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SUBMANIFOLDS WITH PROPER d-PLANAR GEODESICS IMMERSED IN COMPLEX PROJECTIVE SPACES

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Introduction. Recently, several authors studied submaifolds with "simple" geodesics immersed in space forms. For example, planar geodesic immersions were studied in [6], [8], [13], [14], geodesic normal sections in [3] and helical immersions in [15]. In [9], Nakagawa also introduced a notion of cubic geodesic immersions. Let M and $\tilde{M}$ be connected complete Riemannian manifolds of dimensions n and $n+p$, respectively. An isometric immersion ι of M into $\tilde{M}$ is called a *d-planar geodesic immersion* if each geodesic in M is mapped locally under ι into a d-dimesional totally geodesic submanifold of $\tilde{M}$. In particular, if a 3-planar geodesic immersion is isotropic, then it is called a *cubic geodesic immersion*. In this paper, we study a proper d-planar geodesic Kählerian immersion $\iota: M \to CP^m(c)$ of a Kähler manifold M into a complex projective space $CP^m(c)$ of constant holomorphic sectional curvature c and proper cubic geodesic totally real immersion $\iota: M \to CP^m(c)$ of a Riemannian manifold M, where "proper" means that the image of each geodesic in M is not $(d-1)$-planar. Here and elsewhere, m in N^m denotes the complex dimension, if N is a complex manifold.

In a complex projective space $CP^m(c)$ of complex dimension m, an odd-dimensional totally geodesic submanifold is a totally real submanifold $RP^l(c/4)$ of constant sectional curvature $c/4$. In § 2 we show that if $\iota: M^n \to CP^m(c)$ is a proper d-planar geodesic Kählerian immersion of a Kähler manifold M^n and d is odd, then $M^n = CP^n(c/d)$ and ι is equivalent to the d-th Veronese map. Here we recall the definition of k-th Veronese map ($k=1, 2, \cdots$). This is a Kähler imbedding $CP^n(c/k) \to CP^{m'}(c)$ given by

$$[z_i]_{0\leq i\leq n} \mapsto \left[\left(\frac{k!}{k_0! \cdots k_n!}\right)^{1/2} z_0^{k_0} \cdots z_n^{k_n}\right]_{k_0+\cdots+k_n=k},$$

where $[*]$ means the point of the projective space with the homogeneous coordinates $*$ and $m' = \binom{n+k}{k} - 1$. More generally, we prove that if

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the image of each geodesic in M^n is locally properly contained in a d-dimensional totally real totally geodesic submanifold, then $M^n = CP^n(c/d)$ and ι is equivalent to the d-th Veronese map. This result is a geometric characterization of the Veronese map.

In § 3, we consider a proper cubic geodesic totally real immersion $\iota: M^n \to CP^m(c)$ of a Riemannian manifold M^n of dimension n. We shall prove that $\iota(M^n)$ is contained in a totally real submanifold $RP^{n+q}(c/4)$ and apply Nakagawa's theorem:

THEOREM N. *For $n \geqq 3$, let M be an n-dimensional compact simply connected Riemannian manifold and ι a proper cubic geodesic immersion of M into an $(n+p)$-dimensional sphere $S^{n+p}(c)$, where $p \geqq 2$. If ι is minimal, then $M = S^n(nc/3(n+2))$ and ι is equivalent to the immersion $\iota_0 \circ \iota_3$ of S^n into S^{n+p}, where ι_0 is a totally geodesic immersion of $S^{N(3)}(c)$ into S^{n+p}, $N(3)+1$ is the multiplicity of the third eigenvalue of the Laplace operator of S^n and ι_3 is the third standard minimal immersion of S^n into $S^{N(3)}(c)$.*

Here we recall the definition of the k-th standard minimal immersion of S^n into S^{n+p} (cf. [4]). Let $H^{k,n}$ be the eigenspace of the k-th eigenvalue of the Laplace operator on S^n, where $\dim H^{k,n} = (n+2k-1)(n+k-2)!/k!(n-1)! =: N(k)+1$. For an orthonormal basis $\{f_1, \cdots, f_{N(k)+1}\}$ of $H^{k,n}$, an immersion ι_k of S^n into an $(N(k)+1)$-dimensional Euclidean space $E^{N(k)+1}$ defined by $\iota_k(x) = (f_1(x), \cdots, f_{N(k)+1}(x))/(N(k)+1)^{1/2}$ is a minimal isometric immersion into the unit hypersphere $S^{N(k)}(1)$ in $E^{N(k)+1}$ and $\iota_k(S^n)$ is not contained in any great sphere of $S^{N(k)}$ (i.e., full). If $k \leqq 3$, then ι_k is rigid (cf. [23]). The immersion ι_k is called a *k-th standard minimal immersion.*

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1. Preliminaries. Let M and $\tilde{M}$ be connected Riemannian manifolds and $\iota: M \to \tilde{M}$ an isometric immersion. We denote by $\tilde{\nabla}$ the covariant differentiation with respect to the Riemannian metric of $\tilde{M}$. Then we may write

$$\tilde{\nabla}_X Y = \nabla_X Y + H(X, Y) \tag{1.1}$$

for arbitrary tangent vector fields X and Y on M, where $\nabla_X Y$ and $H(X, Y)$ denote the components of $\tilde{\nabla}_X Y$ tangent and normal to M, respectively. Then ∇ becomes the covariant differentiation of the Riemannian manifold M. The symmetric bilinear form H valued in the normal bundle is called the *second fundamental form* of the immersion ι. For a normal vector field C on a neighborhood of $P \in M$, we write

$$\tilde{\nabla}_X C = -A_C X + \nabla^{\perp}_X C , \tag{1.2}$$

$-A_C X$ and $\nabla^{\perp}_X C$ being the components of $\tilde{\nabla}_X C$ tangent and normal to M, respectively, where $\nabla^{\perp}$ is the covariant differentiation with respect to the induced connection in the normal bundle $T^{\perp}M$ which will be called the *normal connection*. Denoting by $\langle\ ,\ \rangle$ the inner product with respect to the Riemannian metric of $\tilde{M}$, we find that A_C and H are related by $\langle A_C X, Y\rangle = \langle H(X, Y), C\rangle$ for any vectors X, Y tangent to M. Thus A_C is a symmetric linear transformation of $T_P M$.

Let the ambient manifold $\tilde{M}$ be a complete, simply connected complex space form with constant holomorphic sectional curvature c. Thus $\tilde{M}$ is a complex projective space $CP^m(c)$. If we denote by $\tilde{J}$ the complex structure, the Riemannian curvature tensor $\tilde{R}$ of $CP^m(c)$ is of the form

$$\begin{aligned}\tilde{R}(\tilde{X}, \tilde{Y})\tilde{Z} = (c/4)\{&\langle \tilde{Y}, \tilde{Z}\rangle \tilde{X} - \langle \tilde{X}, \tilde{Z}\rangle \tilde{Y} + \langle \tilde{J}\tilde{Y}, \tilde{Z}\rangle \tilde{J}\tilde{X} \\ &- \langle \tilde{J}\tilde{X}, \tilde{Z}\rangle \tilde{J}\tilde{Y} - 2\langle \tilde{J}\tilde{X}, \tilde{Y}\rangle \tilde{J}\tilde{Z}\}\end{aligned} \tag{1.3}$$

for all vectors $\tilde{X}$, $\tilde{Y}$, $\tilde{Z}$ tangent to $CP^m(c)$.

We denote by Proj_{TM} and $\mathrm{Proj}_{T^{\perp}M}$ the projections of $T_P\tilde{M}$ to the tangent space $T_P M$ and the normal space $T_P^{\perp}M$, respectively and put $J = \mathrm{Proj}_{TM} \circ \tilde{J} \mid TM$, $J_N = \mathrm{Proj}_{T^{\perp}M} \circ \tilde{J} \mid TM$, $J_T = \mathrm{Proj}_{TM} \circ \tilde{J} \mid T^{\perp}M$ and $J^{\perp} = \mathrm{Proj}_{T^{\perp}M} \circ \tilde{J} \mid T^{\perp}M$. Then we can write

$$\tilde{J}X = JX + J_N X , \qquad \tilde{J}C = J_T C + J^{\perp}C \tag{1.4}$$

for every tangent vector X and normal vector C of M. Taking account of $\tilde{J}^2 = -I$, we find that these tensors satisfy

$$\begin{aligned} J^2 + J_T J_N &= -I , \qquad & J_N J + J^{\perp} J_N &= 0 , \\ J^{\perp 2} + J_N J_T &= -I , \qquad & J J_T + J_T J^{\perp} &= 0 , \end{aligned} \tag{1.5}$$

I being the identity transformation, and also we have

$$\langle J_N X, C\rangle = -\langle X, J_T C\rangle \tag{1.6}$$

with the help of $\langle \tilde{J}\tilde{X}, \tilde{Y}\rangle = -\langle \tilde{X}, \tilde{J}\tilde{Y}\rangle$.

Differentiating covariantly the left hand side of (1.4), and using $\tilde{\nabla}\tilde{J} = 0$ and (1.4) itself, we can easily see that

$$\begin{aligned} (D_X J)Y &= A_{J_N Y} X + J_T H(Y, X) , \\ (D_X J_N)Y &= J^{\perp} H(Y, X) - H(JY, X), \\ (D_X J_T)C &= A_{J^{\perp}C} X - J A_C X , \\ (D_X J^{\perp})C &= -J_N A_C X - H(X, J_T C) , \end{aligned} \tag{1.7}$$

where D denotes the van der Waerden-Bortolotti covariant differentiation.

Let us denote the curvature tensors of the connections ∇ and $\nabla^{\perp}$ by R and $R^{\perp}$, respectively. Then, using (1.3), we find that the structure equations of Gauss, Codazzi and Ricci are respectively given by

$$R(X, Y)Z = (c/4)\{\langle Y, Z\rangle X - \langle X, Z\rangle Y + \langle JY, Z\rangle JX - \langle JX, Z\rangle JY - 2\langle JX, Y\rangle JZ\} + A_{H(Y,Z)}X - A_{H(X,Z)}Y , \tag{1.8}$$

$$(D_XH)(Y, Z) - (D_YH)(X, Z) = (c/4)\{\langle JY, Z\rangle J_NX - \langle JX, Z\rangle J_NY - 2\langle JX, Y\rangle J_NZ\} , \tag{1.9}$$

$$R^{\perp}(X, Y)C = (c/4)\{\langle J_NY, C\rangle J_NX - \langle J_NX, C\rangle J_NY - 2\langle JX, Y\rangle J^{\perp}C\} + H(X, A_CY) - H(A_CX, Y) , \tag{1.10}$$

where $(D_XH)(Y, Z) = \nabla^{\perp}_X(H(Y, Z)) - H(\nabla_XY, Z) - H(Y, \nabla_XZ)$. Therefore, if the submanifold M is complex or totally real, that is, $J_N = 0$ or $J = 0$, then

$$(D_XH)(Y, Z) - (D_YH)(X, Z) = 0 \tag{1.11}$$

because of (1.9). Conversely, if (1.11) is verified at every point of M, then M is complex or totally real. Thus 3-dimensional complete totally geodesic submanifolds in $\boldsymbol{CP}^m(c)$ are $\boldsymbol{RP}^3(c/4)$.

Sometimes we denote $(D_XH)(Y, Z)$ by $(DH)(X, Y, Z)$. It is clear that DH is a normal bundle-valued tensor field of type (0, 3). For $k \geqq 1$, the k-th covariant derivative of H is defined by

$$(D^kH)(X_1, X_2, \cdots, X_{k+2}) = \nabla^{\perp}_{X_1}((D^{k-1}H)(X_2, \cdots, X_{k+2})) - \sum_{i=2}^{k+2}(D^{k-1}H)(X_2, \cdots, \nabla_{X_1}X_i, \cdots, X_{k+2}) , \tag{1.12}$$

where $D^0H = H$. It is clear that D^kH is a normal bundle-valued tensor field of type $(0, k+2)$. By direct computation we have

$$(D^kH)(X_1, X_2, X_3, \cdots, X_{k+2}) - (D^kH)(X_2, X_1, X_3, \cdots, X_{k+2}) = R^{\perp}(X_1, X_2)((D^{k-2}H)(X_3, \cdots, X_{k+2})) - \sum_{i=3}^{k+2}(D^{k-2}H)(X_3, \cdots, R(X_1, X_2)X_i, \cdots, X_{k+2}) \tag{1.13}$$

for $k \geqq 2$.

As for the second fundamental form H, if

$$\| H(X, X)\|^2 = \lambda^2 \tag{1.14}$$

for every unit vector X tangent to M, then the immersion ι is said to be *isotropic* (or λ-*isotropic*). The immersion ι is isotropic if and only if

$$\langle H(X, X), H(X, Y)\rangle = 0 \tag{1.15}$$

for any orthonormal vectors X and Y at every point. The condition is equivalent to

$$\mathfrak{S}_3\langle H(X_1, X_2), H(X_3, Y)\rangle = \lambda^2\mathfrak{S}_3\langle X_1, X_2\rangle\langle X_3, Y\rangle , \tag{1.16}$$

where X_i $(i = 1, 2, 3)$ and Y are unit vectors and $\mathfrak{S}_3$ denotes the cyclic sum with respect to vectors X_1, X_2, X_3.

2. d-planar geodesic Kähler immersions. Let $\iota: M^n \to \boldsymbol{C}P^m(c)$ be a Kähler immersion of a connected complete Kähler manifold M^n into $\boldsymbol{C}P^m(c)$. We first prove:

PROPOSITION 2.1. *Suppose that for each geodesic $\gamma: \boldsymbol{R} \to M^n$ and each $s \in \boldsymbol{R}$, there exist an open interval I_s $(\ni s)$ and a totally real totally geodesic submanifold P_s of $\boldsymbol{C}P^m(c)$ such that $\iota(\gamma(I_s)) \subset P_s$. Then M^n is a compact simply connected Hermitian symmetric space.*

PROOF. Let $x \in M^n$ be any point and X any unit tangent vector at x of M^n. Let γ be the unit speed geodesic satisfying $\gamma(0) = x$ and $\dot\gamma(0) = X$. Since P_0 is totally geodesic, we see that $\dot\tau$, $\tilde\nabla_{\dot\tau}\dot\tau$ and $\tilde\nabla^2_{\dot\tau}\dot\tau$ is tangent to P_0 on I_0, where $\tau = \iota\circ\gamma$. Since γ is geodesic, we have

$$\begin{aligned}\dot\tau(0) &= X ,\\ (\tilde\nabla_{\dot\tau}\dot\tau)(0) &= H(X, X) ,\\ (\tilde\nabla^2_{\dot\tau}\dot\tau)(0) &= -A_{H(X,X)}X + (DH)(X, X, X) .\end{aligned}$$

From the assumption that P_0 is totally real, we find

$$\langle\tilde J H(X, X), (DH)(X, X, X)\rangle = 0 . \tag{2.1}$$

Now we have $J_N = 0$ and $J_T = 0$, since ι is a Kähler immersion. It follows from (1.7) that

$$H(JY, X) = J^\perp H(Y, X) , \qquad H(JY, JX) = -H(Y, X) \tag{2.2}$$

for every $X, Y \in T_xM$. Moreover, Codazzi's equation (1.11) and (2.2) imply that

$$(DH)(JZ, Y, X) = J^\perp(DH)(Z, Y, X) \tag{2.3}$$

for every $Z, Y, X \in T_xM$. Equation (2.1) holds for every $X \in T_xM$. Replacing X by JX in (2.1) and using (2.2) and (2.3), we thus have

$$\langle H(X, X), (DH)(X, X, X)\rangle = 0 \tag{2.4}$$

for every $X \in T_xM$. Let X and Y be orthonormal tangent vectors. Let $X(\theta) = \cos\theta X + \sin\theta Y$. Differentiating $\langle H(X(\theta), X(\theta)), (DH)(X(\theta), X(\theta), X(\theta))\rangle = 0$ at $\theta = 0$, we see that

$$2\langle H(X, Y), (DH)(X, X, X)\rangle + 3\langle H(X, X), (DH)(X, X, Y)\rangle = 0 .$$

This equation holds for all $X, Y \in T_xM$ in virtue of (2.4). Replacing X by JX in the above equation, we have

$$-2\langle H(X, Y), (DH)(X, X, X)\rangle + 3\langle H(X, X), (DH)(X, X, Y)\rangle = 0 ,$$

and hence

$$\langle H(X, Y), (DH)(X, X, X)\rangle = 0 \tag{2.5}$$

for every $X, Y \in T_xM$. Symmetrize (2.5) with respect to X. Then for every X, Y, Z,

$$\langle H(Z, Y), (DH)(X, X, X)\rangle + 3\langle H(X, Y), (DH)(X, X, Z)\rangle = 0 .$$

Replacing Z and Y by JZ, JY respectively, we see from (2.2) that

$$\langle H(Z, Y), (DH)(X, X, X)\rangle = 0$$

for every $X, Y, Z \in T_xM$. By virtue of (1.11), we obtain

$$\langle H(X, Y), (DH)(Z, U, V)\rangle = 0$$

for every $X, Y, Z, U, V \in T_xM$, which shows that M^n is locally symmetric because of the Gauss equation (1.8). In [22, Theorem 2.1 and its Corollary], Takeuchi showed that if a complete locally homogeneous Kähler manifold admits a Kähler immersion into $CP^m(c)$, then it is a compact simply connected homogeneous Kähler manifold. Using this result, we have the assertion. q.e.d.

Let $\tilde{M}$ be a Riemannian manifold. A curve $\tau: I \rightarrow \tilde{M}$ is said to be *d-planar* if there exist an open interval I_s ($s \in I_s \subset I$) and a d-dimensional totally geodesic submanifold P_s for each $s \in I$ such that $\tau(I_s) \subset P_s$. An isometric immersion $\iota: M \rightarrow \tilde{M}$ is called a *d-planar geodesic immersion* if $\tau = \iota \circ \gamma$ is d-planar for each geodesics γ of M.

COROLLARY. *Let $\iota: M^n \rightarrow CP^m(c)$ be a d-planar geodesic Kähler immersion of a Kähler manifold M^n into $CP^m(c)$. If d is odd, then M^n is a compact simply connected Hermitian symmetric space.*

PROOF. The assertion follows from the fact that an odd-dimensional totally geodesic submanifold in $CP^m(c)$ is totally real. q.e.d.

Secondly, we shall characterize the d-th Veronese map by the shape of geodesics in the ambient space. Let M be an irreducible symmetric Kähler manifold of compact type and d a positive integer. In [10], Nakagawa and Takagi constructed a full equivariant Kähler imbedding $f_d: M \rightarrow CP^m(c)$ which is called the *d-th full Kähler imbedding* of M.

Moreover Takagi and Takeuchi [20] constructed a full Kähler imbedding of a (not necessarily irreducible) symmetric Kähler manifold of compact type into a complex projective space as follows. Segre imbedding S_2: $\boldsymbol{CP}^{m_1}(c)\times\boldsymbol{CP}^{m_2}(c)\to\boldsymbol{CP}^m(c)$ $(m=(m_1+1)(m_2+1)-1)$ is defined by the tensor product of the homogeneous coordinates:

$$[z_i]_{0\leqq i\leqq m_1}\times[w_j]_{0\leqq j\leqq m_2}\mapsto[z_iw_j]_{0\leqq i\leqq m_1,0\leqq j\leqq m_2}\,.$$

Similarly, we can define a full Kähler imbedding S_q: $\boldsymbol{CP}^{m_1}(c)\times\cdots\times\boldsymbol{CP}^{m_q}(c)\to\boldsymbol{CP}^m(c)$ $(m=(m_1+1)\times\cdots\times(m_q+1)-1)$ by the multifold tensor product of the homogeneous coordinates. Let M be a compact symmetric Kähler manifold and M_k $(k=1,\cdots,q)$ its irreducible components, i.e., $M=M_1\times\cdots\times M_q$. Let f_{d_k}: $M_k\to\boldsymbol{CP}^{m_k}(c)$ be the d_k-th full Kähler imbedding of M_k. Then the *tensor product* $f_{d_1}\boxtimes\cdots\boxtimes f_{d_q}$: $M\to\boldsymbol{CP}^m(c)$ $(m=\prod_{k=1}^q(m_k+1)-1)$ of f_{d_k} $(k=1,\cdots,q)$ is defined as $S_q\circ(f_{d_1}\times\cdots\times f_{d_q})$. This is a full equivariant Kähler imbedding. In [10] and [22], it was shown that any full Kähler immersion into $\boldsymbol{CP}^m(c)$ of a symmetric Kähler manifold of compact type is obtained in this way (cf. [22, Corollary 2, p. 177]). In particular, we note that if $M=\boldsymbol{CP}^n(c/d)$, then the d-th full Kähler imbedding is the d-th Veronese map whose defining equation is given in the introduction.

A d-planar curve τ in $\tilde{M}$ is said to be *proper d-planar* if τ is not $(d-1)$-planar. A d-planar geodesic immersion ι: $M\to\tilde{M}$ is said to be *proper* if $\tau=\iota\circ\gamma$ is proper d-planar for each geodesic γ of M.

LEMMA 2.2. *The d-th Veronese map V_d^n: $\boldsymbol{CP}^n(c/d)\to\boldsymbol{CP}^{m'}(c)$ is proper d-planar geodesic.*

PROOF. Since the map V_d^n is equivariant and there exists an isometry of $\boldsymbol{CP}^n(c/d)$ which maps γ_1 to γ_2 for any two geodesics γ_1 and γ_2 of $\boldsymbol{CP}^n(c/d)$, we have only to consider the geodesic γ:

$$\gamma(t)=[\cos t,\sin t,0,\cdots,0]$$

in homogeneous coordinates of $\boldsymbol{CP}^n(c/d)$, where t is a parameter proportional to the arc-length parameter. By the d-th Veronese map V_d^n, γ is mapped to the curve

$$\tau(t)=[\alpha_0,\cdots,\alpha_d,0,\cdots,0]\,,$$
$$\alpha_k(t)=\left(\frac{d!}{k!(d-k)!}\right)^{1/2}\cos^k t\sin^{d-k}t\,,\quad(k=0,\cdots,d)$$

in homogeneous coordinates of $\boldsymbol{CP}^{m'}(c)$. Thus τ is contained in the totally real totally geodesic submanifold $\boldsymbol{RP}^d(c/4)=\{[z_i]\in\boldsymbol{CP}^{m'}(c);\ z_i\in\boldsymbol{R}$ for $0\leqq i\leqq d$, $z_i=0$ for $d+1\leqq i\leqq m'\}$. The intersection of two totally geodesic submanifolds in $CP^{m'}(c)$ is totally geodesic. Thus τ is proper d-planar, since $\sum a_k\alpha_k(t) \equiv 0$. $a_k \in \boldsymbol{R}$ easily implies $a_k = 0$ $(k = 0, 1, \cdots, d)$. q.e.d.

THEOREM 2.3. *Let $\iota: M^n \to CP^m(c)$ be a proper d-planar geodesic Kähler immersion of a complete Kähler manifold M^n into $CP^m(c)$. Suppose that for each γ and s, we can take P_s in the definition of d-planar geodesic immersions to be a totally real totally geodesic submanifold. Then $M^n = CP^n(c/d)$ and ι is equivalent to $i \circ V_d^n$ where $i: CP^{m'}(c) \to CP^m(c)$ is a totally geodesic imbedding.*

PROOF. By Proposition 2.1, we see that M^n is a symmetric Kähler manifold of compact type. We shall prove that M^n is of rank one and apply [22, Corollary, p. 203] (cf. [2], [11]). Assume that the rank r of M^n is greater than 2. Let M_k $(k = 1, \cdots, q)$ be the irreducible components of M^n and r_k the rank of M_k, where $r = r_1 + \cdots + r_q \geqq 2$. It is known that there is a totally geodesic Kähler immersion

$$\phi: (CP^1(c/d_1))^{r_1} \times \cdots \times (CP^1(c/d_q))^{r_q} \to M^n ,$$

where $d_1, \cdots, d_q$ are certain positive integers (see [20, the proof of Theorem 2, p. 515]). Since $r \geqq 2$, we thus have a totally geodesic Kähler immersion

$$\psi: CP^1(c/a) \times CP^1(c/b) \to M^n , \quad a, b \in \boldsymbol{Z}_+ .$$

The composite $\iota \circ \psi$ is equivalent to $\tilde{i} \circ (V_a^1 \boxtimes V_b^1): CP^1(c/a) \times CP^1(c/b) \to CP^m(c)$, where $\tilde{i}: CP^{ab+a+b}(c) \to CP^m(c)$ is a totally geodesic imbedding. Let γ_1 (resp. γ_2) be a geodesic of $CP^1(c/a)$ (resp. $CP^1(c/b)$). Then $\psi \circ \gamma_j$ $(j = 1, 2)$ is a geodesic in M^n. By Lemma 2.2, $\iota \circ \psi \circ \gamma_1$ (resp. $\iota \circ \psi \circ \gamma_2$) is proper a-planar (resp. b-planar). Thus the assumption implies that $a = b = d$. Hence we have only to prove that

$$V_d^1 \boxtimes V_d^1: CP^1(c/d) \times CP^1(c/d) \to CP^{d(d+2)}(c)$$

is not proper d-planar. Consider the geodesic γ in $CP^1(c/d) \times CP^1(c/d)$ defined by

$$\gamma(t) = [\cos t, \sin t] \times [\cos t, \sin t]$$

in homogeneous coordinates, where t is a parameter proportional to the arc-length parameter. The curve $\tau = (V_d^1 \boxtimes V_d^1) \circ \gamma$ in $CP^{d(d+2)}(c)$ is given by

$$\tau(t) = [\alpha_k(t)\alpha_l(t)]_{0 \leqq k \leqq d, 0 \leqq l \leqq d} ,$$

where $\alpha_k(t)$ is as defined in the proof of Lemma 2.2. This curve is contained in $RP^{d(d+2)}(c/4) = \{[v_{kl}] \in CP^{d(d+2)}(c);\ v_{kl} \in \boldsymbol{R}$ for $0 \leqq k, l \leqq d\}$. We easily see that functions $\alpha_0(t)\alpha_0(t), \alpha_0(t)\alpha_1(t), \cdots, \alpha_0(t)\alpha_d(t), \alpha_1(t)\alpha_d(t), \cdots, \alpha_d(t)\alpha_d(t)$ are linearly independent over $\boldsymbol{R}$. Suppose that there exists a

$(d-1)$-dimensional totally geodesic submanifold P such that $\tau(I)\subset P$, for some open interval. Then $\tau(I)$ is contained in $\boldsymbol{RP}^{d(d+2)}(c/4)\cap P$ which is a totally real totally geodesic submanifold of dimension not greater than $d-1$. Thus the dimension of the vector space spanned by functions $\alpha_k\alpha_l$ $(0\leqq k, l\leqq d)$ is not greater than d. We thus have a contradiction $2d+1\leqq d$. q.e.d.

COROLLARY. *Let $\iota\colon M^n\to \boldsymbol{CP}^m(c)$ be a proper d-planar geodesic Kähler immersion of a complete Kähler manifold M^n into $\boldsymbol{CP}^m(c)$. If d is odd, then $M^n=\boldsymbol{CP}^n(c/d)$ and ι is equivalent to $i\circ V_d^n$.*

3. Cubic geodesic totally real immersions. Let $\iota\colon M\to \boldsymbol{CP}^m(c)$ be a cubic geodesic immersion of a Riemannian manifold M into $\boldsymbol{CP}^m(c)$, where $\dim M\geqq 3$. Let $x\in M$, X be a unit vector tangent to M at x and γ the unit speed geodesic such that $\gamma(0)=x$, $\dot\gamma(0)=X$. There exists a totally real, totally geodesic submanifold P_0 of dimension 3 such that $\tau(I_0)\subset P_0$ for some open interval I_0 containing 0, where $\tau=\iota\circ\gamma$. We now assume that the isotropy $\lambda(x)$ at x is positive and hence $\lambda>0$ on a neighborhood of x. We take I_0 small enough if necessary and put $\tau_1=\dot\tau$ and $\tau_2=H(\tau_1,\tau_1)/\lambda$. Noting that $\tilde\nabla_{\tau_1}\tau_1=H(\tau_1,\tau_1)$, we see that τ_2 is tangent to P_0. Then $C:=\tilde\nabla_{\tau_1}\tau_2$ is orthogonal to τ_1, τ_2 and tangent to P_0. Using (1.2), we have

$$\lambda C=-\lambda'\tau_2-A_{H(\tau_1,\tau_1)}\tau_1+(DH)(\tau_1,\tau_1,\tau_1)+\lambda^2\tau_1\,,$$

where $\lambda'=d\lambda(\gamma(s))/ds$, from which

$$(DH)(\tau_1,\tau_1,\tau_1)=\lambda'\tau_2+\lambda C \tag{3.1}$$

because of (1.15). The above equation shows that C is normal to M. Covariantly differentiating (3.1) in the direction τ_1, we have

$$(D^2H)(\tau_1,\tau_1,\tau_1,\tau_1)=A_{(DH)(\tau_1,\tau_1,\tau_1)}\tau_1-\lambda\lambda'\tau_1+\lambda''\tau_2+2\lambda'C+\lambda\tilde\nabla_{\tau_1}C\,. \tag{3.2}$$

Since τ_1, τ_2 and C are mutually orthogonal, $\tilde\nabla_{\tau_1}C$ is orthogonal to τ_1. Suppose that $C(0)\neq 0$. If necessary, we choose I_0 so that $C(s)\neq 0$ for every $s\in I_0$. Put $\mu=\|C\|$ and $\tau_3=C/\mu$. Vector fields τ_1, τ_2 and τ_3 are orthonormal and tangent to P_0, Therefore, $\tilde\nabla_{\tau_1}C$ is spanned by τ_2 and τ_3 which are normal to M. It follows from (3.2) that

$$\langle (DH)(X,X,X),H(X,Y)\rangle=0 \tag{3.3}$$

for every $Y\in T_xM$ orthogonal to X. If $C(0)=0$, then (3.1) and (1.15) also imply (3.3).

LEMMA 3.1. *The immersion ι is constant isotropic.*

PROOF. Let $x\in M$, $Y\in T_xM$ with $\|Y\|=1$ be arbitrarily fixed. Let

X be a unit tangent vector orthogonal to Y. We shall prove $Y\cdot\lambda^2=0$. If $\lambda(x)=0$, then λ^2 attains the minimum at x and hence $Y\cdot\lambda^2=0$. Thus we may assume $\lambda(x)>0$. Extend X and Y to orthonormal vector fields X^* and Y^*, respectively, on a neighborhood of x so that $\nabla X^*=\nabla Y^*=0$ at x. We have

$$Y\cdot\lambda^2=Y\cdot\langle H(X^*,X^*),H(X^*,X^*)\rangle=2\langle(DH)(Y,X,X),H(X,X)\rangle .$$

Using (1.9), we obtain

$$Y\cdot\lambda^2=2\langle(DH)(X,X,Y),H(X,X)\rangle-\frac{3}{2}c\langle JY,X\rangle\langle J_NX,H(X,X)\rangle .$$

Since P_0 is totally real, we have $\langle\tilde{J}X,H(X,X)\rangle=0$. Therefore,

$$\begin{aligned}Y\cdot\lambda^2&=2\langle(DH)(X,X,Y),H(X,X)\rangle\\&=2\{X\cdot\langle H(X^*,Y^*),H(X^*,X^*)\rangle-\langle H(X,Y),(DH)(X,X,X)\rangle\}\\&=0\end{aligned}$$

by virtue of (1.15) and (3.3). q.e.d.

In the sequel, we assume that the cubic geodesic immersion $\iota: M\to CP^m(c)$ is *proper and totally real*. By means of Lemma 3.1, we may assume that $\lambda>0$. We next prove that μ is a nonzero constant and independent of the choice of the geodesic γ. From (3.1), we have

$$\|(DH)(X,X,X)\|^2=\lambda^2\mu^2(X) , \tag{3.4}$$

where μ is regarded as a non-negative function on the unit sphere bundle UM of M.

LEMMA 3.2. *The function μ is constant on the unit tangent sphere U_xM for every $x\in M$.*

PROOF. Let x be an arbitrary point. Suppose that there exists a vector $X_0\in U_xM$ such that $\mu(X_0)>0$. Put $S=\{X\in U_xM: \mu(X)>0\}$, which is an open set in U_xM because of the continuity of μ. For each $X\in S$, we consider the unit speed geodesic γ such that $\gamma(0)=x$ and $\dot\gamma(0)=X$. Taking Lemma 3.1 into account, we see that (3.3) holds for every $X, Y\in TM$ and hence $A_{(DH)(X,X,X)}X=0$ for any $X\in TM$. From (3.2), we have $(D^2H)(\tau_1,\tau_1,\tau_1,\tau_1)=\lambda\tilde{\nabla}_{\tau_1}C$. The right hand side is spanned by τ_2 and τ_3. It follows that $(D^2H)(X,X,X,X)$ is spanned by $H(X,X)$ and $(DH)(X,X,X)$ for $X\in S$. Let Y be orthogonal to X. Differentiate

$$\langle(DH)(X^*,X^*,X^*),H(X^*,Y^*)\rangle=0$$

in the direction X where X^* and Y^* are local vector fields used in the proof of Lemma 3.1. Then we have

$$\langle (D^2H)(X, X, X, X), H(X, Y)\rangle + \langle (DH)(X, X, X), (DH)(X, X, Y)\rangle = 0\ ,$$

from which

$$\langle (DH)(X, X, X), (DH)(X, X, Y)\rangle = 0 \tag{3.5}$$

in virtue of (1.15) and (3.3). This means that $\|(DH)(X, X, X)\|^2$ is constant on each connected component of S. Therefore, the component ($\ni X_0$) of S is open and closed. We have proved μ is constant on $S = U_xM$. q.e.d.

By Lemma 3.2, we see that μ is a function defined on M. If $\mu(x) > 0$, then for each $X \in U_xM$

$$\mu(D^2H)(X, X, X, X) = (X\cdot\mu)(DH)(X, X, X) - \mu^3 H(X, X) \tag{3.6}$$

because of $(DH)(X, X, X) \perp H(X, X)$, $\langle (D^2H)(X, X, X, X), H(X, X)\rangle = -\lambda^2\mu^2$ and $2\langle (D^2H)(X, X, X, X), (DH)(X, X, X)\rangle = \lambda^2(X\cdot\mu^2)$.

LEMMA 3.3. *μ is a nonzero constant.*

PROOF. If μ vanishes identically on M, then the image τ of each geodesic γ is a circle in $P = \boldsymbol{RP}^3(c/4)$. Thus τ is contained in a totally geodesic submanifold $\boldsymbol{RP}^2(c/4)$ of $\boldsymbol{RP}^3(c/4)$. This contradicts the assumption that ι is proper cubic geodesic. Put $\tilde{S} = \{x \in M\colon \mu(x) > 0\}$. Let $x \in \tilde{S}$ and $Y \in U_xM$ be fixed. Let $X \in U_xM$ be orthogonal to Y. Then from (3.4), we have

$$\lambda^2(Y\cdot\mu^2) = 2\langle (D^2H)(Y, X, X, X), (DH)(X, X, X)\rangle\ .$$

Making use of (1.10) and (1.13), we find

$$\begin{aligned}
&(D^2H)(Y, X, X, X) - (D^2H)(X, X, X, Y)\\
&\quad = R^\perp(Y, X)H(X, X) - 2H(R(Y, X)X, X)\\
&\quad = \frac{c}{4}\{\langle J_N X, H(X, X)\rangle J_N Y - \langle J_N Y, H(X, X)\rangle J_N X\\
&\qquad - 2\langle JY, X\rangle J^\perp H(X, X)\} + H(Y, A_{H(X,X)}X) - H(A_{H(X,X)}Y, X)\\
&\qquad - 2H(R(Y, X)X, X)\ .
\end{aligned}$$

Using the fact that $\langle J_N X, H(X, X)\rangle = \langle J_N X, (DH)(X, X, X)\rangle = 0$, $J = 0$, $A_{H(X,X)}X = \lambda^2 X$ and (3.3) holds for every $X, Y \in U_xM$, we have

$$\lambda^2(Y\cdot\mu^2) = 2\langle (D^2H)(X, X, X, Y), (DH)(X, X, X)\rangle\ .$$

Differentiate $\langle (DH)(X^*, X^*, X^*), (DH)(X^*, X^*, Y^*)\rangle = 0$ (cf. (3.5)) in the direction X. Then

$$\begin{aligned}
&\langle (D^2H)(X, X, X, X), (DH)(X, X, Y)\rangle\\
&\qquad\qquad + \langle (DH)(X, X, X), (D^2H)(X, X, X, Y)\rangle = 0\ .
\end{aligned}$$

Substitute (3.6) into the above equation and use Lemma 3.1 and (3.5). We obtain $Y\cdot\mu^2=0$. It follows that μ is a nonzero constant on each connected component of $\tilde{S}$. q.e.d.

Next we shall prove that there is a totally real, totally geodesic submanifold Q of $CP^m(c)$ such that $\iota(M)\subset Q$ and $\iota: M\to Q$ is full. In contrast with Erbacher [5], our proof is based on the situation that $\iota: M\to CP^m(c)$ is proper cubic geodesic, totally real immersion.

Since each geodesic is mapped locally into a 3-dimensional totally real, totally geodesic submanifold, the discussion up to this point yields

$$\begin{aligned}&\langle\tilde{J}X, H(X,X)\rangle=0\,, \qquad \langle\tilde{J}X, (DH)(X,X,X)\rangle=0\\&\langle\tilde{J}H(X,X), (DH)(X,X,X)\rangle=0\,.\end{aligned}\tag{3.7}$$

for every $X\in TM$. Moreover we have, from (3.6) and Lemma 3.3,

$$(D^2H)(X,X,X,X)=-\mu^2H(X,X)\langle X,X\rangle\tag{3.8}$$

for every $X\in TM$. Let O_3 denote the third osculating space $\mathrm{Sp}\{X, H(X,X), (DH)(X,X,X): X\in T_xM\}$ at a distinguished point x.

LEMMA 3.4. *The third osculating space* O_3 *is totally real*, i.e., $\tilde{J}O_3\perp O_3$.

PROOF. We must show (1) $\langle\tilde{J}X, Y\rangle=0$, (2) $\langle\tilde{J}X, H(Y,Z)\rangle=0$, (3) $\langle\tilde{J}X, (DH)(Y,Z,W)\rangle=0$, (4) $\langle\tilde{J}H(X,Y), H(Z,W)\rangle=0$, (5) $\langle\tilde{J}H(X,Y), (DH)(Z,W,U)\rangle=0$ and (6) $\langle\tilde{J}(DH)(X,Y,Z), (DH)(W,U,V)\rangle=0$ for any $X, Y, Z, U, V, W\in T_xM$.

(1) is the definition of totally real immersions.

The first equation (1.7) with $J=0$ gives $A_{J_NY}X+J_TH(Y,X)=0$ and, consequently, $\langle J_NX, H(Y,Z)\rangle=\langle J_NY, H(Z,X)\rangle$. On the other hand, the first equation of (3.7) implies $\mathfrak{S}_3\langle\tilde{J}X, H(Y,Z)\rangle=0$. Thus we obtain (2).

(3) is shown as follows. From the second equation of (3.7) it follows that $\mathfrak{S}_4\langle\tilde{J}X, (DH)(Y,Z,W)\rangle=0$. Differentiating $\langle\tilde{J}X^*, H(Y^*,Z^*)\rangle=0$ in the direction W, we have

$$\langle\tilde{J}H(W,X), H(Y,Z)\rangle+\langle\tilde{J}X, (DH)(Y,Z,W)\rangle=0\,.\tag{3.9}$$

The first term on the left hand side is symmetric with respect to W and X. Thus we see that $\langle\tilde{J}X, (DH)(Y,Z,W)\rangle=\langle\tilde{J}W, H(Y,Z,X)\rangle$. Therefore, we have (3).

Combining (3) with (3.9), we have (4).

Differentiating $\langle\tilde{J}H(X^*,Y^*), H(Z^*,W^*)\rangle=0$ in the direction U, we find

$$\langle \tilde{J}(DH)(U, X, Y), H(Z, W)\rangle + \langle \tilde{J}H(X, Y), (DH)(U, Z, W)\rangle = 0 .$$

By virtue of Codazzi's equation (1.11), we see that $\langle \tilde{J}H(\cdot, \cdot), (DH)(\cdot, \cdot, \cdot)\rangle$ is a symmetric 5-form on T_xM. Thus the third equation of (3.7) shows (5).

Finally, we prove (6). Differentiating $\langle \tilde{J}H(X^*, Y^*), (DH)(Z^*, W^*, U^*)\rangle = 0$ in the direction V, we find

$$\langle \tilde{J}(DH)(V, X, Y), (DH)(Z, W, U)\rangle + \langle \tilde{J}H(X, Y), (D^2H)(V, Z, W, U)\rangle = 0 .$$

Thus it suffices to show that $\langle \tilde{J}H(X, Y), (D^2H)(V, Z, Z, Z)\rangle = 0$ for any $X, Y, Z, V \in T_xM$. Equation (3.8) gives

$$(D^2H)(V, Z, Z, Z) + 3(D^2H)(Z, Z, Z, V) = -2\mu^2H(V, Z)\langle Z, Z\rangle - 2\mu^2H(Z, Z)\langle Z, V\rangle .$$

Since $(D^2H)(V, Z, Z, Z) - (D^2H)(Z, Z, Z, V)$ is a linear combination of $H(V, A_{H(Z,Z)}Z)$, $H(A_{H(Z,Z)}V, Z)$ and $H(R(V, Z)Z, Z)$ (see the proof of Lemma 3.3), $(D^2H)(V, Z, Z, Z)$ is a linear combination of vectors $H(\cdot, \cdot)$. Thus (4) implies (6). q.e.d.

LEMMA 3.5. *There exists a totally real, totally geodesic submanifold* $Q \approx \boldsymbol{R}P^{n+q}(c/4)$ *in* $CP^m(c)$ *such that* $\iota(M) \subset Q$ *and the immersion* $\iota: M \to Q$ *is full, where* $n = \dim M$ *and* $q = \dim O_3 - n$.

PROOF. Let $x \in M$ be fixed. Since O_3 is totally real, there exists a unique totally real, totally geodesic submanifold Q such that $x \in Q$ and $T_xQ = O_3$. Let $y \in M$ and γ be a unit speed geodesic from x to y. The curve $\tau = \iota \circ \gamma$ satisfies the Frenet equation:

$$\dot{\tau} = \tau_1 , \quad \tilde{\nabla}_{\tau_1}\tau_1 = \lambda\tau_2 , \quad \tilde{\nabla}_{\tau_1}\tau_2 = -\lambda\tau_1 + \mu\tau_3 , \quad \tilde{\nabla}_{\tau_1}\tau_3 = -\mu\tau_2 ,$$

where λ and μ are constants. Let $\gamma(0) = x$ and $\dot{\gamma}(0) = X$. The initial conditions of the above differential equation are $\tau(0) = x$, $\tau_1(0) = X$, $\tau_2(0) = H(X, X)/\lambda$ and $\tau_3(0) = (DH)(X, X, X)/\lambda\mu$ which are elements of O_3. Consider a helix ω in Q whose curvature and torsion are λ and μ, respectively, and which satisfies $\omega(0) = x$, $\omega_1(0) = X$, $\omega_2(0) = H(X, X)/\lambda$ and $\omega_3(0) = (DH)(X, X, X)/\lambda\mu$, where ω_1, ω_2 and ω_3 are unit tangent, principal normal and binormal vectors, respectively. Since Q is totally geodesic, the fundamental theorem of ordinary differential equation implies $\tau = \omega$. Therefore, we have $y \in Q$. It is clear that $\iota: M \to Q$ is full. q.e.d.

THEOREM 3.6. *Let M be an $n(\geqq 3)$-dimensional compact simply connected Riemannian manifold and $\iota: M \to CP^m(c)$ be a proper cubic geodesic, totally real immersion. If ι is minimal, then M is isometric to a sphere $S^n(nc/12(n+2))$ with curvature $nc/12(n+2)$ and ι is equivalent to* $i \circ \pi \circ \iota_3$, *where* $i: Q \to CP^m(c)$ *is the inclusion,* $\pi: S^{n+q}(c/4) \to Q$ *the covering and* $\iota_3: S^n(nc/12(n+2)) \to S^{n+q}(c/4)$ *the third standard minimal immersion.*

PROOF. By Lemma 3.5, we have only to consider the immersion $\iota: M \to Q \approx RP^{n+q}(c/4)$. We can apply Theorem N stated in the introduction to a lifting $\hat{\iota}: M \to S^{n+q}(c/4)$ of ι, since $\hat{\iota}$ is also proper cubic geodesic ($\hat{\iota}$ is a helical immersion of order 3 in the sense of [15]). Noting that the immersion $\hat{\iota}$ is full, we see that $M = S^n(nc/12(n+2))$ and $\hat{\iota}$ is equivalent to ι_3. Thus clearly ι is equivalent to $\pi \circ \iota_3$. q.e.d.

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