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SUBMANIFOLDS WITH PARALLEL MEAN CURVATURE VECTOR OF A EUCLIDEAN SPACE OR A SPHERE

BY KENTARO YANO

§ 0. Introduction.

Liebmann [11] and Süss [19] proved that the only convex hypersurface with constant mean curvature in a Euclidean space is a sphere. To prove this theorem, we need integral formulas of Minkowski in which the so-called position vector plays a very important rôle.

Recently various attempts have been done to generalize this theorem of Liebmann and Süss to the case of hypersurfaces in a Riemannian manifold. See, for example, Hsiung [3], Katsurada [4, 5], Koyanagi [10], Ōtsuki [17], Tani [20, 26], Yano [21, 22, 26]. In these papers, authors assume the existence of a vector field in the Riemannian manifold or that of a vector field along the hypersurface which plays the rôle of the position vector in the case of hypersurfaces in a Euclidean space and prove that, under certain conditions, the hypersurface under consideration is umbilical.

When the ambient Riemannian manifold is a general one, it is almost impossible to give conditions under which the hypersurface is isometric to a sphere, but when the ambient Riemannian manifold admits a scalar function v such that $\nabla\nabla v=f(v)g$, where g is the Riemannian metric and ∇ the Riemannian connection, we can give conditions under which the hypersurface under consideration is isometric to a sphere. (See Yano [22]).

The attempts have recently been started to generalize the above results to the cases of general submanifolds in a Riemannian manifold by Katsurada [6, 7, 8], Kôjyô [7] and Nagai [8, 12]. They assume that the ambient Riemannian manifold admits a conformal Killing vector field and that this vector field is contained in the linear space spanned by the mean curvature vector of the submanifold and the tangent space of the submanifold.

The present author [24] studied similar problems under conditions a little bit weaker than those of Katsurada, Kôjyô and Nagai.

The main purpose of the present paper is to determine all the submanifolds in a Euclidean space or in a sphere which satisfy the conditions imposed by the present author in [24].

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§ 1. Preliminaries [9].

Let E^m be an m-dimensional Euclidean space in which an orthogonal coordinate system is introduced. A point P of E^m is represented by the so-called position vector X from the origin O to the point P.

Let M^n $(1<n<m)$ be an n-dimensional differentiable submanifold covered by a system of coordinate neighborhoods $\{U; u^a\}$ and imbedded differentiably in E^m by a vector equation

$$X=X(u^a), \tag{1.1}$$

where here and in the sequel the indices a, b, c, d, e run over the range $\{1, 2, \cdots, n\}$. We put

$$X_b=\partial_b X, \qquad \partial_b=\partial/\partial u^b. \tag{1.2}$$

The X_b are n linearly independent vectors tangent to the submanifold M^n. Assuming M^n to be orientable, we choose $m-n$ mutually orthogonal local unit vectors C_x normal to M^n in such a way that $X_1, \cdots, X_n, C_{n+1}, \cdots, C_m$ form a positive orientation of E^m, where here and in the sequel the indices x, y, z run over the range $\{n+1, \cdots, m\}$.

We put

$$g_{cb}=X_c\cdot X_b, \tag{1.3}$$

where the dot denotes the inner product of vectors in E^m and denote by $\{{}_c{}^a{}_b\}$, ∇_c, and $K_{dcb}{}^a$ the Christoffel symbols formed with g_{cb}, the operator of covariant differentiation with respect to $\{{}_c{}^a{}_b\}$, and the curvature tensor of the submanifold M^n respectively.

Then the equations of Gauss of M^n in E^m are

$$\nabla_c X_b=\partial_c X_b-\left\{ {a \atop c\ b} \right\} X_a=H_{cb}{}^x C_x, \tag{1.4}$$

where $H_{cb}{}^x=H_{bc}{}^x$ are components of the second fundamental tensors with respect to the normals C_x and those of Weingarten of M^n in E^m are

$$\nabla_c C_y=\partial_c C_y=-H_c{}^a{}_y X_a+L_{cy}{}^x C_x, \tag{1.5}$$

where

$$H_c{}^a{}_y=H_c{}^{ay}=H_{cb}{}^y g^{ba}, \tag{1.6}$$

g^{ba} being contravariant components of the metric tensor and

$$L_{cy}{}^x=-L_{cx}{}^y \tag{1.7}$$

are components of the third fundamental tensor.

Now, equations of Gauss, Codazzi and Ricci of M^n in E^m are respectively

(1. 8)
$$K_{dcb}{}^a = H_d{}^a{}_x H_{cb}{}^x - H_c{}^a{}_x H_{db}{}^x,$$

(1. 9)
$$\nabla_d H_c{}^a{}_y - \nabla_c H_d{}^a{}_y + H_d{}^a{}_x L_{cy}{}^x - H_c{}^a{}_x L_{dy}{}^x = 0,$$

and

(1. 10)
$$\nabla_d L_{cy}{}^x - \nabla_c L_{dy}{}^x + L_{dz}{}^x L_{cy}{}^z - L_{cz}{}^x L_{dy}{}^z + H_d{}^a{}_y H_{ca}{}^x - H_c{}^a{}_y H_{da}{}^x = 0.$$

Let S^{m-1} be a sphere in E^m with centre at the origin and with radius r and we denote the parametric representation of S^{m-1} by

(1. 11)
$$X = X(x^h),$$

where here and in the sequel, h, i, j, k run over the range $\{1, 2, \cdots, m-1\}$.

We put

(1. 12)
$$X_i = \partial_i X, \qquad G_{ji} = X_j \cdot X_i, \qquad \partial_i = \partial / \partial x^i,$$

and denote by ∇_i the operator of covariant differentiation with respect to Christoffel symbols $\{{}_j{}^h{}_i\}$ formed with G_{ji}. Then equations of Gauss and Weingarten of S^{m-1} in E^m are respectively

(1. 13)
$$\nabla_j X_i = \partial_j X_i - \left\{ \begin{matrix} h \\ j \ i \end{matrix} \right\} X_h = -cG_{ji} C$$

and

(1. 14)
$$\nabla_j C = \partial_j C = cX_j,$$

where $c = 1/r$ and

(1. 15)
$$C = cX$$

is the unit normal to the sphere S^{m-1}.

We take an arbitrary fixed unit vector V in E^m and put

(1. 16)
$$v = V \cdot X,$$

then we have

$$\nabla_i v = V \cdot X_i,$$

$$\nabla_j \nabla_i v = V \cdot \nabla_j X_i = V \cdot (-cG_{ji} C),$$

that is,

(1. 17)
$$\nabla_j \nabla_i v = -c^2 v G_{ji}$$

by virtue of $C = cX$, [14]. In the sequel, we put

(1. 18)
$$v_i = \nabla_i v, \qquad v^h = v_i G^{ih},$$

then we have

$$\nabla_j v^h = -c^2 v \delta_j^h. \tag{1.19}$$

We consider an n-dimensional differentiable submanifold M^n of S^{m-1} covered by a system of coordinate neighborhoods $\{U; u^a\}$ and imbedded differentiably in S^{m-1}. We represent it by

$$x^h = x^h(u^a). \tag{1.20}$$

We put

$$B_b{}^h = \partial_b x^h \tag{1.21}$$

and denote by $C_u{}^h$ $m-1-n$ mutually orthogonal local unit vectors normal to M^n in S^{m-1}, where here and in the sequel u, v, w run over the range $\{n+1, \cdots, m-1\}$.

Then the metric g_{cb} of M^n induced from that of S^{m-1} is given by

$$g_{cb} = G_{ji} B_c{}^j B_b{}^i. \tag{1.22}$$

The equations of Gauss and Weingarten of M^n in S^{m-1} are respectively

$$\begin{aligned} \nabla_c B_b{}^h &= \partial_c B_b{}^h + \left\{ {h \atop j\ i} \right\} B_c{}^j B_b{}^i - B_a{}^h \left\{ {a \atop c\ b} \right\} \\ &= h_{cb}{}^u C_u{}^h, \end{aligned} \tag{1.23}$$

$h_{cb}{}^u$ being components of the second fundamental tensor of M^n with respect to the normals $C_u{}^h$ and

$$\begin{aligned} \nabla_c C_u{}^h &= \partial_c C_u{}^h + \left\{ {h \atop j\ i} \right\} B_c{}^j C_u{}^i \\ &= -h_c{}^a{}_u B_a{}^h + l_{cu}{}^v C_v{}^h, \end{aligned} \tag{1.24}$$

$l_{cu}{}^v$ being components of the third fundamental tensors of M^n with respect to the normals $C_u{}^h$.

The equations of Gauss, Codazzi, and Ricci of M^n in S^{m-1} are respectively

$$K_{dcb}{}^a = c^2(\delta_d^a g_{cb} - \delta_c^a g_{db}) + h_d{}^a{}_u h_{cb}{}^u - h_c{}^a{}_u h_{db}{}^u, \tag{1.25}$$

$$\nabla_d h_{cbv} - \nabla_c h_{dbv} - l_{dv}{}^u h_{cbu} + l_{cv}{}^u h_{dbu} = 0, \tag{1.26}$$

and

$$\nabla_d l_{cv}{}^u - \nabla_c l_{dv}{}^u + l_{dw}{}^u l_{cv}{}^w - l_{cw}{}^u l_{dv}{}^w + h_d{}^a{}_v h_{ca}{}^u - h_c{}^a{}_v h_{da}{}^u = 0. \tag{1.27}$$

Now we regard the submanifold M^n of S^{m-1} as a submanifold of the Euclidean space E^m, then we have

$$X = X(x^h(u^a)), \tag{1.28}$$

(1. 29) $$X_b = B_b{}^h X_h,$$

and consequently

$$\nabla_c X_b = (\nabla_c B_b{}^h) X_h + B_c{}^j B_b{}^i \nabla_j X_i,$$

or

(1. 30) $$\nabla_c X_b = h_{cb}{}^u C_u{}^i X_i - c g_{cb} C$$

by virtue of (1. 13), (1. 22) and (1. 23).

If we regard

(1. 31) $$C_u = C_u{}^i X_i, \qquad \text{and} \qquad C_m = C = cX$$

as normals to M^n in E^m, we have, comparing (1. 4) with (1. 30),

(1. 32) $$H_{cb}{}^u = h_{cb}{}^u, \qquad H_{cb}{}^m = -c g_{cb}.$$

From the first equation of (1. 31), we have

$$\nabla_c C_u = (\nabla_c C_u{}^i) X_i + B_c{}^j C_u{}^i \nabla_j X_i,$$

or

$$\nabla_c C_u = (-h_c{}^a{}_u B_a{}^i + l_{cu}{}^v C_v{}^i) X_i,$$

that is,

(1. 33) $$\nabla_c C_u = -h_c{}^a{}_u X_a + l_{cu}{}^v C_v.$$

Comparing (1. 5) in which $y=u$ with (1. 33), we find

(1. 34) $$H_c{}^a{}_u = h_c{}^a{}_u, \qquad L_{cu}{}^v = l_{cu}{}^v, \qquad L_{cy}{}^m = 0.$$

From the second equation of (1. 31), we have

(1. 35) $$\nabla_c C_m = c X_c.$$

Comparing (1. 5) in which $y=m$, that is,

$$\nabla_c C_m = -H_c{}^a{}_m X_a + L_{cm}{}^x C_x$$

with (1. 35), we find

(1. 36) $$H_c{}^a{}_m = -c\delta_c^a, \qquad L_{cm}{}^x = 0.$$

Thus, equations of Gauss and Weingarten of M^n in S^{m-1} in E^m are respectively

(1. 37) $$\nabla_c X_b = h_{cb}{}^u C_u - c g_{cb} C,$$

(1. 38) $$\nabla_c C_u = -h_c{}^a{}_u X_a + l_{cu}{}^v C_v,$$

$$\nabla_c C = cX_c, \tag{1.39}$$

g_{cb}, $h_{cb}{}^u$ and $l_{cu}{}^v$ being respectively the first, second and third fundamental tensors of M^n in S^{m-1}.

§ 2. Mean curvature vectors of M^n with respect to E^m and S^{m-1}.

The mean curvature vector of M^n with respect to E^m is defined to be

$$\frac{1}{n} g^{cb} \nabla_c X_b = \frac{1}{n} H_a{}^{ax} C_x. \tag{2.1}$$

Thus the mean curvature vector of M^n with respect to E^m is an intrinsic normal to M^n. If this mean curvature vector vanishes identically M^n is said to be minimal in E^m.

Suppose that M^n is a submanifold of a sphere S^{m-1} in E^m, then the mean curvature vector of M^n with respect to S^{m-1} is defined to be

$$\frac{1}{n} g^{cb} \nabla_c B_b{}^h = \frac{1}{n} h_a{}^{au} C_u{}^h. \tag{2.2}$$

Thus the mean curvature vector of M^n with respect to S^{m-1} is an intrinsic normal to M^n. If this mean curvature vector vanishes identically M^n is said to be minimal in S^{m-1}.

Now, for a submanifold M^n of S^{m-1} in E^m, we have

$$X_b = B_b{}^h X_h$$

and consequently

$$\frac{1}{n} g^{cb} \nabla_c X_b = \left(\frac{1}{n} g^{cb} \nabla_c B_b{}^h \right) X_h - c^2 X. \tag{2.3}$$

Thus we have

LEMMA 2.1. *Let M^n be a submanifold of a sphere S^{m-1} in E^m. Then the difference of the mean curvature vector of M^n with respect to E^m and that with respect to S^{m-1} is in the direction of the radius vector of S^{m-1}.*

LEMMA 2.2. *Let M^n be a submanifold of a sphere S^{m-1} in E^m. A necessary and sufficient condition for M^n to be minimal in S^{m-1} is that the mean curvature vector of M^n with respect to E^m lies in the direction of the radius vector of S^{m-1}* [1, 2, 13, 18].

We now assume that the mean curvature vector of M^n with respect to E^m never vanishes and choose the last normal C_m in the direction of the mean curvature vector. We then put

$$C_m = C, \qquad H_{cbm} = H_{cb}, \qquad L_{cm}{}^x = L_c{}^x. \tag{2. 4}$$

Then from

$$\begin{aligned}\frac{1}{n} g^{cb} \nabla_c X_b &= \frac{1}{n} H_a{}^{ax} C_x \\ &= \frac{1}{n} H_a{}^{au} C_u + \frac{1}{n} H_a{}^a C,\end{aligned}$$

we find

$$H_a{}^{au} = 0, \tag{2. 5}$$

and

$$\frac{1}{n} g^{cb} \nabla_c X_b = \frac{1}{n} H_a{}^a C. \tag{2. 6}$$

Thus if the mean curvature vector is parallel with respect to the connection induced in the normal bundle, that is, if the covariant derivative of the mean curvature vector

$$\nabla_c \left(\frac{1}{n} H_a{}^a C \right) = \frac{1}{n} (\nabla_c H_a{}^a) C + \frac{1}{n} H_a{}^a (-H_c{}^b X_b + L_c{}^x C_x)$$

is tangent to M^n, then we have

$$H_a{}^a = \text{const.} \neq 0, \qquad L_c{}^x = 0 \tag{2. 7}$$

by virtue of $L_c{}^m = L_{cm}{}^m = 0$.

We next assume that the mean curvature vector of M^n with respect to S^{m-1} never vanishes and choose the last normal $C_{m-1}{}^h$ in the direction of the mean curvature vector. We put

$$C_{m-1}{}^h = C'^h, \qquad H_{cb\,m-1} = h_{cb}, \qquad l_{c\,m-1}{}^u = l_c{}^u. \tag{2. 8}$$

Then, from

$$\begin{aligned}\frac{1}{n} g^{cb} \nabla_c B_b{}^h &= \frac{1}{n} h_a{}^{au} C_u{}^h \\ &= \frac{1}{n} h_a{}^{ar} C_r{}^h + \frac{1}{n} h_a{}^a C'^h,\end{aligned}$$

we find

$$h_a{}^{ar} = 0 \tag{2. 9}$$

and

$$\frac{1}{n} g^{cb} \nabla_c B_b{}^h = \frac{1}{n} h_a{}^a C'^h, \tag{2. 10}$$

where here and in the sequel r, s, t run over the range $\{n+1, \cdots, m-2\}$.

Thus if the mean curvature vector of M^n with respect to S^{m-1} is parallel with respect to the connection induced in the normal bundle of M^n in S^{m-1}, that is, if the covariant derivative of the mean curvature vector

$$\nabla_c \left(\frac{1}{n} h_a{}^a C'^h \right) = \frac{1}{n} (\nabla_c h_a{}^a) C'^h + \frac{1}{n} h_a{}^a (-h_c{}^b B_b{}^h + l_c{}^u C_u{}^h)$$

is tangent to M^n, then we have

$$h_a{}^a = \text{const.} \neq 0, \qquad l_c{}^r = 0 \tag{2. 11}$$

by virtue of $l_c{}^{m-1} = 0$.

§ 3. Integral formulas.

Let M^n be a submanifold in E^m. We put

$$X = X_a v^a + C_x \alpha^x \tag{3. 1}$$

and differentiate this covariantly along M^n. Then we obtain

$$X_c = (H_{cb}{}^x C_x) v^b + X_a \nabla_c v^a + (-H_c{}^a{}_x X_a + L_{cx}{}^y C_y) \alpha^x + C_x \nabla_c \alpha^x,$$

from which

$$\nabla_c v^a = \delta_c^a + H_c{}^a{}_x \alpha^x \tag{3. 2}$$

and

$$\nabla_c \alpha^x = -H_{cb}{}^x v^b - L_{cy}{}^x \alpha^y, \tag{3. 3}$$

from which

$$\nabla_a v^a = n + H_a{}^a{}_x \alpha^x. \tag{3. 4}$$

Assuming M^n to be compact and orientable and integrating, we have from equation (3. 4)

$$\int_{M^n} (n + H_a{}^a{}_x \alpha^x) dS = 0, \tag{3. 5}$$

where dS is the surface element of M^n.

On the other hand, we have

$$\begin{aligned} \nabla_a (H_b{}^a v^b) &= (\nabla_a H_b{}^a) v^b + H_b{}^a (\delta_a^b + H_a{}^b{}_x \alpha^x) \\ &= (\nabla_a H_b{}^a) v^b + H_a{}^a + H_{cb} H^{cb}{}_x \alpha^x \end{aligned}$$

by virtue of (3. 2), from which, by integration,

$$\int_{M^n} \{(\nabla_a H_b{}^a)v^b + H_a{}^a + H_{cb}H^{cb}{}_x\alpha^x)\} dS = 0. \tag{3. 6}$$

If we assume that the mean curvature vector of M^n in E^m is parallel with respect to the connection induced in the normal bundle, we have (2. 7) and consequently we have from (1. 9)

$$\nabla_a H_b{}^a = \nabla_b H_a{}^a = 0.$$

Thus (3. 6) becomes

$$\int_{M^n} (H_a{}^a + H_{cb}H^{cb}{}_x\alpha^x) dS = 0. \tag{3. 7}$$

Now subtracting (3. 5)$\times(1/n)H_b{}^b$ from (3. 7), we obtain

$$\int_{M^n} \left(H_{cb}H^{cb}{}_x\alpha^x - \frac{1}{n} H_b{}^b H_a{}^a{}_x\alpha^x \right) dS = 0. \tag{3. 8}$$

Let M^n be a submanifold of a sphere S^{m-1} in E^m. We put

$$v^h = B_a{}^h v^a + C_u{}^h \alpha^u \tag{3. 9}$$

and differentiate this covariantly along M^n. Then we obtain

$$-c^2 v B_c{}^h = (h_{cb}{}^u C_u{}^h)v^b + B_a{}^h \nabla_c v^a + (-h_c{}^a{}_u B_a{}^h + l_{cu}{}^v C_v{}^h)\alpha^u + C_u{}^h \nabla_c \alpha^u,$$

from which

$$\nabla_c v^a = -c^2 v \delta_c^a + h_c{}^a{}_u \alpha^u \tag{3. 10}$$

and

$$\nabla_c \alpha^u = -h_{cb}{}^u v^b - l_{cv}{}^u \alpha^v, \tag{3. 11}$$

from which

$$\nabla_c v^c = -nc^2 v + h_a{}^a{}_u \alpha^u. \tag{3. 12}$$

Assuming M^n to be compact and orientable and integrating, we have from equation (3. 12)

$$\int_{M^n} (-nc^2 v + h_a{}^a{}_u \alpha^u) dS = 0. \tag{3. 13}$$

On the other hand, we have

$$\begin{aligned} \nabla_a(h_b{}^a v^b) &= (\nabla_a h_b{}^a)v^b + h_b{}^a(-c^2 v \delta_a^b + h_a{}^b{}_u \alpha^u) \\ &= (\nabla_a h_b{}^a)v^b - c^2 v h_a{}^a + h_{cb}h^{cb}{}_u \alpha^u, \end{aligned}$$

by virtue of (3. 12), from which by integration

$$\int_{M^n}\{(\nabla_a h_b{}^a)v^b - c^2 v h_a{}^a + h_{cb}h^{cb}{}_u\alpha^u\}dS=0. \tag{3. 14}$$

If we assume that the mean curvature vector of M^n in S^{m-1} is parallel with respect to the connection induced in the normal bundle, we have (2. 11) and consequently we have from (1. 26)

$$\nabla_a h_b{}^a = \nabla_b h_a{}^a = 0.$$

Thus (3. 14) becomes

$$\int_{M^n}(-c^2 v h_a{}^a + h_{cb}h^{cb}{}_u\alpha^u)dS=0. \tag{3. 15}$$

Now subtracting (3. 13)×$(1/n)h_a{}^a$ from (3. 15), we find

$$\int_{M^n}\left(h_{cb}h^{cb}{}_u\alpha^u - \frac{1}{n}h_b{}^b h_a{}^a{}_u\alpha^u\right)dS=0. \tag{3. 16}$$

§4. Applications of integral formulas.

Let M^n be a compact and orientable submanifold of E^m whose mean curvature vector is parallel with respect to the connection induced in the normal bundle. We assume that

$$H_{cbx}\alpha^x = H_{cb}\alpha, \qquad (H_{cbm}=H_{cb},\ \alpha^m=\alpha) \tag{4. 1}$$

that is,

$$H_{cb\,n+1}\alpha^{n+1}+\cdots+H_{cb\,m-1}\alpha^{m-1}=0. \tag{4. 2}$$

Since

$$\nabla_c X_b = H_{cb}{}^x C_x \qquad \text{and} \qquad X = X_a v^a + C_x\alpha^x,$$

the assumption (4. 1) is equivalent to

$$(\nabla_c X_b)\cdot X = H_{cb}\alpha. \tag{4. 3}$$

If the assumption (4. 1) is satisfied, then integral formula (3. 8) reduces to

$$\int_{M^n}\alpha\left(H_{cb}H^{cb}-\frac{1}{n}H_b{}^b H_a{}^a\right)dS=0,$$

or to

$$\int_{M^n}\alpha\left(H_{cb}-\frac{1}{n}H_e{}^e g_{cb}\right)\left(H^{cb}-\frac{1}{n}H_d{}^d g^{cb}\right)dS=0. \tag{4. 4}$$

Thus, if α has a fixed sign, we have

$$H_{cb}=\frac{1}{n}H_d{}^d g_{cb} \tag{4.5}$$

or

$$H_{cb}=\frac{1}{\lambda}g_{cb}, \tag{4.6}$$

λ being a constant, that is, the submanifold M^n is umbilical with respect to the mean curvature normal, or M^n is pseudo-umbilical.

Since

$$H_c{}^a=\frac{1}{\lambda}\delta_c^a, \qquad L_c{}^x=0,$$

we have, from equation (1. 5) of Weingarten,

$$\nabla_c C=-\frac{1}{\lambda}X_c,$$

from which

$$C=-\frac{1}{\lambda}X+\frac{1}{\lambda}A$$

or

$$X-A=-\lambda C, \tag{4.7}$$

A being a constant vector, which shows that the submanifold M^n lies on a sphere S^{m-1} with centre at A and radius $|\lambda|$. Thus we have

THEOREM 4. 1. *Suppose that the mean curvature vector of a compact and orientable submanifold M^n of a Euclidean space E^m does not vanish and that we take the last unit normal C_m to M^n in the direction of the mean curvature vector. If the mean curvature vector is parallel with respect to the connection induced in the normal bundle of M^n in E^m, $H_{cbx}\alpha^x=H_{cb}\alpha$, and α has a fixed sign, then the submanifold M^n lies on a sphere S^{m-1}.*

The assumption

$$H_{cbx}\alpha^x=H_{cb}\alpha$$

or

$$H_{cb\,n+1}\alpha^{n+1}+\cdots+H_{cb\,m-1}\alpha^{m-1}=0$$

is satisfied if

$$\text{(A)} \qquad \alpha^{n+1}=0, \cdots, \alpha^{m-1}=0$$

or

$$\text{(B)} \qquad H_{cb\,n+1}=0, \cdots, H_{cb\,m-1}=0$$

is satisfied.

We first assume that condition (A) is satisfied. Then we have

$$X=X_a v^a+C\alpha, \tag{4.8}$$

that is, the position vector X is in the linear space spanned by the tangent plane to M^n and the mean curvature vector of M^n with respect to E^m. This is the condition assumed by Katsurada [6, 7, 8], Kôjyô [7] and Nagai [8] in their study of submanifolds with parallel mean curvature vector.

On the other hand, we have, from (1. 37),

$$\frac{1}{n} g^{cb} \nabla_c X_b = \frac{1}{n} h_a{}^{au} C_u - c^2 X \tag{4.9}$$

and consequently, comparing (4. 8) with (4. 9) we find

$$v^a=0, \qquad h_a{}^{ar}=0, \tag{4.10}$$

which mean that the position vector X is in the direction of the mean curvature vector of M^n with respect to E^m and the submanifold M^n is a minimal submanifold of the sphere S^{m-1}.

Conversely, suppose that the submanifold M^n lies on a sphere S^{m-1} and the radius vector X of S^{m-1} is in the direction of the mean curvature vector of M^n with respect to E^m, then we have

$$X=rC \tag{4.11}$$

r being the radius of S^{m-1}, and consequently

$$\nabla_c C = \frac{1}{r} X_c, \tag{4.12}$$

which shows that the mean curvature vector of M^n with respect to E^m is parallel with respect to the connection induced in the normal bundle of M^n. Equation (4. 11) shows that the condition (A) is satisfied and $\alpha=r$ has a fixed sign. Thus we have

THEOREM 4. 2. *In order that a submanifold M^n satisfying the conditions given in Theorem* 4. 1 *satisfies the additional condition* (A), *it is necessary and sufficient that M^n lies on a sphere S^{m-1} and the radius vector of S^{m-1} is in the direction of the mean curvature vector of M^n with respect to E^m. In this case the submanifold M^n is minimal in S^{m-1}.*

We next assume that the condition (B) is satisfied. In this case, equations of

Gauss become

$$\nabla_c X_b = H_{cb} C \tag{4.13}$$

and equations of Weingarten become

$$\nabla_c C_u = L_{cu}{}^v C_v \tag{4.14}$$

and

$$\nabla_c C = -H_c{}^a X_a. \tag{4.15}$$

Equation (4.14) gives

$$\nabla_d L_{cv}{}^u - \nabla_c L_{dv}{}^u + L_{dw}{}^u L_{cv}{}^w - L_{cw}{}^u L_{dv}{}^w = 0, \tag{4.16}$$

which shows that we can assume that the $L_{cv}{}^u$ vanish and consequently that

$$\nabla_c C_u = 0, \tag{4.17}$$

that is, the unit normals C_u are constant vectors. Since

$$\nabla_c (C_u \cdot X) = 0,$$

that is

$$C_u \cdot X = \text{const.},$$

the submanifold M^n is on an $(n+1)$-dimensional plane E^{n+1} in E^m.

Since $H_{cb} = \lambda g_{cb}$, λ being a constant, we have, from (4.13) and (4.15),

$$\nabla_c X_b = \lambda h_{cb} C \quad \text{and} \quad \nabla_c C = -\lambda X_c$$

respectively, where X can be regarded as the position vector in an $(n+1)$-dimensional Euclidean space, thus, from the second equation above, we have

$$\nabla_c (C + \lambda X) = 0,$$

from which

$$C + \lambda X = \lambda B,$$

B being a constant vector and consequently the submanifold M^n is a sphere S^n lying in an $(n+1)$-dimensional Euclidean space E^{n+1}.

Conversely, a sphere S^n in E^m satisfies all the conditions stated in Theorem 4.1 and (B). Thus we have

THEOREM 4.3. *In order that a submanifold M^n satisfyiug the conditions given in Theorem* 4.1 *satisfies the additional condition* (B), *it is necessary and sufficient that the submanifold is an n-dimensional sphere.*

Next, let M^n be a compact and orientable submanifold of a sphere S^{m-1} in E^m.

We assume that

$$h_{cbu}\alpha^u = h_{cb}\alpha, \tag{4.18}$$

that is,

$$h_{cb\,n+1}\alpha^{n+1} + \cdots + h_{cb\,m-2}\alpha^{m-2} = 0. \tag{4.19}$$

Since

$$\nabla_c B_b{}^h = h_{cb}{}^u C_u{}^h \qquad \text{and} \qquad v^h = B_a{}^h v^a + C_u{}^h \alpha^u,$$

the assumption (4. 18) is equivalent to

$$G_{ji}(\nabla_c B_b{}^j) v^i = h_{cb}\alpha. \tag{4.20}$$

If the assumption (4. 18) is satisfied, then integral formula (3. 16) reduces to

$$\int_{M^n} \alpha \left(h_{cb} h^{cb} - \frac{1}{n} h_b{}^b h_a{}^a \right) dS = 0,$$

or to

$$\int_{M^n} \alpha \left(h_{cb} - \frac{1}{n} h_e{}^e g_{cb} \right) \left(h^{cb} - \frac{1}{n} h_d{}^d g^{cb} \right) dS = 0. \tag{4.21}$$

Thus, if α has a fixed sign, we obtain

$$h_{cb} = \frac{1}{n} h_d{}^d g_{cb}, \tag{4.22}$$

or

$$h_{cb} = \frac{1}{\lambda} g_{cb}, \tag{4.23}$$

λ being a constant, that is, M^n is pseudo-umbilical in S^{m-1}.

Since

$$h_c{}^a = \frac{1}{\lambda} \delta_c^a, \qquad l_c{}^u = 0,$$

we have, from equation (1. 33) of Weingarten

$$\nabla_c C' = -\frac{1}{\lambda} X_c, \tag{4.24}$$

from which

$$\nabla_c (\lambda C' + X) = 0,$$

that is,

$$\lambda C' + X = V,$$

V being a constant vector, which shows that the submanifold M^n lies on the sphere with centre at V and with radius $|\lambda|$. Thus we have

THEOREM 4.4. *Suppose that the mean curvature vector of a compact and orientable submanifold M^n of a sphere S^{m-1} does not vanish and that we take the last unit normal $C_{m-1}{}^h$ in the direction of the mean curvature vector of M^n in S^{m-1}. If the mean curvature vector of M^n in S^{m-1} is parallel with respect to the connection induced in the normal bundle of M^n in S^{m-1}, $h_{cbu}\alpha^u = h_{cb}\alpha$ and α has a fixed sign, then the submanifold M^n lies on a sphere S^{m-2}.*

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Department of Mathematics,
Tokyo Institute of Technology.