathbb{H} = 0)$ *if and only if*

$$\nabla_{\gamma'}\nabla_{\gamma'}\nabla_{\gamma'}\gamma' = 0.$$

When M is the Euclidean space $\mathbb{E}^m$, a curve γ satisfies $\varDelta\mathbb{H} = 0$ if and only if γ is *biharmonic*, i.e., $\varDelta\varDelta\gamma = 0$ since $\varDelta\gamma = -\mathbb{H}$.

The following general result is essentially obtained in [24].

THEOREM 1.1. *Let* γ *be a Frenet curve in a Riemannian* 3*-manifold* (M, g). *Then* γ *satisfies* $\varDelta\mathbb{H} = \lambda\mathbb{H}$ *in* γ^*TM *if and only if* γ *is a geodesic* $(\lambda = 0)$ *or a helix satisfying* $\lambda = \kappa^2 + \tau^2$.

Proof. Let I be an open interval and $\gamma : I \to M$ be a curve parametrised by arclength with Frenet frame field $P = (\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3)$. Direct computation shows that

$$\nabla_{\gamma'}\mathbb{H} = -\kappa^2\mathbf{p}_1 + \kappa'\mathbf{p}_2 + \kappa\tau\mathbf{p}_3. \tag{9}$$

Let us compute the Laplacian of $\mathbb{H}$:

$$-\varDelta\mathbb{H} = \nabla_{\gamma'}\nabla_{\gamma'}\mathbb{H} = -3\kappa\kappa'\mathbf{p}_1 + (\kappa'' - \kappa^3 - \kappa\tau^2)\mathbf{p}_2 + (2\kappa'\tau + \kappa\tau')\mathbf{p}_3.$$

Hence $\varDelta\mathbb{H} = \lambda\mathbb{H}$ if and only if

$$\kappa\tau' = 0, \qquad \kappa^3 + \kappa\tau^2 = \lambda\kappa.$$

These formulae imply that γ is a geodesic or a helix satisfying $\lambda = \kappa^2 + \tau^2$.

Conversely, every geodesic satisfies $\varDelta\mathbb{H} = 0$. Helices satisfy $\varDelta\mathbb{H} = \lambda\mathbb{H}$ with $\lambda = \kappa^2 + \tau^2$. ∎

COROLLARY 1.1 ([19]). *Let γ be a curve in Euclidean* 3*-space* $\mathbb{E}^3$*. Then γ is biharmonic if and only if γ is a straight line.*

On the contrary, in indefinite semi-Euclidean spaces, there exist non-geodesic biharmonic curves. Chen and Ishikawa [15] classified biharmonic spacelike curves in $\mathbb{E}^m_\nu$ (see also [28]).

1.5. *Curves with normal-harmonic mean curvature.* The results in the preceding subsection imply that to characterise curves which are not geodesics we need to use another differential operator.

In this subsection we use the *normal* Laplacian.

Let $\gamma : I \to M$ be a Frenet curve in an oriented Riemannian 3-manifold M parametrised by arclength. Denote by $P = (\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3)$ the Frenet frame field of γ as before. Then the *normal bundle* $T^\perp\gamma$ of the curve γ is given by

$$T^\perp\gamma = \bigcup_{s\in I} T^\perp_s\gamma, \qquad T^\perp_s\gamma = \mathbb{R}\mathbf{p}_2(s) \oplus \mathbb{R}\mathbf{p}_3(s).$$

The *normal connection* $\nabla^\perp$ is the connection in $T^\perp\gamma$ defined by

$$\nabla^\perp_{\gamma'} X = \text{normal component of } \nabla_{\gamma'} X$$

for any section X of the normal bundle $T^\perp\gamma$.

By using the Frenet frame field, $\nabla^\perp$ can be represented as

$$\nabla^\perp_{\gamma'} X = \nabla_{\gamma'} X - g(\nabla_{\gamma'} X, \mathbf{p}_1)\mathbf{p}_1.$$

Denote by $\Delta^\perp$ the Laplace operator acting on the space $\Gamma(T^\perp\gamma)$ of all smooth sections of the normal bundle $T^\perp\gamma$. The operator $\Delta^\perp$ is called the *normal Laplacian* of γ in M. The normal Laplacian $\Delta^\perp$ is given by

$$\Delta^\perp X = -\nabla^\perp_{\gamma'}\nabla^\perp_{\gamma'} X, \qquad X \in \Gamma(T^\perp\gamma).$$

Now we compute $\Delta^\perp\mathbb{H}$. From (9), we have

$$\nabla^\perp_{\gamma'}\mathbb{H} = \kappa'\mathbf{p}_2 + \kappa\tau\mathbf{p}_3.$$

From this equation, we get

$$-\Delta^\perp\mathbb{H} = (\kappa'' - \kappa\tau^2)\mathbf{p}_2 + (2\kappa'\tau + \kappa\tau')\mathbf{p}_3.$$

THEOREM 1.2 (CF. [24]). *A curve γ satisfies $\Delta^\perp\mathbb{H} = \lambda\mathbb{H}$ if and only if*

$$\kappa'' - \kappa\tau^2 = -\lambda\kappa, \quad 2\kappa'\tau + \kappa\tau' = 0.$$

COROLLARY 1.2. *A curve γ satisfies $\Delta^\perp\mathbb{H} = 0$ if and only if*

$$\kappa'' - \kappa\tau^2 = 0, \quad 2\kappa'\tau + \kappa\tau' = 0.$$

We shall apply these general results for curves in Sasaki 3-manifolds in the next section. Note that Barros and Garay classified curves which satisfy $\Delta^\perp\mathbb{H} = \lambda\mathbb{H}$ in space forms [4], [5].

2. Curves and surfaces in 3-dimensional Sasaki manifolds

2.1. *Curves in 3-dimensional Sasaki manifolds.* Now let $M^3 = (M, \eta, \xi, \varphi, g)$ be a contact Riemannian 3-manifold with an associated metric g. A curve $\gamma : I \to M$ parametrised by arclength is said to be a *Legendre curve* if γ is tangent to the contact distribution D of M. It is obvious that γ is Legendre if and only if $\eta(\gamma') = 0$.

Let γ be a Legendre curve in M^3. Then we can take a Frenet frame field $P = (\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3)$ so that $\mathbf{p}_1 = \gamma'$ and $\mathbf{p}_3 = \xi$ (see [2]).

Now we assume that M is a Sasaki manifold. Then by definition, the Frenet–Serret formula for γ is given explicitly by

$$\nabla_{\gamma'} P = P \begin{pmatrix} 0 & -\kappa & 0 \\ \kappa & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Hence every Legendre curve has constant torsion 1 (see [2]).

Now we investigate curves with harmonic or normal-harmonic mean curvature vector field in Sasakian 3-manifolds.

The following two results are direct consequences of Theorems 1.1 and 1.2, respectively.

Corollary 2.1. *Let γ be a Legendre curve in a 3-dimensional Sasaki manifold. Then γ satisfies $\Delta\mathbb{H} = \lambda\mathbb{H}$ in γ^*TM if and only if γ is a Legendre geodesic $(\lambda = 0)$ or a Legendre helix satisfying $\lambda = \kappa^2 + 1$ $(\lambda \neq 0)$.*

Remark 2.1. Sasaki manifolds with a compatible Lorentz metric are called *Sasakian spacetimes* ([20], [40]). On Sasakian spacetimes, the Reeb vector fields are timelike. Every 3-dimensional Sasakian spacetime contains proper biharmonic Legendre curves. In fact, in a 3-dimensional Sasakian spacetime biharmonic Legendre curves are Legendre geodesics or Legendre helices with curvature 1 (cf. [28]).

Proposition 2.1. *Let γ be a Legendre curve in a Sasaki 3-manifold. Then $\Delta^{\perp}\mathbb{H} = \lambda\mathbb{H}$ if and only if γ is a Legendre geodesic $(\lambda = 0)$ or a Legendre helix with constant nonzero curvature $(\lambda \neq 0)$. In the latter case, $\lambda = 1$.*

2.2. *Biharmonic Hopf cylinders.* In this section we study harmonicity and normal-harmonicity of the mean curvature of Hopf cylinders.

Let M^3 be a regular Sasaki manifold with Boothby–Wang fibration $\pi : M \to \overline{M}$.

Take a curve $\bar{\gamma}$ parametrised by arclength s in the base space form $\overline{M}$. Denote by $S = S_{\bar{\gamma}}$ the Hopf cylinder of $\bar{\gamma}$ (see Section 1.3).

Let $\mathbf{t} = (\overline{\mathbf{p}}_1)^*$ be the horizontal lift of $\overline{\mathbf{p}}_1$ with respect to the Boothby–Wang fibering. Then $(\mathbf{t}, \xi)$ gives an orthonormal frame field of M. The unit normal vector field $\mathbf{n}$ is the horizontal lift of $\overline{\mathbf{p}}_2$. Note that $\mathbf{n} = \varphi\mathbf{t}$.

The mean curvature vector field $\mathbb{H}$ of S is $\mathbb{H} = H\mathbf{n} = (\bar{\kappa} \circ \pi)\mathbf{n}/2$.

Now we study harmonicity and normal-harmonicity of $\mathbb{H}$. Denote by ι the inclusion map of S into M. Then the Laplace operator Δ acting on the space $\Gamma(\iota^*TM)$ and the normal Laplacian $\Delta^\perp$ of S are given by

$$\Delta = -(\nabla_{\mathbf{t}}\nabla_{\mathbf{t}} + \nabla_\xi\nabla_\xi), \qquad \Delta^\perp = -(\nabla^\perp_{\mathbf{t}}\nabla^\perp_{\mathbf{t}} + \nabla^\perp_\xi\nabla^\perp_\xi),$$

respectively. Direct computation shows that

$$\nabla_{\mathbf{t}}\mathbb{H} = -2H^2\mathbf{t} + H'\mathbf{n} + H\xi, \qquad \nabla^\perp_{\mathbf{t}}\mathbb{H} = H'\mathbf{n} + H\xi,$$
$$\nabla_\xi\mathbb{H} = H\mathbf{t}, \qquad \nabla^\perp_\xi\mathbb{H} = 0, \qquad \nabla_\xi\nabla_\xi\mathbb{H} = -H\mathbf{n}.$$

Thus we get

$$-\Delta\mathbb{H} = -6HH'\mathbf{t} + (H'' - 4H^3 - 2H)\mathbf{n} + 2H'\xi, \qquad -\Delta^\perp\mathbb{H} = H''\mathbf{n}.$$

THEOREM 2.1. *A Hopf cylinder $S_{\bar{\gamma}}$ in a 3-dimensional regular Sasaki manifold satisfies $\Delta\mathbb{H} = \lambda\mathbb{H}$ in ι^*TM if and only if $\bar{\gamma}$ is a geodesic* $(\lambda = 0)$ *or a Riemannian circle* $(\lambda \neq 0)$. *If $\lambda \neq 0$, then $\lambda = 4H^2 + 2 > 2$.*

REMARK 2.2. Every Hopf cylinder in a 3-dimensional regular Sasaki manifold is *anti-invariant*. Sasahara showed that an anti-invariant surface in $\mathbb{R}^3(-3)$ satisfies $\Delta\mathbb{H} = \lambda\mathbb{H}$ with $\lambda \neq 0$ if and only if it is a Hopf cylinder over a circle with $\lambda > 2$ (see Proposition 11 in [37]).

LEMMA 2.1. *A Hopf cylinder $S_{\bar{\gamma}}$ satisfies $\Delta^\perp\mathbb{H} = \lambda\mathbb{H}$ if and only if γ is defined by one of the following natural equations*:

(1) $\bar{\kappa}(s) = as + b,\ a, b \in \mathbb{R},\ \lambda = 0$;
(2) $\bar{\kappa}(s) = a\cos(\sqrt{\lambda}\, s) + b\sin(\sqrt{\lambda}\, s),\ \lambda > 0$;
(3) $\bar{\kappa}(s) = a\exp(\sqrt{-\lambda}\, s) + b\exp(-\sqrt{-\lambda}\, s),\ \lambda < 0$.

Proof. The Hopf cylinder $S_{\bar{\gamma}}$ satisfies $\Delta^\perp\mathbb{H} = \lambda\mathbb{H}$ if and only if $\bar{\gamma}$ satisfies

$$\bar{\kappa}'' + \lambda\bar{\kappa} = 0.$$

Hence the result follows. ∎

THEOREM 2.2. *A Hopf cylinder $S_{\bar{\gamma}}$ satisfies $\Delta^\perp\mathbb{H} = 0$ if and only if $\bar{\gamma}$ is one of the following*:

(1) *a geodesic*;
(2) *a Riemannian circle*;
(3) *a Riemannian clothoid* (*Cornu spiral*).

Here a Riemannian clothoid is a curve in $\overline{M}^2$ whose curvature is a linear function of arclength.

Remark 2.3. On curves in Riemannian 2-space forms, the following result is obtained [24]:

Theorem 2.3. *Let $\bar{\gamma}$ be a curve in a Riemannian 2-manifold $\overline{M}^2$. To avoid confusion, denote by $\Delta^{\perp}_{\bar{\gamma}}$ and $\mathbb{H}_{\bar{\gamma}}$ the normal Laplacian of $\bar{\gamma}$ and the mean curvature vector in $\overline{M}^2$ respectively. Then $\Delta^{\perp}_{\bar{\gamma}}\mathbb{H}_{\bar{\gamma}} = \lambda\mathbb{H}_{\bar{\gamma}}$ if and only if*

(1) *$\bar{\gamma}$ is a geodesic, a Riemannian circle or a Riemannian clothoid*;
(2) $\bar{\kappa}(s) = a\cos(\sqrt{\lambda}\,s) + b\sin(\sqrt{\lambda}\,s),\ \lambda > 0$;
(3) $\bar{\kappa}(s) = a\exp(\sqrt{-\lambda}\,s) + b\exp(-\sqrt{-\lambda}\,s),\ \lambda < 0$.

Corollary 2.2. *Let M be a 3-dimensional regular Sasaki manifold with Boothby–Wang fibering $\pi : M \to \overline{M}$. Let $\bar{\gamma}$ be a curve in $\overline{M}$. Then the Hopf cylinder $S = S_{\bar{\gamma}}$ satisfies $\Delta^{\perp}\mathbb{H} = \lambda\mathbb{H}$ if and only if $\bar{\gamma}$ satisfies $\Delta^{\perp}_{\bar{\gamma}}\mathbb{H}_{\bar{\gamma}} = \lambda\mathbb{H}_{\bar{\gamma}}$.*

Theorem 2.2 is a generalisation of a result obtained by Barros and Garay [3]. In fact, if we choose $M^3 = S^3$ then we obtain the following.

Theorem 2.4 ([3]). *A Hopf cylinder $S_{\bar{\gamma}}$ in the unit 3-sphere S^3 satisfies $\Delta^{\perp}\mathbb{H} = 0$ if and only if γ is one of the following*:

(1) *a geodesic*;
(2) *a Riemannian circle*;
(3) *a Riemannian clothoid.*

Here a Riemannian clothoid is a curve in the 2-sphere $S^2(1/2)$ of radius $1/2$ whose curvature is a linear function of arclength.

Riemannian clothoids are called “Cornu spirals” in [3].

PART II

3. Polyharmonic maps. Let (M^m, g) and (N^n, h) be Riemannian manifolds and $\phi : M \to N$ a smooth map. The *tension field* $\mathcal{T}(\phi)$ is a section of the vector bundle $\phi^*(TN)$ defined by

$$\mathcal{T}(\phi) := \operatorname{tr}(\nabla d\phi).$$

A smooth map ϕ is said to be a *harmonic map* if its tension field vanishes. It is well known that ϕ is harmonic if and only if ϕ is a critical point of the *energy*

$$E(\phi) = \int \frac{1}{2}|d\phi|^2\, dv_g$$

over every compact region of M.

Now let ϕ be a harmonic map. Then the Hessian $\mathcal{H}_\phi$ of the energy is given by the following second variation formula:

$$\mathcal{H}_\phi(V, W) = \int h(\mathcal{J}_\phi(V), W)\, dv_g, \qquad V, W \in \Gamma(\phi^*TN).$$

Here the operator $\mathcal{J}_\phi$ is the *Jacobi operator* of the harmonic map ϕ defined by

$$\mathcal{J}_\phi(V) := \overline{\Delta}_\phi V - \mathcal{R}_\phi(V), \qquad V \in \Gamma(\phi^*TN),$$

$$\overline{\Delta}_\phi := -\sum_{i=1}^{m}(\nabla^\phi_{e_i}\nabla^\phi_{e_i} - \nabla^\phi_{\nabla_{e_i}e_i}), \qquad \mathcal{R}_\phi(V) = \sum_{i=1}^{m} R^N(V, d\phi(e_i))e_i.$$

Here ∇^ϕ, R^N and $\{e_i\}$ denote the induced connection of ϕ^*TN, the curvature tensor of N and a local orthonormal frame field of M, respectively.

For the general theory of harmonic maps and their Jacobi operators, we refer to [21] and [42].

J. Eells and J. H. Sampson suggested studying *polyharmonic maps* (see [23] and [21, p. 77 (8.7)]). Let $\phi : M \to N$ be a smooth map as before. Then ϕ is said to be *polyharmonic* of order k if it is an extremal of the functional

$$E_k(\phi) = \int |(d + d^*)^k\phi|^2\, dv_g.$$

Here d^* is the codifferential operator. In particular, if $k = 2$, we have

$$E_2(\phi) = \int |\mathcal{T}(\phi)|^2\, dv_g.$$

The functional E_k satisfies condition (C) of Palais–Smale if $k > m/2$. In particular, in the order 2 case, E_2 satisfies (C) if $m \leq 3$.

The Euler–Lagrange equation of the functional E_2 was computed by Caddeo and Oproiu (see [9, p. 867]) and G. Y. Jiang [29]–[30], independently. The Euler–Lagrange equation of E_2 is

$$\mathcal{T}_2(\phi) := -\mathcal{J}_\phi(\mathcal{T}(\phi)) = 0.$$

REMARK 3.1. Let $\phi : M \to N$ be an isometric immersion. Then its tension field is $m\mathbb{H}$. Thus the functional E_2 is given by

$$E_2(\phi) = m^2 \int |\mathbb{H}|^2\, dv_g.$$

In case M is 2-dimensional, $E_2(\phi)$ is the total mean curvature of M up to a constant multiple. See [11, Section 5.3].

In particular, if $N = \mathbb{E}^n$ and ϕ is an isometric immersion, then

$$\mathcal{T}_2(\phi) = -\Delta_M \Delta_M \phi,$$

since $\Delta_M\phi = m\mathbb{H}$. Here Δ_M is the Laplacian of (M, g). Thus the polyharmonicity (of order 2) for an isometric immersion into Euclidean space is equivalent to the biharmonicity in the sense of Chen. For this reason, polyharmonic maps of order 2 are frequently called *biharmonic maps* (or 2-harmonic maps) [9], [29], [30], [34].

Obviously, the notion of p-harmonic map in the sense of [22, p. 397] is different from that of polyharmonic map of order p.

Hereafter we call polyharmonic maps of order 2 “polyharmonic maps” for short.

Caddeo, Montaldo and Oniciuc classified polyharmonic curves in 3-dimensional Riemannian space forms. More precisely they showed the following two results.

THEOREM 3.1 ([9]). *Let N be a 3-dimensional Riemannian space form of nonpositive curvature. Then all the polyharmonic curves are geodesics.*

Next for the study of polyharmonic curves in positively curved space forms, we may assume that N^3 is the unit 3-sphere.

THEOREM 3.2 ([9]). *Let $\gamma : I \to S^3$ be a polyharmonic curve parametrised by arclength. Then $0 \leq \kappa \leq 1$ and γ is one of the following*:

(1) *a geodesic $\kappa = 0$;*
(2) *a Riemannian circle of curvature 1 if $k = 1$;*
(3) *a geodesic of the Clifford minimal torus of S^3 if $0 < \kappa < 1$.*

The preceding theorem implies the following result:

COROLLARY 3.1. *Let $\gamma : I \to S^3$ be a Legendre curve parametrised by arclength. Then γ is polyharmonic if and only if γ is a Legendre geodesic.*

In fact, curves in the last two classes cannot be Legendre. (Recall that every Legendre curve has constant torsion 1.)

Now we study polyharmonic Legendre curves in contact Riemannian 3-manifolds. Let M^3 be a contact Riemannian 3-manifold and $\gamma : I \to M$ a Frenet curve framed by $(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3)$. Then direct computation shows that

$$\mathcal{T}_2(\gamma) = -3\kappa\kappa'\mathbf{p}_1 + (\kappa'' - \kappa^3 - \kappa\tau^2)\mathbf{p}_2 + (2\kappa'\tau + \kappa\tau')\mathbf{p}_3 + \kappa R(\mathbf{p}_2, \mathbf{p}_1)\mathbf{p}_1.$$

Now assume that M is a Sasakian space form of constant holomorphic sectional curvature c. Then

$$\begin{aligned} R(\mathbf{p}_2, \mathbf{p}_1)\mathbf{p}_1 = \frac{c+3}{4}\,\mathbf{p}_2 + \frac{c-1}{4}\,\{&\eta(\mathbf{p}_2)\eta(\mathbf{p}_1)\mathbf{p}_1 \\ &- \eta(\mathbf{p}_1)^2\mathbf{p}_2 - \eta(\mathbf{p}_2)\xi + 3g(\mathbf{p}_2, \varphi\mathbf{p}_1)\varphi\mathbf{p}_1\}. \end{aligned}$$

In particular, if γ is Legendre, then $R(\mathbf{p}_2, \mathbf{p}_1)\mathbf{p}_1 = c\mathbf{p}_2$. Thus a Legendre curve γ in M is polyharmonic if and only if

$$\kappa = \text{constant}, \quad \kappa^3 - (c-1)\kappa = 0, \quad \tau = 1.$$

If we look for nongeodesic polyharmonic Legendre curves, we obtain

$$\kappa = \text{constant}, \quad \kappa^2 = c - 1, \quad \tau = 1.$$

Thus we obtain the following result which is a generalisation of Corollary 3.1.

THEOREM 3.3. *Let $M^3(c)$ be a Sasakian space form of constant holomorphic sectional curvature c and $\gamma : I \to M$ a polyharmonic Legendre curve parametrised by arclength.*

(1) *If $c \leq 1$, then γ is a Legendre geodesic.*

(2) *If* $c > 1$, *then* γ *is a Legendre geodesic or a Legendre helix of curvature* $\sqrt{c-1}$.

Let $\phi : M \to N$ be an isometric immersion. Then ϕ is a critical point of the volume functional if and only if ϕ is minimal. The *Jacobi operator* $\mathcal{J}$ of a minimal immersion ϕ (with respect to the volume functional) appears in the second variation formula of the volume and is given by [39]

$$\mathcal{J}V = \Delta^{\perp}V - \mathcal{S}V + \mathcal{R}(V), \qquad V \in \Gamma(T^{\perp}M).$$

Here the operators $\mathcal{S}$ and $\mathcal{R}$ are defined by

$$h(\mathcal{S}V, W) = \mathrm{tr}(\mathcal{A}_V \circ \mathcal{A}_W), \qquad \mathcal{R}(V) = \sum_{i=1}^{m} (R^N(d\phi(e_i), V)d\phi(e_i))^{\perp}.$$

Here $\mathcal{A}_V$ denotes the Weingarten operator with respect to V.

Arroyo, Barros and Garay studied submanifolds in S^3 whose mean curvature vector fields are eigen-sections of the Jacobi operator with respect to the volume functional [1], [4], [5]. For surfaces in 5-dimensional Sasakian space forms, such a study can be found in [37].

It seems interesting to study similar problems for submanifolds in space forms or Sasakian space forms with respect to the energy functional.

In [9], all the polyharmonic surfaces in S^3 are classified. More precisely, the only nonminimal polyharmonic surfaces are totally umbilical 2-spheres.

Based on this result, we propose the following problem:

Are there nonminimal and nontotally umbilical polyharmonic submanifolds in homogeneous Riemannian manifolds?

To close this paper, we study polyharmonic Hopf cylinders in 3-dimensional Sasakian space forms. Moreover we show the existence of nonminimal and non-totally umbilical polyharmonic surfaces in Sasakian space forms.

First we recall the following result which is a consequence of the main result in [9]:

PROPOSITION 3.1. *There are no nonminimal polyharmonic Hopf cylinders in the unit* 3*-sphere* S^3.

Now we generalise this result to Sasakian space forms.

Let $S = S_{\bar{\gamma}}$ be a Hopf cylinder and $\iota : S \subset M^3(c)$ its inclusion map into a Sasakian space form $M^3(c)$. Then the bitension field $\mathcal{T}_2(\iota)$ is given by

$$\mathcal{T}_2(\iota) = -\mathcal{J}_\iota(\mathcal{T}(\iota)) = -2\mathcal{J}_\iota(\mathbb{H}).$$

We use the orthonormal frame field $\{\mathbf{t}, \xi\}$ as before. Then since S is flat, we have

$$\overline{\Delta}_\iota \mathbb{H} = \Delta\mathbb{H}, \qquad \mathcal{R}(\mathbb{H}) = H(R(\mathbf{n}, \mathbf{t})\mathbf{t} + R(\mathbf{n}, \xi)\xi).$$

Using the curvature formula for a Sasakian space form, we get

$$\mathcal{R}(\mathbb{H}) = (c+1)H\mathbf{n}.$$

Hence

$$\mathcal{J}_\iota(\mathbb{H}) = 6HH'\mathbf{t} - (H'' - 4H^3 + (c-1)H)\mathbf{n} - 2H'\xi.$$

Thus $\mathcal{J}_\iota(\mathbb{H}) = \lambda\mathbb{H}$ if and only if

$$H' = 0, \qquad 4H^3 = (c - 1 + \lambda)H$$

and hence $H = 0$ or $\lambda = 4H^2 + 1 - c,\ H \neq 0$.

THEOREM 3.4. *Let S be a Hopf cylinder in a Sasakian space form $M^3(c)$. Then S satisfies $\mathcal{J}_\iota(\mathbb{H}) = \lambda\mathbb{H}$ if and only if the base curve of S is a Riemannian circle or a geodesic. If it is not a geodesic*, *then $\lambda = 4H^2 + 1 - c$.*

COROLLARY 3.2. *Let $\iota : S_{\bar{\gamma}} \to M^3(c)$ be a polyharmonic Hopf cylinder in a Sasakian space form.*

(1) *If $c \leq 1$ then $\bar{\gamma}$ is a geodesic.*
(2) *If $c > 1$ then $\bar{\gamma}$ is a geodesic or a Riemannian circle of curvature $\bar{\kappa} = \sqrt{c-1}$.*

In particular, *there exist nonminimal polyharmonic Hopf cylinders in Sasakian space forms of holomorphic sectional curvature greater than* 1.

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