

# Submodular Maximization over Multiple Matroids via Generalized Exchange Properties

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## Introduction

In this paper, we consider the problem of maximizing a non-negative submodular function  $f$ , defined on a (finite) ground set  $N$ , subject to matroid constraints. A function  $f : 2^N \rightarrow \mathbb{R}$  is *submodular* if for all  $S, T \subseteq N$ ,  $f(S \cup T) + f(S \cap T) \leq f(S) + f(T)$ . Furthermore, all submodular functions that we deal with are assumed to be non-negative. Throughout, we assume that our submodular function  $f$  is given by a *value oracle*; i.e., for a given set  $S \subseteq N$ , an algorithm can query an oracle to find the value  $f(S)$ . Without loss of generality, we take the ground set  $N$  to be  $[n] = \{1, 2, \dots, n\}$ .

We assume some familiarity with matroids [26] and associated algorithmics [28]. Briefly, a matroid  $\mathcal{M}$  is an ordered pair  $(N, \mathcal{I})$ , where  $N$  is the ground set of  $\mathcal{M}$  and  $\mathcal{I}$  is the set of independent sets of  $\mathcal{M}$ . For a given matroid  $\mathcal{M}$ , the associated *matroid constraint* is  $S \in \mathcal{I}(\mathcal{M})$ . In our usage, we deal with  $k$  matroids  $\mathcal{M}_i = (N, \mathcal{I}_i)$ ,  $i = 1, \dots, k$ , on the common ground set  $N$ . We assume that each matroid is given by an *independence oracle*, answering whether  $S \in \mathcal{I}_i$  or not. It is no coincidence that we use  $N$  for the ground set of our submodular function  $f$  as well as for the ground set of our matroids  $\mathcal{M}_i = (N, \mathcal{I}_i)$ ,  $i = 1, \dots, k$ . Indeed, our optimization problem is

$$\max \{ f(S) : S \in \cap_{i=1}^k \mathcal{I}_i \}.$$

Where necessary, we make some use of other standard matroid notation. For a matroid  $\mathcal{M} = (N, \mathcal{I})$ , we denote its rank function by  $r_{\mathcal{M}}$  and its dual by  $\mathcal{M}^*$ . A base of  $\mathcal{M}$  is a maximal independent set  $J \in \mathcal{I}$ , having cardinality  $r_{\mathcal{M}}(N)$ . For a set  $S \subset N$ , we let  $\mathcal{M} \setminus S$ ,  $\mathcal{M}/S$ , and  $\mathcal{M}|S$  denote deletion of  $S$ , contraction of  $S$ , and restriction to  $S$ , respectively.

*Previous Results.* Optimization of submodular functions is a central topic in combinatorial optimization [22, 28]. While submodular minimization is polynomially solvable [18, 29], maximization variants are usually NP-hard because they include either Max Cut, variants of facility location, and set coverage problems.

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\* This work was done while the last author was at Princeton University.

A classical technique for submodular maximization is the greedy algorithm. The greedy algorithm was first applied to a wide range of submodular maximization problems in the late-70's and early-80's [8, 9, 10, 14, 15, 19, 23, 24]. The most relevant result for our purposes is the proof that the greedy algorithm gives a  $1/(k+1)$ -approximation for the problem of maximizing a monotone submodular function subject to  $k$  matroid constraints [24]. Due to a simple reduction, this problem also encapsulates the problem of maximizing a linear function subject to  $k+1$  matroid constraints.<sup>1</sup> Thus we get a  $1/k$ -approximation for maximizing a linear function subject to  $k$  matroid constraints,  $k \geq 3$  (this result appeared first in [15]). Until recently, the greedy algorithm had the best established performance guarantee for these problems under general matroid constraints.

Recently, improved results have been achieved using the *multilinear extension* of a submodular function and *pipage rounding* [1, 5, 6, 30]. In particular, Vondrák [30] designed the continuous greedy algorithm which achieves a  $(1 - 1/e)$ -approximation for our problem with  $k = 1$ , i.e. monotone submodular maximization subject to a single matroid constraint (see also [6]). This result is optimal in the oracle model even for the case of a uniform matroid constraint [25], and also optimal unless  $P = NP$  for the special case of maximum coverage [11].

Another algorithmic technique that has been used for submodular maximization is local search. Cornuéjols et al. [9] show that a local-search algorithm achieves a constant-factor approximation guarantee for the maximum uncapacitated facility-location problem which is a special case of submodular maximization. Analogously, Nemhauser et al. [23] show a similar result for the problem of maximizing a monotone submodular function subject to a single cardinality constraint (i.e. a uniform matroid constraint). We remark that local search in this case is known to yield only a  $1/2$ -approximation, i.e. it performs worse than the greedy algorithm [23].

The maximum  $k$ -dimensional matching problem is a problem of maximizing a linear function subject to  $k$  special partition matroid constraints. Improved algorithms for maximum  $k$ -dimensional matching have been designed using local search. The best known approximation factors are  $2/(k + \epsilon)$  in the unweighted case (i.e., 0/1 weights), and  $2/(k + 1 + \epsilon)$  for a general linear function, even in the more general cases of weighted set packing [17] and independent set problems in  $(k + 1)$ -claw free graphs [3]. The latter result was obtained after a series of improvements over the basic local-search algorithm [2, 3, 7].

However, general matroid constraints seem to complicate the matter. Prior to this paper, the best approximation for the problem of maximum independent set in the intersection of  $k \geq 3$  matroids was  $1/k$  (for a recent discussion see [27]). On the hardness side, it is known that unless  $P = NP$ , there is no approximation better than  $O(\log k/k)$  for  $k$ -dimensional matching [16], and hence neither for the intersection of  $k$  general matroids. The  $1/(k + 1)$ -approximation for submodular maximization subject to  $k$  matroids [24] can be improved in the case when all  $k$

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<sup>1</sup> Given a problem  $\max\{w(S) : S \in \bigcap_{i=0}^k \mathcal{I}_i\}$  where  $w(S)$  is linear, we can equivalently consider the problem  $\max\{f(S) : S \in \bigcap_{i=1}^k \mathcal{I}_i\}$ , where  $f(S) = \max\{w(I) : I \subseteq S, I \in \mathcal{I}_0\}$ , the *weighted rank function* of  $\mathcal{M}_0$ , is known to be monotone submodular.

constraints correspond to partition matroids. For any fixed  $k \geq 2$  and  $\varepsilon > 0$ , a simple local-search algorithm gives a  $1/(k + \varepsilon)$ -approximation for this variant of the problem [21]. The analysis strongly uses the properties of partition matroids. It is based on relatively simple exchange properties of partition matroids that do not hold in general.

Local-search algorithms were also designed for non-monotone submodular maximization. The best approximation guarantee known for unconstrained submodular maximization is  $2/5 - \varepsilon$  [12]. For the problem of non-monotone submodular maximization subject to  $k$  matroid constraints, the best known approximation is  $1/(k + 2 + 1/k + \varepsilon)$  (for any constant  $k \geq 1$  and  $\varepsilon > 0$ ) [21].

*Our Results and Techniques.* In this paper we analyze a natural local-search algorithm: Given a feasible solution, i.e. a set  $S$  that is independent in each of the  $k$  matroids, our local-search algorithm tries to add at most  $p$  elements and delete at most  $kp$  elements from  $S$ . If there is a local move that generates a feasible solution and improves the objective value, our algorithm repeats the local-search procedure with that new solution, until no improvement is possible. Our main result is that for  $k \geq 2$ , every locally-optimal feasible solution  $S$  satisfies the inequality

$$(k + 1/p) \cdot f(S) \geq f(S \cup C) + (k - 1 + 1/p) \cdot f(S \cap C),$$

for every feasible solution  $C$ . We also provide an approximate variant of the local-search procedure that finds an approximate locally-optimal solution in polynomial time, while losing a factor of  $1 + \varepsilon$  on the left-hand side of the above inequality (Lemma 11). Therefore, for any fixed  $k \geq 2$  and  $\varepsilon > 0$ , we obtain a polynomial-time algorithm with approximation guarantee  $1/(k + \varepsilon)$  for the problem of maximizing a monotone non-decreasing submodular function subject to  $k$  matroid constraints. This algorithm gives a  $1/(k - 1 + \varepsilon)$ -approximation in the case when the objective function is linear. These results are tight for our local search algorithm, which follows from [2].

We also obtain an approximation algorithm for non-monotone submodular functions. In this case, one round of local search is not enough, but applying the local search iteratively, as in [21], one can obtain an approximation algorithm with performance guarantee of  $1/(k + 1 + 1/(k - 1) + \varepsilon)$ .

The main technical contributions of this paper are two new exchange properties for matroids. One is a generalization of the classical Rota Exchange Property (Lemma 8) and another is an exchange property for the intersection of two matroids (Lemma 5), which generalizes an exchange property based on augmenting paths which was used in [21] for partition matroids. We believe that both properties and their proofs are interesting in their own right.

In §1, we establish some useful properties of submodular functions. In §2, we establish our exchange properties for matroids. In §3, we describe and analyze our local-search algorithm.

# 1 Some Useful Properties of Submodular Functions

**Lemma 1.** *Let  $f$  be a submodular function on  $N$ . Let  $S, C \subseteq N$  and let  $\{T_i\}_{i=1}^t$  be a collection of subsets of  $C \setminus S$  such that each element of  $C \setminus S$  appears in exactly  $k$  of these subsets. Then*

$$\sum_{i=1}^t [f(S \cup T_i) - f(S)] \geq k (f(S \cup C) - f(S)).$$

*Proof.* Let  $s = |S|$  and  $c = |C \cup S|$ . We will use the notation  $[n]$  to denote the set  $\{1, \dots, n\}$  (by convention  $[0] = \emptyset$ ). Without loss of generality, we can assume that  $S = \{1, 2, \dots, s\}$  and that  $C \setminus S = \{s+1, s+2, \dots, c\}$ . Then for any  $T \subseteq C \setminus S$ , by submodularity:  $f(S \cup T) - f(S) \geq \sum_{p \in T} [f([p]) - f([p-1])]$ . Summing up over all sets  $T_i$ , we get

$$\begin{aligned} \sum_{i=1}^t [f(S \cup T_i) - f(S)] &\geq \sum_{i=1}^t \sum_{p \in T_i} [f([p]) - f([p-1])] \\ &= k \sum_{p=s+1}^c [f([p]) - f([p-1])] = k [f(S \cup C) - f(S)]. \end{aligned}$$

The first equality follows from the fact that each element in  $\{s+1, \dots, c\}$  appears in exactly  $k$  sets  $T_i$ , and the second equality follows from a telescoping summation.  $\square$

**Lemma 2.** *Let  $f$  be a submodular function on  $N$ . Let  $S' \subseteq S \subseteq N$ , and let  $\{T_i\}_{i=1}^t$  be a collection of subsets of  $S \setminus S'$  such that each element of  $S \setminus S'$  appears in exactly  $k$  of these subsets. Then*

$$\sum_{i=1}^t (f(S) - f(S \setminus T_i)) \leq k (f(S) - f(S')).$$

*Proof.* Let  $s = |S|$  and  $c = |S'|$ . Without loss of generality, we can assume that  $S' = \{1, 2, \dots, c\} = [c] \subseteq \{1, 2, \dots, s\} = [s] = S$ . For any  $T \subseteq S$ ,  $f(S) - f(S \setminus T) \leq \sum_{p \in T} (f([p]) - f([p-1]))$  by submodularity. Using this we obtain

$$\begin{aligned} \sum_{i=1}^t (f(S) - f(S \setminus T_i)) &\leq \sum_{i=1}^t \sum_{p \in T_i} (f([p]) - f([p-1])) \\ &= k \sum_{i=c+1}^s (f([i]) - f([i-1])) = k (f(S) - f(S')). \end{aligned}$$

The first equality follows from  $S \setminus C = \{c+1, \dots, s\}$  and the fact that each element of  $S \setminus C$  appears in exactly  $k$  of the sets  $\{T_i\}_{i=1}^t$ . The last equality is due to a telescoping summation.  $\square$

## 2 New Exchange Properties of Matroids

### 2.1 Intersection of Two Matroids

An exchange digraph is a well-known construct for devising efficient algorithms for exact maximization of linear functions over the intersection of two matroids (for example, see [28]). We are interested in submodular maximization,  $k$  matroids and approximation algorithms; nevertheless, we are able to make use of such exchange digraphs, once we establish some new properties of them.

Let  $\mathcal{M}_l = (N, \mathcal{I}_l)$ ,  $l = 1, 2$ , be two matroids on ground set  $N$ . For  $I \in \mathcal{I}_1 \cap \mathcal{I}_2$ , we define two digraphs  $D_{\mathcal{M}_1}(I)$  and  $D_{\mathcal{M}_2}(I)$  on node set  $N$  as follows:

- For each  $i \in I, j \in N \setminus I$  with  $I - i + j \in \mathcal{I}_1$ , we have an arc  $(i, j)$  of  $D_{\mathcal{M}_1}(I)$ ;
- For each  $i \in I, j \in N \setminus I$  with  $I - i + j \in \mathcal{I}_2$ , we have an arc  $(j, i)$  of  $D_{\mathcal{M}_2}(I)$ .

The arcs in  $D_{\mathcal{M}_l}(I), l = 1, 2$ , encode valid swaps in  $\mathcal{M}_l$ .

In what follows, we assume that  $I$  is our current solution and  $J$  is the optimal solution. We also assume that  $|I| = |J|$ . If not, we extend  $I$  or  $J$  by dummy elements so that we maintain independence in both matroids (more details later). When we refer to a matching (or perfect matching) in  $D_{\mathcal{M}_l}(I)$  for  $l = 1, 2$  we mean a matching in an undirected graph where the arcs of the graph  $D_{\mathcal{M}_l}(I)$  are treated as undirected edges. We use two known lemmas from matroid theory.

**Lemma 3 ([28, Corollary 39.12a]).** *If  $|I| = |J|$  and  $I, J \in \mathcal{I}_l$  ( $l = 1$  or  $2$ ), then  $D_{\mathcal{M}_l}(I)$  contains a perfect matching between  $I \setminus J$  and  $J \setminus I$ .*

**Lemma 4 ([28, Theorem 39.13]).** *Let  $|I| = |J|, I \in \mathcal{I}_1$ , and assume that  $D_{\mathcal{M}_1}(I)$  has a unique perfect matching between  $I \setminus J$  and  $J \setminus I$ . Then  $J \in \mathcal{I}_1$ .*

Next, we define a digraph  $D_{\mathcal{M}_1, \mathcal{M}_2}(I)$  on node set  $N$  as the union of  $D_{\mathcal{M}_1}(I)$  and  $D_{\mathcal{M}_2}(I)$ . A dicycle in  $D_{\mathcal{M}_1, \mathcal{M}_2}(I)$  corresponds to a chain of feasible swaps. However, observe that it is not necessarily the case that the entire cycle gives a valid exchange in both matroids.

If  $|I| = |J|$  and  $I, J \in \mathcal{I}_1 \cap \mathcal{I}_2$ , this means we have two perfect matchings on  $I \Delta J$  which together form a collection of dicycles in  $D_{\mathcal{M}_1, \mathcal{M}_2}(I)$ . However, only the uniqueness of a perfect matching assures us that we can legally perform the exchange. This motivates the following definition.

**Definition 1.** *We call a dicycle  $C$  in  $D_{\mathcal{M}_1, \mathcal{M}_2}(I)$  irreducible if  $C \cap D_{\mathcal{M}_1}(I)$  is the unique perfect matching in  $D_{\mathcal{M}_1}(I)$  and  $C \cap D_{\mathcal{M}_2}(I)$  is the unique perfect matching in  $D_{\mathcal{M}_2}(I)$  on their vertex set  $V(C)$ . Otherwise, we call  $C$  reducible.*

The following, which is our main technical lemma, allows us to consider only irreducible cycles. The proof follows the ideas of matroid intersection (see [28, Lemma 41.5a]). This lemma holds trivially for partition matroids with  $s = 0$ .

**Lemma 5.** *Let  $\mathcal{M}_l = (N, \mathcal{I}_l), l = 1, 2$ , be matroids on ground set  $N$ . Suppose that  $I, J \in \mathcal{I}_1 \cap \mathcal{I}_2$  and  $|I| = |J|$ . Then there is  $s \geq 0$  and a collection of irreducible dicycles  $\{C_1, \dots, C_m\}$  (allowing repetition) in  $D_{\mathcal{M}_1, \mathcal{M}_2}(I)$ , using only elements of  $I \Delta J$ , so that each element of  $I \Delta J$  appears in exactly  $2^s$  of the dicycles.*

*Proof.* Consider  $D_{\mathcal{M}_1, \mathcal{M}_2}(I) = D_{\mathcal{M}_1}(I) \cup D_{\mathcal{M}_2}(I)$ . By Lemma 3, there is a perfect matching between  $I \setminus J$  and  $J \setminus I$ , both in  $D_{\mathcal{M}_1}(I)$  and  $D_{\mathcal{M}_2}(I)$ . We denote these two perfect matchings by  $M_1, M_2$ . The union  $M_1 \cup M_2$  forms a subgraph of out-degree 1 and in-degree 1 on  $I \Delta J$ . Therefore, it decomposes into a collection of dicycles  $C_1, \dots, C_m$ . If they are all irreducible, we are done and  $s = 0$ .

If  $C_i$  is not irreducible, it means that either  $M'_1 = C_i \cap D_{\mathcal{M}_1}(I)$  or  $M'_2 = C_i \cap D_{\mathcal{M}_2}(I)$  is not a unique perfect matching on  $V(C_i)$ . Let us assume, without loss of generality, that there is another perfect matching  $M''_1$  in  $D_{\mathcal{M}_1}(I)$ . We consider the disjoint union  $M'_1 + M''_1 + M'_2 + M'_2$ , duplicating arcs where necessary.

This is a subgraph of out-degree 2 and in-degree 2 on  $V(C_i)$ , which decomposes into dicycles  $C_{i1}, \dots, C_{it}$ , covering each vertex of  $C_i$  exactly twice:

$$V(C_{i1}) + V(C_{i2}) + \dots + V(C_{it}) = 2V(C_i).$$

Because  $M'_1 \neq M''_1$ , we have a chord of  $C_i$  in  $M''_1$ , and we can choose the first dicycle so that it does not cover all of  $V(C_i)$ . So we can assume that we have  $t \geq 3$  dicycles, and at most one of them covers all of  $V(C_i)$ . If there is such a dicycle among  $C_{i1}, \dots, C_{it}$ , we remove it and duplicate the remaining dicycles. Either way, we get a collection of dicycles  $C_{i1}, \dots, C_{it'}$  such that each of them is shorter than  $C_i$  and together they cover each vertex of  $C_i$  exactly twice.

We repeat this procedure for each reducible dicycle  $C_i$ . For irreducible dicycles  $C_i$ , we just duplicate  $C_i$  to obtain  $C_{i1} = C_{i2} = C_i$ . This completes one stage of our procedure. After the completion of the first stage, we have a collection of dicycles  $\{C_{ij}\}$  covering each vertex in  $I\Delta J$  exactly twice.

As long as there exists a reducible dicycle in our current collection of dicycles, we perform another stage of our procedure. This means decomposing all reducible dicycles and duplicating all irreducible dicycles. In each stage, we double the number of dicycles covering each element of  $I\Delta J$ . To see that this cannot be repeated indefinitely, observe that every stage decreases the size of the longest reducible dicycle. All dicycles of length 2 are irreducible, and therefore the procedure terminates after a finite number of stages  $s$ . Then, all cycles are irreducible and together they cover each element of  $I\Delta J$  exactly  $2^s$  times.  $\square$

We remark that of course the procedure in the proof of Lemma 5 is very inefficient, but it is not part of our algorithm — it is only used for this proof.

Next, we extend this Lemma 5 to sets  $I, J$  of different size, which forces us to deal with dipaths as well as dicycles.

**Definition 2.** We call a dipath or dicycle  $A$  feasible in  $D_{\mathcal{M}_1, \mathcal{M}_2}(I)$ , if

- $I\Delta V(A) \in \mathcal{I}_1 \cap \mathcal{I}_2$ , and
- For any sub-dipath  $A' \subset A$  such that each endpoint of  $A'$  is either an endpoint of  $A$  or an element of  $I$ , we also have  $I\Delta V(A') \in \mathcal{I}_1 \cap \mathcal{I}_2$ .

First, we establish that irreducible dicycles are feasible.

**Lemma 6.** Any irreducible dicycle in  $D_{\mathcal{M}_1, \mathcal{M}_2}(I)$  is also feasible in  $D_{\mathcal{M}_1, \mathcal{M}_2}(I)$ .

*Proof.* An irreducible dicycle  $C$  consists of two matchings  $M_1 \cup M_2$ , which are the unique perfect matchings on  $V(C)$ , in  $D_{\mathcal{M}_1}(I)$  and  $D_{\mathcal{M}_2}(I)$  respectively. Therefore, we have  $I\Delta V(C) \in \mathcal{I}_1 \cap \mathcal{I}_2$  by Lemma 4.

Consider any sub-dipath  $A' \subset C$  whose endpoints are in  $I$ . ( $C$  has no endpoints, so the other case in Definition 2 does not apply.) This means that  $A'$  has even length. Suppose that  $a_1 \in V(A')$  is the endpoint incident to an edge in  $M_1 \cap A'$  and  $a_2 \in V(A')$  is the other endpoint, incident to an edge in  $M_2 \cap A'$ . Note that any subset of  $M_1$  or  $M_2$  is again a unique perfect matching on its respective vertex set, because otherwise we could produce a different perfect matching on  $V(C)$ . We can view  $I\Delta V(A')$  in two possible ways:

- $I\Delta V(A') = (I - a_1)\Delta(V(A') - a_1)$ ; because  $V(A') - a_1$  has a unique perfect matching  $M_2 \cap A'$  in  $D_{\mathcal{M}_2}(I)$ , this shows that  $I\Delta V(A') \in \mathcal{I}_2$ .
- $I\Delta V(A') = (I - a_2)\Delta(V(A') - a_2)$ ; because  $V(A') - a_2$  has a unique perfect matching  $M_1 \cap A'$  in  $D_{\mathcal{M}_1}(I)$ , this shows that  $I\Delta V(A') \in \mathcal{I}_1$ .  $\square$

Finally, we establish the following property of possible exchanges between arbitrary solutions  $I, J$  (not necessarily of the same size).

**Lemma 7.** *Let  $\mathcal{M}_1 = (N, \mathcal{I}_1)$  and  $\mathcal{M}_2 = (N, \mathcal{I}_2)$  be two matroids and let  $I, J \in \mathcal{I}_1 \cap \mathcal{I}_2$ . Then there is  $s \geq 0$  and a collection of dipaths/dicycles  $\{A_1, \dots, A_m\}$  (possibly with repetition), feasible in  $D_{\mathcal{M}_1, \mathcal{M}_2}(I)$ , using only elements of  $I\Delta J$ , so that each element of  $I\Delta J$  appears in exactly  $2^s$  dipaths/dicycles  $A_i$ .*

*Proof.* If  $|I| = |J|$ , we are done by Lemmas 5 and 6. If  $|I| \neq |J|$ , we extend the matroids by new “dummy elements”  $E$ , independent of everything else (in both matroids), and add them to  $I$  or  $J$ , to obtain sets of equal size  $|\tilde{I}| = |\tilde{J}|$ . We denote the extended matroids by  $\tilde{\mathcal{M}}_1 = (N \cup E, \tilde{\mathcal{I}}_1), \tilde{\mathcal{M}}_2 = (N \cup E, \tilde{\mathcal{I}}_2)$ . We consider the graph  $D_{\tilde{\mathcal{M}}_1, \tilde{\mathcal{M}}_2}(\tilde{I})$ . Observe that the dummy elements do not affect independence among other elements, so the graphs  $D_{\mathcal{M}_1, \mathcal{M}_2}(I)$  and  $D_{\tilde{\mathcal{M}}_1, \tilde{\mathcal{M}}_2}(\tilde{I})$  are identical on  $I \cup J$ .

Applying Lemma 5 to  $\tilde{I}, \tilde{J}$ , we obtain a collection of irreducible dicycles  $\{C_1, \dots, C_m\}$  on  $\tilde{I}\Delta\tilde{J}$  such that each element appears in exactly  $2^s$  dicycles. Let  $A_i = C_i \setminus E$ . Obviously, the sets  $V(A_i)$  cover  $I\Delta J$  exactly  $2^s$  times. We claim that each  $A_i$  is either a feasible dicycle, a feasible dipath, or a collection of feasible dipaths (in the original digraph  $D_{\mathcal{M}_1, \mathcal{M}_2}(I)$ ).

First, assume that  $C_i \cap E = \emptyset$ . Then  $A_i = C_i$  is an irreducible cycle in  $D_{\mathcal{M}_1, \mathcal{M}_2}(I)$  (the dummy elements are irrelevant). By Lemma 6, we know that  $A_i = C_i$  is a feasible dicycle.

Next, assume that  $C_i \cap E \neq \emptyset$ .  $C_i$  is still a feasible dicycle, but in the extended digraph  $D_{\tilde{\mathcal{M}}_1, \tilde{\mathcal{M}}_2}(\tilde{I})$ . We remove the dummy elements from  $C_i$  to obtain  $A_i = C_i \setminus E$ , a dipath or a collection of dipaths. Consider any sub-dipath  $A'$  of  $A_i$ , possibly  $A' = A_i$ , satisfying the assumptions of Definition 2.  $A_i$  does not contain any dummy elements. If both endpoints of  $A'$  are in  $I$ , it follows from the feasibility of  $C_i$  that  $\tilde{I}\Delta V(A') \in \tilde{\mathcal{I}}_1 \cap \tilde{\mathcal{I}}_2$ , and hence  $I\Delta V(A') = (\tilde{I}\Delta V(A')) \setminus E \in \mathcal{I}_1 \cap \mathcal{I}_2$ .

If an endpoint of  $A'$  is outside of  $I$ , then it must be an endpoint of  $A_i$ . This means that it has a dummy neighbor in  $\tilde{I} \cap C_i$  that we deleted. (Note that this case can occur only if we added dummy elements to  $I$ , i.e.  $|I| < |J|$ .) In that case, extend the path to  $A''$ , by adding the dummy neighbor(s) at either end. We obtain a dipath from  $\tilde{I}$  to  $\tilde{I}$ . By the feasibility of  $C_i$ , we have  $\tilde{I}\Delta V(A'') \in \tilde{\mathcal{I}}_1 \cap \tilde{\mathcal{I}}_2$ , and therefore  $I\Delta V(A') = (\tilde{I}\Delta V(A'')) \setminus E \in \mathcal{I}_1 \cap \mathcal{I}_2$ .  $\square$

## 2.2 A Generalized Rota-Exchange Property

Next, we establish a very useful property for a pair of bases of one matroid.

**Lemma 8.** *Let  $\mathcal{M} = (N, \mathcal{I})$  be a matroid and  $A, B$  bases in  $\mathcal{M}$ . Let  $A_1, \dots, A_m$  be subsets of  $A$  such that each element of  $A$  appears in exactly  $q$  of them. Then*

there are sets  $B_1, \dots, B_m \subseteq B$  such that each element of  $B$  appears in exactly  $q$  of them, and for each  $i$ ,  $A_i \cup (B \setminus B_i) \in \mathcal{I}$ .

*Remark 1.* A very special case of Lemma 8, namely when  $m = 2$  and  $q = 1$ , attracted significant interest when it was conjectured by G.-C. Rota and proved in [4, 13, 31]; see [28, (39.58)].

*Proof.* We can assume for convenience that  $A$  and  $B$  are disjoint (otherwise we can make  $\{B_i\}$  equal to  $\{A_i\}$  on the intersection  $A \cap B$  and continue with a matroid where  $A \cap B$  is contracted).

For each  $i$ , we define a matroid  $\mathcal{N}_i = (\mathcal{M}/A_i)|B$ , where we contract  $A_i$  and restrict to  $B$ . In other words,  $S \subseteq B$  is independent in  $\mathcal{N}_i$  exactly when  $A_i \cup S \in \mathcal{I}$ . The rank function of  $\mathcal{N}_i$  is

$$r_{\mathcal{N}_i}(S) = r_{\mathcal{M}}(A_i \cup S) - r_{\mathcal{M}}(A_i) = r_{\mathcal{M}/A_i}(S).$$

Let  $\mathcal{N}_i^*$  be the dual matroid to  $\mathcal{N}_i$ . Recall that the ground set is now  $B$ . By definition,  $T \subseteq B$  is a spanning set in  $\mathcal{N}_i^*$  if and only if  $B \setminus T$  is independent in  $\mathcal{N}_i$ , i.e. if  $A_i \cup (B \setminus T) \in \mathcal{I}$ . The bases of  $\mathcal{N}_i^*$  are minimal such sets  $T$ ; these are the candidate sets for  $B_i$ , which can be exchanged for  $A_i$ . The rank function of the dual matroid  $\mathcal{N}_i^*$  is (by [28, (Theorem 39.3)])

$$\begin{aligned} r_{\mathcal{N}_i^*}(T) &= |T| - r_{\mathcal{N}_i}(B) + r_{\mathcal{N}_i}(B \setminus T) = |T| - r_{\mathcal{M}}(A_i \cup B) + r_{\mathcal{M}}(A_i \cup (B \setminus T)) \\ &= |T| - |B| + r_{\mathcal{M}}(A_i \cup (B \setminus T)) = r_{\mathcal{M}/(B \setminus T)}(A_i). \end{aligned}$$

Observe that the rank of  $\mathcal{N}_i^*$  is  $r_{\mathcal{N}_i^*}(B) = |A_i|$ .

Now, we consider a new ground set  $\hat{B} = B \times [q]$ . We view the elements  $\{(i, j) : j \in [q]\}$  as parallel copies of  $i$ . For  $T \subseteq \hat{B}$ , we define its *projection* to  $B$  as

$$\pi(T) = \{i \in B \mid \exists j \in [q] \text{ with } (i, j) \in T\}.$$

A natural extension of  $\mathcal{N}_i^*$  to  $\hat{B}$  is a matroid  $\hat{\mathcal{N}}_i^*$  where a set  $T$  is independent if  $\pi(T)$  is independent in  $\mathcal{N}_i^*$ . The rank function of  $\hat{\mathcal{N}}_i^*$  is

$$r_{\hat{\mathcal{N}}_i^*}(T) = r_{\mathcal{N}_i^*}(\pi(T)) = r_{\mathcal{M}/(B \setminus \pi(T))}(A_i). \tag{1}$$

The question now is whether  $\hat{B}$  can be partitioned into  $B'_1, \dots, B'_m$  so that  $B'_i$  is a base in  $\hat{\mathcal{N}}_i^*$ . If this is true, then we are done, because each  $B_i = \pi(B'_i)$  would be a base of  $\mathcal{N}_i^*$  and each element of  $B$  would appear in  $q$  sets  $B_i$ . To prove this, consider the union of our matroids,  $\hat{\mathcal{N}}^* := \hat{\mathcal{N}}_1^* \vee \hat{\mathcal{N}}_2^* \vee \dots \vee \hat{\mathcal{N}}_m^*$ . By the matroid union theorem ([28, (Corollary 42.1a)]), this matroid has rank function

$$r_{\hat{\mathcal{N}}^*}(\hat{B}) = \min_{T \subseteq \hat{B}} \left( |\hat{B} \setminus T| + \sum_{i=1}^m r_{\hat{\mathcal{N}}_i^*}(T) \right).$$

We claim that for any  $T \subseteq \hat{B}$ ,

$$\sum_{i=1}^m r_{\hat{\mathcal{N}}_i^*}(T) = \sum_{i=1}^m r_{\mathcal{M}/(B \setminus \pi(T))}(A_i) \geq q \cdot r_{\mathcal{M}/(B \setminus \pi(T))}(A) = q|\pi(T)|.$$



The first equality follows from our rank formula (1). The inequality follows from Lemma 1 applied to the submodular function  $r_{\mathcal{M}/(B \setminus \pi(T))}$ , with  $S = \emptyset$  and  $C = A$ . The last equality holds because both  $A$  and  $B$  are bases of  $\mathcal{M}$  and the rank of the matroid  $\mathcal{M}/(B \setminus \pi(T))$  is  $|\pi(T)|$ . We also have  $|T| \leq q|\pi(T)|$ , hence  $\sum_{i=1}^m r_{\hat{\mathcal{N}}_i^*}(T) \geq q|\pi(T)| \geq |T|$  for any  $T \subseteq \hat{B}$ . Therefore the rank of  $\hat{\mathcal{N}}^*$  is

$$r_{\hat{\mathcal{N}}^*}(\hat{B}) = \min_{T \subseteq \hat{B}} \left( |\hat{B} \setminus T| + \sum_{i=1}^m r_{\hat{\mathcal{N}}_i^*}(T) \right) = |\hat{B}|.$$

This means that  $\hat{B}$  can be partitioned into sets  $B'_1, \dots, B'_m$ , where  $B'_i$  is independent in  $\hat{\mathcal{N}}_i^*$ . However, the ranks of  $\hat{B}$  in the  $\hat{\mathcal{N}}_i^*$  sum up to  $\sum_{i=1}^m r_{\hat{\mathcal{N}}_i^*}(\hat{B}) = \sum_{i=1}^m |A_i| = |\hat{B}|$ , so this implies that each  $B'_i$  is a base of  $\hat{\mathcal{N}}_i^*$ . Then, each  $B_i = \pi(B'_i)$  is a base of  $\mathcal{N}_i^*$ , and these are the sets demanded by the lemma.  $\square$

Finally, we give a version of Lemma 8 where the two sets need not be bases.

**Lemma 9.** *Let  $\mathcal{M} = (N, \mathcal{I})$  be a matroid and  $I, J \in \mathcal{I}$ . Let  $I_1, \dots, I_m$  be subsets of  $I$  such that each element of  $I$  appears in at most  $q$  of them. Then there are sets  $J_1, \dots, J_m \subseteq J$  such that each element of  $J$  appears in at most  $q$  of them, and for each  $i$ ,  $I_i \cup (J \setminus J_i) \in \mathcal{I}$ .*

*Proof.* We reduce this statement to Lemma 8. Let  $A, B$  be bases such that  $I \subseteq A$  and  $J \subseteq B$ . Let  $q_e$  be the number of appearances of an element  $e \in I$  in the subsets  $I_1, \dots, I_m$  and let  $q' = \max_{e \in I} q_e$ . Obviously,  $q' \leq q$ . We extend  $I_i$  arbitrarily to  $A_i$ ,  $I_i \subseteq A_i \subseteq A$ , so that each element of  $A$  appears in exactly  $q'$  of them. By Lemma 8, there are sets  $B_i \subseteq B$  such that each element of  $B$  appears in exactly  $q'$  of them, and  $A_i \cup (B \setminus B_i) \in \mathcal{I}$  for each  $i$ . We define  $J_i = J \cap B_i$ . Then, each element of  $J$  appears in at most  $q' \leq q$  sets  $J_i$ , and

$$I_i \cup (J \setminus J_i) \subseteq A_i \cup (B \setminus B_i) \in \mathcal{I}. \quad \square$$

### 3 Local-Search Algorithm

At each iteration of our local-search algorithm, given a current feasible solution  $S \in \cap_{j=1}^k \mathcal{I}_j$ , our algorithm seeks an improved solution by looking at a polynomial number of options to change  $S$ . If the algorithm finds a better solution, it moves to the next iteration, otherwise the algorithm stops. Specifically, given a current solution  $S \in \cap_{j=1}^k \mathcal{I}_j$ , the local moves that we consider are:

**$p$ -exchange Operation:** If there is  $S' \subseteq N$  and  $S' \in \cap_{j=1}^k \mathcal{I}_j$  such that **(i)**  $|S' \setminus S| \leq p$ ,  $|S \setminus S'| \leq kp$ , and **(ii)**  $f(S') > f(S)$ , then  $S \leftarrow S'$ .

The  $p$ -exchange operation for  $S' \subseteq S$  is called a *delete operation*. Our main result is the following lower bound on the value of the locally-optimal solution.

**Lemma 10.** *For every  $k \geq 2$  and every  $C \in \cap_{j=1}^k \mathcal{I}_j$ , a locally-optimal solution  $S$  under  $p$ -exchanges, satisfies*

$$(k + 1/p) \cdot f(S) \geq f(S \cup C) + (k - 1 + 1/p) \cdot f(S \cap C).$$

*Proof.* Our proof is based on the new exchange properties of matroids: Lemmas 7 and 9. By applying Lemma 7 to the independent sets  $C$  and  $S$  in matroids  $\mathcal{M}_1$  and  $\mathcal{M}_2$ , we obtain a collection of dipaths/dicycles  $\{A_1, \dots, A_m\}$  (possibly with repetition), feasible in  $D_{\mathcal{M}_1, \mathcal{M}_2}(S)$ , using only elements of  $C \Delta S$ , so that each element of  $C \Delta S$  appears in exactly  $2^s$  paths/cycles  $A_i$ .

We would like to define the sets of vertices corresponding to the exchanges in our local-search algorithm, based on the sets of vertices in paths/cycles  $\{A_1, \dots, A_m\}$ . The problem is that these paths/cycles can be much longer than the maximal cardinality of a set allowable in a  $p$ -exchange operation. To handle this, we index vertices of the set of  $C \setminus S$  in each path/cycle  $A_i$  for  $i = 1, \dots, m$ , in such a way that vertices along any path or cycle are numbered consecutively. The vertices of  $S \setminus C$  remain unlabeled. Because one vertex appears in  $2^s$  paths/cycles, it might get different labels corresponding to different appearances of that vertex. So one vertex could have up to  $2^s$  different labels.

We also define  $p + 1$  copies of the index sets  $\{A_1, \dots, A_m\}$ . For each copy  $q = 0, \dots, p$  of labeled  $\{A_1, \dots, A_m\}$ , we throw away appearances of vertices from  $C \setminus S$  that were labeled by  $q$  modulo  $p + 1$  from each  $A_i$ . By throwing away some appearances of the vertices, we are changing our set of paths in each copy of the original sets  $\{A_1, \dots, A_m\}$ . Let  $\{A_{q1}, \dots, A_{qm_q}\}$  be the resulting collection of paths for  $q = 0, \dots, p$ . Now each path  $A_{qi}$  contains at most  $p$  vertices from  $C \setminus S$  and at most  $p + 1$  vertices from  $S \setminus C$ .

Because our original collection of paths/cycles was feasible in  $D_{\mathcal{M}_1, \mathcal{M}_2}(S)$  (see definition 2), each of the paths in the new collections correspond to feasible exchanges for matroids  $\mathcal{M}_1$  and  $\mathcal{M}_2$ , i.e.  $S \Delta V(A_{qi}) \in \mathcal{I}_1 \cap \mathcal{I}_2$ . Consider now the collection of paths  $\{A_{qi} | q = 0, \dots, p, i = 1, \dots, m_q\}$ . By construction, each element of the set  $S \setminus C$  appears in exactly  $(p + 1)2^s$  paths, and each element of  $C \setminus S$  appears in exactly  $p2^s$  paths, because each vertex has  $2^s(p + 1)$  appearances in total, and each appearance is thrown away in exactly one out of  $p + 1$  copies of the original sets  $\{A_1, \dots, A_m\}$ . Let  $L_{qi} = S \cap V(A_{qi})$  denote the set of vertices in the path  $A_{qi}$  belonging to the locally-optimal solution  $S$ , and let  $W_{qi} = C \cap V(A_{qi})$  denote the set of vertices in the path  $A_{qi}$  belonging to the set  $C$ .

For each matroid  $\mathcal{M}_i$  for  $i = 3, \dots, k$ , independent sets  $S \in \mathcal{I}_i$  and  $C \in \mathcal{I}_i$ , and collection of sets  $\{W_{qi} | q = 0, \dots, p; i = 1, \dots, m_q\}$  (note that some of these sets might be empty), we apply Lemma 9. For convenience, we re-index the collection of sets  $\{W_{qi} | q = 0, \dots, p, i = 1, \dots, m_q\}$ . Let  $W_1, \dots, W_t$  be that collection, after re-indexing, for  $t = \sum_{q=0}^p m_q$ . By Lemma 9, for each  $i = 3, \dots, k$  there exist a collection of sets  $X'_{1i}, \dots, X'_{ti}$  such that  $W_j \cup (S \setminus X'_{ji}) \in \mathcal{I}_i$ . Moreover, each element of  $S$  appears in at most  $p2^s$  of the sets from collection  $X'_{1i}, \dots, X'_{ti}$ .

We consider the set of  $p$ -exchanges that correspond to adding the elements of the set  $W_j$  to the set  $S$  and removing the set of elements  $A_j = L_j \cup (\cup_{i=3}^k X'_{ji})$  for  $j = 1, \dots, t$ . Note that,  $|A_j| \leq (p + 1) + (k - 2)p = (k - 1)p + 1 \leq kp$ . By Lemmas 7 and 9, the sets  $W_j \cup (S \setminus A_j)$  are independent in each of the matroids  $\mathcal{M}_1, \dots, \mathcal{M}_k$ . By the fact that  $S$  is a locally-optimal solution, we have

$$f(S) \geq f((S \setminus A_j) \cup W_j), \quad \forall j = 1, \dots, t. \tag{2}$$

Using inequalities (2) together with submodularity for  $j = 1, \dots, t$ , we have

$$f(S \cup W_j) - f(S) \leq f((S \setminus A_j) \cup W_j) - f(S \setminus A_j) \leq f(S) - f(S \setminus A_j). \quad (3)$$

Moreover, we know that each element of the set  $C \setminus S$  appears in exactly  $p2^s$  sets  $W_j$ , and each element  $e \in S \setminus C$  appears in  $n_e \leq (p + 1)2^s + (k - 2)p2^s$  sets  $A_j$ .

Consider the sum of  $t$  inequalities (3), and add  $(p + 1)2^s + (k - 2)p2^s - n_e$  inequalities

$$f(S) \geq f(S \setminus \{e\}) \quad (4)$$

for each element  $e \in S \setminus C$ . These inequalities correspond to the delete operations. We obtain

$$\begin{aligned} \sum_{j=1}^t [f(S \cup W_j) - f(S)] &\leq \sum_{j=1}^t [f(S) - f(S \setminus A_j)] + \\ \sum_{e \in S \setminus C} ((p + 1)2^s + (k - 2)p2^s - n_e) [f(S \setminus \{e\}) - f(S)]. \end{aligned} \quad (5)$$

Applying Lemma 2 to the right-hand side of the inequality (5) and Lemma 1 to the left-hand side of the inequality (5), we have

$$p2^s [f(S \cup C) - f(S)] \leq ((p + 1)2^s + (k - 2)p2^s) [f(S) - f(S \cap C)],$$

which is equivalent to

$$(k + 1/p) \cdot f(S) \geq f(S \cup C) + (k - 1 + 1/p) \cdot f(S \cap C).$$

The result follows. □

Simple consequences of Lemma 10 are bounds on the value of a locally-optimal solution when the submodular function  $f$  has additional structure.

**Corollary 1.** *For  $k \geq 2$ , a locally-optimal solution  $S$ , and any  $C \in \cap_{j=1}^k \mathcal{I}_j$ , the following inequalities hold:*

1.  $f(S) \geq f(C)/(k + 1/p)$  if function  $f$  is monotone,
2.  $f(S) \geq f(C)/(k - 1 + 1/p)$  if function  $f$  is linear.

The local-search algorithm defined at the beginning of this section could run for an exponential amount of time before reaching a locally-optimal solution. To ensure polynomial runtime, we follow the standard approach of approximate local search under a suitable (small) parameter  $\varepsilon > 0$  as described in Figure 1. The following is a simple extension of Lemma 10.

**Lemma 11.** *For an approximate locally-optimal solution  $S$  and any  $C \in \cap_{j=1}^k \mathcal{I}_j$ ,*

$$(1 + \varepsilon)(k + 1/p) \cdot f(S) \geq f(S \cup C) + (k - 1 + 1/p) \cdot f(S \cap C),$$

where  $\varepsilon > 0$  is the parameter used in the procedure of Figure 1.

**Input:** Finite ground set  $N := [n]$ , value-oracle access to submodular function  $f : 2^N \rightarrow \mathbb{R}$ , and matroids  $\mathcal{M} = (N, \mathcal{I}_i)$ , for  $i \in [k]$ .

1. Set  $v \leftarrow \arg \max\{f(u) \mid u \in N\}$  and  $S \leftarrow \{v\}$ .
2. While the following local operation is possible, update  $S$  accordingly:  

**$p$ -exchange operation.** If there is a feasible  $S'$  such that

  - (i)  $|S' \setminus S| \leq p$ ,  $|S \setminus S'| \leq kp$ , and
  - (ii)  $f(S') \geq (1 + \varepsilon/n^4)f(S)$ ,

then  $S \leftarrow S'$ .

**Output:**  $S$ .

**Fig. 1.** The approximate local-search procedure

*Proof.* The proof of this lemma is almost identical to the proof of the Lemma 10 — the only difference is that left-hand sides of inequalities (2) and inequalities (4) are multiplied by  $1 + \varepsilon/n^4$ . Therefore, after following the steps in the proof of Lemma 10, we obtain the inequality:

$$(k + 1/p + \varepsilon\lambda/n^4p2^s) \cdot f(S) \geq f(S \cup C) + (k - 1 + 1/p) \cdot f(S \cap C),$$

where  $\lambda = t + \sum_{e \in S \setminus C} [(p + 1)2^s + (k - 2)p2^s - n_e]$  is the total number of inequalities (2) and (4). because  $t \leq |C|p2^s$  we obtain that  $\lambda \leq (n + k)p2^s$ . Assuming that  $n^4 \gg n + k$ , we obtain the result.  $\square$

Lemma 11 implies the following:

**Theorem 1.** *For any fixed  $k \geq 2$  and fixed constant  $\delta > 0$ , there exists a polynomial  $1/(k + \delta)$ -approximation algorithm for maximizing a non-negative non-decreasing submodular function subject to  $k$  matroid constraints. This bound improves to  $1/(k - 1 + \delta)$  for linear functions.*

*Remark 2.* Combining techniques from this paper with the iterative local-search from [21], we can improve the performance guarantees of the approximation algorithms for maximizing a general (non-monotone) submodular function subject to  $k \geq 2$  matroid constraints from  $k + 2 + \frac{1}{k} + \delta$  to  $k + 1 + \frac{1}{k-1} + \delta$  for any  $\delta > 0$ .

## 4 Tightness of Analysis

Next, we demonstrate that our analysis of local search for maximizing monotone submodular functions is tight. By local search, we mean for fixed  $p > 0$ , adding  $\leq p$  elements and removing  $\geq kp$  elements at a time. It was known [2] that such an algorithm cannot give better than  $1/(k - 1 + 1/p)$ -approximation for the weighted  $k$ -set packing problem ( $k \geq 3$ ). From the example of [2], the same bound follows also for weighted  $k$ -dimensional matching and hence also for the more general problems of maximizing a linear function subject to  $k$  matroid constraints, or a monotone submodular function subject to  $k - 1$  matroid constraints.

**Proposition 1.** *For any  $k, p \geq 2$ , there are instances of maximizing a linear function subject to  $k$  partition matroids, where a local optimum with respect to  $p$ -exchanges has value  $OPT/(k - 1 + 1/p)$ .*

*Proof.* Let  $G = (V, E)$  be a  $k$ -regular bipartite graph of girth at least  $2p + 2$  (see [20] for a much stronger result), with bipartition  $V = A \cup B$ . We define vertex weights  $w_i = 1$  for  $i \in A$  and  $w_j = k - 1 + 1/p$  for  $j \in B$ . Being a  $k$ -regular bipartite graph,  $G$  can be decomposed into  $k$  matchings,  $E = M_1 \cup M_2 \cup \dots \cup M_k$ . For each  $M_i$ , we define a partition matroid  $\mathcal{M}_i = (V, \mathcal{I}_i)$  where  $S \in \mathcal{I}_i$  iff  $S$  contains at most one vertex from each edge in  $M_i$ . We maximize  $w(S)$  over  $S \in \bigcap_{i=1}^k \mathcal{I}_i$ . Equivalently, we seek a maximum-weight independent set in  $G$ .

Clearly,  $A$  and  $B$  are both feasible solutions. Because  $|A| = |B|$ , we have  $w(A)/w(B) = 1/(k - 1 + 1/p)$ . We claim that  $A$  is a local optimum. Consider any set obtained by a local move,  $A' = (A \setminus K) \cup L$  where  $K \subseteq A$ ,  $L \subseteq B$  and  $|L| \leq p$ . For  $A'$  to be independent,  $K \cup L$  must contain all edges incident with  $L$ . (Otherwise, there is an edge contained in  $A'$ .) Also,  $K \cup L$  cannot contain any cycle, because every cycle in  $G$  has at least  $p + 1$  vertices on each side. Therefore,  $K \cup L$  induces a forest with  $k|L|$  edges. Hence  $|K \cup L| \geq k|L| + 1$ , i.e.  $|K| \geq (k - 1)|L| + 1$ . The value of  $A'$  is

$$w(A') = w(A) - |K| + (k - 1 + 1/p)|L| \leq w(A) - |K| + (k - 1)|L| + 1 \leq w(A). \quad \square$$

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